DEGENERATION OF FERMAT HYPERSURFACES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. Let A be an $(n+1) \times (n+1)$ matrix with coefficients a_{ij} in k, and let X_A be a hypersurface of degree $q+1$ in the projective space $\mathbb{P}^n$ defined by $\sum a_{ij} x_i x_j^q = 0$. It is well-known that if the rank of A is $n+1$, the hypersurface X_A is projectively isomorphic to the Fermat hypersuface of degree $q+1$. We investigate the hypersurfaces X_A when the rank of A is n, and determine their projective isomorphism classes.

1. INTRODUCTION

We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. Let n be a positive integer. We denote by $M_{n+1}(k)$ the set of square matrices of size $n+1$ with coefficients in k. For a nonzero matrix $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$, we denote by X_A the hypersurface of degree $q+1$ defined by the equation

$$\sum a_{ij} x_i x_j^q = 0$$

in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. The following is well-known ([2], [10], [14], see also §4 of this paper).

Proposition 1.1. *Let $A = (a_{ij})_{0 \le i,j \le n} \in M_{n+1}(k)$ and $X_A \subset \mathbb{P}^n$ be as above. Then the following conditions are equivalent:*

(i) $\operatorname{rank}(A) = n+1$,

(ii) *X_A is smooth,*

(iii) *X_A is isomorphic to the Fermat hypersurface of degree $q+1$, and*

(iv) *there exists a linear transformation of coordinates $T \in GL_{n+1}(k)$ such that ${}^tTAT^{(q)} = I_{n+1}$, where ${}^tT$ is the transpose of T, $T^{(q)}$ is the matrix obtained from T by raising each coefficient to its q-th power, and I_{n+1} is the identity matrix.*

The Fermat hypersurface of degree $q+1$ defined over an algebraically closed field of positive characteristic p has been a subject of numerous papers. It has many interesting properties, such as supersingularity ([15], [16], [17]), or unirationality ([13], [15], [16]). Moreover, the hypersurface X_A associated with the matrix A with coefficients a_{ij} in the

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finite field $\mathbb{F}_{q^2}$, which is called a Hermitian variety, has also been studied for many applications, such as coding theory ([8]). (The general results on Hermitian varieties are due to Segre [11]; see also [6]). Therefore it is important to extend these studies to degenerate cases.

In the case where characteristic $p \neq 2$, the following is well-known and can be found in any standard textbook on quadratic forms: the hypersurface defined by the quadratic form $\sum a_{ij}x_ix_j = 0$ is projectively isomorphic to the hypersurface defined by

$$x_0^2 + \cdots + x_{r-1}^2 = 0,$$

where r is the rank of $A = (a_{ij})$. This result has been extended the case of characteristic 2 (see [3]). Therefore we have a question what is the normal form of the hypersurfaces defined by a form $\sum a_{ij}x_ix_j^q = 0$. When A satisfies ${}^tA = A^{(q)}$ and hence this form is the Hermitian form over $\mathbb{F}_q$, the hypersurface X_A is projectively isomorphic over $\mathbb{F}_{q^2}$ to

$$x_0^{q+1} + \cdots + x_{r-1}^{q+1} = 0,$$

where r is the rank of A ([5]).

In this paper, we classify the hypersurfaces X_A associated with the matrices A of rank n *over an algebraically closed field.* Note that two hypersurfaces $X_A,\ X_{A'}$ associated with the matrices A, A' are projectively isomorphic if and only if there exists a linear transformation $T \in GL_{n+1}(k)$ such that $A' = {}^tTAT^{(q)}$. In this case, we denote $A \sim A'$.

We define I_s to be the $s \times s$ identity matrix, and E_r to be the $r \times r$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In particular, $E_1 = (0)$ and E_0 is the 0×0 matrix. Throughout this paper, a blank in a block decomposition of a matrix means that all the components of the block are 0. Our main result is as follow.

Theorem 1.2. *Let $A = (a_{ij})_{0 \leq i,j \leq n}$ be a nonzero matrix in $M_{n+1}(k)$, and let X_A be the hypersurface of degree $q+1$ defined by $\sum a_{ij}x_ix_j^q = 0$ in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Suppose that the rank of A is n. Then the hypersurface X_A is projectively isomorphic to one of the hypersurfaces X_s associated with the matrices*

$$W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array}\right),$$

where $0 \leq s \leq n$. Moreover, if $s \neq s'$, then X_s and $X_{s'}$ are not projectively isomorphic.

Corollary 1.3. *If A is a general point of $\{A \in M_{n+1}(k) |\ rank(A) = n\}$, then $A \sim W_{n-1}$.*

Corollary 1.4. *Suppose that $n \geq 2, s < n$ and $(n, s) \neq (2, 0)$. Then X_s is rational.*

We also determine the automorphism group

$$\mathrm{Aut}(X_s) = \{g \in PGL_{n+1}(k) \mid g(X_s) = X_s\},$$

of the hypersurface X_s for each s. For $M \in GL_{n+1}(k)$, we denote by $[M] \in PGL_{n+1}(k)$ the image of M by the natural projection.

Theorem 1.5. *Let X_s be the hypersurface associated with the matrix W_s in the projective space $\mathbb{P}^n$. The projective automorphism group $\mathrm{Aut}(X_s)$ with $s \leq n-2$ is the group consisting of $[M]$, with*

$$M = \left(\begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline 0 & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array}\right),$$

where $T \in GL_{n-1}(k)$, $\mathbf{a}, \mathbf{c}$ are row vectors of dimension $n-1$, and $d, e \in k$, and they satisfy the following conditions:

(i) $[T] \in \mathrm{Aut}(X_s^{n-2})$, ${}^tTW_s'T^{(q)} = \delta W_s'$, $\delta = \delta^q \neq 0$, *where X_s^{n-2} is the hypersurface defined in $\mathbb{P}^{n-2}$ by the matrix*

$$W_s' = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array}\right)$$

(ii) $d = \delta$,

(iii) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot T^{(q)} = \delta(0, \cdots, 0, 1)$,

(iv) ${}^tTW_s' \cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0$,

(v) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot {}^t\mathbf{a}^{(q)} + ed^q = 0$.

Moreover, we have

$$\mathrm{Aut}(X_n) = \left\{ \left[\begin{array}{c|c} T_n & \\ \hline \mathbf{u} & 1 \end{array}\right] \;\middle|\; \begin{array}{l} {}^tT_nT_n^{(q)} = \lambda I_n, T_n \in GL_n(k),\ \lambda \neq 0, \\ \mathbf{u} \text{ is a row vector of dimension } n \end{array} \right\},$$

and

$$\mathrm{Aut}(X_{n-1}) = \left\{ \left[\begin{array}{c|c|c} T_{n-1} & & \\ \hline & \beta & \\ \hline & & 1 \end{array}\right] \;\middle|\; \begin{array}{l} {}^tT_{n-1}T_{n-1}^{(q)} = \beta^q I_{n-1}, \\ T_{n-1} \in GL_{n-1}(k),\ 0 \neq \beta \in k \end{array} \right\}$$

We give a brief outline of our paper. In §2, we prove Theorem 1.2 and its corollaries. In §3, we prove Theorem 1.5. In §4, we recall the proof of Proposition 1.1 because this proposition plays an important role in the proof of Theorem 1.2. In §5, we investigate the plane curve X_A associated with the matrix A of rank ≤ 2 in the projective plane $\mathbb{P}^2$, and recover Homma's unpublished work [9] (see Remark 5.2).

2. Proofs of Theorem 1.2 and its corollaries

We present several preliminary lemmas. The following remark may be helpful in reading the proof of lemmas.

Remark 2.1. *Let*

$$T = \begin{pmatrix} t_{00} & \cdots & t_{0n} \\ \vdots & & \vdots \\ t_{n0} & \cdots & t_{nn} \end{pmatrix}$$

be an invertible matrix. Suppose that $\sum a_{ij} x_i x_j^q = 0$ *is the equation associated to the matrix* $A = (a_{ij})_{0 \le i,j \le n}$. *Then the operation*

$$A \mapsto^t TAT^{(q)}$$

on the matrix is equivalent to the transformation of the coordinates

$$x_i \mapsto \sum_{j=0}^{n} t_{ij} x_j,$$

where $0 \le i \le n$.

Lemma 2.2. *Put*

$$G_{s,r} = \left(\begin{array}{c|c|c} I_s & & \\ \hline & E_r & \\ \hline \mathbf{a} & 0 \cdots 0\, 1 & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s-r+1} \\ 0 & 0 & \end{array} \right),$$

and

$$G_{s,r+2} = \left(\begin{array}{c|c|c} I_s & & \\ \hline & E_{r+2} & \\ \hline \mathbf{a}^{(q^2)} & 0 \cdots 0\, 1 & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s-r-1} \\ 0 & 0 & \end{array} \right),$$

where $s \ge 1, r \ge 0, n - s - r - 1 \ge 0$, *and* $\mathbf{a}$ *is a nonzero row vector of dimension* s. *Then*

$$G_{s,r} \sim G_{s,r+2}.$$

Proof. By the transformation

$$T_G = \left(\begin{array}{c|c|c|c|c} I_s & & -{}^t\mathbf{a} & & \\ \hline & I_r & & & \\ \hline & & 1 & & \\ \hline \mathbf{a}^{(q)} & & & 1 & \\ \hline & & & & I_{n-s-r-1} \end{array} \right),$$

we have

$${}^tT_G G_{s,r} T_G^{(q)} = G_{s,r+2}.$$

$\square$

Remark 2.3. *Lemma 2.2 holds when $r = 0$ or $n - s - r - 1 = 0$. In particular, when $n - s - r - 1 = 0$, we have $G_{s,r+2} = W_s$.*

Lemma 2.4. *Put*

$$H_{s,r} = \left(\begin{array}{c|c|c|c} D_{s-1} & -{}^t\mathbf{a}''\ 0\cdots 0 & & \\ \hline -\mathbf{a}' & & & \\ 0 & & & \\ \vdots & E_r & & \\ 0 & & & \\ \hline & 0\cdots 0\ 1 & 1 & \\ \hline & & 1 & \\ & & 0 & \\ & & \vdots & E_{n-s-r+1} \\ & & 0 & \end{array}\right),$$

where $s \geq 1, r \geq 2, n - s - r - 1 \geq 1$, $D_{s-1} \in M_{s-1}(k)$, $\mathbf{a}'$ and $\mathbf{a}''$ are row vectors of dimension $s - 1$. Then

$$H_{s,r} \sim H_{s,r+2}.$$

Proof. By the transformation

$$T_H = \left(\begin{array}{c|c|c|c|c} I_{s+r-1} & & & & \\ \hline & 1 & & & \\ \hline & -1 & 1 & 1 & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-r-1} \end{array}\right),$$

we have

$${}^tT_H H_{s,r} T_H^{(q)} = H_{s,r+2}.$$

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Lemma 2.5. *Put*

$$H'_{s,r} = \left(\begin{array}{c|c|c|c|c|c}
D_{s-1} & & & & & \\ \hline
-\mathbf{a}' & 0 & & & & \\ \hline
 & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_r & & & \\ \hline
 & & 0 \cdots 0\ 1 & 0 & 1 & \\ \hline
 & & & 1 & 0 & \\ \hline
 & & & & \begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s-r-1}
\end{array}\right),$$

where $s \geq 1, r \geq 2, n-s-r-3 \geq 1$, $D_{s-1} \in M_{s-1}(k)$, *and* $\mathbf{a}'$ *is a row vector of dimension* $s-1$. *Then*

$$H'_{s,r} \sim H'_{s,r+2}.$$

Proof. By the transformation

$$T_{H'} = \left(\begin{array}{c|c|c|c|c|c}
I_{s+r} & & & & & \\ \hline
 & 1 & & & & \\ \hline
 & & 1 & & 1 & \\ \hline
 & -1 & & 1 & & \\ \hline
 & & & & 1 & \\ \hline
 & & & & & I_{n-s-r-3}
\end{array}\right),$$

we have

$${}^{t}T_{H'} H'_{s,r} T_{H'}^{(q)} = H'_{s,r+2}.$$

$\square$

Remark 2.6. *Lemma 2.4 and 2.5 will be used only in the case where* $n-s+1$ *is odd. Hence we do not need to prove the case* $n-s-1=0$ *in Lemma 2.4 nor the case* $n-s-3=0$ *in Lemma 2.5.*

Lemma 2.7. *Put*

$$P_s = \left(\begin{array}{c|c}
I_s & \\ \hline
\begin{matrix} \mathbf{a} \\ 0 \\ \vdots \\ 0 \end{matrix} & E_{n-s+1}
\end{array}\right),$$

where $s \geq 1, n-s+1 \geq 1$, *and* $\mathbf{a}$ *is a nonzero row vector of dimension* s. *Then*

(1) *If* $n-s+1$ *is even, then* $P_s \sim W_s$.

(2) *If* $n-s+1$ *is odd, then*

$$P_s \sim B_{s-1} = \left(\begin{array}{c|c} D_{s-1} & \\ \hline \mathbf{b}_{s-1} & \\ 0 & \\ \vdots & E_{n-s+2} \\ 0 & \end{array}\right),$$

where $D_{s-1} \in M_{s-1}(k)$, $\mathbf{b}_{s-1}$ *is the row vector of dimension* $s-1$. *In particular, if* $s=1$ *and* n *is odd, then* $P_1 \sim W_0$.

Proof. (1) Suppose that $n-s+1$ is even. Using Lemma 2.2 and Remark 2.3, we have

$$P_s = G_{s,0} \sim G_{s,n-s+1} = W_s.$$

(2) Next suppose that $n-s+1$ is odd. By interchanging the coordinates $x_0, \cdots, x_{s-1}$, and scalar multiplication of the coordinates $x_s, \cdots, x_n$ if nessesary, we can show that

$$P_s \sim P'_s = \left(\begin{array}{c|c|c|c} I_{s-1} & & & \\ \hline & 1 & & \\ \hline \mathbf{a}' & 1 & 0 & \\ \hline & & 1 & \\ & & 0 & \\ & & \vdots & E_{n-s} \\ & & 0 & \end{array}\right),$$

with $\mathbf{a}'$ being a row vector of dimension $s-1$. By the transformation

$$T_1 = \left(\begin{array}{c|c|c|c} I_{s-1} & & & \\ \hline -\mathbf{a}'' & 1 & & \\ \hline & & 1 & \\ \hline & & & I_{n-s} \end{array}\right),$$

with $\mathbf{a}''^{(q)} = \mathbf{a}'$, we have

$$Q_s = {}^tT_1 P'_s T_1^{(q)} = \left(\begin{array}{c|c|c|c} D_{s-1} & -{}^t\mathbf{a}'' & & \\ \hline -\mathbf{a}' & 1 & & \\ \hline & 1 & 0 & \\ \hline & & 1 & \\ & & 0 & \\ & & \vdots & E_{n-s} \\ & & 0 & \end{array}\right),$$

where $D_{s-1} = I_{s-1} + {}^t\mathbf{a}'' \cdot \mathbf{a}'$. If $n-s+1=1$, by the transformation

$$T_2 = \left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & 1 & \\ \hline \mathbf{a}'' & -1 & 1 \end{array}\right),$$

we have

$$ {}^tT_2Q_nT_2^{(q)} = B_{n-1}. $$

Suppose that $n - s + 1 > 1$. Note that, since we are in the case where $n - s + 1$ is odd, we have $n - s + 1 \geq 3$. By the transformation

$$ T_3 = \left(\begin{array}{c|c|c|c|c} I_{s-1} & & & & \\ \hline & 1 & & & \\ \hline & -1 & 1 & 1 & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-1} \end{array}\right), $$

we have

$$ Q'_s = {}^tT_3Q_sT_3^{(q)} = \left(\begin{array}{c|c|c|c|c} D_{s-1} & -{}^t\mathbf{a}'' & & & \\ \hline -\mathbf{a}' & 0 & & & \\ \hline & 1 & 0 & & \\ \hline & & 1 & 1 & \\ \hline & & & \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} & E_{n-s-1} \end{array}\right) = H_{s,2}. $$

Using Lemma 2.4, we have

$$ Q'_s = H_{s,2} \sim H_{s,n-s} = Q''_s = \left(\begin{array}{c|c|c|c} D_{s-1} & -{}^t\mathbf{a}''\ 0\cdots 0 & & \\ \hline \begin{array}{c} -\mathbf{a}' \\ 0 \\ \vdots \\ 0 \end{array} & E_{n-s} & & \\ \hline & 0\cdots 0\ 1 & 1 & \\ \hline & & 1 & 0 \end{array}\right). $$

Then by the transformation

$$ T_4 = \left(\begin{array}{c|c|c} I_{n-1} & & \\ \hline & 1 & \\ \hline & -1 & 1 \end{array}\right), $$

we have

$$ R_s = {}^tT_4Q''_sT_4^{(q)} = \left(\begin{array}{c|c} D_{s-1} & -{}^t\mathbf{a}''\ 0\cdots 0 \\ \hline \begin{array}{c} -\mathbf{a}' \\ 0 \\ \vdots \\ 0 \end{array} & E_{n-s+2} \end{array}\right). $$

If $s = 1$, $R_1 \sim W_0$. Suppose that $s > 1$. By the transformation

$$T_5 = \left(\begin{array}{c|c|c|c|c} I_{s-1} & & & & \\ \hline & 1 & & 1 & \\ \hline \mathbf{a}'' & & 1 & & \\ \hline & & & 1 & \\ \hline & & & & I_{n-s-1} \end{array}\right),$$

we obtain

$$R'_s = {}^tT_5 R_s T_5^{(q)} = \left(\begin{array}{c|c|c|c|c} D_{s-1} & & & & \\ \hline -\mathbf{a}' & 0 & & & \\ \hline & 1 & 0 & 1 & \\ \hline & & 1 & 0 & \\ \hline & & & \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} & E_{n-s-1} \end{array}\right).$$

If $n - s - 1 = 1$, by the tranformation

$$T_6 = \left(\begin{array}{c|c|c|c} I_{n-2} & & & \\ \hline & 1 & & \\ \hline & & 1 & \\ \hline & -1 & & 1 \end{array}\right),$$

we have

$${}^tT_6 R'_{n-2} T_6^{(q)} = B_{n-3}.$$

Suppose that $n - s - 1 > 1$. Then by the transformation

$$T_7 = \left(\begin{array}{c|c|c|c|c|c} I_s & & & & & \\ \hline & 1 & & & & \\ \hline & & 1 & & 1 & \\ \hline & -1 & & 1 & & \\ \hline & & & & 1 & \\ \hline & & & & & I_{n-s-3} \end{array}\right),$$

we have

$$R''_s = {}^tT_7 R'_s T_7^{(q)} = \left(\begin{array}{c|c|c|c|c|c} D_{s-1} & & & & & \\ \hline -\mathbf{a}' & 0 & & & & \\ \hline & 1 & & & & \\ & 0 & E_2 & & & \\ \hline & & 0\,1 & 0 & 1 & \\ \hline & & & 1 & 0 & \\ \hline & & & & 1 & \\ & & & & 0 & \\ & & & & \vdots & E_{n-s-3} \\ & & & & 0 & \end{array}\right) = H'_{s,2}.$$

Using Lemma 2.5, we have

$$R''_s = H'_{s,2} \sim H'_{s,n-s-2} = R'''_s = \left(\begin{array}{c|c|c|c|c|c} D_{s-1} & & & & & \\ \hline -\mathbf{a}' & 0 & & & & \\ \hline & 1 & & & & \\ & 0 & & & & \\ & \vdots & E_{n-s-2} & & & \\ & 0 & & & & \\ \hline & & 0\cdots 0\,1 & 0 & 1 & \\ \hline & & & 1 & 0 & \\ \hline & & & & 1 & 0 \end{array}\right).$$

It is easy to see that

$${}^tT_6 R'''_s T_6^{(q)} = B_{s-1}.$$

$\square$

Lemma 2.8. *Put*

$$B_s = \left(\begin{array}{c|c} D_s & \\ \hline \mathbf{b}_s & \\ 0 & \\ \vdots & E_{n-s+1} \\ 0 & \end{array}\right),$$

where $s \geq 1,\ n-s+1 \geq 1$, $D_s \in M_s(k)$, and $\mathbf{b}_s$ is a row vector of dimension s. Suppose that the rank of B_s is n. Then

$$B_s \sim W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array}\right),$$

or

$$B_s \sim B_{s-1} = \left(\begin{array}{c|c} D_{s-1} & \\ \hline \mathbf{b}_{s-1} & \\ 0 & \\ \vdots & E_{n-s+2} \\ 0 & \end{array}\right),$$

where $D_{s-1} \in M_{s-1}(k)$, and $\mathbf{b}_{s-1}$ is a row vector of dimension $s-1$.

Proof. Suppose that $\det D_s \neq 0$. By Proposition 1.1, there exists a linear transformation of coordinates $T_D \in GL_s(k)$ such that ${}^tT_D D_s T_D^{(q)} = I_s$. By the transformation

$$T = \left(\begin{array}{c|c} T_D & \\ \hline & I_{n-s+1} \end{array}\right),$$

we have

$${}^tTB_sT^{(q)} = \left(\begin{array}{c|c} I_s & \\ \hline \mathbf{b}'_s & \\ 0 & \\ \vdots & E_{n-s+1} \\ 0 & \end{array}\right),$$

where $\mathbf{b}'_s = \mathbf{b}_s T_D^{(q)}$. If $\mathbf{b}'_s = 0$, then $B_s \sim W_s$. Suppose that $\mathbf{b}'_s \neq 0$. By Lemma 2.7, we have $B_s \sim W_s$, or $B_s \sim B_{s-1}$.

Suppose that $\det D_s = 0$. Then one row of the matrix D_s is a linear combination of the other rows. By interchanging coordinates $x_0, \cdots, x_{s-1}$ if nessesary, we can assume that the s-th row is a linear combination of the other rows. We write the matrix D_s as

$$D_s = \left(\begin{array}{c|c} P & {}^t\mathbf{g} \\ \hline \mathbf{h} & d \end{array}\right),$$

where $P \in M_{s-1}(k)$, $\mathbf{g}, \mathbf{h}$ are row vectors of dimension $s-1$, $d \in k$, and that satisfy $\mathbf{h} = \mathbf{w}P, d = \mathbf{w}{}^t\mathbf{g}$ with $\mathbf{w}$ being a row vector of dimension $s-1$. Then

$$B_s \sim B'_s = \left(\begin{array}{c|c|c} P & {}^t\mathbf{g} & \\ \hline \mathbf{h} & d & \\ \hline \mathbf{f} & e & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \end{array}\right),$$

where $\mathbf{f}$ is a row vector of dimension $s-1$, and $e \in k$. By the tranformation

$$T' = \left(\begin{array}{c|c|c} I_{s-1} & -{}^t\mathbf{w} & \\ \hline & 1 & \\ \hline & & I_{n-s+1} \end{array}\right),$$

we obtain

$$B''_s = {}^tT'B'_sT'^{(q)} = \left(\begin{array}{c|c|c} P & -P\cdot {}^t\mathbf{w}^{(q)} + {}^t\mathbf{g} & \\ \hline & & \\ \hline \mathbf{f} & -\mathbf{f}\cdot {}^t\mathbf{w}^{(q)} + e & \\ 0 & 0 & \\ \vdots & \vdots & E_{n-s+1} \\ 0 & 0 & \end{array}\right).$$

Put

$$Q = \left(\begin{array}{c|c} P & -P\cdot {}^t\mathbf{w}^{(q)} + {}^t\mathbf{g} \\ \hline \mathbf{f} & -\mathbf{f}\cdot {}^t\mathbf{w}^{(q)} + e \end{array}\right).$$

Because the rank of B'_s is n, we have $\det Q \neq 0$. Let $Q' \in GL_s(k)$ such that $QQ'^{(q)} = I_s$,

$$Q' = \left(\begin{array}{c|c} P' & {}^t\mathbf{g}' \\ \hline \mathbf{f}' & e' \end{array}\right),$$

where $P' \in M_{s-1}(k)$, $\mathbf{g}', \mathbf{f}'$ are row vectors of dimension $s-1$, $e' \in k$. By the transformation

$$T'' = \left(\begin{array}{c|c|c} P' & {}^t\mathbf{g}' & \\ \hline \mathbf{f}' & e' & \\ \hline & & I_{n-s+1} \end{array}\right),$$

we obtain

$${}^tT''B''_sT''^{(q)} = \left(\begin{array}{c|c|c} {}^tP' & & \\ \hline \mathbf{g}' & 0 & \\ \hline & 1 & \\ & 0 & \\ & \vdots & E_{n-s+1} \\ & 0 & \end{array}\right).$$

Putting $D_{s-1} = {}^tP'$ and $\mathbf{b}_{s-1} = \mathbf{g}'$, we have $B''_s \sim B_{s-1}$. $\square$

Remark 2.9. *When $s = 1$, we have*

$$B_{s-1} = B_0 = E_{n+1} = W_0.$$

Now we prove Theorem 1.2 and Corollary 1.3.

Proof. Because the rank of the matrix A is n, Proposition 1.1 implies that the hypersurface X_A is singular. By using a linear transformation of coordinates if nessesary, we can assume that X_A has a singular point $(0, \cdots, 0, 1)$. Then we have $a_{in} = 0$ for any $0 \leq i \leq n$. The matrix A is now of the form

$$A = \left(\begin{array}{c|c} D_n & \\ \hline \mathbf{b}_n & \end{array}\right) = B_n,$$

where $D_n \in M_n(k)$, and $\mathbf{b}_n$ is a row vector of dimension n. Using Lemma 2.8 repeatedly and Remark 2.9, we have that the hypersurface X_A is isomorphic to one of the hypersurfaces defined by the matrixes W_s with $0 \leq s \leq n$.

If A is general, then $\det(D_n) \neq 0$, and hence by the first paragraph of the proof of Lemma 2.8 and Lemma 2.7, we have $A \sim W_{n-1}$.

Next we prove that $s \neq s'$ implies $W_s \not\sim W_{s'}$. For this, we introduce some notions. Let X_s^n be the hypersurface defined by the matrix W_s in the projective space $\mathbb{P}^n$. The defining equation of X_s^n can be written as

$$F_q x_n + F_{q+1} = 0,$$

where

$$F_q = \begin{cases} 0 & \text{if } s = n \\ x_{n-1}^q & \text{if } s < n, \end{cases}$$

and

$$F_{q+1} = \begin{cases} x_0^{q+1} + \cdots + x_{n-1}^{q+1} & \text{if } s = n \\ x_0^{q+1} + \cdots + x_{s-1}^{q+1} + x_s^q x_{s+1} + \cdots + x_{n-2}^q x_{n-1} & \text{if } s < n. \end{cases}$$

It is easy to see that X_s^n has only one singular point $P_0 = (0, \cdots, 0, 1)$. The variety of lines in $\mathbb{P}^n$ passing through P_0 can be naturally identified with the hypersurface $\mathcal{H}_\infty = \{x_n = 0\}$ in $\mathbb{P}^n$ by the correspondence $Q \in \mathcal{H}_\infty$ to the line $\overline{QP_0}$. Let φ be the map defined by

$$\begin{aligned} \varphi : \mathbb{P}^n \setminus \{P_0\} &\longrightarrow \mathbb{P}^{n-1} \\ P &\longmapsto \overline{PP_0}. \end{aligned}$$

Let $\overline{X_s^n} = \varphi(X_s^n \setminus \{P_0\})$. For $Q = (y_0, \cdots, y_{n-1}, 0) \in \mathcal{H}_\infty$, we consider the line

$$l = \overline{QP_0} = \{(\lambda y_0, \cdots, \lambda y_{n-1}, \mu) \mid (\lambda, \mu) \in \mathbb{P}^1\}.$$

We have $l \in \overline{X_s^n}$ if and only if there exists $P = (p_0, \cdots, p_{n-1}, p_n) \in X_s^n \setminus \{P_0\}$ satisfying $P \in l$, i.e. there exists an element $\mu \in k$ such that

$$(p_0, \cdots, p_{n-1}, p_n) = (y_0, \cdots, y_{n-1}, \mu),$$

for some $P \in X_s^n \setminus \{P_0\}$, or equivalently there exists an element $\mu \in k$ such that

$$F_q(y_0, \cdots, y_{n-1})\mu + F_{q+1}(y_0, \cdots, y_{n-1}) = 0.$$

Then

$$\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\}) = \begin{cases} \emptyset & \text{if } F_q(y_0, \ldots, y_{n-1}) = 0 \text{ and} \\ & \quad F_{q+1}(y_0, \ldots, y_{n-1}) \neq 0, \\ \{\text{a single point}\} & \text{if } F_q(y_0, \ldots, y_{n-1}) \neq 0, \\ l \setminus \{P_0\} & \text{if } F_q(y_0, \ldots, y_{n-1}) = 0 \text{ and} \\ & \quad F_{q+1}(y_0, \ldots, y_{n-1}) = 0. \end{cases}$$

Putting $V_s = \{F_q = 0,\ F_{q+1} = 0\} \subset \mathbb{P}^{n-1}$, and $H_s = \{F_q = 0\} \subset \mathbb{P}^{n-1}$, we have

$$V_s = \begin{cases} X_s^{n-2} & \text{if } s \leq n-2, \\ \text{nonsingular Fermat hypersurface in } \mathbb{P}^{n-1} & \text{if } s = n, \\ \text{nonsingular Fermat hypersurface in } \mathbb{P}^{n-2} & \text{if } s = n-1, \end{cases}$$

where X_s^{n-2} is the hypersurface in $\mathbb{P}^{n-2}$ associated with the matrix

$$\left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array}\right).$$

For any $s \neq s'$, suppose that X_s^n and $X_{s'}^n$ are isomorphic and let $\psi : X_s^n \longrightarrow X_{s'}^n$ be an isomorphism. Because each of X_s^n and $X_{s'}^n$ has only one singular point P_0, we have $\psi(P_0) = P_0$, and hence ψ induces an isomorphism $\overline{\psi}$ from $\overline{X_s^n}$ to $\overline{X_{s'}^n}$. For any line $l \in \overline{X_s^n}$ and $l' \in \overline{X_{s'}^n}$ such that $\overline{\psi}(l) = l'$, we have

$$\sharp(\varphi^{-1}(l) \cap (X_s^n \setminus \{P_0\})) = \sharp(\varphi^{-1}(l') \cap (X_{s'}^n \setminus \{P_0\})).$$

Thus $V_s \cong V_{s'}$ and $H_s \cong H_{s'}$. Hence for any $s \neq s'$, if $V_s \ncong V_{s'}$ or $H_s \ncong H_{s'}$ then $X_s^n \ncong X_{s'}^n$.

In the case $n = 1$, we have that X_0^1 consists of two points, and X_1^1 consists of a single point. In the case $n = 2$, we have that X_0^2 consists of two irreducible components, X_1^2 is irreducible, and X_2^2 consists of $(q+1)$ lines. Hence, in the case $n = 1$ and $n = 2$, we see that $s \neq s'$ implies $W_s \not\sim W_{s'}$. By induction on n, we have the proof. □

Next we prove Corollary 1.4.

Proof. Under the condition $n \geq 2, s < n$ and $(n, s) \neq (2, 0)$, we have x_{n-1} does not divide F_{q+1}, and hence V_s is of codimension 2 in $\mathbb{P}^{n-1}$. By induction on n, X_s^n is irreducible. The morphism

$$\varphi|_{X_s^n \setminus \{P_0\}} : X_s^n \setminus \{P_0\} \longrightarrow \mathcal{H}_\infty \cong \mathbb{P}^{n-1}$$

is birational with the inverse rational map

$$Q = (y_0, \cdots, y_{n-1}, 0) \longmapsto \left(y_0, \cdots, y_{n-1}, -\frac{F_{q+1}(y_0, \cdots, y_{n-1})}{y_{n-1}^q}\right).$$

□

3. Proof of Theorem 1.5

For any $s \leq n-2$, the matrix W_s can be written

$$W_s = \left(\begin{array}{c|c|c} W'_s & & \\ \hline 0 \cdots 0\ 1 & 0 & \\ \hline & 1 & 0 \end{array}\right).$$

For any $g \in \mathrm{Aut}(X_s)$, we have $g(P_0) = P_0$ because X_s has only one singular point $P_0 = (0, \cdots, 0, 1)$. The automorphism g is defined by a matrix of the form

$$M = \left(\begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline \mathbf{b} & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array} \right),$$

where $T \in M_{n-1}(k)$, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are row vectors of dimension $n-1$, $d, e \in k$. We have ${}^tMW_sM^{(q)} = \delta W_s$ for some $0 \neq \delta \in k$ implies

$$\begin{cases} {}^tTW_s'T^{(q)} = \delta W_s' & (1) \\ [\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot T^{(q)} = \delta(0, \cdots, 0, 1) & (2) \\ {}^tTW_s' \cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0 & (3) \\ [\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot {}^t\mathbf{a}^{(a)} + ed^q = 0 & (4) \\ \mathbf{b} = 0 & (5) \\ d^q = \delta & (6) \end{cases}$$

By (1), we see that T is a matrix defining an automorphism of X_s^{n-2} in $\mathbb{P}^{n-2}$. Because $s \leq n-2$, by (2) we have $d = \delta$. Hence we can calculate T by induction on n. The vector $\mathbf{a}, \mathbf{c}$ and d, e can be find by using the equations (2)-(6). Conversely, it is easy to show that if the matrix M satifies the conditions (i)-(v) then it define a projective automorphism of X_s. The projective automorphism group of X_n and X_{n-1} is easy to calculate. □

4. Proof of Propostion 1.1

For the reader's convenience, we give a proof of Proposition 1.1, which is based on the argument of [12], chapter VI. The implications (iv)⇒(iii)⇒(ii)⇒(i) are clear. We will prove (i)⇒(iv). For $B \in GL_{n+1}(k)$, consider the map f_B defined by

$$\begin{array}{rcl} f_B : GL_{n+1}(k) & \longrightarrow & GL_{n+1}(k) \\ T & \longmapsto & {}^tTBT^{(q)}. \end{array}$$

Because the differential of the Frobenius map $F : T \longmapsto T^{(q)}$ is identically zero, we can deduce that

$$d(f_B) = d({}^tT)BT^{(q)}.$$

Therefore, the tangent map of f_B is surjective for any $B \in GL_{n+1}(k)$. Hence, f_B is generically surjective, and the image of f_B contains a non-empty open subset U_B. Let A be any matrix of $M_{n+1}(k)$ such that the hypersurface X_A is nonsingular, i.e. $A \in GL_{n+1}(k)$. Because $GL_{n+1}(k)$ is irreducible, we have $U_A \cap U_I \neq \emptyset$, where I is identity matrix of size $n+1$. There exist $T_1, T_2 \in GL_m(k)$ such that $f_A(T_1) = f_I(T_2)$. Putting $T = T_1T_2^{-1}$, we have ${}^tTAT^{(q)} = I$. □

5. The case of plane curves

Next we will study the plane curves X_A associated with matrices A of rank ≤ 2 in the projective plane $\mathbb{P}^2$.

Theorem 5.1. *Let $A = (a_{ij})_{0\leq i,j\leq 2} \in M_3(k)$ be a nonzero matrix and let X_A be the curve defined by $\sum a_{ij}x_i x_j^q = 0$ in $\mathbb{P}^2$. Suppose that the rank of A is smaller than 3.*

(i) *When the rank of A is 1, the curve X_A is projectively isomorphic to one of the following curves*

$$Z_0 : x_0^{q+1} = 0, \text{ or } Z_1 : x_0^q x_1 = 0.$$

(ii) *When the rank of A is 2, the curve X_A is projectively isomorphic to one of the following curves*

$$X_0 : x_0^q x_1 + x_1^q x_2 = 0, \text{ or } X_1 : x_0^{q+1} + x_1^q x_2 = 0, \text{ or } X_2 : x_0^{q+1} + x_1^{q+1} = 0.$$

Proof. In the case the rank of A is 2. By Theorem 1.2, the plane curve X_A is projectively isomorphic to one of the plane curves X_0, or X_1, or X_2.

In the case rank of A is 1. With the same argument of the proof of Theorem 1.2, we can assume that the matrix A is as following form

$$A = \begin{pmatrix} a_{00} & a_{01} & 0 \\ a_{10} & a_{11} & 0 \\ a_{20} & a_{21} & 0 \end{pmatrix}.$$

By interchanging with x_0 and x_1 if nessesary, we can assume that $(a_{01}, a_{11}, a_{21}) \neq (0,0,0)$. Because rank of A is 1, there exists $\lambda \in k$ such that $(a_{00}, a_{10}, a_{20}) = \lambda(a_{01}, a_{11}, a_{21})$. The curve X_A is defined by the equation

$$(a_{00}x_0 + a_{10}x_1 + a_{20}x_2)(x_0^q + \lambda x_1^q) = 0.$$

It is easy to show that X_A is projectively isomorphic to the curve Z_0 or Z_1. □

Remark 5.2. *In fact, the case when the plane curve X_A of degree $p+1$ has been proved by Homma in [9].*

Note that the plane curve X_1 has a special property such that the tangent line of X_1 at every smooth point passes through the point $(0,1,0)$. Therefore the plane curve X_1 is strange. Moreover this curve is irreducible and nonreflexive. In [1], Ballico and Hefez proved that a reduced irreducible nonreflexive plane curve of degree $q+1$ is isomorphic to one of the following curves:

(1) $X_I : x_0^{q+1} + x_1^{q+1} + x_2^{q+1} = 0$,
(2) a nodal curve whose defining equation is given in [4] and [7],
(3) strange curves.

Let $\mathcal{L}$ be the space of all reduced irreducible projective plane curves of degree $q+1$, which is open in the space $\mathcal{P} \cong \mathbb{P}^{\binom{q+3}{2}}$ of all projective plane curves of degree $q+1$.

Let $\mathcal{L}_*$ be the locus of $\mathcal{P}$ consisting of curves isomorphic to X_I, and let $\mathcal{L}_1$ be the locus of $\mathcal{P}$ consisting of strange curves. Let (ξ_J) be the homogeneous coordinates of $\mathcal{P}$ where $J = (j_0, j_1, j_2)$ ranges over the set of all ordered triples on non-negative integer such that $j_0 + j_1 + j_2 = q + 1$. The point (ξ_J) corresponds to the curve $\sum \xi_J x^J = 0$ where $x^J = x_0^{j_0} x_1^{j_1} x_2^{j_2}$. Then the locus of all curves defined by the equation of the form $\sum a_{ij} x_i x_j^q = 0$ is the linear subspace of $\mathcal{P}$ defined by $\xi_J = 0$, unless $J \in \{(q+1,0,0),(0,q+1,0),(0,0,q+1),(q,1,0),(q,0,1),(1,q,0),(1,0,q),(0,q,1),(0,1,q)\}$. By Theorem 5.1, we have that because Z_0, Z_1, X_0, X_2 are reducible, the closure $\overline{\mathcal{L}_*}$ of $\mathcal{L}_*$ in $\mathcal{L}$ consists of curves isomorphic to X_I or to X_1, and the intersection of $\overline{\mathcal{L}_*}$ and $\mathcal{L}_1$ consist of curves isomorphic to X_1.

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