Rational curves on a smooth Hermitian surface

(非特異エルミート曲面上の有理曲線)
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主論文
RATIONAL CURVES ON A SMOOTH HERMITIAN SURFACE

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Abstract. We study the set $R$ of nonplanar rational curves of degree $d < q + 2$ on a smooth Hermitian surface $X$ of degree $q + 1$ defined over an algebraically closed field of characteristic $p > 0$, where $q$ is a power of $p$. We prove that $R$ is the empty set when $d < q + 1$. In the case where $d = q + 1$, we count the number of elements of $R$ by showing that the group of projective automorphisms of $X$ acts transitively on $R$ and by determining the stabilizer subgroup. In the special case where $X$ is the Fermat surface, we present an element of $R$ explicitly.

1. Introduction

Let $q$ be a power of a prime $p$, and $k$ an algebraic closure of the finite field $\mathbb{F}_q$. For a matrix $m$ with entries in $k$, we denote by $m^{(q)}$ the matrix whose entries are the $q$-th power of those of $m$. We denote by a column vector $x = (x_0, x_1, x_2, x_3)$ a point in the $k$-projective space $\mathbb{P}^3$. Let $A$ be a nonzero 4-by-4 matrix with entries in $k$. A $k$-Hermitian surface $X_A$ is defined by

$$X_A := \{ x \in \mathbb{P}^3 \mid \langle x, Ax^{(q)} \rangle = 0 \}.$$ 

If $A$ is a Hermitian matrix, namely $A$ has the entries in $\mathbb{F}_{q^2}$ and $A = A^{(q)}$, the surface $X_A$ is called a Hermitian surface. It is easily shown that $X_A$ is smooth if and only if $A$ is invertible.

The geometry of Hermitian varieties was systematically investigated by B. Segre in [8]. Especially, the number of linear spaces lying on a Hermitian variety and their configuration were considered. It was shown that the numbers of points and lines on a smooth Hermitian surface in $\mathbb{P}^3(\mathbb{F}_{q^2})$ are equal to $(q^3 + 1)(q^2 + 1)$ and $(q^3 + 1)(q + 1)$ respectively, and no plane is contained. Further, the set of points and lines on a smooth Hermitian surface forms a block design, see also [3]. In recent years, the number of rational normal curves totally tangent to a smooth Hermitian variety $X$ has been determined in [10] by considering the action of the automorphism group of $X$ on the set of the curves. In [11], non-singular conics totally tangent to the smooth Hermitian curve of degree 6 in characteristic 5 were utilized for a geometric construction of strongly regular graphs. On the other hand, projective isomorphism classes of degenerate Hermitian varieties of
corank 1 and the automorphism group of each isomorphism class have been determined in [7].

Let $A$ be an invertible 4-by-4 matrix with entries in $k$. We will be concerned with rational curves of degree $> 1$ on a smooth $k$-Hermitian surface $X_A$. Let $d$ be a positive integer and $F$ a 4-by-$(d + 1)$ matrix of rank$(F) \geq 2$ with entries in $k$. A rational curve $C_F$ of degree $d$ in $\mathbb{P}^3$ is the image of a rational map

$$\mathbb{P}^1 \ni (s, t) \mapsto F{(s^d, s^{d-1}t, \ldots, st^{d-1}, t^d)} \in \mathbb{P}^3.$$  

We call rank$(F)$ the rank of the curve $C_F$. If rank$(F) = 2$, then $C_F$ degenerates to a line. If rank$(F) = 3$, then $C_F$ degenerates to a plane curve of degree $\geq 2$. When rank$(F) = 4$, the curve $C_F$ is nondegenerate and is a space curve of degree $\geq 3$. Then $C_F$ is said to be nonplanar, namely $C_F$ is not contained in any plane. Thus the study of rational curves of rank 2 on $X_A$ is reduced to that of lines on $X_A$. Further, an algebraic curve of rank 3 on $X_A$ is a smooth $k$-Hermitian curve of degree $q + 1$, which is of genus $q(q - 1)/2 > 0$. Hence we may restrict ourselves to the case of rank 4.

Our results are as follows:

**Theorem 1.1.** There is no nonplanar rational curve of degree $\leq q$ on a smooth $k$-Hermitian surface.

Let $R$ be the set of nonplanar rational curves of degree $q + 1$ on a smooth $k$-Hermitian surface $X_A$. As will be seen later, the set $R$ is nonempty and each element is projectively isomorphic over $k$ to the smooth curve

$$C_0 := \{ (s^{q+1}, s^qt, s^qt^q, t^{q+1}) \in \mathbb{P}^3 \mid (s, t) \in \mathbb{P}^1 \}.$$  

We denote by Aut$(X_A)$ the group of projective automorphisms of $X_A$. Let $n$ be a positive integer. We deal with the group PGU$_n(F_{q^2})$ defined by

$$\{ Q \in \text{GL}_n(F_{q^2}) \mid Q^{-1}Q^{(q)}(q) = I \}/\mu_{q+1}I,$$

where $\mu_{q+1}$ denotes the group of $(q+1)$-th roots of unity and $I$ denotes the unit matrix. As is well-known, the group Aut$(X_A)$ is isomorphic to PGU$_4(F_{q^2})$. Then we shall prove the following theorem.

**Theorem 1.2.** The group Aut$(X_A)$ acts transitively on the set $R$, and the stabilizer subgroup is isomorphic to PGU$_2(F_{q^4})$.

By Theorem 1.2, the cardinality of $R$ is equal to $|\text{PGU}_4(F_{q^2})|/|\text{PGU}_2(F_{q^4})|$. We know by [6, pp.64-65] that

$$|\text{PGU}_4(F_{q^2})| = q^6(q^4 - 1)(q^3 + 1)(q^2 - 1) \text{ and } |\text{PGU}_2(F_{q^4})| = q^2(q^4 - 1).$$

Thus we have the following.

**Corollary 1.3.** $|R| = q^4(q^3 + 1)(q^2 - 1)$.

The number $|R|$ is 432, 18144, 249600, 1890000, 39645312, 383162400, ... as $q = 2, 3, 4, 5, 7, 9, \ldots$. 
In the special case where \( A = I \), that is, where the surface \( X_A \) is the Fermat surface, we can explicitly give an element \( C_F \) of \( R \) such as
\[
\{(\eta^{-q}q^s q^t + 1 - \eta^{-q}q^t + 1, s q^t, t s^q, \omega \eta^{-1}q^s t^q + \omega \eta^{-1}q^t + 1) \in \mathbb{P}^3 \mid (s, t) \in \mathbb{P}^1 \},
\]
where \( \omega, \xi, \) and \( \eta \) are elements of \( \mathbb{F}_{q^2} \) satisfying \( \omega q^2 = -1, \xi q^2 = 1 \) with \( \xi^2 \neq -1 \), and \( \eta^q + 1 = q^q + \xi \). Note that \( \eta \neq 0 \) because \( \xi^2 \neq 0, -1 \). The curve \( C_{F, \xi} \) is smooth since it is projectively isomorphic to the smooth curve \( C_0 \). On the other hand, a complete set of representatives for \( \text{Aut}(X_I) \) can be taken from \( \text{GL}_4(\mathbb{F}_{q^2}) \) (see Lemma 4.1). Therefore we have the following.

**Corollary 1.4.** All nonplanar rational curves of degree \( q + 1 \) on \( X_I \) are projectively isomorphic over \( \mathbb{F}_{q^2} \) to the smooth curve \( C_{F, \xi} \).

In the case where \( q = 2 \), we have \( |X_I(\mathbb{F}_{q^2})| = 45 \) where \( X_I(\mathbb{F}_{q^2}) \) denotes the set of \( \mathbb{F}_{q^2} \)-rational points of \( X_I \), and \( \text{Aut}(X_I) \) is of order 25920. Then \( |C_F(\mathbb{F}_{q^2})| = 5 \) for each nonplanar cubic \( C_F \) on \( X_I \). We can actually obtain by computation 432 nonplanar cubics on \( X_I \) and the stabilizer subgroup of \( \text{Aut}(X_I) \) fixing \( C_{F, \xi} \) of order 60. By restricting \( X_I \) to \( X_I(\mathbb{F}_{q^2}) \), we can verify that each cubic intersects 150 other cubics at a single point, 40 other cubics at two points and another cubic at five points. Here, when we say two cubics \( C_F, C_{F'} \) intersect at \( n \) points we mean \( |C_F(\mathbb{F}_{q^2}) \cap C_{F'}(\mathbb{F}_{q^2})| = n \). We can also verify that \( \text{Aut}(X_I) \) acts transitively on \( X_I(\mathbb{F}_{q^2}) \) and the stabilizer subgroup is of order 576, and furthermore, there are 48 cubics passing through each point of \( X_I(\mathbb{F}_{q^2}) \). These computational data files obtained by using \( \text{GAP} \) [4] are available upon request addressed to the author.

We give a brief outline of our paper. In the next section, we prove Theorem 1.1. By the same argument, we show directly that each irreducible conic, which is a rational curve of rank 3, is not contained in \( X_A \). In section 3, we give a bijection between the set \( R \) and the quotient of certain sets consisting of invertible 4-by-4 matrices, by showing basic lemmas. In section 4, we first prove two lemmas which are necessary for our proof of Theorem 1.2. We prove Theorem 1.2 in the last of the section.

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**2. Proof of Theorem 1.1**

*Proof of Theorem 1.1.* Suppose that a nonplanar rational curve \( C_F \) defined by (1) is contained in a smooth \( k \)-Hermitian surface \( X_A \). Denoting by \( b_{i,j} \) the entries of the \((d + 1)\)-by-\((d + 1)\) matrix \( t F A F^q \), one has the identity
\[
\sum_{i,j=0}^d b_{i,j}s^{d-i+q(d-j)} t^i q^j = 0
\]
(2)
Therefore if \( d < q \), all the coefficients \( b_{i,j} \) must vanish because the exponents \((i + qj)\)'s are all different. This implies that \( t F A F^q = O \), but it is a contradiction. In fact, since \( \text{rank}(F) = 4 \) by definition, we can take an
invertible matrix $F^*$ consisting of linearly independent 4 column vectors of $F$. Then, however, $t^*F^*AF^*(q)$ must be $O$. If $d = q$, the coefficients $b_{i,j}$ must vanish except for $b_{q,l-1} = -b_{0,l}$ with $1 \leq l \leq q$. This implies that rank($t^*F^*AF^*(q)$) $\leq 2$, but it is a contradiction by the argument above. Hence we conclude that $C_F \not\subset X_A$.

\[\square\]

Remark 2.1. We can similarly give a proof for the case of irreducible conics. In fact, since an irreducible conic $C_F$ is of rank 3, we can make an invertible matrix $F^*$ consisting of linearly independent 3 column vectors of $F$ and a vector linearly independent to those vectors. Suppose that $C_F \subset X_A$. Since $d = 2 \leq q$, one has rank($t^*F^*AF^*(q)$) $\leq 2$ in the same argument as the above proof. Therefore the 4-by-4 matrix $t^*F^*AF^*(q)$ must be of rank 3 at the most, but $t^*F^*AF^*(q)$ is of rank 4 by definition. This is a contradiction. As we have seen, this proof is valid for rational curves which are of rank $\geq 3$ and degree $\leq q$.

3. Basic lemmas

In this section, we will prove some basic lemmas to prepare for our proof of Theorem 1.2. The following lemma gives a necessary and sufficient condition for a nonplanar rational curve of degree $q+1$ to be on a smooth $k$-Hermitian surface.

Lemma 3.1. Let $C_F$ be a nonplanar rational curve of degree $q+1$ defined by (1). The curve $C_F$ is contained in a smooth $k$-Hermitian surface $X_A$ if and only if the $(q+2)$-by-$(q+2)$ matrix $t^*F^*AF^*(q)$ is of the form

$$
\begin{pmatrix}
0 & b_{0,1} & 0, \ldots, 0 & 0 & b_{0,q+1} \\
0 & b_{1,1} & 0, \ldots, 0 & 0 & b_{1,q+1} \\
0 & 0 & 0, \ldots, 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0, \ldots, 0 & 0 & 0 \\
-b_{0,1} & 0 & 0, \ldots, 0 & -b_{0,q+1} & 0 \\
-b_{1,1} & 0 & 0, \ldots, 0 & -b_{1,q+1} & 0 
\end{pmatrix}.
$$

If the above condition is satisfied, the matrix $F$ is of the form

$$(f_{0}, f_{1}, 0, \ldots, 0, f_{q}, f_{q+1}).$$

Proof. As was seen above, the curve $C_F$ is contained in $X_A$ if and only if one has (2). In the present case where $d = q + 1$, if $C_F \subset X_A$ then the coefficients $b_{i,j}$ must vanish except for $b_{q,l-1} = -b_{0,l}$, $b_{q+1,l-1} = -b_{1,l}$ with $1 \leq l \leq q+1$. Since rank($F$) = 4, there are 4 column vectors $f_x, f_y, f_z, f_w$ of $F$ with $0 \leq x < y < z < w \leq q+1$ such that the matrix $F^* := (f_x, f_y, f_z, f_w)$ is invertible. Then none of $x, y, z, w$ is from 2 to $q - 1$ because $t^*F^*AF^*(q)$ is also invertible, and thus $x = 0, y = 1, z = q, w = q + 1$. Let $f_i$ be the $i$-th
column vector with $2 \leq i \leq q - 1$ of $F$. Then one has

$$\langle \mathbf{f}_i, \mathbf{A} \mathbf{F}^{(q)} \rangle = (0, 0, 0),$$

and thus $\mathbf{f}_i = \mathbf{0}$. Hence $F$ and $\mathbf{F}^{(q)} \mathbf{A}$ are of the form described above. The converse is obvious since (2) holds automatically.

A rational curve $C_F$ defined by (1) is also obtained by replacing $F$ by $\lambda F \varphi(q)$, where $\lambda$ is an element of the multiplicative group $k^\times$ and $\varphi$ is a homomorphism from $\text{GL}_2(k)$ to $\text{GL}_{d+1}(k)$ defined by the following: for each $\langle s, t \rangle \in k^2$ with $\langle s, t \rangle \neq \langle 0, 0 \rangle$ and $g \in \text{GL}_2(k)$, put $\langle u, v \rangle := g \langle s, t \rangle$, then

$$\varphi : \text{GL}_2(k) \rightarrow \text{GL}_{d+1}(k)$$

$$(g : \langle s, t \rangle \mapsto \langle u, v \rangle) \mapsto (\varphi(g) : \langle s', d-1 \rangle \mapsto \langle u', d-1 \rangle).$$

Indeed, it is obvious by definition that $\varphi(I) = I$. Putting $\langle x, y \rangle := h \langle u, v \rangle$ for each $h \in \text{GL}_2(k)$, one has

$$\varphi(hg) \langle s', d-1 \rangle = \varphi(h) \langle u', d-1 \rangle = \varphi(h) \varphi(g) \langle s', d-1 \rangle.$$

Hence $\varphi(hg) = \varphi(h) \varphi(g)$, and thus $\varphi(g) \in \text{GL}_{d+1}(k)$.

Conversely if there is a matrix $F'$ such that $C_F = C_{F'}$, then one has

$$F \langle s', d-1 \rangle = F' \langle u', d-1 \rangle \in \mathbb{P}^3.$$ 

This implies that there are homogeneous polynomials $f$, $f'$ of degree $d$ such that $f(s, t) = f'(u, v)$. Therefore there is an element $g$ of $\text{GL}_2(k)$ such that $\langle s', d-1 \rangle = g \langle u', d-1 \rangle$, and thus $F' = \lambda F \varphi(q)$ for some $\lambda \in k^\times$. Hence, denoting by $\text{Im}(\varphi)$ the image of $\varphi$, we see that the set $k^\times \mathbf{F} \text{Im}(\varphi)$ corresponds one-to-one with $C_F$.

Let $S$ be the set of matrices $F$ such that $\mathbf{F}^{(q)} \mathbf{A}$ satisfies the condition of Lemma 3.1. Then by Lemma 3.1, for each $F \in S$ the set $k^\times \mathbf{F} \text{Im}(\varphi)$ corresponds one-to-one with the nonplanar rational curve $C_F$ on $X_A$. Therefore one has the following bijection

$$k^\times \setminus S/\text{Im}(\varphi) \ni k^\times \mathbf{F} \text{Im}(\varphi) \mapsto C_F \in R.$$

By Lemma 3.1, we define the map

$$^*: S \ni F = (f_0, f_1, 0, \ldots, 0, f_q, f_{q+1}) \mapsto F^* = (f_0, f_1, f_q, f_{q+1}) \in S^*,$$

where $S^*$ is written as

$$S^* = \{ F^* \in \text{GL}_4(k) \mid ^* \mathbf{F}^* \mathbf{A} \mathbf{F}^{(q)} = D_B, \ B \in \text{GL}_2(k) \},$$

and $D_B$ is a matrix defined by

$$D_B := \begin{pmatrix} 0 & b_1 & 0 & b_2 \\ -b_1 & 0 & -b_2 & 0 \end{pmatrix} \in \text{GL}_4(k) \text{ for } B = (b_1, b_2) \in \text{GL}_2(k).$$

Further, we define the map \( \ast \) from \( \text{Im}(\varphi) \subset \text{GL}_{q+2}(k) \) to \( \text{Im}(\varphi) \ast \subset \text{GL}_4(k) \) as follows:

for every \( g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_2(k) \),

\[
\varphi(g) = \begin{pmatrix} \alpha^q+1 & \alpha^q \beta & \ldots & \alpha \beta^q & \beta^q+1 \\ \alpha^q \gamma & \alpha^q \delta & \ldots & \gamma \beta^q & \delta \beta^q \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha^{q+1} \beta \gamma & \ldots & \alpha \delta^{q+1} & \beta \delta^{q+1} & \delta \beta^{q+1} \\ \gamma^{q+1} \delta \gamma & \ldots & \gamma \delta^{q+1} & \delta \gamma^{q+1} & \delta^{q+1} \end{pmatrix} \rightarrow \varphi(g) \ast = \begin{pmatrix} \alpha^q+1 & \alpha^q \beta & \alpha \beta^q & \beta^q+1 \\ \alpha^q \gamma & \alpha^q \delta & \gamma \beta^q & \delta \beta^q \\ \alpha \gamma^q & \beta \gamma^q & \alpha \delta^q & \beta \delta^q \\ \gamma^{q+1} \delta \gamma & \gamma \delta^{q+1} & \delta \gamma^{q+1} & \delta^{q+1} \end{pmatrix},
\]

where \( \text{Im}(\varphi) \ast \) is written as

\[
\text{Im}(\varphi) \ast = \left\{ \begin{pmatrix} \alpha^q g & \beta^q g \\ \gamma^q g & \delta^q g \end{pmatrix} \in \text{GL}_4(k) \mid g \in \text{GL}_2(k) \right\}.
\]

Indeed, it is easy to see that \( \det(\varphi(g) \ast) = \det(g)^{2q+2} \) for every \( g \in \text{GL}_2(k) \), and thus \( \varphi(g) \ast \in \text{GL}_4(k) \).

We denote by \( \varphi \ast \) the composition of \( \varphi \) and \( \ast \), namely \( \varphi \ast(g) = \varphi(g) \ast \) for every \( g \in \text{GL}_2(k) \).

**Lemma 3.2.** The map \( \varphi \ast \) is a homomorphism from \( \text{GL}_2(k) \) to \( \text{GL}_4(k) \). There is the following natural bijection

\[
k^\times \backslash S/\text{Im}(\varphi) \longrightarrow k^\times \backslash S^*/\text{Im}(\varphi) \ast.
\]

**Proof.** For each

\[
g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad h = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \text{GL}_2(k),
\]

one has

\[
gh = \begin{pmatrix} \alpha x + \beta z & \alpha y + \beta w \\ \gamma x + \delta z & \gamma y + \delta w \end{pmatrix}.
\]

Therefore

\[
\varphi \ast(gh) = \begin{pmatrix} (\alpha x + \beta z)^q gh & (\alpha y + \beta w)^q gh \\ (\gamma x + \delta z)^q gh & (\gamma y + \delta w)^q gh \end{pmatrix}.
\]

On the other hand,

\[
\varphi \ast(g) \varphi \ast(h) = \begin{pmatrix} \alpha^q g & \beta^q g \\ \gamma^q g & \delta^q g \end{pmatrix} \begin{pmatrix} x^q h & y^q h \\ z^q h & w^q h \end{pmatrix}
\]
\[
= \begin{pmatrix} \alpha^q x^q gh + \beta^q z^q gh & \alpha^q y^q gh + \beta^q w^q gh \\ \gamma^q x^q gh + \delta^q z^q gh & \gamma^q y^q gh + \delta^q w^q gh \end{pmatrix}
\]
\[
= \begin{pmatrix} (\alpha^q x^q + \beta^q z^q) gh & (\alpha^q y^q + \beta^q w^q) gh \\ (\gamma^q x^q + \delta^q z^q) gh & (\gamma^q y^q + \delta^q w^q) gh \end{pmatrix}.
\]

Since the \( q \)-th power is an automorphism of \( k \), one has \( \varphi \ast(gh) = \varphi \ast(g) \varphi \ast(h) \) and thus \( \varphi \ast \) is a homomorphism from \( \text{GL}_2(k) \) to \( \text{GL}_4(k) \).
For each $F \in S$, $q \in \text{GL}_2(k)$, denoting by $a_{i,j}$ the entries of $\varphi(g)$, we can write the $j$-th column vector $g_j$ with $j \in \{0, 1, q, q + 1\}$ of $F\varphi(g)$ as
\[
g_j = \sum_{i \in \{0, q, q+1\}} a_{i,j} f_i,
\]
since $f_i = 0$ for $2 \leq i \leq q - 1$. Then it is immediate from definition that
\[
F^*\varphi_s(g) = (g_0, g_1, g_q, g_{q+1}),
\]
and thus $(F\varphi(g))^* = F^*\varphi_s(g)$. This implies that there is the natural map from $k^s/S/\text{Im}(\varphi)$ to $k^s/S^*/\text{Im}(\varphi)_*$. The bijectivity is obvious since by definition the map $S \to S^*$ is bijective.

By (3) and Lemma 3.2, one has the bijection
\[
k^s \setminus S^*/\text{Im}(\varphi)_* \ni k^s F^s/\text{Im}(\varphi)_* \mapsto C_F \in R.
\]

The following well-known proposition is useful. The readers may find a proof for example in [2] and [9, Proposition 2.5.].

**Proposition 3.3.** For each element $A$ of $\text{GL}_n(k)$, there is an element $B$ of $\text{GL}_n(k)$ such that $A = B B^q(q)$. If $A$ is a Hermitian matrix, then the matrix $B$ can be taken from $\text{GL}_n(\mathbb{F}_q \varphi)$.

By Proposition 3.3, it follows immediately that a smooth $k$-Hermitian (resp. Hermitian) surface is projectively isomorphic over $k$ (resp. $\mathbb{F}_q \varphi$) to the Fermat surface $X_I$.

We define the set
\[
M := \left\{ D_B := \begin{pmatrix} 0 & b_1 & 0 & b_2 \\ -b_1 & 0 & -b_2 & 0 \end{pmatrix} \in \text{GL}_4(k) \mid B = (b_1 \ b_2) \in \text{GL}_2(k) \right\}.
\]
Then the following map is surjective:
\[
S^* \ni F^* \mapsto 4^s A F^s(q) \in M.
\]
In fact, by Proposition 3.3 there is an element $D$ of $\text{GL}_4(k)$ such that $D_B = 4^s D D^q(q)$ for each $D_B \in M$. Similarly there is an element $A^*$ of $\text{GL}_4(k)$ such that $A = 4^s A^* A^q(q)$. Hence putting $F^* := A^* D$, one has $4^s F^* A F^s(q) = D_B$, and thus $F^* \in S^*$.

**Lemma 3.4.** The set $R$ is nonempty, and each element of $R$ is projectively isomorphic over $k$ to the smooth curve
\[
C_0 := \left\{ 4^q(s^{q+1}, s^q t, s^q t^{q+1}) \in \mathbb{P}^3 \mid 4^q(s, t) \in \mathbb{P}^1 \right\}.
\]

**Proof.** The set $S^*$ is nonempty by the surjectivity of the map (5). Hence by (4) the set $R$ is nonempty. For each element $C_F$ of $R$, it is obvious by definition that
\[
F^{-1} F = (e_1, e_2, 0, \ldots, 0, e_3, e_4) \text{ with } (e_1, e_2, e_3, e_4) = I.
\]
This implies that $C_F$ is projectively isomorphic over $k$ to $C_0$. Then by definition, the curve $C_0$ is smooth clearly.
Remark 3.5. It is known that each nonplanar nonreflexive curve of degree $q + 1$ is projectively isomorphic to the curve $C_0$ (cf. [1, Theorem 2]). For nonreflexive curves, see also [5]. Hence by Lemma 3.4, each element of $R$ is projectively isomorphic to each nonplanar nonreflexive curve of degree $q + 1$.

Remark 3.6. In the case where $A = I$, we can find an element of $R$. We put

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$  

Then the matrix $D_J$ is a Hermitian matrix. Hence by Proposition 3.3, there is an element $F_J^*$ of $\text{GL}_4(\mathbb{F}_{q^2})$ such that $tF_J^*F_J^*(q) = D_J$. Actually taking $F_J^*$ such as

$$\begin{pmatrix} \eta^{-q}\xi^q & 0 & 0 & -\eta^{-q} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega\eta^{-1}\xi & 0 & 0 & \omega\eta^{-1} \end{pmatrix}$$

for $\omega, \xi$ and $\eta$ as mentioned in Introduction, one has by (4) the corresponding curve $C_{F_J}$ lying on $X_I$.

4. Proof of Theorem 1.2

The group $\text{Aut}(X_A)$ of projective automorphisms of $X_A$ is equal to

$$\{Q \in \text{GL}_4(k) \mid {}^tQAQ(q) = \lambda A, \, \lambda \in k^\times \}/k^\times I.$$  

By Proposition 3.3, the group $\text{Aut}(X_A)$ is conjugate to $\text{Aut}(X_I)$ in $\text{PGL}_4(k)$.

We prove the following lemma on matrix groups of arbitrary rank because we need the lemma to our proof of Theorem 1.2.

Lemma 4.1. Let $n$ be a positive integer. The group $\text{PGU}_n(\mathbb{F}_{q^2})$ is isomorphic to

$$G := \{Q \in \text{GL}_n(k) \mid {}^tQQ(q) = \lambda I, \, \lambda \in k^\times \}/k^\times I.$$  

Proof. We consider the map

$$G \ni Qk^\times \mapsto \xi_\lambda Q\mu_{q+1} \in \text{PGU}_n(\mathbb{F}_{q^2}),$$

where $\lambda$ is the element of $k^\times$ satisfying ${}^tQQ(q) = \lambda I$ and $\xi_\lambda$ is an element of $k^\times$ satisfying $\xi_\lambda^{q+1} = \lambda^{-1}$. Then the map is well-defined. In fact, it is obvious that $(\xi_\lambda Q)(\xi_\lambda Q)(q) = I$, and the matrix $\xi_\lambda Q$ has the entries in $\mathbb{F}_{q^2}$ because $I$ is a Hermitian matrix. Hence $\xi_\lambda Q\mu_{q+1}$ belongs to $\text{PGU}_n(\mathbb{F}_{q^2})$.

Further, putting $P := \alpha Q$ for each $\alpha \in k^\times$, one has $^tPP(q) = \alpha^{q+1}\lambda I$. It is easily shown by definition that

$$\xi_{\alpha^{q+1}\lambda}\mu_{q+1} = \xi_{\alpha^{q+1}\lambda}\mu_{q+1}$$

and

$$\alpha\xi_{\alpha^{q+1}\lambda}\mu_{q+1} = \mu_{q+1}.$$  

Therefore we conclude that

$$\xi_{\alpha^{q+1}\lambda}P\mu_{q+1} = \xi_\lambda Q\mu_{q+1}. $$
Thus the map is independent of the choice of representatives for $G$.

Let $Q'k^\times$ be an element of $G$ with $Q'Q'(q) = \eta I$ for some $\eta \in k^\times$. Then one has

$$(\xi_\eta Q'\mu_{q+1})(\xi_\eta Q \mu_{q+1}) = \xi_\eta \lambda Q' Q \mu_{q+1},$$

since $\xi_\eta \xi_\eta \mu_{q+1} = \xi_\eta \lambda \mu_{q+1}$.

Hence the map is a homomorphism from $G$ to $\text{PGU}_n(F_{q^2})$. The injectivity and the surjectivity are immediate from definition.

By Lemma 4.1, the group $\text{Aut}(X_A)$ isomorphic to $\text{PGU}_4(F_{q^2})$.

The following lemma is a key ingredient in our proof of Theorem 1.2.

**Lemma 4.2.** For every $g, B \in \text{GL}_2(k)$, one has

$$t \varphi_*(g)D_B \varphi_*(g)(q) = \det(g)^q D_{gBg^{-1}}(q).$$

**Proof.** The proof is due to straightforward computation. We put

$$g := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad B := (b_1, b_2).$$

Then one has

$$\begin{align*}
t \varphi_*(g)D_B \varphi_*(g)(q) & = \begin{pmatrix} \alpha^q t^g & \gamma^q t^g \\ \beta^q t^g & \delta^q t^g \end{pmatrix} \begin{pmatrix} 0 & b_1 & 0 & b_2 \\ -b_1 & 0 & -b_2 & 0 \end{pmatrix} \begin{pmatrix} \alpha^q g(q) & \beta^q g(q) \\ \gamma^q g(q) & \delta^q g(q) \end{pmatrix} \\
& = \begin{pmatrix} -\gamma^q t^g b_1 & \alpha^q t^g b_1 & -\gamma^q t^g b_2 & \alpha^q t^g b_2 \\ -\delta^q t^g b_1 & \beta^q t^g b_1 & -\delta^q t^g b_2 & \beta^q t^g b_2 \end{pmatrix} \begin{pmatrix} \alpha^q + \gamma^q & \alpha^q \beta^q & \alpha^q \delta^q & \beta^q + \gamma^q \\ \alpha^q \gamma^q & \alpha^q \delta^q & \gamma^q \beta^q & \delta^q \beta^q \\ \alpha^q \beta^q \gamma^q & \alpha^q \beta^q \delta^q & \gamma^q \beta^q \delta^q & \beta^q \delta^q \gamma^q \\ \gamma^q + \delta^q & \gamma^q \delta^q & \delta^q \gamma^q & \delta^q + \gamma^q \end{pmatrix}
\end{align*}$$

Putting

$$t \varphi_*(g)D_B \varphi_*(g)(q) := \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \end{pmatrix}.$$
one has
\[
c_1 = -\alpha^q + q^q \gamma \beta^q \gamma b_1 + \alpha^q \gamma \alpha^q \gamma \gamma b_2 - \alpha^q \gamma \gamma \gamma b_2 + \gamma^q + q^q \gamma \gamma b_2 = 0,
\]
\[
c_2 = -\alpha^q \beta^q \gamma \gamma b_1 + \alpha^q \delta^q \alpha^q \gamma \gamma b_2 - \beta^q \gamma^q \gamma b_2 + \delta^q \gamma^q \alpha^q \gamma b_2 = \det(g)^q (\alpha^q \gamma b_1 + \gamma^q \gamma b_2) = \det(g)^q (\alpha^q, \gamma^q,)
\]
\[
c_3 = -\alpha^q \beta^q \gamma^q b_1 + \gamma^q \beta^q \alpha^q \gamma \gamma b_2 - \alpha^q \delta^q \gamma \gamma b_2 + \gamma^q \delta^q \alpha^q \gamma b_2 = 0,
\]
\[
c_4 = -\beta^q + q^q \gamma \beta \gamma b_1 + \delta^q \beta^q \alpha^q \gamma \gamma b_2 - \beta^q \delta^q \gamma \gamma b_2 + \delta^q + q^q \gamma \gamma b_2 = \det(g)^q (\beta^q \gamma b_1 + \gamma^q \gamma b_2) = \det(g)^q (\gamma b_1, b_2) \cdot (\gamma^q,)
\]
\[
c_5 = -\alpha^q + q^q \gamma \delta \gamma b_1 + \alpha^q \gamma^q \beta^q \gamma \gamma b_2 - \alpha^q \gamma \delta^q \gamma b_2 + \gamma^q + q^q \gamma \gamma b_2 = -\det(g)^q (\alpha^q \gamma b_1 + \gamma^q \gamma b_2) = -\det(g)^q (b_1, b_2) \cdot (\alpha^q, \gamma^q,)
\]
\[
c_6 = -\alpha^q \beta^q \delta \gamma \gamma b_1 + \alpha^q \delta^q \beta^q \gamma \gamma b_2 - \beta^q \gamma^q \delta \gamma \gamma b_2 + \delta^q \gamma^q \beta^q \gamma \gamma b_2 = 0,
\]
\[
c_7 = -\alpha^q \beta^q \delta \gamma \gamma b_1 + \gamma^q \beta^q \beta^q \gamma \gamma b_2 - \alpha^q \delta^q \delta^q \gamma b_2 + \gamma^q \delta^q \beta^q \gamma \gamma b_2 = -\det(g)^q (\beta^q \gamma b_1 + \gamma^q \gamma b_2) = -\det(g)^q (b_1, b_2) \cdot (\beta^q, \delta^q,)
\]
\[
c_8 = -\beta^q + q^q \gamma \beta \gamma b_1 + \delta^q \beta^q \beta^q \gamma \gamma b_2 - \beta^q \delta^q \delta \gamma \gamma b_2 + \delta^q + q^q \beta^q \gamma \gamma b_2 = 0.
\]
Hence one has
\[
(c_2, c_4) = \det(g)^q \gamma b g(q^2) = -(c_3, c_7), \quad c_1 = c_3 = c_6 = c_8 = 0.
\]
This completes the proof.

Proof of Theorem 1.2. We define an equivalence relation \(\sim\) on the set \(M\) as follows: \(D_B \sim D_B'\) for \(D_B, D_B' \in M\) if there is an element \(g \in \text{GL}_2(k)\) such that \(D_B' = c_\varphi^*(g) D_B c_{\varphi^*}(g)^q\). We denote by \(D_B^\varphi^*\) an equivalence class containing \(D_B\). On the other hand, the group \(\text{Aut}(X_A)\) acts on \(k^\times \backslash S^*/\text{Im}(\varphi)_s\) by multiplication from the left. Then the following map is bijective:
\[
\begin{align*}
\text{Aut}(X_A)_k^\times \backslash S^*/\text{Im}(\varphi)_s & \longrightarrow k^\times \backslash M/ \sim \\
\text{Aut}(X_A)_k^\times \text{F}^* \text{Im}(\varphi)_s & \longrightarrow k^\times (\text{F}^* \text{AF}^{*(q)})^\varphi^*.
\end{align*}
\]
Indeed, the surjectivity is obvious since the map (5) is surjective. If we assume that \(k^\times (\text{F}^* \text{AF}^{*(q)})^\varphi^* = k^\times (\text{F}_1^* \text{AF}_1^{*(q)})^\varphi^*\) for some \(F_1^* \in S^*\), then
we have
\[ t(F_1^*\varphi_*(g)F_1^*F^{-1})A(F_1^*\varphi_*(g)F_1^*F^{-1})(q) = \lambda A \]
for some \( g \in \text{GL}_2(k) \) and \( \lambda \in k^\times \). Therefore \( k^\times F_1^*\varphi_*(g)F_1^*F^{-1} \) belongs to \( \text{Aut}(X_A) \). This implies the injectivity, and thus bijectivity. By Proposition 3.3, there is an element \( B' \) of \( \text{GL}_2(k) \) such that \( B = B'B'^*(q^2) \) for each \( DB \in M \). Then by Lemma 4.2, one has
\[ t\varphi_*(B'^{-1})DB\varphi_*(B'^{-1})(q) = \det(B'^{-1})qD_1. \]
This implies that \( k^\times D_1^{\varphi_*(q)} = k^\times D_1^{\varphi_*(q^2)} \). Hence \( |k^\times M/\sim \rangle = 1 \) and thus \( |\text{Aut}(X_A)k^\times \text{S}^*/\text{Im}(\varphi)_*\rangle = 1 \), and by (4) one has \( |\text{Aut}(X_A)\text{R}^\times|=1 \). This proves half of our theorem.

Let \( \Gamma/k^\times I \) be the stabilizer subgroup of \( \text{Aut}(X_A) \) fixing the element \( k^\times F_1^*\text{Im}(\varphi)_* \) of \( k^\times \text{S}^*/\text{Im}(\varphi)_* \) such that \( tF_1^*AF_1^*(q) = D_I \). Then it follows immediately that
\[ \Gamma = F_1^*\text{Im}(\varphi)_*F_1^{-1} \cap \{ Q \in \text{GL}_2(k) \mid tQAQ(q) = \lambda A, \ \lambda \in k^\times \}. \]
Hence each element of \( \Gamma \) can be written as \( F_1^*\varphi_*(g)F_1^{-1} \) for some element \( g \) of \( \text{GL}_2(k) \) satisfying
\[ t(F_1^*\varphi_*(g)F_1^{-1})A(F_1^*\varphi_*(g)F_1^{-1})(q) = \lambda A \]
or equivalently,
\[ t\varphi_*(g)D_I\varphi_*(g)(q) = \lambda D_I \]
for \( \lambda \in k^\times \).

By Lemma 4.2, this equality is equivalent to \( tgg(q^2) = \lambda I \) for \( \lambda \in k^\times \). Consequently, one has the following isomorphism:
\[
\{ g \in \text{GL}_2(k) \mid tgg(q^2) = \lambda I, \ \lambda \in k^\times \}/k^\times I \xrightarrow{\psi} \Gamma/k^\times I \xrightarrow{\psi} F_1^*\varphi_*(g)F_1^{-1}k^\times.
\]
By Lemma 4.1, we conclude that \( \text{PGU}_2(\mathbb{F}_q^4) \simeq \Gamma/k^\times I \).

\[
\square
\]

References


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