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A Characterization of Subpluriharmonicity for a Function of Several Complex Variables

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Abstract
We give a characterization of a subpluriharmonic function of several complex variables in the sense of Fujita (J. Math. Kyoto Univ., 30:637–649, 1990) by using polynomial functions of degree at most two.

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1. Introduction

Let $D$ be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. According to Fujita \cite{Fujita}, we say that $u$ is subpluriharmonic if for every relatively compact domain $G$ in $D$ and for every real-valued pluriharmonic function $h$ defined near $\overline{G}$, the inequality $u \leq h$ on $\partial G$ implies the inequality $u \leq h$ on $\overline{G}$. If $n = 1$, then an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where $D$ is an open set of $\mathbb{C} = \mathbb{R}^2$, is subpluriharmonic if and only if $u$ is subharmonic.

By Yasuoka \cite[Theorem 1]{Yasuoka}, an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where $D$ is an open set of $\mathbb{C}$, is subharmonic if and only if for every open disk $B$ relatively compact in $D$ and for every polynomial $P(z)$ of a complex variable $z$ of degree at most two, the inequality $u(z) \leq \Re(P(z))$ on $\partial B$ implies the inequality $u(z) \leq \Re(P(z))$ on $\overline{B}$.

In this paper, we generalize this fact to several complex variables. That is to say, we prove that an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where $D$ is an open set of $\mathbb{C}^n$, is subpluriharmonic if and only if for every open ball $B$ relatively compact in $D$ and for every polynomial $P(z_1, z_2, ..., z_n)$ of $n$ complex variables $z_1, z_2, ..., z_n$ of degree at most two, the inequality $u(z) \leq \Re(P(z))$ on $\partial B$ implies the inequality $u(z) \leq \Re(P(z))$ on $\overline{B}$, where $z = (z_1, z_2, ..., z_n)$ (see Theorem 3.2).

2. Preliminaries

Let $N \in \mathbb{N}$ and let $F$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We denote by $\| \cdot \|$ the Euclidean norm on $F^N$, that is,

$$\| x \| = \left( \sum_{v=1}^{N} |x_v|^2 \right)^{1/2}$$

for every $x = (x_1, x_2, ..., x_N) \in F^N$. For every $c \in F^N$ and for every $r \in (0, +\infty)$, the set $B(c, r) := \{ x \in F^N \mid \| x - c \| < r \}$
is said to be the open ball of radius \( r \) with center \( c \) in \( \mathbb{F}^N \). For every point \( c \in \mathbb{F}^N \) and for every subset \( E \) of \( \mathbb{F}^N \), the number
\[
\text{dist}(c, E) := \inf \{ \| x - c \| \mid x \in E \}
\]
is said to be the distance from \( c \) to \( E \).

We denote by \( z_1, z_2, \ldots, z_n \) the complex coordinates of \( \mathbb{C}^n \). Let \( N \in \mathbb{N} \) and let \( D \) be an open set of \( \mathbb{C}^n \). A \( C^2 \) function \( u : D \to \mathbb{R} \) is said to be pluriharmonic if
\[
\frac{\partial^2 u}{\partial z_\mu \partial \bar{z}_\nu} = 0
\]
on \( D \) for every \( \mu, \nu = 1, 2, \ldots, n \) (see, for instance, Fritzsche-Grauert [1, p. 318]). An upper semicontinuous function \( u : D \to [-\infty, +\infty) \) is said to be subpluriharmonic if for every relatively compact open set \( G \) of \( D \) and for every pluriharmonic function \( h \) defined near \( \bar{G} \), the inequality \( u \leq h \) on \( \partial G \) implies the inequality \( u \leq h \) on \( \bar{G} \) (cf. Fujita [2, 3]). As is noted in Fujita [2, p. 638] (see also Fujita [3, Proposition 2]), the subpluriharmonic functions on \( D \) exactly coincide with the \( (n - 1) \) plurisubharmonic functions on \( D \) in the sense of Hunt-Murray [5, Definition 2.3].

By the second statement of Słodkowski [6, Lemma 4.4], we have the following proposition (see also Sugiyama [8, Proposition 2.1]).

**Proposition 2.1** (Słodkowski). Let \( D \) be an open set of \( \mathbb{C}^n \) and let \( u : D \to [-\infty, +\infty) \) be an upper semicontinuous function. If \( u \) is not subpluriharmonic on \( D \), then there exist \( c \in D, r \in (0, \text{dist}(c, \partial D)) \), \( K > 0 \), and a function \( f \) holomorphic near \( \bar{B}(c, R) \) such that \( u(c) = \Re(f(c)) \) and
\[
|Q(z)| \leq M \| z - c \|^2
\]
on \( \bar{B}(c, R) \).

3. Results on Subpluriharmonic Functions

We denote by \( \mathbb{C}[z_1, z_2, \ldots, z_n] \) the algebra of polynomial functions of \( n \) complex variables \( z_1, z_2, \ldots, z_n \) with coefficients in \( \mathbb{C} \). We have the following lemma which refines Proposition 2.1.

**Lemma 3.1.** Let \( D \) be an open set of \( \mathbb{C}^n \) and let \( u : D \to [-\infty, +\infty) \) be an upper semicontinuous function. If \( u \) is not subpluriharmonic on \( D \), then there exist \( c \in D, r \in (0, \text{dist}(c, \partial D)) \), \( K > 0 \), and \( P \in \mathbb{C}[z_1, z_2, \ldots, z_n] \) with \( \deg P \leq 2 \) such that
\[
u(c) = \Re(P(c)) \quad \text{and} \quad \nu \leq \Re(f) - K \| z - c \|^2
\]
on \( \bar{B}(c, R) \).

**Proof.** By Proposition 2.1, there exist \( c \in D, R \in (0, \text{dist}(c, \partial D)) \), \( L > 0 \), and a function \( f \) holomorphic near \( \bar{B}(c, R) \) such that \( u(c) = \Re(f(c)) \) and
\[

|Q(z)| \leq M \| z - c \|^2
\]
on \( \bar{B}(c, R) \). Then, we have \( P \in \mathbb{C}[z_1, z_2, \ldots, z_n] \), \( \deg P \leq 2 \), \( f = P + Q \) on \( \bar{B}(c, R) \) and there exists \( M > 0 \) such that
\[
|Q(z)| \leq M \| z - c \|^2
\]
on \( \bar{B}(c, R) \). Take an arbitrary \( K \in (0, L) \). For any \( r \in (0, \min[R, (L - K)/M]) \), we have
\[
\Re(P - K \| z - c \|^2) = \Re(f) - \Re(Q) - K \| z - c \|^2 \\
\geq u + L \| z - c \|^2 - M \| z - c \|^2 - K \| z - c \|^2 = u + (L - K) \| z - c \|^2 \geq u
\]
on \( \bar{B}(c, r) \). On the other hand, we have that
\[
\Re(P(c)) = \Re(f(c)) = u(c).
\]

We have the following theorem, which generalizes Yaukoka [9, Theorem 1] to several complex
variables and also refines the first statement of Slodkowski [6, Lemma 4.4].

**Theorem 3.2.** Let $D$ be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following two conditions are equivalent.

1. $u$ is subpluriharmonic.
2. For every $c \in D$, there exists $R \in (0, \text{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ with $\deg P \leq 2$, the inequality $u \leq \Re(P)$ on $\overline{B(c, r)}$ implies the inequality $u \leq \Re(P)$ on $B(c, r)$.

**Proof.** (1) $\Rightarrow$ (2). Since the real part of a holomorphic function is pluriharmonic, the assertion follows.

(2) $\Rightarrow$ (1). Suppose that $u$ is not subpluriharmonic. Take an arbitrary $R > 0$. Then, by Lemma 3.1, there exist $c \in D$, $r \in (0, \min(\text{dist}(c, \partial D)))$, $K > 0$, and $Q \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ with $\deg Q \leq 2$ such that $u(c) = \Re(Q(c))$ and $u \leq \Re(Q) - K\|z - c\|^2$ on $\overline{B(c, r)}$. Then, $P := Q - Kr^2 \in \mathbb{C}[z_1, z_2, \ldots, z_n]$, $\deg P \leq 2$, and $u \leq \Re(Q) - Kr^2 = \Re(P)$ on $\partial B(c, r)$ while $u(c) > \Re(P(c))$, which is a contradiction. 

**Corollary 3.3** (cf. Yasuoka [9, Theorem 1]). Let $D$ be an open set of $\mathbb{C}$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following two conditions are equivalent.

1. $u$ is subharmonic.
2. For every $c \in D$, there exists $R \in (0, \text{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{C}[z_1, z_2, \ldots, z_n]$ with $\deg P \leq 2$, the inequality $u \leq \Re(P)$ on $\overline{B(c, r)}$ implies the inequality $u \leq \Re(P)$ on $B(c, r)$.

**Remark 3.4.** For a related characterization of pseudoconvexity of a domain in $\mathbb{C}^n$ or over $\mathbb{C}^n$ by using polynomial functions of degree at most two, see Sugiyama [7, Theorem 3.1] as well as Yasuoka [9, Theorem 2].

**Example 3.6.** Let $n \in \mathbb{N}$ and let

$$u(z) := x_1^2 - 2y_1^2 - \sum_{v=2}^{n} |z_v|^2$$

for every $z \in D = \mathbb{C}^n$, where $z_1 = x_1 + iy_1$. Then, $u$ is not subpluriharmonic while $u$ satisfies condition (2)' in Remark 3.5.

**Proof.** By Fujita [2, Proposition 5], a $C^2$ function is subpluriharmonic if and only if its complex Hessian matrix has at least one nonnegative eigenvalue at any point. Since

$$\left( \frac{\partial^2 u}{\partial z_{\mu} \partial \overline{z}_v} \right) = \begin{pmatrix} -\frac{1}{2} & -1 & \cdots & 0 \\ -1 & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & -1 \end{pmatrix},$$

the function $u$ is not subpluriharmonic on $\mathbb{C}^n$. Let $G$ be an arbitrary relatively compact open set of $\mathbb{C}^n$. Let

$$P(z) := \sum_{v=1}^{n} c_vz_v + d,$$

where $c_1, c_2, \ldots, c_n, d \in \mathbb{C}$, and assume that $u \leq \Re(P)$ on $\partial G$. Let

$$\mu : \mathbb{C}^n \to \mathbb{C}^n, \quad z \mapsto \left( x_1 + \frac{i}{\sqrt{2}} y_1, z_2, \ldots, z_n \right),$$

which is an $\Re$-linear isomorphism of $\mathbb{C}^n$. Since
We have the following characterization of the subharmonic functions of several real variables, which resembles Theorem 3.2.

**Theorem 4.2.** Let \( D \) be an open set of \( \mathbb{R}^N \) and let \( u : D \to [-\infty, +\infty) \) be an upper semicontinuous function. Then, the following two conditions are equivalent.

(1) \( u \) is subharmonic.

(2) For every \( c \in D \), there exists \( R \in (0, \text{dist}(c, \partial D)) \) such that for every \( r \in (0, R] \) and for every \( P \in \mathbb{R}[x_1, x_2, \ldots, x_N] \) with \( \deg P \leq 2 \) which is harmonic on \( \mathbb{R}^N \), the inequality \( u \leq P \) on \( \partial B(c, r) \) implies the inequality \( u \leq P \) on \( B(c, r) \).

**Proof.** (1) \( \rightarrow \) (2). The assertion is clear.

(2) \( \rightarrow \) (1). Suppose that \( u \) is not subharmonic. Take an arbitrary \( R > 0 \). Then, by Lemma 4.1, there exist \( c \in D \),

\[ r \in (0, \min\{R, \text{dist}(c, \partial D)\}) \]

and \( Q \in \mathbb{R}[x_1, x_2, \ldots, x_N] \) with \( \deg Q \leq 2 \) such that \( u(c) = Q(c) \), \( \Delta Q < 0 \), and \( u \leq Q \) on \( \overline{B(c, r)} \).

Then, \( \Delta Q = -2NK \) on \( \mathbb{R}^N \) for some constant \( K > 0 \). Let \( P := Q + K|\|x - c\|^2 - r^2| \).

Then, \( P \in \mathbb{R}[x_1, x_2, \ldots, x_N] \), \( \deg P \leq 2 \), \( \Delta P = 0 \) on \( \mathbb{R}^n \), and \( P = Q \geq u \) on \( \partial B(c, r) \), although

\[ P(c) = Q(c) - Kr^2 = u(c) - Kr^2 < u(c) \],

which is a contradiction.

**Remark 4.3.** As Example 4.4 below shows, if \( N \geq 2 \), then we cannot replace condition (2) by the following condition (2)' in Theorem 4.2.

(2)' For every \( c \in D \), there exists \( R \in (0, \text{dist}(c, \partial D)) \) such that for every \( r \in (0, R] \) and for every \( P \in \mathbb{R}[x_1, x_2, \ldots, x_N] \) with \( \deg P \leq 1 \), the inequality \( u \leq P \) on \( \partial B(c, r) \) implies the inequality \( u \leq P \) on \( B(c, r) \).

**Example 4.4.** Let \( N \geq 2 \) and let

\[ u(x) := x_1^2 - 2x_2^2 \]
for every \( x = (x_1, x_2, ..., x_N) \in D = \mathbb{R}^N \). Then, \( u \) is not subharmonic while \( u \) satisfies condition (2)' in Remark 4.3.

**Proof.** Since \( \Delta u = -2 \) on \( \mathbb{R}^N \), the function \( u \) is not subharmonic on \( \mathbb{R}^N \) (see, for instance, Hörmander [4, p. 146]). Take an arbitrary relatively compact open set \( G \) of \( \mathbb{R}^N \) and arbitrary \( a_1, a_2, ..., a_N, b \in \mathbb{R} \). Let

\[
P(x) := \sum_{k=1}^{N} a_k x_k + b
\]

and assume that \( u \leq P \) on \( \partial G \). Let

\[
\mu : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto \left( x_1, \frac{x_2}{\sqrt{2}}, x_3, ..., x_N \right),
\]

which is an \( \mathbb{R} \)-linear isomorphism of \( \mathbb{R}^N \). Since \((u \circ \mu)(x) = x_1^2 - x_2^2\), the function \( u \circ \mu \) is harmonic on \( \mathbb{R}^N \) and therefore is subharmonic on \( \mathbb{R}^N \).

Since the function

\[
(P \circ \mu)(x) = a_1 x_1 + \frac{a_2}{\sqrt{2}} x_2 + \sum_{k=3}^{N} a_k x_k + b
\]

is harmonic on \( \mathbb{R}^N \) and satisfies \( u \circ \mu \leq P \circ \mu \) on \( \mu^{-1}(\partial G) = \partial (\mu^{-1}(G)) \), we have that \( u \circ \mu \leq P \circ \mu \) on \( \mu^{-1}(G) = \mu^{-1}(G) \). It follows that \( u \leq P \) on \( G \) and therefore condition (2)' in Remark 4.3 is satisfied.

\[\blacksquare\]

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