Time-Reversal Symmetries in Reversible Elementary Square and Triangular Partitioned Cellular Automata, and Their Data

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Abstract

Time-reversal symmetry (T-symmetry) in a reversible cellular automaton (CA) is the property in which forward and backward evolutions of configurations are governed by the same local transition function. We show that the framework of partitioned cellular automata (PCAs) is useful to study T-symmetries of reversible CAs. Here, we investigate reversible elementary square PCAs (ESPCAs) and reversible elementary triangular PCAs (ETPCAs), and prove that a large number of reversible ESPCAs and all reversible ETPCAs are T-symmetric under some kinds of simple transformations on configurations. As applications, these results are used to find and analyse backward evolution processes in reversible PCAs. For example, for a given functional module implemented in a reversible PCA, such as a reversible logic element, we can obtain its inverse functional module very easily using its T-symmetry.

1 Introduction

The notion of time-reversal symmetry (T-symmetry, for short) comes from physics. It is the symmetry of an evolution law of a dynamical system under the transformation of reversal of time (see, e.g., a survey paper [5]). For example, in the classical mechanics, its law for the negative time direction is exactly the same as the one for the positive time direction. Assume a classical mechanical system starts to evolve from a given initial state. At some time, if we transform the momentum vector \( p \) of every particle to \(-p\) simultaneously, the whole evolving process is exactly traced back. Namely, it goes back to the initial state by the same evolution law.

There are various kinds of dynamical systems having such a property. A cellular automaton (CA) is a discrete dynamical system, in which configurations (i.e., whole states of the cellular space) evolve by applying a local transition function to all the cells in parallel. A reversible CA is one whose evolution process can be traced back uniquely (but not necessarily by the same local function). In [2, 4, 6, 17], it is argued that some kinds of reversible CAs have T-symmetry, i.e., the backward transition is performed by the same local function. For example, the ‘block CA’ of Margolus is known to be T-symmetric [2, 6]. In fact, by applying a simple transformation to a configuration, the block CA evolves to the reverse direction by the same local function. On the other hand, it is also known that there are reversible CAs that are not T-symmetric [2].

Here, we pose the question: Which reversible CAs are T-symmetric? We study this problem using the framework of two-dimensional reversible partitioned cellular automata (PCAs). A PCA was introduced as a special subclass of CAs for making it easy to design a reversible CA [15]. Each cell of a PCA is divided into several parts, whose number is equal to the neighbourhood size. The next state of a cell is determined by the present states of the corresponding parts of the neighbour cells, not by the states of the whole neighbour cells. It has been shown that in a PCA injectivity of a local function is equivalent to injectivity of the global function that determines evolutions of configurations [15, 8] (see Lemmas 2.4 and 5.4). By this property, we can obtain a reversible PCA very easily by designing a PCA so that its local function is injective.

The framework of reversible PCAs also has an advantage for studying T-symmetry. Each part of a cell can be regarded as an output port to the corresponding neighbour cell, and thus its state is interpreted as a signal moving to the neighbour cell. Therefore, reversing the moving directions of signals, which corresponds to changing the momentum vector of each particle from \( p \) to \(-p\) in the classical mechanics, is easily performed by a simple transformation on configurations (defined by \( H^{rev} \) in Sections 3.1 and 5.3). By this, T-symmetries for reversible PCAs are naturally defined.

In this paper, we investigate T-symmetries of reversible four-neighbour elementary square partitioned CAs (ESPCAs), which are rotation-symmetric and each part of a cell has only two states. We also extend the results on T-symmetries of reversible three-neighbour elementary triangular partitioned CAs (ETPCAs) given in [13]. Here, we define two sorts of T-symmetries for these reversible PCAs. The first one is strict T-symmetry. If a PCA is strictly T-symmetric, then its backward evolution of configurations is governed by exactly the same local function for the forward evolution. The

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second one is a weaker version of T-symmetry, where the backward evolution of configuration is governed by a local function that is ‘similar’ to the forward local function. As ‘similar’ ones, the local functions obtained by taking a mirror image, taking 0-1 complementation, and taking both of these operations to the forward local function are used. Hence, we consider three kinds of weaker T-symmetries here.

In the following, we show that a large number of reversible ESPCAs and all reversible ETPCAs are strictly or weakly T-symmetric in the above sense. These results are useful for finding or analysing their backward evolution processes. In particular, it makes it easy to design an ‘inverse functional module’ that undoes the forward function of a given module. We give several examples of applications of T-symmetries.

In Section 2, definitions on ESPCAs are given. In Section 3, T-symmetries in reversible ESPCAs are defined and their properties are studied. In Section 4, several applications of T-symmetries in ESPCAs are shown. In Section 5, ETPCAs are defined, and their T-symmetries are clarified. In Section 6, applications of T-symmetries of ETPCAs are shown. Section 7 gives concluding remarks and open problems. In Appendix, all 1536 reversible ESPCAs are listed, and the data on their T-symmetries are given.

2 Elementary Square Partitioned Cellular Automata (ESPCAs)

In this section, we give basic definitions on a four-neighbour square partitioned cellular automaton (SPCA). Figure 1 (a) is the cellular space of SPCA. Its square cell is divided into four parts. In SPCA, a cell changes its state depending on their T-symmetries are given.

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We give several examples of applications of T-symmetries.

2.1 Definitions on ESPCAs

Definition 2.1 A four-neighbour square partitioned cellular automaton (SPCA) is a system defined by

\[ P = (\mathbb{Z}^2, (T, R, B, L), ((0, -1), (-1, 0), (0, 1), (1, 0)), f). \]

Here, \( \mathbb{Z}^2 \) is the set of all points with integer coordinates where cells are placed. The items \( T, R, B, \) and \( L \) are non-empty finite sets of states of the top, right, bottom, and left parts of a cell. The set of states of a cell is thus \( Q = T \times R \times B \times L \). The quadruple \( ((0, -1), (-1, 0), (0, 1), (1, 0)) \) is a neighbourhood of each cell, and \( f : Q \rightarrow Q \) is a local (transition) function.

If \( f(t, r, b, l) = (t', r', b', l') \) holds for \( (t, r, b, l), (t', r', b', l') \in Q \), this relation is called a local transition rule of \( P \). It is also indicated as in Figure 1 (b). The local function \( f \) is thus defined by a set of local transition rules.

Definition 2.2 Let \( P = (\mathbb{Z}^2, (T, R, B, L), ((0, -1), (-1, 0), (0, 1), (1, 0)), f) \) be a four-neighbour SPCA. A configuration of \( P \) is a function \( \alpha : \mathbb{Z}^2 \rightarrow Q \). The set of all configurations of \( P \) is denoted by \( \text{Conf}(P) \), i.e., \( \text{Conf}(P) = \{ \alpha | \alpha : \mathbb{Z}^2 \rightarrow Q \} \).

Let \( \text{pr}_T : Q \rightarrow T \) be the projection function that satisfies \( \text{pr}_T(t, r, b, l) = t \) for all \( (t, r, b, l) \in Q \). The projection functions \( \text{pr}_R : Q \rightarrow R, \text{pr}_B : Q \rightarrow B \) and \( \text{pr}_L : Q \rightarrow L \) are defined similarly. The global function \( F : \text{Conf}(P) \rightarrow \text{Conf}(P) \) of \( P \) is defined as the one that satisfies the following.

\[ \forall \alpha \in \text{Conf}(P), \forall x \in \mathbb{Z}^2 : F(\alpha)(x) = f(\text{pr}_T(\alpha(x + (0, -1))), \text{pr}_R(\alpha(x + (-1, 0))), \text{pr}_B(\alpha(x + (0, 1))), \text{pr}_L(\alpha(x + (1, 0)))) \]
In this paper, reversibility of an SPCA is defined as follows. Note that a detailed discussion on the definition of a reversible CA is found in Section 10.3 of [8].

**Definition 2.3** An SPCA \( P \) is called reversible if its global function is injective.

The next Lemma shows that, in a PCA, injectivity of the global function is equivalent to injectivity of the local function [8, 15]. By this, we can easily obtain a reversible CA by giving a PCA whose local function is injective.

**Lemma 2.4** Let \( P \) be an SPCA. Its global function \( F \) is injective if and only if its local function \( f \) is injective.

Next, we define a subclass of SPCAs such that its local function is rotation-symmetric, and each of four parts has only two states. It is called an elementary SPCA (ESPCA) as in the case of a one-dimensional elementary cellular automaton (ECA) [20]. We first define the notion of rotation-symmetry.

**Definition 2.5** Let \( P = (\mathbb{Z}^2, (T, R, B, L)), ((0, -1), (-1, 0), (0, 1), (1, 0)), f) \) be an SPCA. The SPCA \( P \) is called rotation-symmetric (or isotropic) if the following conditions (1) and (2) hold.

1. \( T = R = B = L \)
2. \( \forall (r, t, b, l), (t', r', b', l') \in T \times R \times B \times L : \\
f((r, t, b, l)) = f((t', r', b', l') = (r', t', b', l')

**Definition 2.6** Let \( P = (\mathbb{Z}^2, (T, R, B, L)), ((0, -1), (-1, 0), (0, 1), (1, 0)), f) \) be an SPCA. We say \( P \) is an elementary triangular partitioned cellular automaton (ESPCA), if \( T = R = B = L = \{0, 1\} \), and it is rotation-symmetric.

Since an ESPCA is rotation-symmetric, its local function \( f : \{0, 1\}^4 \rightarrow \{0, 1\}^4 \) is defined by only six local transition rules that are described by the following six values.

\[
f(0,0,0,0), f(0,0,1,0), f(0,0,1,1), f(1,0,1,0), f(1,0,1,1), f(1,1,1,1)
\]

Here, \( f(0,0,1,0), f(0,0,1,1), f(0,1,1,1) \in \{0, 1\}^4 \). On the other hand, \( f(1,0,1,0) \in \{(0,0,0,0), (0,1,0,1), (1,0,1,0), (1,1,1,1)\} \) and \( f(0,0,0,0), f(1,1,1,1) \in \{(0,0,0,0), (1,1,1,1)\} \), since it is rotation-symmetric. Hence, there are \( 16^3 \times 2^2 = 65,536 \) ESPCAs in total.

Reading the 4-bit values of \( f(0,0,0,0), f(0,0,1,0), f(0,0,1,1), f(1,0,1,0), f(1,0,1,1), f(1,1,1,1) \) as six binary numbers, we express an ESPCA by a 6-digit hexadecimal identification (ID) number \( u\,w\,x\,y\,z\,f \) as in Figure 2. An ESPCA with the ID number \( u\,w\,x\,y\,z\,f \) is denoted by ESPCA-\( u\,w\,x\,y\,z\,f \). Its local and global functions are denoted by \( f_{u\,w\,x\,y\,z} \) and \( F_{u\,w\,x\,y\,z} \), respectively. For example, Figure 3 shows the six local transition rules of ESPCA-01c57f, which define \( f_{01c57f} \).

**Figure 2:** Expressing an ESPCA by a 6-digit hexadecimal ID number \( u\,w\,x\,y\,z\,f \). States 0 and 1 are represented by a blank and ●. Vertical bars indicate alternatives of the right-hand side of each local transition rule

**Figure 3:** Local function \( f_{01c57f} \) of ESPCA-01c57f defined by the six local transition rules
Definition 2.7 An ESPCA \( P \) is called conservative if the number of state 1’s (i.e., particles) is conserved in each of its local transition rules.

Conservativeness of an ESPCA is an analog of various conservation laws in physics such as conservation of mass, energy, and so on. From Definitions 2.3 and 2.7, it is easy to see the following proposition.

Proposition 2.8 Let \( P \) be an ESPCA with an ID number \(uvwxyz\).

1. \( P \) is reversible if and only if the following condition holds.
\[
(u, z) \in \{(0, 0), (1, 0)\} \land x \in \{5, a\} \land (v, w, y) \in \{(A \times A \times B \cup A \times C \times B \cup B \times A \times C \cup B \times C \times A \cup C \times A \times B \cup C \times B \times A)\},
\]
where \( A = \{1, 2, 4, 8\}, B = \{3, 6, 9, c\}, C = \{7, b, d, e\} \)

2. \( P \) is conservative if and only if the following condition holds.
\[
u = 0 \land v \in \{1, 2, 4, 8\} \land w \in \{3, 5, 6, 9, a, c\} \land x \in \{5, a\} \land y \in \{7, b, d, e\} \land z = f
\]

3. \( P \) is reversible and conservative if and only if the following condition holds.
\[
u = 0 \land v \in \{1, 2, 4, 8\} \land w \in \{3, 6, 9, c\} \land x \in \{5, a\} \land y \in \{7, b, d, e\} \land z = f
\]

From the above proposition, we can see the total numbers of reversible, conservative, and reversible and conservative ESPCAs are 1536, 192, and 128, respectively.

2.2 Dualities in ESPCA

We consider two kinds of dualities among ESPCAs, which are the ones under reflection and complementation. These notions are given in [19] for one-dimensional elementary cellular automata (ECAs). The dual ESPCAs are essentially the same as the original one in the sense that any evolution process is simulated in the dual ESPCA after taking a simple transformation to the initial configuration.

Definition 2.9 Let \( P \) be an ESPCA and \( f : \{0, 1\}^4 \rightarrow \{0, 1\}^4 \) be its local function. Define \( f^r : \{0, 1\}^4 \rightarrow \{0, 1\}^4 \) as follows.
\[
\forall (t, r, d, l), (t', r', d', l') \in \{0, 1\}^4 : \quad f(t, r, d, l) = (t', r', d', l') \Leftrightarrow f^r(t, r, d, l) = (t', l', d', r')
\]
Then, the ESPCA \( P^r \) having the local function \( f^r \) is called the dual ESPCA of \( P \) under reflection.

From this definition, we can see that the local transition rules of \( P^r \) are the mirror images of those of \( P \). It means that any evolution process in \( P \) is simulated in \( P^r \) in a straightforward manner by taking the mirror image of the initial configuration (see Lemma 3.5). Note that, in the above definition, the mirror images are taken with respect to the vertical axis (i.e., \( r \) and \( l \) and \( r' \) and \( l' \) are exchanged). However, since ESPCA \( P \) is rotation-symmetric (Definition 2.5), it is equivalent to the case where the mirror images are taken with respect to the horizontal axis.

Definition 2.10 Let \( P \) be an ESPCA and \( f : \{0, 1\}^4 \rightarrow \{0, 1\}^4 \) be its local function. For \( x \in \{0, 1\} \), let \( \overline{x} = 1 - x \), i.e., \( \overline{x} \) is the complement of \( x \). Define \( f^c : \{0, 1\}^4 \rightarrow \{0, 1\}^4 \) as follows.
\[
\forall (t, r, d, l), (t', r', d', l') \in \{0, 1\}^4 : \quad f(t, r, d, l) = (t', r', d', l') \Leftrightarrow f^c(t, r, d, l) = (t', \overline{r}, \overline{d}, \overline{l})
\]
Then, the ESPCA \( P^c \) having the local function \( f^c \) is called the dual ESPCA of \( P \) under complementation.

From this definition, we can see that the local transition rules of \( P^c \) are obtained from those of \( P \) by exchanging 0 and 1. Therefore, any evolution process in \( P \) is simulated in \( P^c \) in a straightforward manner by taking the complement of the initial configuration (see Lemma 3.7). For an ESPCA \( P \) with a local function \( f \), there is an ESPCA \( P^c \) whose local function is \( (f^c)^r = (f^r)^c \). It can also be regarded as a kind of a dual ESPCA. We write the local function of \( P^c \) by \( f^c \) shortly.

Let \( f_{\text{owxyz}} \) be a reversible ESPCA. We denote the ID numbers of \( f^c_{\text{owxyz}}, f_{\text{owxyz}}^c, f_{\text{owxyz}}^r, f_{\text{owxyz}}^l \), and \( f_{\text{owxyz}}^{-1} \) by \( (\text{owxyz}), \text{c(owxyz)}, \text{rc(owxyz)}, \text{inv(owxyz)}, \text{rc(inv(owxyz))}) \), respectively. Namely, \( f_{\text{owxyz}} = f_{\text{rc(owxyz)}}, f_{\text{owxyz}}^c = f_{\text{rc(owxyz)}}^r, f_{\text{owxyz}}^c = f_{\text{inv(owxyz)}}, f_{\text{owxyz}}^r = f_{\text{inv(owxyz)}}^c \), and \( f_{\text{owxyz}}^{-1} = f_{\text{inv(owxyz)}} \).

Table 1 shows the list of ID numbers of local functions (\( f \)) of 128 reversible and conservative ESPCAs, their dual ones (\( f^c, f^r \) and \( f^c \)), and their inverses (\( f^{-1} \)). We included inverse local functions besides dual ones in the table, since they will be used in Section 3. For example, if we consider ESPCA-01357f, then \( f_{01357f}^c = f_{01357f}^r = f_{01357f}^{-1} = f_{01357f} \).

Table 1 shows that, since the total number of all reversible ESPCAs is 1536, their complete list is given in Appendix A.
Table 1: Identification numbers of 128 reversible and conservative ESPCAs, their dual ones (under reflection, complementation, and both) and inverses. In each ESPCA, the IDs of local functions among $f, f^2, f^3$, and $f^4$ that are equal to $f^{-1}$ are marked by *. It means that the ESPCA is T-symmetric under the corresponding involutions (see Section 3)

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3 Time-Reversal Symmetries in Reversible ESPCAs

In this section we define two sorts of time-reversal symmetries (T-symmetries) in reversible ESPCAs. The first one is a stronger version of T-symmetry where the backward evolution is governed by exactly the same law as the forward one. The second is a weaker version of T-symmetry where the backward evolution is governed by a ‘similar’ law. As we shall see in Section 4, the both versions of T-symmetries are useful for finding or analysing the backward evolution process for a given evolution process.

3.1 Basic property of reversible ESPCAs

Before defining T-symmetries on ESPCAs, we first show a basic property on their backward evolution. Let \( P \) be a reversible ESPCA-uvwxyz with the local function \( f \). Then \( P \) can be interpreted as the one that reverses the moving directions of all the particles in the cellular space. In the classical mechanics, the operation \( H \in L(\mathbb{R}^p) \) corresponds to the transformation of the momentum vector \( p \) of each particle to \( -p \). It is easy to verify that \( H^{-1} \) is an involution.

![Figure 4: Local function \( f_{01357f} \), and its duals and inverse](image)

The following lemma shows that a backward evolution of reversible ESPCA-uvwxyz is performed by \( F_{\text{inv}(uvwxyz)} \) applying \( H \) just before and after \( F_{\text{inv}(uvwxyz)} \). However, it does not mean T-symmetry of the ESPCA, since \( F_{\text{inv}(uvwxyz)} \) may be very different from \( F_{\text{inv}(uvwxyz)} \). In the special case where \( F_{\text{inv}(uvwxyz)} = F_{\text{inv}(uvwxyz)} \), we can say that the backward evolution is carried out by exactly the same global function as the forward one, which we call strict T-symmetry given in Section 3.2. This lemma is also used for defining weaker T-symmetries as discussed in Section 3.3. Note that this lemma is proved in a similar manner as in the case of reversible elementary triangular PCAs (ETPCA) [13].

**Lemma 3.1** Let \( P \) be a reversible ESPCA-uvwxyz with the local function \( f_{\text{inv}(uvwxyz)} \) and the global function \( F_{\text{inv}(uvwxyz)} \). Let \( P' \) be a reversible ESPCA with the ID number \( \text{inv}(uvwxyz) \). The local and global functions of \( P' \) are thus \( f_{\text{inv}(uvwxyz)} = F_{\text{inv}(uvwxyz)}^{-1} \) and \( F_{\text{inv}(uvwxyz)} \), respectively. Then the following holds.

\[
F_{\text{inv}(uvwxyz)}^{-1} = H^{-1} \circ F_{\text{inv}(uvwxyz)} \circ H
\]

**Proof.** Let \( \alpha \in \text{Conf}_E \) be any configuration, and \( (x_0, y_0) \in \mathbb{Z}^2 \) be any point. Let \( (t_1, r_1, b_1, l_1) \in \{0, 1\}^4 \) be as follows: \( \alpha_1(x_0, y_0) = (t_1, r_1, b_1, l_1) \). See Figure 6 that shows the process of state-changes by the operations given below. First, we can see the following relations.

\[
\begin{align*}
\text{pr}_1(H^{-1} f_{\text{inv}(uvwxyz)}(x_0, y_0 - 1)) &= b_1 \\
\text{pr}_1(H^{-1} f_{\text{inv}(uvwxyz)}(x_0 - 1, y_0)) &= l_1 \\
\text{pr}_1(H^{-1} f_{\text{inv}(uvwxyz)}(x_0, y_0 + 1)) &= t_1 \\
\text{pr}_1(H^{-1} f_{\text{inv}(uvwxyz)}(x_0 + 1, y_0)) &= r_1
\end{align*}
\]
Assume \( f_{uvwxyz}^{-1}(t_1, r_1, b_1, l_1) = (t_0, r_0, b_0, l_0) \) (i.e., \( f_{uvwxyz}(t_0, r_0, b_0, l_0) = (t_1, r_1, b_1, l_1) \)). Then, the following holds, since \( f_{uvwxyz}^{-1} \) (and thus \( f_{uvwxyz}^{-1} \)) is rotation-symmetric.

\[
(F_{\text{inv}(uvwxyz)} \circ H_{\text{rev}}(\alpha_1))(x_0, y_0) = (b_0, l_0, t_0, r_0)
\]

Let \( \alpha_0 = F_{\text{inv}(uvwxyz)} \circ H_{\text{rev}}(\alpha_1) \). Then, the following relations hold.

\[
\begin{align*}
\text{pr}_x(H_{\text{rev}}(\alpha_0)(x_0, y_0 - 1)) &= t_0 \\
\text{pr}_y(H_{\text{rev}}(\alpha_0)(x_0 - 1, y_0)) &= r_0 \\
\text{pr}_b(H_{\text{rev}}(\alpha_0)(x_0, y_0 + 1)) &= b_0 \\
\text{pr}_l(H_{\text{rev}}(\alpha_0)(x_0 + 1, y_0)) &= l_0
\end{align*}
\]

Hence,

\[
(F_{\text{uvwxyz}} \circ H_{\text{rev}}(\alpha_0))(x_0, y_0) = (t_1, r_1, b_1, l_1) = \alpha_1(x_0, y_0).
\]

By above, the following holds for all \((x_0, y_0) \in \mathbb{Z}^2\).

\[
(F_{\text{uvwxyz}} \circ H_{\text{rev}} \circ F_{\text{inv}(uvwxyz)} \circ H_{\text{rev}}(\alpha_1))(x_0, y_0) = \alpha_1(x_0, y_0)
\]

Thus, \( F_{\text{uvwxyz}} \circ H_{\text{rev}} \circ F_{\text{inv}(uvwxyz)} \circ H_{\text{rev}}(\alpha_1) = \alpha_1 \) for all \( \alpha_1 \in \text{Config} \). Therefore,

\[
F_{\text{uvwxyz}}^{-1} = H_{\text{rev}} \circ F_{\text{inv}(uvwxyz)} \circ H_{\text{rev}}.
\]

This completes the proof.

\[\Box\]

**Figure 6:** Process of the state-changes around the cell \((x_0, y_0)\) in Lemma 3.1

### 3.2 Strict T-symmetry

We now define the notion of strict T-symmetry for reversible ESPCAs. It basically follows the definition given in [2].

**Definition 3.2** Let \( P \) be a reversible ESPCA whose global function is \( F \). If \( F^{-1} = H_{\text{rev}} \circ F \circ H_{\text{rev}} \), then \( P \) is called strictly time-reversal symmetric (or strictly T-symmetric for short).

The above definition means that in a strictly T-symmetric reversible ESPCA its backward transition is carried out by exactly the same global function as the one for the forward evolution provided that the moving directions of all the particles are reversed before and after the global function is applied.

By Lemma 3.1, we have the following theorem.

**Theorem 3.3** A reversible ESPCA with the ID number \( uvwxyz \) is strictly T-symmetric, if \( \text{inv}(uvwxyz) = uvwxyz \). In this case, the following holds.

\[
F_{\text{uvwxyz}}^{-1} = H_{\text{rev}} \circ F_{\text{uvwxyz}} \circ H_{\text{rev}}
\]

From Theorem 3.3 and Table 1, we can see that 16 reversible and conservative ESPCAs are strictly T-symmetric.

**Example 1** We consider ESPCA-02c5df (Figure 7). It is reversible and conservative. Since \( \text{inv}(02c5df) = 02c5df \) as shown in Table 1, it is strictly T-symmetric by Theorem 3.3. Thus the following holds.

\[
F_{02c5df}^{-1} = H_{\text{rev}} \circ F_{02c5df} \circ H_{\text{rev}}
\]

The diagram given in Figure 8 illustrates this relation. It shows that the backward transition from a configuration \( \alpha(t) \) to \( \alpha(t - 1) \) is performed by the global function \( F_{02c5df} \) for the forward transition. Only the additional operation \( H_{\text{rev}} \), which reverses the moving direction of all particles, is needed before and after applying \( F_{02c5df} \).
3.3 T-symmetry under a general involution $H$

Next, we define a weaker version of T-symmetry by replacing the particular involution $H^{rev}$ in Definition 3.2 by an arbitrary involution $H$. Note that the notion of ‘weak’ T-symmetry in this definition is expressed by the phrase ‘under the involution $H$’.

**Definition 3.4** Let $P$ be a reversible ESPCA whose global function is $F$. If there is an involution $H : \text{Conf}_E \rightarrow \text{Conf}_E$ that satisfies $F^{-1} = H \circ F \circ H$, then $P$ is called time-reversal symmetric under the involution $H$ (or T-symmetric under $H$ for short).

We do not restrict the involution $H$ in this definition. However, in the following, we consider the case where $H$ is expressed by $H = H^{rev} \circ H' = H' \circ H^{rev}$ for some involution $H'$. In this case, the backward evolution is performed by $H' \circ F \circ H'$ applying $H^{rev}$ just before and after $H' \circ F \circ H'$. If $H'$ is a simple involution, we can say that the backward evolution is carried out by a ‘similar’ law as the forward one.

We show that ESPCA-$uvwxyz$ satisfying inv($uvwxyz$) = $r(uvwxyz)$ is T-symmetric under a certain simple involution. First, define a function $\text{refl}_I : \{0,1\}^4 \rightarrow \{0,1\}^4$ as follows: $\text{refl}_I(t,r,b,l) = (t,l,b,r)$ for any $(t,r,b,l) \in \{0,1\}^4$. Next define an involution $H^{refl} : \text{Conf}_E \rightarrow \text{Conf}_E$ as follows. For all $\alpha \in \text{Conf}_E$ and $(x_0,y_0) \in \mathbb{Z}^2$:

$$H^{refl}(\alpha)(x_0,y_0) = \text{refl}_I(\alpha(-x_0,y_0))$$

It gives the mirror image of a configuration with respect to the $y$-axis.

**Lemma 3.5** The next relation holds for any ESPCA-$uvwxyz$.

$$F_{(uvwxyz)} = H^{refl} \circ F_{(uvwxyz)} \circ H^{refl}$$

**Proof.** First, we show $F_{(uvwxyz)} = H^{refl} \circ F_{(uvwxyz)} \circ H^{refl}$. Let $\alpha \in \text{Conf}_E$ be any configuration, and $(x_0,y_0) \in \mathbb{Z}^2$ be any point. Let $(t_0,r_0,b_0,l_0) \in \{0,1\}^4$ be as follows.

- $pr_T(\alpha(x_0,y_0 - 1)) = t_0$
- $pr_R(\alpha(x_0 - 1,y_0)) = r_0$
- $pr_B(\alpha(x_0,y_0 + 1)) = b_0$
- $pr_L(\alpha(x_0 + 1,y_0)) = l_0$

See Figure 9 that shows the process of state-changes by the operations given below. In the next step, we apply $H^{refl}$, and have the following.

- $pr_T(H^{refl}(\alpha)(-x_0,y_0 - 1)) = t_0$
- $pr_R(H^{refl}(\alpha)(-x_0 - 1,y_0)) = l_0$
- $pr_B(H^{refl}(\alpha)(-x_0,y_0 + 1)) = b_0$
- $pr_L(H^{refl}(\alpha)(-x_0 + 1,y_0)) = r_0$
Here we assume \( f_{uvwxyz}(t_0, r_0, b_0, l_0) = (t_1, r_1, b_1, l_1) \). Since \( f_{uvwxyz}(t_0, l_0, b_0, r_0) = (t_1, l_1, b_1, r_1) \),
\( (F_{uvwxyz}) \circ H^{refl}(\alpha))(\neg x_0, y_0) = (t_1, l_1, b_1, r_1) \).

Finally, we have the following relation for all \( \alpha \in \text{Conf}_E \) and \((x_0, y_0)\).
\[
(H^{refl} \circ F_{uvwxyz}) \circ H^{refl}(\alpha)(x_0, y_0) = (t_1, r_1, b_1, l_1) = F_{uvwxyz}(\alpha)(x_0, y_0)
\]

Therefore, \( F_{uvwxyz} = H^{refl} \circ F_{uvwxyz} \circ H^{refl} \) holds, and thus
\[
H^{refl} \circ F_{uvwxyz} \circ H^{refl} = H^{refl} \circ H^{refl} \circ F_{uvwxyz} \circ H^{refl} \circ H^{refl} = F_{uvwxyz}.
\]

This completes the proof. \( \square \)

![Diagram](image)

Figure 9: Process of the state-changes around the cells \((x_0, y_0)\) and \((-x_0, y_0)\) in Lemma 3.5

By Lemmas 3.1 and 3.5, we have the following theorem, since it is easy to see \( H^{rev} \circ H^{refl} = H^{refl} \circ H^{rev} \).

**Theorem 3.6** A reversible ESPCA with the ID number \( uwwxyz \) is T-symmetric under the involution \( H^{rev} \circ H^{refl} \), if \( \text{inv}(uwwxyz) = \tau(uwwxyz) \). In this case, the following holds.

\[
F^{-1}_{uvwxyz} = H^{rev} \circ F_{uvwxyz} \circ H^{rev}
\]

From Theorem 3.6 and Table 1, we can see that all the 128 reversible and conservative ESPCAs are T-symmetric under the involution \( H^{rev} \circ H^{refl} \).

**Example 2** We consider ESPCA-02c5bf (Figure 10). It is reversible and conservative. Since \( \text{inv}(02c5bf) = \tau(02c5bf) \) as shown in Table 1, it is T-symmetric under the involution \( H^{rev} \circ H^{refl} \) (Theorem 3.6). Thus the following holds.

\[
F^{-1}_{02c5bf} = H^{rev} \circ F_{02c5bf} \circ H^{rev} = H^{rev} \circ H^{refl} \circ F_{02c5bf} \circ H^{refl} \circ H^{rev}
\]

The diagram given in Figure 11 illustrates this relation. Namely, the backward transition from a configuration \( \alpha(t) \) to \( \alpha(t - 1) \) is performed by the global function \( F_{02c5bf} \) for the forward transition, provided that the operation \( H^{rev} \circ H^{refl} \) is applied before and after \( F_{02c5bf} \). The diagram can also be interpreted that the backward transition is performed by the 'similar' global function \( F_{102c5bf} \), provided that \( H^{rev} \) is applied before and after \( F_{102c5bf} \). Here, 'similar' means that each local transition rule for \( F_{102c5bf} \) is a mirror image of the corresponding rule for \( F_{02c5bf} \).

![Diagram](image)

Figure 10: Local function \( f_{02c5bf} \) of ESPCA-02c5bf

Next, we show Lemma 3.7 stating that ESPCA-uvwxyz satisfying \( \text{inv}(uvwxyz) = c(uvwxyz) \) is T-symmetric under a certain simple involution. First, define a function \( \text{comp}_4 : \{0, 1\}^4 \to \{0, 1\}^4 \) as follows: \( \text{comp}_4(t, r, b, l) = (T, \tau, B, L) \) for any \( (t, r, b, l) \in \{0, 1\}^4 \). Next define an involution \( H^{comp} : \text{Conf}_E \to \text{Conf}_E \) as follows. For all \( \alpha \in \text{Conf}_E \) and \((x_0, y_0) \in \mathbb{Z}^2\):

\[
H^{comp}(\alpha)(x_0, y_0) = \text{comp}_4(\alpha(x_0, y_0))
\]

The involution \( H^{comp} \) gives the complement image of a configuration.

**Lemma 3.7** The next relation holds for any ESPCA-uvwxyz.
\[
F_{c(uvwxyz)} = H^{comp} \circ F_{uvwxyz} \circ H^{comp}
\]
Finally, we have the following relation for all \( \alpha \in \text{Conf}_E \) be any configuration, and \((x_0, y_0) \in \mathbb{Z}^2\) be any point. Let \((t_0, r_0, b_0, l_0) \in \{0, 1\}^4\) as be follows.

\[
\begin{align*}
pr_T(\alpha(x_0, y_0 - 1)) &= t_0 \\
pr_R(\alpha(x_0 - 1, y_0)) &= r_0 \\
pr_B(\alpha(x_0, y_0 + 1)) &= b_0 \\
pr_L(\alpha(x_0 + 1, y_0)) &= l_0
\end{align*}
\]

See Figure 12 that shows the process of state-changes by the operations given below. In the next step, we apply \(H^{\text{comp}}\), and have the following.

\[
\begin{align*}
pr_T(H^{\text{comp}}(\alpha)(x_0, y_0 - 1)) &= \overline{t_0} \\
pr_R(H^{\text{comp}}(\alpha)(x_0 - 1, y_0)) &= \overline{r_0} \\
pr_B(H^{\text{comp}}(\alpha)(x_0, y_0 + 1)) &= \overline{b_0} \\
pr_L(H^{\text{comp}}(\alpha)(x_0 + 1, y_0)) &= \overline{l_0}
\end{align*}
\]

Here we assume \(f_{uvwxyz}(t_0, r_0, b_0, l_0) = (t_1, r_1, b_1, l_1)\). Since \(f_c(uvwxyz)(t_0, r_0, b_0, l_0) = (\overline{t}, \overline{r}, \overline{b}, \overline{l})\),

\[
(F_c(uvwxyz) \circ H^{\text{comp}}(\alpha))(x_0, y_0) = (\overline{t}, \overline{r}, \overline{b}, \overline{l})
\]

Finally, we have the following relation for all \( \alpha \in \text{Conf}_E \) and \((x_0, y_0)\).

\[
(H^{\text{comp}} \circ f_{uvwxyz} \circ H^{\text{comp}})(\alpha)(x_0, y_0) = (t_1, r_1, b_1, l_1) = F_{uvwxyz}(\alpha)(x_0, y_0)
\]

Therefore, \(F_{uvwxyz} = H^{\text{comp}} \circ f_{uvwxyz} \circ H^{\text{comp}}\) holds, and thus

\[
H^{\text{comp}} \circ F_{uvwxyz} \circ H^{\text{comp}} = F_c(uvwxyz).
\]

This completes the proof. \(\square\)

![Figure 11: Diagram that illustrates T-symmetry of ESPCA-02c5bf under the involution \(H^{\text{rev}} \circ H^{\text{refl}}\)](image)

![Figure 12: Process of the state-changes around the cell \((x_0, y_0)\) in Lemma 3.7](image)
By Lemmas 3.1 and 3.7, we have the following theorem, since it is easy to see $H_{\text{rev}} \circ H_{\text{comp}} = H_{\text{comp}} \circ H_{\text{rev}}$.

**Theorem 3.8** A reversible ESPCA with the ID number $uvwxyz$ is $T$-symmetric under the involution $H_{\text{rev}} \circ H_{\text{comp}}$, if $\text{inv}(uvwxyz) = c(uvwxyz)$. In this case, the following holds.

$$F_{uvwxyz}^{-1} = H_{\text{rev}} \circ F_{c(uvwxyz)} \circ H_{\text{rev}} = H_{\text{rev}} \circ H_{\text{comp}} \circ F_{uvwxyz} \circ H_{\text{comp}} \circ H_{\text{rev}}$$

From Theorem 3.8 and Table 1, we can see that 16 reversible and conservative ESPCAs are $T$-symmetric under the involution $H_{\text{rev}} \circ H_{\text{comp}}$.

Combining Lemmas 3.5 and 3.7, we also obtain the next lemma.

**Lemma 3.9** The next relation holds for any ESPCA-uvwxyz.

$$F_{r(c(uvwxyz))} = H_{\text{refl}} \circ H_{\text{comp}} \circ F_{uvwxyz} \circ H_{\text{comp}} \circ H_{\text{refl}}$$

**Proof.** By Lemma 3.7, we have

$$F_{c(uvwxyz)} = H_{\text{comp}} \circ F_{uvwxyz} \circ H_{\text{comp}}.$$  

Therefore, by Lemma 3.5, we have

$$F_{r(c(uvwxyz))} = H_{\text{refl}} \circ F_{c(uvwxyz)} \circ H_{\text{refl}} = H_{\text{refl}} \circ H_{\text{comp}} \circ F_{uvwxyz} \circ H_{\text{comp}} \circ H_{\text{refl}}.$$  

Since $F_{r(c(uvwxyz))} = F_{r(c(uvwxyz))}$, the lemma holds.

By Lemmas 3.1 and 3.9, we have the following theorem, since it is easy to see $H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}} = H_{\text{comp}} \circ H_{\text{refl}} \circ H_{\text{rev}}$.

**Theorem 3.10** A reversible ESPCA with the ID number $uvwxyz$ is $T$-symmetric under the involution $H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}}$, if $\text{inv}(uvwxyz) = r(c(uvwxyz))$. In this case, the following holds.

$$F_{uvwxyz}^{-1} = H_{\text{rev}} \circ F_{r(c(uvwxyz))} \circ H_{\text{rev}} = H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}} \circ F_{uvwxyz} \circ H_{\text{comp}} \circ H_{\text{refl}} \circ H_{\text{rev}}$$

From Theorem 3.10 and Table 1, we can see that 32 reversible and conservative ESPCAs are $T$-symmetric under the involution $H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}}$.

Table 1 shows all reversible and conservative ESPCAs. We can see every reversible and conservative ESPCA is $T$-symmetric under the corresponding involution. On the other hand, there are many reversible but non-conservative ESPCAs. A complete list of all reversible ESPCAs is given in Appendix A. Since the number of such ESPCAs is large, we give here the total numbers of $T$-symmetric reversible (but may not be conservative) ESPCAs under $H_{\text{rev}}$, $H_{\text{rev}} \circ H_{\text{refl}}$, $H_{\text{rev}} \circ H_{\text{comp}}$, and $H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}}$ in Table 2. The number of non-$T$-symmetric reversible ESPCAs under these involutions is also given in this table. It is not known whether each of these 640 ESPCAs becomes $T$-symmetric under some other involution $H$.

<table>
<thead>
<tr>
<th>Types of ESPCAs</th>
<th>Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reversible ESPCAs</td>
<td>1536</td>
</tr>
<tr>
<td>T-symmetric reversible ESPCAs under $H_{\text{rev}}$ (i.e., strictly T-symmetric)</td>
<td>128</td>
</tr>
<tr>
<td>T-symmetric reversible ESPCAs under $H_{\text{rev}} \circ H_{\text{refl}}$</td>
<td>448</td>
</tr>
<tr>
<td>T-symmetric reversible ESPCAs under $H_{\text{rev}} \circ H_{\text{comp}}$</td>
<td>128</td>
</tr>
<tr>
<td>T-symmetric reversible ESPCAs under $H_{\text{rev}} \circ H_{\text{refl}} \circ H_{\text{comp}}$</td>
<td>448</td>
</tr>
<tr>
<td>Non-T-symmetric reversible ESPCAs under the above involutions</td>
<td>640</td>
</tr>
</tbody>
</table>
4 Applications of T-Symmetries in Reversible ESPCAs

A reversible ESPCA was first proposed in [16]. There, computational universality of ESPCA-02c5df and ESPCA-02c5bf was studied. It was shown that both in these ESPCAs, a switch gate and an inverse switch gate, which are reversible logic gates, are realisable in their cellular spaces [16, 8]. Since a Fredkin gate, a universal reversible logic gate, can be composed of these gates [1], these ESPCAs and their dual ones are computationally universal. It means that any reversible Turing machine can be realised in these reversible ESPCAs [8]. Although T-symmetry is not mentioned in [16, 8], construction of an inverse switch gate has been, in fact, done in a symmetric way as that of a switch gate. In Sections 4.1 and 4.2, we show that such construction of the inverse functional module is explained using T-symmetries of these ESPCAs.

In [14], ESPCA-01caef was investigated, and its computational universality was shown by implementing a reversible logic element with 1-bit memory (RLEM). In ESPCA-01caef, various kinds of space-moving patterns exist. In Section 4.3 we show that a backward evolution process for a given process on space-moving patterns is easily obtained by using T-symmetry of the ESPCA.

We now give the following lemma to make it easy to show application examples of T-symmetries.

Lemma 4.1 Let P be a reversible ESPCA with the global function $F_{uvwxyz}$. Assume P is T-symmetric under an involution $H$, i.e., $F_{uvwxyz}^{-1} = H \circ F_{uvwxyz} \circ H$. Then the following holds for any $n \in \{1, 2, \ldots\}$.

$$(F_{uvwxyz}^{-1})^n = H \circ (F_{uvwxyz})^n \circ H$$

Proof. It is easily proved by a mathematical induction. The case $n = 1$ is obvious. Assume it holds for $n = k$. Then, $(F_{uvwxyz}^{-1})^{k+1} = H \circ (F_{uvwxyz})^k \circ H \circ F_{uvwxyz} \circ H = H \circ (F_{uvwxyz})^k \circ F_{uvwxyz} \circ H = H \circ (F_{uvwxyz})^{k+1} \circ H$. \[\square\]

4.1 ESPCA-02c5df: strictly T-symmetric

Consider ESPCA-02c5df given in Example 1. Its local function is shown in Figure 7. It is reversible and conservative. Note that it is isomorphic to the block-update function of Margolus’ CA [6]. Here, we explain how an inverse switch gate is obtained from a switch gate by using strict T-symmetry of ESPCA-02c5df.

A switch gate is a 2-input 3-output reversible gate having the logical function $f_S(c, x) = (c, cx, \overline{cx})$. We can interpret it as the operation where the input $c$ switches the output port of the input $x$. It is reversible in the sense that $f_S : \{0, 1\}^2 \rightarrow \{0, 1\}^3$ is an injection.

An inverse switch gate is a 3-input 2-output reversible gate having the partial logical function $f_S^{-1}(y_1, y_2, y_3) = (c, x)$, where $c = y_1$, and $x = y_2 + y_3$ under the assumption of $(y_2 \Rightarrow y_1) \land (y_3 \Rightarrow \overline{y_1})$. Namely, it is defined only on the set \{(0, 0, 0), (0, 0, 1), (1, 0, 0), (1, 1, 0)\}.

In [16, 8], a switch gate is implemented in ESPCA-02c5df in the following way. First, a signal, which represents the logical value 1, is given by a space-moving pattern consisting of two particles shown in Figure 13. Colliding two signals as in Figure 14 their trajectories are changed. By this, a kind of logical operation is performed.

A left-turn of a signal is realised by a reflector composed of two blocks as shown in Figure 15, where a block is a stable pattern consisting of eight particles. Note that a right-turn of a signal is possible by mirror images of the configurations in Figure 15.

![Figure 13: Signal in ESPCA-02c5df. It consists of two particles](image)

![Figure 14: Collision of two signals in ESPCA-02c5df [16, 8]. A small circle shows the virtual collision point](image)
Using these phenomena, we can compose a switch gate. The configuration $\alpha(0)$ in Figure 16 is a switch gate pattern [16, 8]. In this figure, input signals are given to both $c$ and $x$ at $t = 0$. In this case, the signals come out from the output ports $c$ and $cx$ as in $\alpha(28)$ at $t = 28$. If only one signal is given to the input port $c$ (or $x$, respectively), it comes out from $c$ (or $cx$). Thus, the pattern correctly simulates a switch gate.

From Lemma 4.1 we can easily obtain an inverse switch gate. Since ESPCA-02c5df is strictly T-symmetric, the following relation holds by Lemma 4.1.

$$(F_{02c5df}^{-1})^{28}(\alpha(28)) = H^{rev} \circ (F_{02c5df})^{28} \circ H^{rev}(\alpha(28))$$

As shown in Figure 16, this relation means that a 28-step backward evolution starting from $\alpha(28)$ is simulated by a 28-step forward evolution starting from $H^{rev}(\alpha(28))$. Its resulting configuration is $(F_{02c5df})^{28} \circ H^{rev}(\alpha(28)) = H^{rev}(\alpha(0))$.

Since blocks do not change the patterns by $H^{rev}$, the configurations $H^{rev}(\alpha(28))$ and $H^{rev}(\alpha(0))$ are obtained from $\alpha(28)$ and $\alpha(0)$ by reversing the move directions of signals. Thus, the switch gate pattern itself works as an inverse switch gate by simply swapping the roles of input and output ports.

Figure 16: We can make an inverse switch gate ($H^{rev}(\alpha(28))$) from a switch gate ($\alpha(0)$) in ESPCA-02c5df based on its strict T-symmetry.
4.2 ESPCA-02c5bf: T-symmetric under $H_{\text{rev}} \circ H_{\text{refl}}$

Next, consider ESPCA-02c5bf given in Example 2. Its local function is in Figure 10. It is reversible and conservative. A signal (Figure 13), and collision of two signals (Figure 14) are exactly the same as in ESPCA-02c5df (Section 4.1). But, a left-turn of a signal is different. As shown in Figure 17, it is performed by a single block. However, a right-turn is not possible by a single block. If we start from a mirror image of the configuration of $t = 0$ in Figure 17, then the signal and the block will be broken (Figure 18). Hence, a right-turn should be implemented by three left-turns.

Using the above phenomena, we obtain a switch gate in ESPCA-02c5bf as in the configuration $\beta(0)$ [16, 8] in Figure 19. This configuration shows that input signals are given to both $c$ and $x$ at $t = 0$. Then, they will come out from the output ports $c$ and $cx$ at $t = 25$. It is also easy to verify other cases, and thus the pattern correctly simulates a switch gate.
As shown in Figure 19, it means that a 25-step backward evolution starting from $\beta(25)$ is simulated by a 25-step forward evolution starting from $H^{rev} \circ H^{refl}(\beta(25))$. Its resulting configuration is $(F_{02c5bf})^{25} \circ H^{rev} \circ H^{refl}(\beta(25)) = H^{rev} \circ H^{refl}(\beta(0))$. Note that $H^{rev} \circ H^{refl}(\beta(25))$ and $H^{rev} \circ H^{refl}(\beta(0))$ are obtained from $\beta(25)$ and $\beta(0)$ by taking mirror images of them, and reversing the move directions of signals. Therefore, the mirror image of the switch gate pattern works as an inverse switch gate by swapping the roles of input and output ports.

Note that ESPCA-02c5df (Section 4.1) is T-symmetric under $H^{rev} \circ H^{refl}$, as well as T-symmetric under $H^{rev}$ (see Table 1 and Theorems 3.3 and 3.6). Therefore, $H^{rev} \circ H^{refl}(\alpha(28))$ also works as an inverse switch gate in ESPCA-02c5df.

### 4.3 ESPCA-01caef: T-symmetric under $H^{rev} \circ H^{refl}$ and $H^{rev} \circ H^{comp}$

We consider ESPCA-01caef, whose local function is shown in Figure 20. It is a reversible and conservative ESPCA studied in [11, 14]. It was shown that any reversible logic element with memory (RLEM), which is a kind of a reversible finite automaton, is implemented in its cellular space. Since reversible Turing machines (RTMs) can be composed of RLEMs, we can see ESPCA-01caef is computationally universal. Construction of RTMs becomes much simpler by using RLEMs than by using reversible logic gates [11, 14].

Note that, despite the simplicity of its local function, evolution processes of ESPCA-01caef are generally very complex, and thus it is difficult to follow them by paper and pencil. We created an emulator for ESPCA-01caef on the general purpose CA simulator Golly [18] for viewing evolution processes. The emulator file and pattern files are available in [10].

First, we give a simple example of using its T-symmetry. In ESPCA-01caef, there exist many kinds of space-moving patterns [11]. Figure 21 is one such example having period 12, which we call here a glider-12. It flies in the cellular space in a diagonal direction. Figure 22 is another example of a space-moving pattern called a glider-44, which is of period 44. It moves horizontally or vertically. Here we consider a process that transforms the former glider to the latter. From it, we can obtain two kinds of inverse transformation processes by two kinds of T-symmetries of ESPCA-01caef.

Consider the configuration $\gamma(0)$ in Figure 23. There are a glider-12 moving to the north-east and a particle. They interact, and produce a glider-44 moving to the east and a particle as shown in $\gamma(145)$. Thus, a glider-12 is converted into a glider-44 by colliding it with a particle.

Applying $H^{rev} \circ H^{refl}$ to $\gamma(145)$, we obtain a configuration that gives the inverse process of the above. In $H^{rev} \circ H^{refl}(\gamma(145))$ there are a glider-44 moving to the east and a particle. Note that the pattern of the glider-44 is the same as the one at $t = 43$ in Figure 22. Namely, the phase of the glider-44 is shifted by the application of $H^{rev} \circ H^{refl}$. These two objects interact in ESPCA-01caef, and finally produce a glider-12 moving to the south-east direction and a particle.
as shown in $H^{\text{rev}} \circ H^{\text{refl}}(\gamma(0))$ of Figure 23. Thus, a glider-44 is converted into a glider-12. Note that the pattern of the glider-12 is the one obtained by rotating the one at $t = 11$ in Figure 21 by 90 degrees clockwise. Namely, the phase and the direction of the glider-12 are changed by the application of $H^{\text{rev}} \circ H^{\text{refl}}$.

Since ESPCA-01caef is $T$-symmetric under $H^{\text{rev}} \circ H^{\text{comp}}$ as well as $H^{\text{rev}} \circ H^{\text{refl}}$ (see Table 1), we can obtain a backward evolution process also by this involution, where the glider-12 and glider-44 are represented by ‘holes’, as shown in Figure 24. In this case, however, the resulting configuration is infinite (i.e., it contains an infinite number of non-blank cells). If we want to have a finite configuration that undoes the evolution process of a given finite configuration, this method is not usable.

Figure 23: Using the process of converting a glider-12 ($\gamma(0)$) to a glider-44 ($\gamma(145)$), we can convert a glider-44 ($H^{\text{rev}} \circ H^{\text{refl}}(\gamma(145))$) to a glider-12 ($H^{\text{rev}} \circ H^{\text{refl}}(\gamma(0))$) in ESPCA-01caef. It is based on its $T$-symmetry under $H^{\text{rev}} \circ H^{\text{refl}}$.

Figure 24: Using the process of converting a complemented glider-44 (shown in $H^{\text{rev}} \circ H^{\text{comp}}(\gamma(145))$ as ‘holes’) to a complemented glider-12 ($H^{\text{rev}} \circ H^{\text{comp}}(\gamma(0))$) in ESPCA-01caef. It is based on its $T$-symmetry under $H^{\text{rev}} \circ H^{\text{comp}}$. Note that, in $H^{\text{rev}} \circ H^{\text{comp}}(\gamma(0))$ and $H^{\text{rev}} \circ H^{\text{comp}}(\gamma(145))$, the state 0 is represented by a small circle.
Next, we give another example. We show that for a given pattern that simulates an RLEM, a pattern that simulates its inverse RLEM is easily obtained by using T-symmetry of ESPCA-01caef.

Here, we make some preparations (see [8, 14] for the details). A sequential machine is a kind of a finite automaton having output symbols as well as input symbols. It is defined by $M = (Q, \Sigma, \Gamma, \delta)$, where $Q$ is a finite set of states, $\Sigma$ and $\Gamma$ are finite sets of input and output symbols, and $\delta : Q \times \Sigma \rightarrow Q \times \Gamma$ is a move function. If $\delta$ is injective, it is called a reversible sequential machine (RSM). A reversible logic element with memory (RLEM) is an RSM that satisfies $|\Sigma| = |\Gamma|$.

If $n = |Q|$ and $k = |\Sigma| = |\Gamma|$, it is called an $n$-state $k$-symbol RLEM. 2-state RLEMs are particularly important, since it is known that any 2-state $k$-symbol RLEM is universal if $k \geq 3$, i.e., any RSM can be composed only of it [8].

We consider a 2-state RLEM No. 2-3, where ‘2’ stands for 2-symbol and ‘3’ is the serial number in the class of 2-state RLEMs. It is defined by $M_{2-3} = (\{0, 1\}, \{a, b\}, \{x, y\}, \delta_{2-3})$, where $\delta_{2-3}$ is as follows.

$$\delta_{2-3}(0,a) = (0,x), \quad \delta_{2-3}(0,b) = (1,x), \quad \delta_{2-3}(1,a) = (1,y), \quad \delta_{2-3}(1,b) = (0,y)$$

A 2-state RLEM No. 2-4 is defined by $M_{2-4} = (\{0, 1\}, \{a, b\}, \{x, y\}, \delta_{2-4})$, where $\delta_{2-4}$ is as follows.

$$\delta_{2-4}(0,a) = (0,x), \quad \delta_{2-4}(0,b) = (1,y), \quad \delta_{2-4}(1,a) = (0,y), \quad \delta_{2-4}(1,b) = (1,x)$$

It is easy to see that $\delta_{2-4}$ is isomorphic to $\delta_{2-3}^{-1}$. In this sense, RLEM 2-4 is the inverse of RLEM 2-3. It has been shown that the set $\{RLEM 2-3, RLEM 2-4\}$ is universal, though each of RLEMs 2-3 and 2-4 is non-universal [8]. Namely, any RSM can be constructed out of RLEMs 2-3 and 2-4.

These RLEMs are implemented in ESPCA-01caef using a glider-12 (Figure 21) and a periodic pattern called a blinker (Figure 25 (a)) [14]. Here, a glider-12 is used as a signal. Note that a stable pattern called a block (Figure 25 (b)) is also used for writing comments and indicating a border of a logic element in the cellular space. Hence, a block has no functional role for composing the RLEMs.

![Figure 25: (a) Blinker, a periodic pattern of period 2, and (b) block, a stable pattern, in ESPCA-01caef](image)

By colliding a glider-12 with a blinker appropriately, a right-turn and a U-turn of a glider-12, and shifting a blinker is possible. First, colliding a glider-12 with a blinker as in Figure 26, a right-turn of a glider-12 is realised.

![Figure 26: Right-turn of a glider-12 in ESPCA-01caef](image)

A U-turn of a glider-12 is performed as in Figure 27. It is used to test if a blinker exists or not at a specified position. It is also used to reversibly merge two signal paths into one (it is explained later).

![Figure 27: U-turn of a glider-12 in ESPCA-01caef](image)
Finally, colliding a glider-12 with a blinker as in Figure 28, the position of the blinker is shifted by 6 cells, and the glider-12 makes a right-turn. Using this phenomenon, a kind of memory device is realized, where the memory states are kept by the positions of the blinker. At the same time, it can test if a blinker exists at a specified position, and can merge two signal paths into one.

Figure 28: Shifting a blinker by a glider-12 in ESPCA-01caef [11, 14]

The pattern shown in Figure 29 simulates RLEM 2-3. There are many blinkers in this pattern. One is used as a position marker for keeping the memory state 0 or 1, while others are used for turning a signal. Two small circles near the center of the pattern show possible positions of the position marker. If the marker is at the left (right, respectively) position, we regard that the RLEM is in the state 0 (1).

First, consider the case where the state is 0 and an input signal is given to the port \( a \). The signal makes a U-turn at the U-turn gadget \( U_1 \), since the state is 0. Then it goes to the gadget \( U_2 \), and again makes a U-turn passing through \( Q \). Note that \( U_2 \) is used to reversibly merge the path with that of the second case. Finally the signal goes out from the port \( x \).

Second, consider the case where the state is 0 and an input signal is given to the port \( b \). At \( P \) the signal shifts the position marker to the right, and makes a right-turn. Thus, the state changes to 1. Then, the signal goes out from the output port \( x \) via the point \( Q \). This signal path is merged with that of the first case at \( Q \).

Third, consider the case where the state is 1 and an input signal is given to the port \( a \). In this case, the signal goes out from the output port \( y \) via \( S \) and \( R \) without interacting the position marker.

Fourth, consider the case where the state is 1 and an input signal is given to the port \( b \). The signal goes straight ahead at the point \( P \). Then, it shifts the position marker to the left and makes a right-turn at \( R \). Thus, the state changes to 0. Finally it goes out from \( y \). This signal path is merged with that of the third case at \( R \).

Note that, in an RLEM, an incoming signal interacts with the state of the RLEM, not with other signals. Therefore, there is no need of synchronizing two or more signals as in the case of logic gates. Therefore, it greatly simplifies implementation of RLEMs and connecting them in ESPCA-01caef.

The pattern shown in Figure 30 simulates RLEM 2-4. It is obtained by applying the involution \( H^{\text{rev}} \circ H^{\text{eff}} \) to the pattern of RLEM 2-4 given in Figure 29. By the T-symmetry of ESPCA-01caef, the pattern for RLEM 2-4 undoes the operations of the pattern for RLEM 2-3. As in the case of RLEM 2-3, one blinker near the center of the pattern is used as a position marker for keeping the memory state 0 or 1. If the marker is in the right (left, respectively) small circle, we regard that the RLEM is in the state 0 (1).

First, consider the case where the state is 0 and an input signal is given to the port \( a \). The signal makes a U-turn at \( U_1 \). Then it goes to \( U_1 \), and again makes a U-turn passing through \( T \). Finally the signal goes out from the port \( x \).

Second, consider the case where the state is 1 and an input signal is given to the port \( b \). At \( R \) the signal goes straight ahead. Then it passes through the points \( S \) and \( T \). Finally it goes out from the port \( x \). This signal path is merged with that of the first case at \( T \).

Third, consider the case where the state is 1 and an input signal is given to the port \( a \). The signal goes straight ahead at \( Q \). Then, at \( P \) it shifts the position marker to the right, and makes a right-turn. By this, the state changes to 0. Finally it goes out from the port \( y \).

Fourth, consider the case where the state is 0 and an input signal is given to the port \( b \). At \( R \) the signal shifts the position marker to the left, and makes a right-turn. By this, state changes to 1. Then, it passes through \( P \), and finally goes out from \( y \). In this case, the signal path is merged with that of the third case at \( P \).

In [14], reversible Turing machines are constructed using the above patterns that simulate RLEMs 2-3 and 2-4. Their whole computing processes in the ESPCA space can be seen on the CA simulator Golly [18] using the emulator file and the pattern files given in [10].
Figure 29: RLEM 2-3 implemented in ESPCA-01caef [14]

Figure 30: RLEM 2-4 implemented in ESPCA-01caef [14]. It is obtained by applying the involution $H^{\text{rev}} \circ H^{\text{refl}}$ to the pattern given in Figure 29 (except the comment part written by blocks)
5 Elementary Triangular Partitioned Cellular Automata (ETPCAs) and Their T-symmetries

A three-neighbour triangular partitioned cellular automaton (TPCA) proposed in [3] is a PCA whose cell is triangular and is divided into three parts. Since its cell has only three neighbour cells, its ‘elementary’ version, which is rotation-symmetric and each part of whose cell has only two states, is simpler than ESPCA. Despite their simplicity, several kinds of reversible ETPCAs have been known to be computationally universal, i.e., any reversible Turing machine can be constructed in their cellular spaces [3, 8, 13].

T-symmetries in reversible ETPCAs were first investigated in [13]. In this section, we describe the previous results, and add some results that have not been given before. We compare T-symmetries in reversible ETPCAs with those in reversible ESPCAs, and show examples of their applications.

5.1 Definitions on ETPCAs

The cellular space of a TPCA is shown in Fig 31. All the cells are identical in their logical functions. However, there are two kinds of directions, i.e., upward and downward (Figure 32). Therefore, the neighbour cells of an up-triangle cell are different from those of a down-triangle cell. A local transition rule for an up-triangle cell is depicted in Figure 33. For a down-triangle cell, the rule obtained by rotating both sides of the rule given in Figure 33 by 180 degrees is used. In the following, we assume up-triangle (down-triangle, respectively) cells are placed at a coordinates \((x, y) \in \mathbb{Z}^2\) such that \(x + y\) is even (odd).

**Definition 5.1** A three-neighbour triangular partitioned cellular automaton (TPCA) is a system defined by

\[
P = (\mathbb{Z}^2; (L, D, R), (\{(−1, 0), (0, −1), (1, 0)\}, \{(1, 0), (0, 1), (−1, 0)\}, f)).
\]

Here, \(\mathbb{Z}^2\) is the set of all two-dimensional points with integer coordinates at which cells are placed, and \(L, D\) and \(R\) are non-empty finite sets of states of the left, downward and right parts of a cell. The state set \(Q\) of a cell is thus \(Q = L \times D \times R\). The triplet \((\{(−1, 0), (0, −1), (1, 0)\} \) is a neighbourhood for up-triangle cells, and \((\{(1, 0), (0, 1), (−1, 0)\}\) is a neighbourhood for down-triangle cells. The item \(f : Q \to Q\) is a local (transition) function.

If \(f(l, d, r) = (l', d', r')\) holds for \((l, d, r), (l', d', r') \in Q\), then this relation is called a local transition rule of the TPCA \(P\). It is written pictorially as in Figure 33. The local function \(f\) is thus defined by a set of local transition rules.

Configurations of a TPCA, and the global function are defined as below.
Definition 5.2 Let $P = (\mathbb{Z}^2, (L, D, R), ((-1, 0), (0, -1), (1, 0)), ((1, 0), (0, 1), (-1, 0)), f)$ be a TPCA. A configuration of $P$ is a function $\alpha : \mathbb{Z}^2 \rightarrow Q$. The set of all configurations of $P$ is denoted by $\text{Conf}(P)$, i.e., $\text{Conf}(P) = \{ \alpha | \alpha : \mathbb{Z}^2 \rightarrow Q \}$. Let $\text{pr}_L : Q \rightarrow L$ be the projection function such that $\text{pr}_L(l, d, r) = l$ for all $(l, d, r) \in Q$. The projection functions $\text{pr}_D : Q \rightarrow D$ and $\text{pr}_R : Q \rightarrow R$ are defined similarly. The global function $F : \text{Conf}(P) \rightarrow \text{Conf}(P)$ of $P$ is defined as the one that satisfies the following:

$$\forall \alpha \in \text{Conf}(T), \forall (x, y) \in \mathbb{Z}^2 :$$

$$F(\alpha)(x, y) = \begin{cases} f(\text{pr}_L(\alpha(x - 1, y)), \text{pr}_D(\alpha(x, y - 1)), \text{pr}_R(\alpha(x + 1, y))) & \text{if } x + y \text{ is even} \\ f(\text{pr}_L(\alpha(x + 1, y)), \text{pr}_D(\alpha(x, y + 1)), \text{pr}_R(\alpha(x - 1, y))) & \text{if } x + y \text{ is odd} \end{cases}$$

Reversibility of TPCA is defined similarly to the case of SPCA (Definition 2.3).

Definition 5.3 A TPCA $P$ is called reversible if its global function $F$ is injective.

As in the case of SPCA (Lemma 2.4), injectivity of the global function is equivalent to injectivity of the local function in a TPCA [8].

Lemma 5.4 Let $P$ be a TPCA. Its global function $F$ is injective if and only if its local function $f$ is injective.

An elementary triangular partitioned cellular automaton (ETPCA) is also defined similarly as in the case of ESPCA (Definition 2.6).

Definition 5.5 Let $P = (\mathbb{Z}^2, (L, D, R), ((-1, 0), (0, -1), (1, 0)), ((1, 0), (0, 1), (-1, 0)), f)$ be a TPCA. The TPCA $P$ is called rotation-symmetric (or isotropic) if the following conditions (1) and (2) holds.

1. $L = D = R$
2. $\forall (l, d, r), (l', d', r') \in L \times D \times R : f(l, d, r) = (l', d', r') \Rightarrow f(d, r, l) = (d', r', l')$

Definition 5.6 Let $P = (\mathbb{Z}^2, (L, D, R), ((-1, 0), (0, -1), (1, 0)), ((1, 0), (0, 1), (-1, 0)), f)$ be a TPCA. The TPCA $P$ is called an elementary triangular partitioned cellular automaton (ETPCA), if $L = D = R = \{0, 1\}$, and it is rotation-symmetric.

Since an ETPCA is rotation-symmetric, its local function $f : \{0, 1\}^3 \rightarrow \{0, 1\}^3$ is described by only four local transition rules, which are obtained by giving the following four values:

$$f(0, 0, 0), f(0, 1, 0), f(1, 0, 1), f(1, 1, 1)$$

Here, $f(0, 1, 0), f(1, 0, 1) \in \{0, 1\}^3$, but $f(0, 0, 0), f(1, 1, 1) \in \{(0, 0, 0), (1, 1, 1)\}$ since it is rotation-symmetric.

Reading the values of $f(0, 0, 0), f(0, 1, 0), f(1, 0, 1)$ and $f(1, 1, 1)$ as four binary numbers, we can express an ETPCA by a 4-digit octal identification (ID) number $wxyz$ as shown in Figure 34. Thus there are 256 ETPCAs in total.

![Figure 34: Expressing an ETPCA by a 4-digit octal ID number wxyz. Vertical bars indicate alternatives of the right-hand side of each local transition rule](image)

An ETPCA with the ID number $wxyz$ is denoted by $\text{ETPCA}-wxyz$. Its local function and global function are represented by $f_{wxyz}$ and $F_{wxyz}$, respectively. Figure 35 shows the set of local transition rules of ETPCA-0137.

![Figure 35: Local transition rules of ETPCA-0137, which define the local function $f_{0137}$](image)
Table 3: Identification numbers of 36 reversible ETPCAs, their dual ones (under reflection, complementation, and both), and their inverses. In each ETPCA, the IDs of local functions among \(f, f^r, f^c\) and \(f^{rc}\) that are equal to \(f^{-1}\) are marked by \(*\). It means that the ETPCA is T-symmetric under the corresponding involutions.

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7160 & 7430 & 7160 & 7430 & 7430 \\
7230 & 7260 & 7450 & 7150 & 7260 \\
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7260 & 7230 & 7150 & 7450 & 7230 \\
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7540 & 7510 & 7320 & 7620 & 7620 \\
7610 & 7340 & 7610 & 7340 & 7340 \\
7620 & 7320 & 7510 & 7540 & 7540 \\
7640 & 7310 & 7310 & 7640 & 7640 \\
\end{array} \]
5.3 T-symmetries in reversible ETPCAs

Let \( \text{Conf}_{E3} = \{ \alpha | \alpha : \mathbb{Z}^2 \to \{0, 1\}^3 \} \) denote the set of all configurations of ETPCA. We define the involution \( H_3^{\text{rev}} : \text{Conf}_{E3} \to \text{Conf}_{E3} \) by the reversible ETPCA-0257:

\[
H_3^{\text{rev}} = F_{0257}
\]

As shown in Figure 36, the involution \( H_3^{\text{rev}} \) is interpreted as the one that reverses the moving directions of all the particles in the cellular space. Though \( H_3^{\text{rev}} \) is different from \( H_3^{\text{rev}} \) for ESPCAs defined in Section 3, it has a similar meaning with the latter. Therefore, in the following, we use the notation \( H_3^{\text{rev}} \) in place of \( H_3^{\text{rev}} \), since no confusion occurs.

![Figure 36](image)

Figure 36: Local function of ETPCA-0257 by which \( H_3^{\text{rev}} \) for ETPCAs is defined. The involution \( H_3^{\text{rev}} \) makes every particle turn backward. Hereafter, we use the notation \( H_3^{\text{rev}} \) in place of \( H_3^{\text{rev}} \).

**Lemma 5.11** [13] Let \( P \) be a reversible ETPCA-wxyz with the local function \( f_{\text{wxyz}} \) and the global function \( F_{\text{wxyz}} \). Let \( P' \) be a reversible ETPCA having the ID number \( \text{inv}(\text{wxyz}) \). Hence, the local and global functions of \( P' \) are \( f_{\text{inv}(\text{wxyz})} = f_{\text{wxyz}}^{-1} \) and \( F_{\text{inv}(\text{wxyz})} \), respectively. Then, the following holds.

\[
F_{\text{inv}(\text{wxyz})}^{-1} = H_3^{\text{rev}} \circ F_{\text{inv}(\text{wxyz})} \circ H_3^{\text{rev}}
\]

**Proof.** Let \( \alpha_1 \in \text{Conf}_{E3} \) be any configuration. Let \((x_0, y_0) \in \mathbb{Z}^2 \) be any point, and \((l_1, d_1, r_1) \in \{0, 1\}^3 \) be as follows: \( \alpha_1(x_0, y_0) = (l_1, d_1, r_1) \). See Figure 37 that shows the process of state-changes by the operations given below. We consider only the case where \( x_0 + y_0 \) is even, since the other case is similar. First, we can see the following relations.

\[
\begin{align*}
\text{pr}_l(H_3^{\text{rev}}(\alpha_1))(x_0 - l_1, y_0) &= l_1 \\
\text{pr}_d(H_3^{\text{rev}}(\alpha_1))(x_0, y_0 - 1) &= d_1 \\
\text{pr}_r(H_3^{\text{rev}}(\alpha_1))(x_0 + 1, y_0) &= r_1
\end{align*}
\]

Assume \( f_{\text{wxyz}}^{-1}(l_1, d_1, r_1) = (l_0, d_0, r_0) \) (thus, \( f_{\text{wxyz}}(l_0, d_0, r_0) = (l_1, d_1, r_1) \)). Then,

\[
(F_{\text{inv}(\text{wxyz})} \circ H_3^{\text{rev}}(\alpha_1))(x_0, y_0) = (l_0, d_0, r_0),
\]

Let \( \alpha_0 = F_{\text{inv}(\text{wxyz})} \circ H_3^{\text{rev}}(\alpha_1) \). Then, the following relations hold.

\[
\begin{align*}
\text{pr}_l(H_3^{\text{rev}}(\alpha_0))(x_0 - l_1, y_0) &= l_0 \\
\text{pr}_d(H_3^{\text{rev}}(\alpha_0))(x_0, y_0 - 1) &= d_0 \\
\text{pr}_r(H_3^{\text{rev}}(\alpha_0))(x_0 + 1, y_0) &= r_0
\end{align*}
\]

Hence,

\[
(F_{\text{wxyz}} \circ H_3^{\text{rev}}(\alpha_0))(x_0, y_0) = (l_1, d_1, r_1) = \alpha_1(x_0, y_0).
\]

By above, the following holds for all \((x_0, y_0) \in \mathbb{Z}^2\).

\[
(F_{\text{wxyz}} \circ H_3^{\text{rev}}(\alpha_0))(x_0, y_0) = \alpha_1(x_0, y_0)
\]

Thus, \( F_{\text{wxyz}} \circ H_3^{\text{rev}}(\alpha_0) = H_3^{\text{rev}}(\alpha_1) \) for all \( \alpha_1 \in \text{Conf}_{E3} \). Therefore,

\[
F_{\text{wxyz}}^{-1} = H_3^{\text{rev}} \circ F_{\text{inv}(\text{wxyz})} \circ H_3^{\text{rev}}.
\]

This completes the proof. \( \Box \)

![Figure 37](image)

Figure 37: Process of the state-changes around the cell at \((x_0, y_0)\) in Lemma 5.11

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**Definition 5.12** Let $P$ be a reversible ETPCA-wxyz whose global function is $F_{wxyz}$. If $F_{wxyz}^{-1} = H^{rev} \circ F_{wxyz} \circ H^{rev}$, then $P$ is called strictly T-symmetric.

From Lemma 5.11 we have the following theorem.

**Theorem 5.13** [13] A reversible ETPCA with the ID number wxyz is strictly T-symmetric, if $inv(wxyz) = wxyz$. In this case, the following holds.

$$F_{wxyz}^{-1} = H^{rev} \circ F_{wxyz} \circ H^{rev}$$

From Table 3 we can see the following.

**Corollary 5.14** [13] The 8 reversible ETPCAs $w25z$, $w31z$, $w52z$ and $w64z$ are strictly T-symmetric, where $(w, z) \in \{(0,7), (7,0)\}$.

We now define a weaker version of T-symmetry for reversible ETPCAs.

**Definition 5.15** Let $P$ be a reversible ETPCA-wxyz whose global function is $F_{wxyz}$. If there is an involution $H : \text{Conf}_{3} \rightarrow \text{Conf}_{3}$ that satisfies $F_{wxyz}^{-1} = H \circ F_{wxyz} \circ H$, then $P$ is called T-symmetric under the involution $H$.

Define a function $refl : \{0, 1\}^3 \rightarrow \{0, 1\}^3$ as follows: $refl(l, d, r) = (r, d, l)$ for any $(l, d, r) \in \{0, 1\}^3$. Next define an involution $H_{3}^{refl} : \text{Conf}_{3} \rightarrow \text{Conf}_{3}$ as follows: $H_{3}^{refl}(\alpha)(x_0, y_0) = refl(\alpha(-x_0, y_0))$ for all $\alpha \in \text{Conf}_{3}$ and $(x_0, y_0) \in \mathbb{Z}^2$. The involution $H_{3}^{refl}$ gives the mirror image of a configuration with respect to the y-axis. As in the case of $H_{3}^{rev}$, we hereafter use the notation $H_{3}^{refl}$ in place of $H_{3}^{refl}$.


$$F_{f(wxyz)} = H_{3}^{refl} \circ F_{wxyz} \circ H_{3}^{refl}$$

**Proof.** First, we show $F_{wxyz} = H_{3}^{refl} \circ F_{f(wxyz)} \circ H_{3}^{refl}$. Let $\alpha \in \text{Conf}_{3}$ be any configuration, and $(x_0, y_0) \in \mathbb{Z}^2$ be any point. We consider only the case where $x_0 + y_0$ is even. Let $(l_0, d_0, r_0) \in \{0, 1\}^3$ be as follows.

$$\begin{align*}
pr_l(\alpha(x_0-1, y_0)) &= l_0 \\
pr_d(\alpha(x_0, y_0-1)) &= d_0 \\
pr_r(\alpha(x_0+1, y_0)) &= r_0
\end{align*}$$

See Figure 38 that shows the process of state-changes by the operations given below. In the next step, we have the following.

$$\begin{align*}
pr_l(H_{3}\alpha(-x_0-1, y_0)) &= r_0 \\
pr_d(H_{3}\alpha(-x_0, y_0-1)) &= d_0 \\
pr_r(H_{3}\alpha(-x_0+1, y_0)) &= l_0
\end{align*}$$

Assume $f_{wxyz}(l_0, d_0, r_0) = (l_1, d_1, r_1)$. Since $f_{f(wxyz)}(r_0, d_0, l_0) = (r_1, d_1, l_1)$,

$$(F_{f(wxyz)} \circ H_{3}^{refl})(-x_0, y_0) = (r_1, d_1, l_1).$$

Finally, we have the following relation for all $\alpha$ and $(x_0, y_0)$.

$$(H_{3}^{refl} \circ F_{f(wxyz)} \circ H_{3}^{refl})(x_0, y_0) = (l_1, d_1, r_1) = F_{wxyz}(\alpha)(x_0, y_0)$$

Therefore, $F_{wxyz} = H_{3}^{refl} \circ F_{f(wxyz)} \circ H_{3}^{refl}$ holds, and thus

$$H_{3}^{refl} \circ F_{wxyz} \circ H_{3}^{refl} = H_{3}^{refl} \circ H_{3}^{refl} \circ F_{f(wxyz)} \circ H_{3}^{refl} \circ H_{3}^{refl} = F_{f(wxyz)}.$$

This completes the proof. □

![Figure 38](image.png)

Figure 38: Process of the state-changes around the cells at $(x_0, y_0)$ and $(-x_0, y_0)$ in Lemma 5.16

From Lemmas 5.16 and 5.16 we have the following.

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Theorem 5.17 [13] A reversible ETPCA with the ID number wxyz is T-symmetric under the involution \(H^{\text{refl}} \circ H^{\text{rev}}\), if \(\text{inv}(wxyz) = r(wxyz)\). In this case, the following holds.

\[
F_{wxyz}^{-1} = H^{\text{rev}} \circ F_{r(wxyz)} \circ H^{\text{rev}}
\]

From Table 3 we can see the following.

Corollary 5.18 [13] The 24 reversible ETPCAs w13z, w15z, w16z, w23z, w25z, w26z, w34z, w43z, w45z, w46z, w52z and w61z are T-symmetric under the involution \(H^{\text{rev}} \circ H^{\text{refl}}\), where \((w,z) \in \{(0,7),(7,0)\}\).

Next, define a function \(\text{comp}_3 : \{0,1\}^3 \to \{0,1\}^3\) as follows: \(\text{comp}_3(l,d,r) = (\overline{l}, \overline{d}, \overline{r})\) for any \((l,d,r) \in \{0,1\}^3\). Define an involution \(H^{\text{comp}}_3 : \text{Conf}_{E3} \to \text{Conf}_{E3}\) as follows. For all \(\alpha \in \text{Conf}_{E3}\) and \((x_0,y_0) \in \mathbb{Z}^2\):

\[
H^{\text{comp}}_3(\alpha)(x_0,y_0) = \text{comp}_3(\alpha(x_0,y_0))
\]

The involution \(H^{\text{comp}}_3\) gives the complement image of a configuration. Hereafter, we use the notation \(H^{\text{comp}}\) in place of \(H^{\text{comp}}_3\).

Lemma 5.19 The next relation holds for any ETPCA-wxyz.

\[
F_{c(wxyz)} = H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}}
\]

Proof. First, we show \(F_{wxyz} = H^{\text{comp}} \circ F_{c(wxyz)} \circ H^{\text{comp}}\). Let \(\alpha \in \text{Conf}_{E3}\) be any configuration, and \((x_0,y_0) \in \mathbb{Z}^2\) be any point. We consider only the case where \(x_0 + y_0\) is even. Let \((l_0,d_0,r_0) \in \{0,1\}^3\) be as follows.

\[
\begin{align*}
pr_L(\alpha(x_0-1,y_0)) &= l_0 \\
pr_D(\alpha(x_0,y_0-1)) &= d_0 \\
pr_R(\alpha(x_0+1,y_0)) &= r_0
\end{align*}
\]

See Figure 38 that shows the process of state-changes by the operations given below. In the next step, we have the following.

\[
\begin{align*}
pr_L(H^{\text{comp}}(\alpha)(x_0-1,y_0)) &= \overline{l_0} \\
pr_D(H^{\text{comp}}(\alpha)(x_0,y_0-1)) &= \overline{d_0} \\
pr_R(H^{\text{comp}}(\alpha)(x_0+1,y_0)) &= \overline{r_0}
\end{align*}
\]

Assume \(f_{wxyz}(l_0,d_0,r_0) = (l_1,d_1,r_1)\). Since \(f_{c(wxyz)}(\overline{l_0},\overline{d_0},\overline{r_0}) = (\overline{l_1},\overline{d_1},\overline{r_1})\),

\[
(F_{c(wxyz)} \circ H^{\text{comp}}(\alpha))(x_0,y_0) = (\overline{l_1},\overline{d_1},\overline{r_1}).
\]

Finally, we have the following relation for all \(\alpha\) and \((x_0,y_0)\).

\[
(H^{\text{comp}} \circ F_{c(wxyz)} \circ H^{\text{comp}}(\alpha))(x_0,y_0) = (l_1,d_1,r_1) = F_{wxyz}(\alpha)(x_0,y_0)
\]

Therefore, \(F_{wxyz} = H^{\text{comp}} \circ F_{c(wxyz)} \circ H^{\text{comp}}\) holds, and thus

\[
H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}} = F_{c(wxyz)}.
\]

This completes the proof.

![Figure 39: Process of the state-changes around the cell \((x_0,y_0)\) in Lemma 5.19](image)

From Lemmas 5.11 and 5.19 we have the following.

Theorem 5.20 A reversible ETPCA with the ID number wxyz is T-symmetric under the involution \(H^{\text{rev}} \circ H^{\text{comp}}\), if \(\text{inv}(wxyz) = c(wxyz)\). In this case, the following holds.

\[
F_{wxyz}^{-1} = H^{\text{rev}} \circ F_{r(wxyz)} \circ H^{\text{rev}}
\]

\[
= H^{\text{rev}} \circ H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}} \circ H^{\text{rev}}
\]

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From Table 3 we can see the following.

**Corollary 5.21** The 8 reversible ETPCAs \(w_{13z}, w_{25z}, w_{46z} \text{ and } w_{52z}\) are T-symmetric under \(H^{\text{rev}} \circ H^{\text{comp}}\), where \((w, z) \in \{(0, 7), (7, 0)\}\).

Combining Lemmas 5.16 and 5.19, we also obtain the next lemma.

**Lemma 5.22** The next relation holds for any ETPCA-\(wxyz\).

\[
F_{r(c(wxyz))} = H^{\text{refl}} \circ F_{wxyz} \circ H^{\text{comp}}.
\]

**Proof.** By Lemma 5.19, we have

\[
F_{c(wxyz)} = H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}}.
\]

Therefore, by Lemma 5.16, we have

\[
F_{r(c(wxyz))} = H^{\text{refl}} \circ F_{c(wxyz)} \circ H^{\text{refl}} = H^{\text{refl}} \circ H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}} \circ H^{\text{refl}}.
\]

Since \(F_{r(c(wxyz))} = F_{r(c(wxyz))}\), the lemma holds. \(\square\)

From Lemmas 5.11 and 5.22 we have the following.

**Theorem 5.23** A reversible ETPCA with the ID number \(wxyz\) is T-symmetric under the involution \(H^{\text{rev}} \circ H^{\text{refl}} \circ H^{\text{comp}}\), if \(\text{inv}(wxyz) = r(c(wxyz))\). In this case, the following holds.

\[
F^{-1}_{wxyz} = H^{\text{rev}} \circ F_{wxyz} \circ H^{\text{rev}} = H^{\text{rev}} \circ H^{\text{refl}} \circ H^{\text{comp}} \circ F_{wxyz} \circ H^{\text{comp}} \circ H^{\text{refl}} \circ H^{\text{rev}}.
\]

From Table 3 we can see the following.

**Corollary 5.24** The 24 reversible ETPCAs \(w_{16z}, w_{25z}, w_{31z}, w_{32z}, w_{34z}, w_{43z}, w_{51z}, w_{52z}, w_{54z}, w_{61z}, w_{62z} \text{ and } w_{64z}\) are T-symmetric under the involution \(H^{\text{rev}} \circ H^{\text{refl}} \circ H^{\text{comp}}\), where \((w, z) \in \{(0, 7), (7, 0)\}\).

From Corollaries 5.14, 5.18, 5.21 and 5.24, we can see that ‘every’ reversible ETPCA is T-symmetric under either of the the involutions \(H^{\text{rev}}\) (i.e., strictly T-symmetric), \(H^{\text{rev}} \circ H^{\text{refl}}, H^{\text{rev}} \circ H^{\text{comp}}, \text{ or } H^{\text{rev}} \circ H^{\text{refl}} \circ H^{\text{comp}}\).

## 6 Applications of T-Symmetries in Reversible ETPCAs

It has been shown that in reversible ETPCAs 0137, 0157 and 0347 a switch gate and an inverse switch gate are realised, and then a Fredkin gate is composed of them [3, 8, 12]. Hence, these ETPCAs and their dual ones are computationally universal. An inverse switch gate is obtained from a switch gate by using T-symmetry of these ETPCAs. But, since the method is similar to the one given in Section 4.2, we do not describe it here. Instead, we give some other examples for finding a backward evolution process for a given process.

The following lemma is the ETPCA version of Lemma 4.1. It is also easily proved.

**Lemma 6.1** Let \(P\) be a reversible ETPCA with the global function \(F_{wxyz}\). Assume \(P\) is T-symmetric under an involution \(H\), i.e., \(F_{wxyz}^{-1} = H \circ F_{wxyz} \circ H\). Then the following holds for any \(n \in \{1, 2, \ldots\}\).

\[
(F_{wxyz}^{-1})^n = H \circ (F_{wxyz})^n \circ H.
\]

### 6.1 ETPCA-0527: strictly T-symmetric

Consider ETPCA-0527, whose local function is shown in Figure 40. It is reversible, but not conservative. By Corollary 5.14, it is strictly T-symmetric.

![Figure 40: Local function of reversible and non-conservative ETPCA-0527](image)

If we start from only one particle in ETPCA-0527, an expanding hexagonal pattern is created as shown in Figure 41. At \(t = 6\) an isolated particle appears again inside the hexagon, and hence a new hexagon is created every 6 steps to form concentric hexagons as shown in \(\delta(18)\) in Figure 42.
The process of generating indefinite number of concentric hexagons can be reversed by simply applying $H^{rev}$ to a configuration. As in Figure 42, the configuration $H^{rev}(\delta(18))$ will become a single particle by applying $(F_{0527})^{18}$. From this, we can see that a one-particle pattern $\delta(0)$ generates concentric hexagons both in the positive and the negative time directions. Note that, since $H^{rev}(\delta(0))$ is the rotated configuration of $\delta(0)$ by 180 degrees, at $t = -18$ the rotated configuration of $H^{rev}(\delta(18))$ by 180 degrees appears.

![Figure 41](image-url)

Figure 41: From a one-particle pattern an expanding hexagonal pattern appears in ETPCA-0527. This process is repeated indefinitely, and a large number of concentric hexagons are generated as in $\delta(18)$ of Figure 42.

![Figure 42](image-url)

Figure 42: Using the generating process of concentric hexagons ($\delta(18)$) from a one-particle pattern ($\delta(0)$), we can shrink the concentric hexagons ($H^{rev}(\delta(18))$) to a one-particle pattern ($H^{rev}(\delta(0))$) in ETPCA-0527. It is based on its strict T-symmetry.

### 6.2 ETPCA-0347: T-symmetric under $H^{rev} \circ H^{refl}$

Consider ETPCA-0347, whose local function is shown in Figure 43. It is reversible, but not conservative. By Corollary 5.18, it is T-symmetric under $H^{rev} \circ H^{refl}$. It was investigated in [9, 13]. In this cellular space, there is a space-moving pattern called a *glider* of period 6 (Figure 44). In ETPCA-0347, interactions of gliders and other patterns show fascinating phenomena. It has also been shown that any reversible Turing machine can be realised in its cellular space. Its emulator that works on *Golly* [18] is available in [7].

---

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Here, we first consider an evolution process of colliding gliders and its inverse. If we collide two gliders as in $\zeta(0)$ of Figure 45, three gliders are generated after 30 steps ($\zeta(30)$). By this, the number of gliders is increased by one. Its inverse process is obtained by T-symmetry under $H^{rev} \circ H^{refl}$. Namely, colliding three gliders as in $H^{rev} \circ H^{refl}(\zeta(30))$, we get two gliders ($H^{rev} \circ H^{refl}(\zeta(0))$). By this, the number of gliders is decreased by one.

Using these symmetric phenomena, a glider gun, which generates gliders periodically, and a glider absorber, which reversibly erases gliders periodically, can be constructed [13]. Furthermore, the glider gun and the glider absorber themselves are symmetrically composed by using T-symmetry of ETPCA-0347.

![Figure 43: Local function of reversible and non-conservative ETPCA-0347](image)

![Figure 44: Glider of period 6 in ETPCA-0347](image)

![Figure 45: Using the process of generating three gliders ($\zeta(30)$) from two ($\zeta(0)$), we can generate two gliders ($H^{rev} \circ H^{refl}(\zeta(0))$) from three ($H^{rev} \circ H^{refl}(\zeta(30))$) in ETPCA-0347. Thus, both increasing and decreasing the number of gliders are possible. It is based on its T-symmetry under $H^{rev} \circ H^{refl}$](image)
Next, we consider an evolution process starting from a one-particle pattern in ETPCA-0347. Figure 46 shows that, if we start from a configuration containing only one particle ($t = 0$), then a disordered pattern appears ($t = 63$), and it expands bigger and bigger ($t = 320$) as if an explosion occurs.

However, if we apply $H^{\text{rev}} \circ H^{\text{refl}}$ to any configuration in the explosion process, it immediately starts to shrink. As seen in Figure 47, the configuration $H^{\text{rev}} \circ H^{\text{refl}}(\eta(64))$ goes to the one-particle configuration $H^{\text{rev}} \circ H^{\text{refl}}(\eta(0))$ after 64 steps. Therefore, in ETPCA-0347, both the explosion process and the implosion process from/to a one-particle pattern exist.

We can also observe that a one-particle pattern $\eta(0)$ generates random-like patterns both in the positive and the negative time directions. In fact, if we go to the negative time direction from $\eta(0)$, then at $t = -64$ the configuration obtained by rotating $H^{\text{rev}} \circ H^{\text{refl}}(\eta(64))$ clockwise by 60 degrees will appear.

![Figure 46: Evolution process like an explosion starting from a one-particle pattern in ETPCA-0347](image)

![Figure 47: Using the expanding process from a one-particle pattern ($\eta(0)$) to a disordered pattern ($\eta(64)$), we can shrink the disordered pattern ($H^{\text{rev}} \circ H^{\text{refl}}(\eta(64))$) to a one-particle pattern ($H^{\text{rev}} \circ H^{\text{refl}}(\eta(0))$) in ETPCA-0347. It is based on its T-symmetry under $H^{\text{rev}} \circ H^{\text{refl}}](image)
7 Concluding Remarks

In this paper, we investigated T-symmetries in reversible ESPCAs and reversible ETPCAs. The framework of PCAs is useful for formalising T-symmetries in reversible CAs. This is because the operation corresponding to the transformation of a momentum vector from $\mathbf{p}$ to $-\mathbf{p}$ in classical mechanics is simply expressed in reversible PCAs (see $H^\text{rev}$ in Sections 3.1 and 5.3). We have shown that a large number of reversible ESPCAs (except 640 ESPCAs) and all reversible ETPCAs are T-symmetric under simple involutions. As applications, the results are used for finding and analysing backward evolution processes of them. It is open whether the remaining 640 ESPCAs are T-symmetric under some involutions.

In this paper, we investigated only reversible PCAs such that they have specific neighbourhoods, and each part of a cell has two states. To investigate the cases where the neighbourhood is different, or each part has more than two states is left for the future study.

References

Appendix

A List of All Reversible ESPCAs, Their Dual Ones, and Inverses

In this appendix, we give identification numbers of all 1536 reversible ESPCAs, their dual ones (under reflection, complementation, and both), and their inverses. In each ESPCA, the IDs of local functions among \( f, f^r, f^c \) and \( f^{rc} \) that are equal to \( f^{-1} \) are marked by *. As shown in Theorems 3.3, 3.6, 3.8, and 3.10, if \( f^{-1} = f \), \( f^{-1} = f^r \), \( f^{-1} = f^c \), and \( f^{-1} = f^{rc} \), respectively, then the ESPCA is T-symmetric under the involution \( H^{rev} \circ H^{refl} \circ H^{comp} \) and \( H^{rev} \circ H^{refl} \). If there is no such function, then the ID of \( f^{-1} \) is marked by #. The mark c means that it is a conservative ESPCA. See also Table 2.
| F | F^2 | F^3 | F^4 | F^5 | C | F | F^2 | F^3 | F^4 | F^5 | C |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0d153f | 0d256f | 0b358f | 0b758f | 0b758f | 0e153f | 0b256f | 0b354f | 0b754f | 0b754f | 0b754f | 0b754f | 0b754f |
| 0d156f | 0d253f | 0b658f | 0b758f | 0b758f | 0e156f | 0b253f | 0b654f | 0b754f | 0b754f | 0b754f | 0b754f | 0b754f |
| 0d159f | 0d25cf | 0b958f | 0c758f | 0b758f | 0e159f | 0b25cf | 0b954f | 0c751f | 0b651f | 0b651f | 0b651f | 0b651f |
| 0d15cf | 0d25cf | 0bc58f | 0b958f | 0b758f | 0e15cf | 0b25cf | 0bc54f | 0b975f | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f |
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| 0da1af | 0da2af | 0b3a8f | 0b7a8f | 0b7a8f | 0e1af | 0b2af | 0b3a4f | 0b67a1f | 0b67a1f | 0b67a1f | 0b67a1f | 0b67a1f |
| 0da1cf | 0da2af | 0bc8f | 0b97a8 | 0b6a8f | 0e19f | 0b2acf | 0b94af | 0c7af | 0b6a1f | 0b6a1f | 0b6a1f | 0b6a1f |
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| 0da2cf | 0da1df | 0bca8f | 0b97a8 | 0b6a8f | 0e1acf | 0bcaaf | 0b97af | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f |
| 0da2af | 0da1df | 0bca8f | 0b97a8 | 0b6a8f | 0e1acf | 0bcaaf | 0b97af | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f |
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| 0da2af | 0da1df | 0bca8f | 0b97a8 | 0b6a8f | 0e1acf | 0bcaaf | 0b97af | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f | 0b6d1f |

Note: The table appears to be incomplete and contains various combinations of letters and numbers, possibly indicating a specific format or code.
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