Commensurability between once-punctured torus groups and once-punctured Klein bottle groups

Mikio Furokawa
(Received Xxx 00, 0000)

Abstract. The once-punctured torus and the once-punctured Klein bottle are topologically commensurable, in the sense that both of them are doubly covered by the twice-punctured torus. In this paper, we give a condition for a faithful type-preserving PSL(2, C)-representation of the fundamental group of the once-punctured Klein bottle to be “commensurable” with that of the once-punctured torus. We also show that such a pair of PSL(2, C)-representations extend to a representation of the fundamental group of a common quotient orbifold. Finally, we give an application to the study of the Ford domains.

1. Introduction

The combinatorial structures of the Ford domains of quasi-fuchsian once-punctured torus groups are characterized by Jorgensen [6] (cf. [1]). It is natural to expect that there is an analogue of Jorgensen’s theory for quasi-fuchsian once-punctured Klein bottle groups, because its deformation space also has complex dimension 2. In fact, for fuchsian once-punctured Klein bottle groups, we can completely describe the structures of their Ford domains (see [3, Theorem 5.7]). However, as shown in [3, Section 6], the Ford domains of general quasi-fuchsian once-punctured Klein bottle groups seem to have much more complicated structures than those of quasi-fuchsian once-punctured torus groups.

On the other hand, the once-punctured torus, $\Sigma_{1,1}$, and the once-punctured Klein bottle, $N_{2,1}$, are topologically commensurable, in the sense that they are doubly covered by the twice-punctured torus, $\Sigma_{1,2}$. Thus we can introduce a notion of commensurability between type-preserving PSL(2, C)-representations of $\pi_1(\Sigma_{1,1})$ and $\pi_1(N_{2,1})$ (see Definitions 2.1 and 2.2). Moreover, we can easily observe that mutually commensurable discrete PSL(2, C)-representations of $\pi_1(\Sigma_{1,1})$ and $\pi_1(N_{2,1})$ have the same Ford domain (see Proposition 6.3). Hence a natural problem now arises: which type-preserving PSL(2, C)-representation of $\pi_1(N_{2,1})$ is commensurable with a type-preserving PSL(2, C)-representation of $\pi_1(\Sigma_{1,1})$ (see Problem 2.3)?

The main purpose of this paper is to give a partial answer to this problem (see Theorem 5.1). This enable us to understand the Ford domains of the
Kleinian groups obtained as the images of discrete faithful type-preserving representations of $\pi_1(N_{2,1})$ which are commensurable with those of $\pi_1(\Sigma_{1,1})$ (see Example 6.4).

The rest of this paper is organized as follows. In Section 2, we recall relation among the once-punctured torus, the once-punctured Klein bottle, the twice-punctured torus and their quotient orbifolds $O_{\Sigma_{1,1}}$, $O_{N_{2,1}}$, $O_{\Sigma_{1,2}}$. We also recall type-preserving representations of their fundamental groups (see Definition 2.1). Then we introduce the concept of commensurability between type-preserving representations of their fundamental groups (see Definition 2.2). In Section 3, we recall the definition and basic properties of “geometric” generator systems of $\pi_1(O_{\Sigma_{1,1}})$ which are called elliptic generator triples. We also introduce geometric generator systems of $\pi_1(O_{N_{2,1}})$, which are also called elliptic generator triples, and describe their basic properties. In Section 4, we study type-preserving representations. In particular, we recall the definition of the complex probabilities of type-preserving representations of $\pi_1(O_{\Sigma_{1,1}})$ and describe a conceptual geometric construction of a type-preserving representation from a given complex probability (see Proposition 4.8). We also introduce the concept of complex probabilities of type-preserving representations of $\pi_1(O_{N_{2,1}})$ and establish a similar geometric construction of a type-preserving representation from a given complex probability (see Proposition 4.11). At the end of Section 4, we study type-preserving $\text{PSL}(2, \mathbb{C})$-representations of $\pi_1(\Sigma_{1,2})$ extending to those of $\pi_1(\Sigma_{1,1})$ or $\pi_1(N_{2,1})$ (see Lemma 4.15). In Section 5, we give a partial answer to the commensurability problem for representations of $\pi_1(O_{\Sigma_{1,1}})$ and $\pi_1(O_{N_{2,1}})$ in terms of complex probabilities (see Theorem 5.1). We also study what happens if we drop the assumption in Theorem 5.1. In Section 6, we give an application to the study of Ford domains.

2. Once-punctured torus, once-punctured Klein bottle and their friends

Let $\Sigma_{1,1}$, $N_{2,1}$ and $\Sigma_{1,2}$, respectively, be the once-punctured torus, the once-punctured Klein bottle and the twice-punctured torus. Their fundamental groups have the following presentations:

\[
\begin{align*}
\pi_1(\Sigma_{1,1}) &= \langle X_1, X_2 \mid - \rangle, \\
\pi_1(N_{2,1}) &= \langle Y_1, Y_2 \mid - \rangle, \\
\pi_1(\Sigma_{1,2}) &= \langle Z_1, Z_2, Z_3 \mid - \rangle.
\end{align*}
\]

Here the generators are represented by the based simple loops in Figure 1. It should be noted that $Y_2$ is represented by the unique non-separating simple orientable loop in $N_{2,1}$. Set $K_{\Sigma_{1,1}} = [X_1, X_2] = X_1X_2X_1^{-1}X_2^{-1}$, $K_{N_{2,1}} = (Y_1Y_2Y_1^{-1}Y_2)^{-1}$, $K_{\Sigma_{1,2}} = Z_1Z_2Z_3$ and $K_{\Sigma_{1,2}}' = Z_2Z_1Z_3$. Then they are represented by the punctures of the surfaces.
The once-punctured torus and the once-punctured Klein bottle are topologically commensurable, in the sense that both of them are doubly covered by the twice-punctured torus. To be precise, the following hold.

1. There are three double coverings \( p_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) up to equivalence. In fact, there are three epimorphisms from \( \pi_1(\Sigma_{1,1}) \) to \( \mathbb{Z}/2\mathbb{Z} \), and the double covering corresponding to each of them is homeomorphic to \( \Sigma_{1,2} \).

2. There is a unique orientation double covering \( p_2 : \Sigma_{1,2} \to N_{2,1} \) up to equivalence. This corresponds to the epimorphism to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generator \( Y_1 \) of \( \pi_1(N_{2,1}) \) to the generator 1 of \( \mathbb{Z}/2\mathbb{Z} \) and maps the generator \( Y_2 \) of \( \pi_1(N_{2,1}) \) to the identity element 0 of \( \mathbb{Z}/2\mathbb{Z} \).

For each \( F = \Sigma_{1,1}, N_{2,1} \) or \( \Sigma_{1,2} \), let \( \iota_F : F \to F \) be the involution illustrated in Figure 2. We denote the quotient orbifold \( F/\iota_F \) by the symbol \( O_F \) and denote the covering projection from \( F \) to \( O_F \) by the symbol \( p_F \). Then we have the following under the notation of [8] (see Figure 2).

1. \( O_{\Sigma_{1,1}} = (2, 2, 2, \infty) \) is the orbifold with underlying space a punctured sphere and with three cone points of cone angle \( \pi \), and \( \pi_1(O_{\Sigma_{1,1}}) \) has the following presentation:
   \[
   \pi_1(O_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 | P_0^2 = P_1^2 = P_2^2 = 1 \rangle.
   \]

2. \( O_{N_{2,1}} = (2, 2; \infty) \) is the orbifold with underlying space a disk and with two cone points of cone angle \( \pi \) and a corner reflector of order \( \infty \), and \( \pi_1(O_{N_{2,1}}) \) has the following presentation:
   \[
   \pi_1(O_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 | Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle.
   \]

3. \( O_{\Sigma_{1,2}} = (2, 2, 2, 2, \infty) \) is the orbifold with underlying space a punctured sphere and with four cone points of cone angle \( \pi \), and \( \pi_1(O_{\Sigma_{1,2}}) \) has the following presentation:
   \[
   \pi_1(O_{\Sigma_{1,2}}) = \langle R_0, R_1, R_2, R_3 | R_0^2 = R_1^2 = R_2^2 = R_3^2 = 1 \rangle.
   \]
Figure 2. The commutative diagram of the fundamental groups and coverings.

For each $F = \Sigma_{1,1}$, $N_{2,1}$ or $\Sigma_{1,2}$, the orbifold $O_F$ admits a complete hyperbolic structure and hence $\pi_1(O_F)$ is identified with a discrete subgroup of $\text{Isom}(\mathbb{H}^2)$ (if we fix a hyperbolic structure). Then the generator $Q_1$ is a reflection and the other generators are order 2 elliptic transformations. Set $K = (P_0P_1P_2)^{-1}$, $K_0 = Q_1^{Q_0}$, $K_2 = Q_1^{Q_2}$ and $K_{O_{\Sigma_{1,2}}} = R_0R_1R_2R_3$, where $A^B = BAB^{-1}$. Then $K$ and $K_{O_{\Sigma_{1,2}}}$, respectively, are represented by the punctures of $O_{\Sigma_{1,1}}$ and $O_{\Sigma_{1,2}}$, and $K_0$ and $K_2$ are represented by the reflector lines which generate the corner reflector of order $\infty$. We identify $\pi_1(F)$ with the image of the inclusion $\pi_1(F) \to \pi_1(O_F)$ induced by the projection $p_F$. Then we have the following relations among the generators of the fundamental groups:

\[
X_1 = P_2P_1, \quad X_2 = P_0P_1, \quad K_{\Sigma_{1,1}} = K^2, \\
Y_1 = Q_0Q_1, \quad Y_2 = Q_0Q_2, \quad K_{N_{2,1}} = K_{O_{\Sigma_{1,2}}}, \\
Z_1 = R_0R_1, \quad Z_2 = R_2R_1, \quad Z_3 = R_1R_3, \quad K_{\Sigma_{1,2}} = K_{O_{\Sigma_{1,2}}}, \quad K_{\Sigma_{1,2}}' = (K_{O_{\Sigma_{1,2}}}^{-1})R_1.
\]

Note that $O_{\Sigma_{1,1}}$ and $O_{N_{2,1}}$ are also topologically commensurable, namely, both of them are doubly covered by $O_{\Sigma_{1,2}}$. To be precise, the following hold.

1. There are three double coverings $p_1 : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}}$ up to equivalence. Each of such covering corresponds to an epimorphism from $\pi_1(O_{\Sigma_{1,1}})$ to $\mathbb{Z}/2\mathbb{Z}$ which maps one of the generators $P_0, P_1, P_2$ to 1 and maps the remaining generators to 0. Each double covering $p_1 : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}}$ uniquely determines a double covering $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ such that the diagram in the left hand side of Figure 2 is commutative.
Thus we obtain the assertion (1).

(2) There is a unique orientation double covering \( p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}} \) up to equivalence. This corresponds to the epimorphism from \( \pi_1(\mathcal{O}_{N_{2,1}}) \) to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generators \( Q_0, Q_1 \) and \( Q_2 \) to 0, 1 and 0, respectively. For the orientation double coverings \( p_2 : \Sigma_{1,2} \to N_{2,1} \) and \( p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}} \), the diagram in the right hand side of Figure 2 is commutative.

The assertion (2) is obvious and the assertion (1) is proved as follows. For a given double covering \( p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}} \), we can check, by the relations among the generators of the fundamental groups, that \( (p_1 \circ p_{\Sigma_{1,2}})(\pi_1(\Sigma_{1,2})) \subset (p_{\Sigma_{1,1}})(\pi_1(\Sigma_{1,1})) \). Hence, by the unique lifting property, there is a unique double covering \( \tilde{p}_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) such that \( p_{\Sigma_{1,1}} \circ \tilde{p}_1 = p_1 \circ p_{\Sigma_{1,2}} \), modulo post composition of \( \iota_{\Sigma_{1,1}} \). Since \( \iota_{\Sigma_{1,1}} \circ \tilde{p}_1 = \tilde{p}_1 \circ \iota_{\Sigma_{1,2}} \), the coverings \( \tilde{p}_1 \) and \( \iota_{\Sigma_{1,1}} \circ \tilde{p}_1 \) are equivalent. Thus the double covering \( p_1 \) uniquely determines the double covering \( \tilde{p}_1 \).

Conversely, for the double covering \( p_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) associated with an epimorphism \( \phi : \pi_1(\Sigma_{1,1}) \to \mathbb{Z}/2\mathbb{Z} \), we can see, by the relations among the generators of the fundamental groups, that there is a unique epimorphism \( \phi : \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \to \mathbb{Z}/2\mathbb{Z} \) such that it maps only one of generators \( P_0, P_1 \) and \( P_2 \) of \( \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \) to the generator 1 of \( \mathbb{Z}/2\mathbb{Z} \) and satisfies \( \phi = \phi \circ (p_{\Sigma_{1,1}}) \). Hence there is a unique double covering \( \tilde{p}_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}} \) such that \( p_{\Sigma_{1,1}} \circ \tilde{p}_1 = p_1 \circ p_{\Sigma_{1,2}} \). Thus we obtain the assertion (1).

The orbifolds \( \mathcal{O}_{\Sigma_{1,1}} \) and \( \mathcal{O}_{N_{2,1}} \) have two distinct common quotient orbifolds, \( \mathcal{O}_\alpha \) and \( \mathcal{O}_\beta \), as described in the following (see Figure 3).

(1) \( \mathcal{O}_\alpha = [2; 2, \infty] \) is the orbifold with underlying space a disk and with a cone point of cone angle \( \pi \) and with a corner reflector of order 2 and a corner reflector of order \( \infty \), and \( \pi_1(\mathcal{O}_\alpha) \) has the following presentation:
\[
\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = (S_1S_2)^2 = 1 \rangle.
\]
Here \( S_0 \) is an order 2 elliptic transformation, and \( S_1 \) and \( S_2 \) are reflections.

(1-1) There is a unique double covering \( p^{(\alpha)}_{\Sigma_{1,1}} : \mathcal{O}_{\Sigma_{1,1}} \to \mathcal{O}_\alpha \). This corresponds to the epimorphism from \( \pi_1(\mathcal{O}_\alpha) \) to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generators \( S_1, S_2 \) to 1 and maps \( S_0 \) to 0. Then we have the following identities:
\[
P_0 = S_0^2, \quad P_1 = S_1S_2, \quad P_2 = S_0.
\]

(1-2) There is a unique double covering \( p^{(\alpha)}_{N_{2,1}} : \mathcal{O}_{N_{2,1}} \to \mathcal{O}_\alpha \). This corresponds to the epimorphism from \( \pi_1(\mathcal{O}_\alpha) \) to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generator \( S_2 \) to 1 and maps \( S_0, S_1 \) to 0. Then we have the following identities:
\[
Q_0 = S_0^2, \quad Q_1 = S_1, \quad Q_2 = S_0.
\]

(2) \( \mathcal{O}_\beta = [2, 2, 2, \infty] \) is the orbifold with underlying space a disk and with three corner reflectors of order 2 and a corner reflector of order \( \infty \), and
\( \pi_1(\mathcal{O}_\beta) \) has the following presentation:

\[
\pi_1(\mathcal{O}_\beta) = \left\langle T_0, T_1, T_2, T_3 \mid T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, \\
(T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1 \right\rangle.
\]

Here the generators \( T_0, T_1, T_2, T_3 \) are reflections.

(2-1) There is a unique double covering \( p^{(\alpha)}_{\Sigma^{1,1}} : \mathcal{O}_{\Sigma^{1,1}} \to \mathcal{O}_\beta \). This corresponds to the epimorphism from \( \pi_1(\mathcal{O}_\beta) \) to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generators \( T_0, T_1, T_2, T_3 \) to 1. Then we have the following identities:

\[
P_0 = T_0T_1, \quad P_1 = T_1T_2, \quad P_2 = T_2T_3.
\]

(2-2) There is a unique double covering \( p^{(\beta)}_{N^{2,1}} : \mathcal{O}_{N^{2,1}} \to \mathcal{O}_\beta \). This corresponds to the epimorphism from \( \pi_1(\mathcal{O}_\beta) \) to \( \mathbb{Z}/2\mathbb{Z} \) which maps the generators \( T_0, T_1, T_2 \) to 1 and maps \( T_3 \) to 0. Then we have the following identities:

\[
Q_0 = T_1T_2, \quad Q_1 = T_3^{T_0}, \quad Q_2 = T_0T_1.
\]
In summary, we have the commutative diagram of double coverings as shown in Figure 4. Every arrow represents a double covering. There are three types of coverings $p_1$ from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $\Sigma_{1,1}$ (resp. $\mathcal{O}_{\Sigma_{1,1}}$) up to equivalence, and the other coverings are unique up to equivalence.

**Definition 2.1.** (1) For $F = \Sigma_{1,1}$, $N_{1,1}$, $\Sigma_{1,2}$, $\mathcal{O}_{\Sigma_{1,1}}$, $\mathcal{O}_{\Sigma_{1,2}}$, $\mathcal{O}_\alpha$ or $\mathcal{O}_\beta$, a representation $\rho : \pi_1(F) \to \text{PSL}(2, \mathbb{C})$ is type-preserving if it is irreducible (equivalently, it does not have a common fixed point in $\partial \mathbb{H}^3$) and sends peripheral elements to parabolic transformations.

(2) Type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho$ and $\rho'$ are equivalent if $i_g \circ \rho = \rho'$, where $i_g$ is the inner automorphism, $i_g(h) = ghg^{-1}$, of $\text{PSL}(2, \mathbb{C})$ determined by $g$.

In the above definition, if $F$ is an orbifold with reflector lines, an element of $\pi_1(F)$ is said to be peripheral if it is (the image of) a peripheral element of $\pi_1(\tilde{F})$, where $\tilde{F}$ is the orientation double covering of $F$.

**Definition 2.2.** Let $\rho_1$ be a type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$). Let $\rho_2$ be a type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(N_{1,1})$ (resp. $\pi_1(\mathcal{O}_{N_{1,1}})$). The representations $\rho_1$ and $\rho_2$ are commensurable.
if there exist a double covering \( p_1 \) from \( \Sigma_{1,2} \) (resp. \( \mathcal{O}_{\Sigma_{1,2}} \)) to \( \Sigma_{1,1} \) (resp. \( \mathcal{O}_{\Sigma_{1,1}} \)) and a double covering \( p_2 \) from \( \Sigma_{1,2} \) (resp. \( \mathcal{O}_{\Sigma_{1,2}} \)) to \( N_{2,1} \) (resp. \( \mathcal{O}_{N_{2,1}} \)) such that \( p_1 \circ (p_1)_* \) and \( p_2 \circ (p_2)_* \) are equivalent, namely \( i_g \circ p_1 \circ (p_1)_* = p_2 \circ (p_2)_* \) for some \( g \in \text{PSL}(2, \mathbb{C}) \). After replacing \( p_1 \) with \( i_g \circ p_1 \), without changing the equivalence class, the last identity can be replaced with the identity \( p_1 \circ (p_1)_* = p_2 \circ (p_2)_* \).

In this paper, we study the following problem:

**Problem 2.3.** For a given type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representation \( \rho_2 \) of \( \pi_1(N_{2,1}) \) (resp. \( \pi_1(\mathcal{O}_{N_{2,1}}) \)), when does there exist a type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representation \( \rho_1 \) of \( \pi_1(\Sigma_{1,1}) \) (resp. \( \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \)) which is commensurable with \( \rho_2 \)?

We will give a partial answer to this problem for a certain family of type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representations of \( \pi_1(N_{2,1}) \) in terms of “complex probabilities” introduced in Section 4 (see Theorem 5.1).

**Remark 2.4.** Recall that there are three equivalence classes of double coverings \( \Sigma_{1,2} \to \Sigma_{1,1} \) and there is a unique equivalence class of double coverings \( \Sigma_{1,2} \to N_{2,1} \). The three classes of double coverings \( \Sigma_{1,2} \to \Sigma_{1,1} \) become equivalent after a post composition of a self-homeomorphism of \( \Sigma_{1,1} \). Hence, by considering compositions of \( p_1 \) with the automorphism of \( \pi_1(\Sigma_{1,1}) \) induced by a self-homeomorphism of \( \Sigma_{1,1} \), we may arbitrarily fix the equivalence classes of the coverings \( \Sigma_{1,2} \to \Sigma_{1,1} \). However, we must be careful in the choices of a representative \( p_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) and a representative \( p_2 : \Sigma_{1,2} \to N_{2,1} \) of the equivalence classes of the coverings, by the following reason. Assume that \( \rho_1 : \pi_1(\Sigma_{1,1}) \to \text{PSL}(2, \mathbb{C}) \) and \( \rho_2 : \pi_1(N_{2,1}) \to \text{PSL}(2, \mathbb{C}) \) are commensurable via coverings \( p_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) and \( p_2 : \Sigma_{1,2} \to N_{2,1} \), i.e., \( \rho_1 \circ (p_1)_* \) and \( \rho_2 \circ (p_2)_* \) are equivalent. Pick a self-homeomorphism \( f \) of \( \Sigma_{1,2} \) and replace \( p_1 \) with another covering \( p'_1 := p_1 \circ f : \Sigma_{1,2} \to \Sigma_{1,1} \). Then the representation \( \rho_1 \circ (p'_1)_* \) is not necessarily equivalent to the representation \( \rho_1 \circ (p_1)_* \), and hence it is not necessarily equivalent to the representation \( \rho_2 \circ (p_2)_* \). In fact, we also need to replace \( p_2 \) with another covering \( p'_2 := p_2 \circ f : \Sigma_{1,2} \to N_{2,1} \), which is equivalent to \( p_2 \).

### 3. Elliptic generators

In this section, we first recall the definition and basic properties of elliptic generators of \( \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \) (see [1, Section 2] for details). We also introduce the concept of elliptic generators of \( \pi_1(\mathcal{O}_{N_{2,1}}) \), and then we establish similar basic properties.

Recall that the (orbifold) fundamental group of \( \mathcal{O}_{\Sigma_{1,1}} \) has the following presentation:

\[
\pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle.
\]

and that \( K = (P_0P_1P_2)^{-1} \) is represented by the puncture of \( \mathcal{O}_{\Sigma_{1,1}} \).
Definition 3.1. An ordered triple \((P_0, P_1, P_2)\) of elements of \(\pi_1(O_{\Sigma_1})\) is called an elliptic generator triple of \(\pi_1(O_{\Sigma_1})\) if its members generate \(\pi_1(O_{\Sigma_1})\) and satisfy \(P_0^2 = P_1^2 = P_2^2 = 1\) and \((P_0 P_1 P_2)^{-1} = K\). A member of an elliptic generator triple of \(\pi_1(O_{\Sigma_1})\) is called an elliptic generator of \(\pi_1(O_{\Sigma_1})\).

Remark 3.2. In the above definition, the condition that the members of the triple generate \(\pi_1(O_{\Sigma_1})\) is actually a consequence of the other conditions. This can be seen from the proof of [1, Lemma 2.1.7].

Proposition 3.3. [1, Proposition 2.1.6] The elliptic generator triples of \(\pi_1(O_{\Sigma_1})\) are characterized as follows.

1. For any elliptic generator triple \((P_0, P_1, P_2)\) of \(\pi_1(O_{\Sigma_1})\), the following hold:

   1.1. The triple of any three consecutive elements in the following bi-infinite sequence is also an elliptic generator triple of \(\pi_1(O_{\Sigma_1})\):

   \[\ldots, P_2^{-K^{-2}}, P_0^{-K^{-1}}, P_1^{-K^{-1}}, P_2^{-K^{-1}}, P_0, P_1, P_2, P_0^K, P_1^K, P_2^K, P_0^{-K^2}, \ldots\]

   1.2. \((P_0, P_2, P_1 P_2)\) and \((P_1 P_0, P_0, P_2)\) are also elliptic generator triples of \(\pi_1(O_{\Sigma_1})\).

2. Conversely, any elliptic generator triple of \(\pi_1(O_{\Sigma_1})\) is obtained from a given elliptic generator triple of \(\pi_1(O_{\Sigma_1})\) by successively applying the operations in (1).

Definition 3.4. For an elliptic generator triple \((P_0, P_1, P_2)\) of \(\pi_1(O_{\Sigma_1})\), the bi-infinite sequence \(\{P_j\}\) in Proposition 3.3(1.1) is called the sequence of elliptic generators of \(\pi_1(O_{\Sigma_1})\) (associated with \((P_0, P_1, P_2)\)).

Recall that the (orbifold) fundamental group of \(O_{N_{2,1}}\) has the following presentation:

\[\pi_1(O_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \mid Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle,\]

and that \(K_0 = Q_0 Q_0\) and \(K_2 = Q_1 Q_2\) are represented by the reflections in the lines which generate the corner reflector of order \(\infty\). It should be noted that \(Q_0\) and \(Q_2\) act on the universal cover of \(O_{N_{2,1}}\) orientation preservingly, and \(Q_1\) acts on the universal cover of \(O_{N_{2,1}}\) orientation reversingly.

Definition 3.5. An ordered triple \((Q_0, Q_1, Q_2)\) of elements of \(\pi_1(O_{N_{2,1}})\) is called an elliptic generator triple of \(\pi_1(O_{N_{2,1}})\) if its members generate \(\pi_1(O_{N_{2,1}})\) and satisfy \(Q_0^2 = Q_1^2 = Q_2^2 = 1\) and \(Q_1 Q_0 Q_1 Q_0 = K_2 K_0\). A member of an elliptic generator triple of \(\pi_1(O_{N_{2,1}})\) is called an elliptic generator of \(\pi_1(O_{N_{2,1}})\).

Remark 3.6. In the above definition, the condition that the members of the triple generate \(\pi_1(O_{N_{2,1}})\) is actually a consequence of the other conditions. This can be seen from the proof of Proposition 3.7 (see [4]).

Proposition 3.7. The elliptic generator triples of \(\pi_1(O_{N_{2,1}})\) are characterized as follows.
(1) For any elliptic generator triple \((Q_0, Q_1, Q_2)\) of \(\pi_1(O_{N_2,1})\), the following hold:

(1.1) The triples in the following bi-infinite sequence are also elliptic generator triples of \(\pi_1(O_{N_2,1})\):

\[
\ldots, (Q_0^{K_0 K_1} Q_1^{K_1 K_2} Q_2^{K_2 K_3}, Q_0^{K_0} Q_1^{K_1} Q_2^{K_2}, Q_0 Q_1 Q_2),
\]

\[
(Q_2^{K_2} Q_1^{K_1} Q_0^{K_0}, Q_0^{K_0} Q_1^{K_1} Q_2^{K_2}, Q_0^{K_0} Q_1^{K_1} Q_2^{K_2} Q_3^{K_3}, \ldots)
\]

To be precise, the following holds. Let \(\{Q_j\}\) be the sequence of elements of \(\pi_1(O_{N_2,1})\) obtained from \((Q_0, Q_1, Q_2)\) by applying the following rule:

\[
Q_j^{K_0} = Q_{-j-1}, \quad Q_j^{K_2} = Q_{-j+5}.
\]

Then the triple \((Q_{3k}, Q_{3k+1}, Q_{3k+2})\) is also an elliptic generator triple of \(\pi_1(O_{N_2,1})\) for any \(k \in \mathbb{Z}\).

(1.2) \((Q_2, Q_1, Q_0, Q_0^{Q_0})\) is also an elliptic generator triple of \(\pi_1(O_{N_2,1})\).

(2) Conversely, any elliptic generator triple of \(\pi_1(O_{N_2,1})\) is obtained from a given elliptic generator triple of \(\pi_1(O_{N_2,1})\) by successively applying the operations in (1).

The proof of (1) is obvious, and the proof of (2) is given in [4]. In this paper, we need only (1).

**Definition 3.8.** For an elliptic generator triple \((Q_0, Q_1, Q_2)\) of \(\pi_1(O_{N_2,1})\), the bi-infinite sequence \(\{Q_j\}\) in Proposition 3.7(1.1) is called the sequence of elliptic generators of \(\pi_1(O_{N_2,1})\) (associated with \((Q_0, Q_1, Q_2)\)).

It should be noted that \(Q_j\) is conjugate to the following element (cf. Proposition 4.11(1.1)):

\[
Q_0 \text{ if } j \equiv 0 \text{ or } 5 \pmod{6},
\]

\[
Q_1 \text{ if } j \equiv 1 \pmod{3},
\]

\[
Q_2 \text{ if } j \equiv 2 \text{ or } 4 \pmod{6}.
\]

In particular, \(Q_j\) acts on the universal cover of \(O_{N_2,1}\) orientation reversingly or orientation preservingly according to whether \(j \equiv 1 \pmod{3}\) or not.

### 4. Type-preserving representations

Let \(\rho_1\) be a type-preserving \(\text{PSL}(2, \mathbb{C})\)-representation of \(\pi_1(O_{\Sigma_1,1})\). Fix a sequence of elliptic generators \(\{P_j\}\) of \(\pi_1(O_{\Sigma_1,1})\). Set

\[
(x_1, x_{12}, x_2) = (\text{tr}(\rho_1(X_1)), \text{tr}(\rho_1(X_1 X_2)), \text{tr}(\rho_1(X_2))),
\]

where \(X_1 = P_2 P_1\) and \(X_2 = P_0 P_1\). As the trace of an element in \(\text{PSL}(2, \mathbb{C})\) is only defined up to sign, we are free to choose the signs of \(x_1\) and \(x_2\) independently. Once we have done this though, the sign of \(x_{12}\) is determined. It is well-known
that the triple \((x_1, x_{12}, x_2)\) is a Markoff triple, namely, it satisfies the Markoff identity (see [2, 1]):

\[x_1^2 + x_{12}^2 + x_2^2 = x_1 x_{12} x_2\]

and that the triple \((x_1, x_{12}, x_2)\) is non-trivial, namely, it is different from \((0,0,0)\). Moreover, the equivalence class of the triple \((x_1, x_{12}, x_2)\) is uniquely determined by the equivalence class of the type-preserving representation \(\rho_1\), and vice versa (see [1, Proposition 2.3.6 and 2.4.2] for details). Here, two triples \((x_1, x_{12}, x_2)\) and \((x_1', x_{12}', x_2')\) are said to be equivalent if the latter is equal to \((x_1, x_{12}, x_2), (x_1, -x_{12}, -x_2), (-x_1, x_{12}, -x_2)\) or \((-x_1, -x_{12}, x_2)\). We call the triple \((x_1, x_{12}, x_2) = (\text{tr}(\rho_1(X_1)), \text{tr}(\rho_1(X_1 X_2)), \text{tr}(\rho_1(X_2)))\) the Markoff triple associated with \(\{\rho_1(P_j)\}\).

Let \(\rho_2\) be a type-preserving \(\text{PSL}(2, \mathbb{C})\)-representation of \(\pi_1(\mathcal{O}_{N,1})\). Fix a sequence of elliptic generators \(\{Q_j\}\) of \(\pi_1(\mathcal{O}_{N,1})\). Set

\[(y_1, y_{12}, y_2) = (\text{tr}(\rho_2(Y_1))/i, \text{tr}(\rho_2(Y_1 Y_2))/i, \text{tr}(\rho_2(Y_2))),\]

where \(Y_1 = Q_0 Q_1\) and \(Y_2 = Q_0 Q_2\) and \(i = \sqrt{-1}\). Note that \(\rho_2(K_{N,1}) = \rho_2((Y_1 Y_2 Y_1^{-1} Y_2^{-1})^{-1})\) is a parabolic element of \(\text{PSL}(2, \mathbb{C})\) with a trace that has a well-defined sign (independent of the signs chosen for the traces of \(\rho(Y_1)\) and \(\rho(Y_2)\)), which is equal to \(y_1^2 + y_{12}^2 - y_1 y_{12} y_2 + 2\). Hence \((y_1, y_{12}, y_2)\) satisfies one of the following identities:

\[y_1^2 + y_{12}^2 + 4 = y_1 y_{12} y_2 \quad \text{if} \quad \text{tr}(\rho_2(K_{N,1})) = -2, \quad \text{(Eq1)}\]

\[y_1^2 + y_{12}^2 = y_1 y_{12} y_2 \quad \text{if} \quad \text{tr}(\rho_2(K_{N,1})) = +2. \]

In addition, the triple \((y_1, y_{12}, y_2)\) is non-trivial, namely, it is different from \((0,0,0)\) (see, for example [3, Remark 4.3]). It is well-known that any two generator subgroup \(\langle A, B \rangle\) of \(\text{PSL}(2, \mathbb{C})\) is irreducible if and only if \(\text{tr}(\langle A, B \rangle) \neq 2\) (see, for example [5, Proposition 2.3.1]). Since \(\rho_2\) is irreducible, it satisfies one of the following identities:

\[y_2 \neq 0 \quad \text{if} \quad \text{tr}(\rho_2(K_{N,1})) = -2, \quad \text{(Eq2)}\]
\[y_2 \neq \pm 2 \quad \text{if} \quad \text{tr}(\rho_2(K_{N,1})) = +2. \]

Moreover, the equivalence class of the triple \((y_1, y_{12}, y_2)\) is uniquely determined by the equivalence class of the type-preserving representation \(\rho_2\), and vice versa (see [3, Propositions 4.4 and 4.6] for details). Here, two triples \((y_1, y_{12}, y_2)\) and \((y_1', y_{12}', y_2')\) are said to be equivalent if the latter is equal to \((y_1, y_{12}, y_2), (y_1, -y_{12}, -y_2), (-y_1, y_{12}, -y_2)\) or \((-y_1, -y_{12}, y_2)\). We call the triple \((y_1, y_{12}, y_2) = (\text{tr}(\rho_2(Y_1))/i, \text{tr}(\rho_2(Y_1 Y_2))/i, \text{tr}(\rho_2(Y_2)))\) the pseudo-Markoff triple associated with \(\{\rho_2(Q_j)\}\).

**Proposition 4.1.** (1) The restriction of any type-preserving \(\text{PSL}(2, \mathbb{C})\)-representation of \(\pi_1(\mathcal{O}_{N,1})\) (resp. \(\pi_1(\mathcal{O}_{N,1})\)) to \(\pi_1(\Sigma_{1,1})\) (resp. \(\pi_1(N_{2,1})\)) is type-preserving.
(2) Conversely, every type-preserving $\text{PSL}(2, \mathbb{C})$-representation $\rho_1$ (resp. $\rho_2$) of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(N_{2,1})$) extends to a unique type-preserving $\text{PSL}(2, \mathbb{C})$-representation $\tilde{\rho}_1$ (resp. $\tilde{\rho}_2$) of $\pi_1(O_{\Sigma_{1,1}})$ (resp. $\pi_1(O_{N_{2,1}})$). Moreover, if $\rho_1$ (resp. $\rho_2$) is faithful, then $\tilde{\rho}_1$ (resp. $\tilde{\rho}_2$) is also faithful.

**Proof.** The assertion (1) is obvious from the definition. The first assertion in (2) is well-known (cf. [10, Section 5.4] and [1, Proposition 2.2.2]). The second assertion in (2) is proved as follows. Suppose to the contrary that $\rho_1$ is faithful but that $\tilde{\rho}_1$ is not faithful. Pick a nontrivial element $\gamma$ of $\text{Ker} \tilde{\rho}_1$. Since $\pi_1(O_{\Sigma_{1,1}})$ is the free product of three cyclic groups and since $\pi_1(\Sigma_{1,1})$ is an index 2 subgroup of $\pi_1(O_{\Sigma_{1,1}})$, we can see that the normal closure of $\gamma$ in $\pi_1(O_{\Sigma_{1,1}})$ has a nontrivial intersection with $\pi_1(\Sigma_{1,1})$. This means that $\rho_1$ is not faithful, a contradiction. The same argument works for the pair of representations $\rho_2$ and $\tilde{\rho}_2$. □

By this proposition, the following are well-defined.

**Definition 4.2.** (1) For $F = \Sigma_{1,1}$ or $O_{\Sigma_{1,1}}$, the symbol $\Omega(\Sigma_{1,1})$ denotes the space of all type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho_1$ of $\pi_1(F)$.

(2) For $F = N_{2,1}$ or $O_{N_{2,1}}$, the symbol $\Omega(N_{2,1})$ (resp. $\Omega'(N_{2,1})$) denotes the space of all type-preserving $\text{PSL}(2, \mathbb{C})$-representations $\rho_2$ of $\pi_1(F)$ such that $\text{tr}(\rho_2(K_{N_{2,1}})) = -2$ (resp. $\text{tr}(\rho_2(K_{N_{2,1}})) = +2$).

**Remark 4.3.** For any $\rho_2 \in \Omega'(N_{2,1})$, the isometries $\rho_2(Q_0Q_2) = \rho_2(Y_2)$ and $\rho_2(K_{N_{2,1}})$ have a common fixed point (see [3, Lemma 4.5(ii)]), and hence $\rho_2$ is indiscrete or non-faithful (see [3, Lemma 4.7]).

The following lemma gives a (local) section of the projection from $\Omega(\Sigma_{1,1})$ (resp. $\Omega(N_{2,1})$) to the space of the equivalence classes of the non-trivial Markoff triples (resp. pseudo-Markoff triple) (cf. [6, Section 2], [9, Section 3], [1, Lemma 2.3.7] and [3, Lemma 4.5]).

**Lemma 4.4.** (1) Let $(x_1, x_{12}, x_2) \in \mathbb{C}^3$ be a triple satisfying $x_1^2 + x_2^2 + x_{12}^2 = x_1x_{12}x_2$ and $x_{12} \neq 0$, and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(O_{\Sigma_{1,1}})$.

(1.1) Let $\rho_1 : \pi_1(\Sigma_{1,1}) \to \text{PSL}(2, \mathbb{C})$ be a representation defined by

$$
\rho_1(X_1) = \begin{pmatrix}
-1 & x_{12}^2 & x_1x_{12}^2 \\
1 & x_{12} & x_1x_{12}
\end{pmatrix},
\rho_1(X_2) = \begin{pmatrix}
x_{12} & x_1 & -1 \\
x_{12} & -x_1 & 0
\end{pmatrix},
\rho_1(K_{\Sigma_{1,1}}) = \begin{pmatrix}
0 & 1 & -2 \\
-1 & 0 & 1
\end{pmatrix},
$$

where $X_1 = P_2P_1$ and $X_2 = P_0P_1$. Then $\rho_1 \in \Omega(\Sigma_{1,1})$ such that the Markoff triple associated with $\{\rho_1(P_j)\}$ is equal to $(x_1, x_{12}, x_2)$ up to equivalence.
(1.2) The above representation $\rho_1$ extends to a type-preserving representation of $\pi_1(O_{\Sigma_1,1})$ satisfying the following identities:

$$\rho_1(P_0) = \begin{pmatrix} x_2/x_{12} & (x_{12}x_2 - x_1)/x_{12}^2 \\ -x_1 & -x_2/x_{12} \end{pmatrix}, \quad \rho_1(P_1) = \begin{pmatrix} 0 & -1/x_{12} \\ x_{12} & 0 \end{pmatrix},$$

$$\rho_1(P_2) = \begin{pmatrix} -x_1/x_{12} & (x_1x_{12} - x_2)/x_{12}^2 \\ -x_2 & x_1/x_{12} \end{pmatrix}, \quad \rho_1(K) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(2) Let $(y_1, y_{12}, y_2) \in \mathbb{C}^3$ be a triple satisfying $y_1^2 + y_{12}^2 + 4 = y_1y_{12}y_2$ and $y_2 \neq 0$, and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(O_{N_{2,1}})$. Then $\rho_2 : \pi_1(O_{N_{2,1}}) \rightarrow \text{PSL}(2, \mathbb{C})$ be a representation defined by

$$\rho_2(Y_1) = \begin{pmatrix} y_1i/2 & -y_12i/2y_2 \\ -(y_1y_2 - y_{12})y_2i/2 & y_1i/2 \end{pmatrix},$$

$$\rho_2(Y_2) = \begin{pmatrix} y_12i/2 & -(y_1y_2 - y_{12})i/2y_2 \\ -y_1y_{12}i/2 & y_12i/2 \end{pmatrix},$$

$$\rho_2(Y_2) = \begin{pmatrix} 0 & 1/y_2 \\ -y_2 & y_2 \end{pmatrix}, \quad \rho_2(K_{N_{2,1}}) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix},$$

where $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Then $\rho_2 \in \Omega(N_{2,1})$ such that the pseudo-Markoff triple associated with $\{\rho_2(Q_j)\}$ is equal to $(y_1, y_{12}, y_2)$ up to equivalence.

(2.1) Let $\rho_2 : \pi_1(O_{N_{2,1}}) \rightarrow \text{PSL}(2, \mathbb{C})$ be a representation defined by

$$\rho_2(Q_0) = \begin{pmatrix} y_1/2 & -y_12y_2/2 \\ (y_1y_2 - y_{12})y_2/2 & y_1/2 \end{pmatrix},$$

$$\rho_2(Q_1) = \begin{pmatrix} -(y_1^2 + 2i)/2 & y_1y_{12}i/2y_2 \\ -y_1y_{12}(y_1y_2 - y_{12})i/2 & (y_1^2 + 2i)/2 \end{pmatrix},$$

$$\rho_2(Q_2) = \begin{pmatrix} -y_12/2 & (y_{12}y_2 - y_1)/2y_2 \\ -y_1y_{12}/2 & y_12/2 \end{pmatrix},$$

$$\rho_2(K) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho_2(K_2) = \begin{pmatrix} i & -2i \\ 0 & -i \end{pmatrix}.$$

Convention 4.5. (1) For any element $\rho_1 \in \Omega(\Sigma_{1,1})$, after taking conjugate of $\rho_1$ by some element of $\text{PSL}(2, \mathbb{C})$, we always assume that $\rho_1$ is normalized so that $\rho_1(K)$ is given by the identity in Lemma 4.4(1.2) without changing the equivalence class.

(2) For any element $\rho_2 \in \Omega(N_{2,1})$, after taking conjugate of $\rho_2$ by some element of $\text{PSL}(2, \mathbb{C})$, we always assume that $\rho_2$ is normalized so that $\rho_2(K_0)$ and $\rho_2(K_2)$ are given by the identities in Lemma 4.4(2.2) without changing the equivalence class.

Pick an element $\rho_1 \in \Omega(\Sigma_{1,1})$ and a sequence of elliptic generators $\{P_i\}$ of $\pi_1(O_{\Sigma_{1,1}})$. Let $(x_1, x_{12}, x_2) \in \mathbb{C}^3$ be the Markoff triple associated with $\{\rho_1(P_i)\}$. Suppose $x_1x_{12}x_2 \neq 0$. Then the identity $x_1^2 + x_{12}^2 + x_2^2 = x_1x_{12}x_2$ implies the
following identity:

\[ a_0 + a_1 + a_2 = 1, \text{ where } a_0 = \frac{x_1}{x_{12}x_2}, \quad a_1 = \frac{x_{12}}{x_2x_1}, \quad a_2 = \frac{x_2}{x_1x_{12}}. \]

We call the triple \((a_0, a_1, a_2) \in (\mathbb{C}^*)^3\) the complex probability associated with \(\{\rho_1(P_j)\}\), where \(\mathbb{C}^* = \mathbb{C} - \{0\}\). We note that the Markoff triple \((x_1, x_{12}, x_2)\) with \(x_1x_{12}x_2 \neq 0\) up to sign (that is, up to equivalence) is recovered from the complex probability by the following identities:

\[ x_1^2 = \frac{1}{a_1a_2}, \quad x_{12}^2 = \frac{1}{a_2a_0}, \quad x_2^2 = \frac{1}{a_0a_1}. \]

Moreover, there is a nice geometric construction of a type-preserving representation from the corresponding complex probability.

To introduce the geometric construction of the representations, we prepare some notations. Throughout this paper, \(\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+\) denotes the upper half space model of the 3-dimensional hyperbolic space.

**Definition 4.6.** Let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be an element of \(\text{PSL}(2, \mathbb{C})\) such that \(A(\infty) \neq \infty\), namely \(c \neq 0\). Then the isometric hemisphere \(I(A)\) of \(A\) is the hyperplane of the upper half space \(\mathbb{H}^3\) bounded by

\[ \{ z \in \mathbb{C} \mid |A'(z)| = 1 \} = \{ z \in \mathbb{C} \mid |cz + d| = 1 \}. \]

Thus \(I(A)\) is a Euclidean hemisphere orthogonal to \(C = \partial \mathbb{H}^3\) with center \(c(A) = A^{-1}(\infty) = -d/c\) and radius \(r(A) = 1/|c|\). We denote by \(E(A)\) the closed half space of \(\mathbb{H}^3\) with boundary \(I(A)\) which is of infinite diameter with respect to the Euclidean metric.

**Lemma 4.7.** [1, Lemma 4.1.1] Let \(A\) be an element of \(\text{PSL}(2, \mathbb{C})\) which does not fix \(\infty\) and let \(W\) be an element of \(\text{PSL}(2, \mathbb{C})\) which preserves \(\infty\) and acts on \(C = \partial \mathbb{H}^3\) as a Euclidean isometry. Then

\[ I(AW) = W^{-1}(I(A)), \quad I(WA) = I(A). \]

In particular, \(I(WAW^{-1}) = WI(A)\).

Now we introduce a nice geometric construction of a type-preserving representation from the corresponding complex probability (cf. [1, Proposition 2.4.4]).

**Proposition 4.8.** Under Convention 4.5, the following hold:

1. For any triple \((a_0, a_1, a_2) \in (\mathbb{C}^*)^3\) such that \(a_0 + a_1 + a_2 = 1\) and for any sequence of elliptic generators \(\{P_j\}\) of \(\pi_1(\Sigma_{1,1})\), there is an element \(\rho_1 \in \Omega(\Sigma_{1,1})\) such that the complex probability associated with \(\{\rho_1(P_j)\}\) is equal to \((a_0, a_1, a_2)\). Moreover, \(\rho_1\) satisfies the following conditions (see Figure 5).

1.1 The centers of isometric hemispheres of \(\rho_1(P_j)\) satisfy the following conditions:

- \(c(\rho_1(P_{3k+2})) - c(\rho_1(P_{3k+1})) = a_0\).
- \(c(\rho_1(P_{3k+3})) - c(\rho_1(P_{3k+2})) = a_1\).
- \(c(\rho_1(P_{3k+4})) - c(\rho_1(P_{3k+3})) = a_2\).
(1.2) The isometries $\rho_1(P_j)$ satisfy the following conditions.
- The isometry $\rho_1(P_{3k+2})$ is the $\pi$-rotation about the geodesic with end-points $\sigma(\rho_1(P_{3k+2})) \pm \sqrt{a_0(-a_1)}$.
- The isometry $\rho_1(P_{3k})$ is the $\pi$-rotation about the geodesic with end-points $\sigma(\rho_1(P_{3k})) \pm \sqrt{a_1(-a_2)}$.
- The isometry $\rho_1(P_{3k+1})$ is the $\pi$-rotation about the geodesic with end-points $\sigma(\rho_1(P_{3k+1})) \pm \sqrt{a_2(-a_0)}$.

(2) Conversely, under Convention 4.5, any element $\rho_1 \in \Omega(\Sigma_{1,1})$ with the complex probability $(a_0, a_1, a_2)$ associated with $\rho_1(P_j)$ for some sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ satisfies the above conditions.

Figure 5. Isometric hemispheres of elliptic generators of $\pi_1(\Sigma_{1,1})$

NOTATION 4.9. Under Convention 4.5, let $\rho_1$ be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let $\xi$ be the automorphism of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ given by the following (cf. Proposition 3.3):

$$(\xi(P_0), \xi(P_1), \xi(P_2)) = (P_0^{P_1}, P_1, P_0^{P_1}).$$

If the complex probability associated with $\rho_1(\xi^k(P_j))$ is well-defined, then we denote it by $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$.

The following lemma can be verified by simple calculation (cf. [1, Lemma 2.4.1]).

LEMMA 4.10. Under Convention 4.5, let $\rho_1$ be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Suppose that the complex probability $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$ associated with $\rho_1(\xi^k(P_j))$ is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 6):

$$a_0^{(k+1)} = 1 - a_2^{(k)}, \quad a_1^{(k+1)} = \frac{a_1^{(k)} a_2^{(k)}}{1 - a_2^{(k)}}, \quad a_2^{(k+1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_2^{(k)}},$$

$$a_0^{(k-1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_0^{(k)}}, \quad a_1^{(k-1)} = \frac{a_0^{(k)} a_1^{(k)}}{1 - a_0^{(k)}}, \quad a_2^{(k-1)} = 1 - a_0^{(k)}.$$
Moreover, we have the following proposition.

For any triple \((y_1, y_{12}, y_2)\) such that the complex probability associated with \(\{(p_2(Q_j))\}\) satisfies the following conditions (see Figure 7). We note that the pseudo-Markoff triple \((y_1, y_{12}, y_2)\) with \(y_1y_{12}y_2\neq 0\) up to sign (that is, up to equivalence) is recovered from the complex probability by the following identities:

\[
y_1^2 = \frac{4b_0}{b_1}, \quad (y_{12})^2 = \frac{4b_2}{b_1}, \quad y_2^2 = \frac{1}{b_2b_0}.
\]

Moreover, we have the following proposition.

**Proposition 4.11.** Under Convention 4.5, the following hold:

1. For any triple \((b_0, b_1, b_2)\) in \((\mathbb{C}^*)^3\) such that \(b_0 + b_1 + b_2 = 1\) and for any sequence of elliptic generators \(\{Q_j\}\) of \(\pi_1(O_{N_2,1})\), there is an element \(p_2 \in \Omega(N_{2,1})\) such that the complex probability associated with \(\{(p_2(Q_j))\}\) is equal to \((b_0, b_1, b_2)\). Moreover, \(p_2\) satisfies the following conditions (see Figure 7).

   1. The centers of isometric hemispheres of \(p_2(Q_j)\) satisfy the following conditions.
      - \(c(p_2(Q_{4k})) - c(p_2(Q_{4k+3}Q_{6k-1})) = b_0\).
      - \(c(p_2(Q_{4k+3}) - c(p_2(Q_{4k}))) = b_1\).
      - \(c(p_2(Q_{6k}Q_{6k+2})) - c(p_2(Q_{6k+2})) = b_2\).
      - \(c(p_2(Q_{6k+3})) - c(p_2(Q_{6k+2})) = b_2\).
      - \(c(p_2(Q_{6k+5})) - c(p_2(Q_{6k+2})) = b_1\).
      - \(c(p_2(Q_{4k+3}Q_{6k+5})) - c(p_2(Q_{6k+3}Q_{6k+5})) = b_0\).
      - \(c(p_2(Q_{3k+1})) = \frac{1}{2}(c(p_2(Q_{3k}))) + c(p_2(Q_{3k+2}))\).

2. The isometries \(p_2(Q_j)\) satisfy the following conditions.
   - For any \(j\) with \(j \equiv 0\) or \(5\) (mod 6), the isometry \(p_2(Q_j)\) is the \(\pi\)-rotation about the geodesic with endpoints \(c(p_2(Q_j)) \pm \sqrt{b_0(-b_1)}\).
   - For any \(j\) with \(j \equiv 2\) or \(3\) (mod 6), the isometry \(p_2(Q_j)\) is the \(\pi\)-rotation about the geodesic with endpoints \(c(p_2(Q_j)) \pm \sqrt{b_1(-b_2)}\).
For any $k$, the isometry $\rho_2(Q_{3k}Q_{3k+2})$ is the composition of the $\pi$-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k}Q_{3k+2})) \pm \sqrt{b_2(-b_0)}$ and the horizontal translation $z \mapsto z - 1$. In particular, the isometry $\rho_2(Q_{3k+2}Q_{3k})$ is the composition of the $\pi$-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k+2}Q_{3k})) \pm \sqrt{b_2(-b_0)}$ and the horizontal translation $z \mapsto z + 1$.

For any $k$, the isometry $\rho_2(Q_{3k+1})$ is the $\pi$-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k}))$ and $c(\rho_2(Q_{3k+2}))$.

(2) Conversely, under Convention 4.5, any element $\rho_2 \in \Omega(N_{2,1})$ with the complex probability $(b_0, b_1, b_2)$ associated with $\{\rho_2(Q_j)\}$ for some sequence of elliptic generators $\{Q_j\}$ of $\pi_1(O_{N_{2,1}})$ satisfies the above conditions.

![Figure 7. Isometric hemispheres of elliptic generators of $\pi_1(N_{2,1})$](image)

**Figure 7. Isometric hemispheres of elliptic generators of $\pi_1(N_{2,1})$**

**Proof.** (1) Pick a triple $(b_0, b_1, b_2) \in (\mathbb{C}^*)^3$ satisfying $b_0 + b_1 + b_2 = 1$ and fix a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(O_{N_{2,1}})$. Let $(y_1, z, y_2) \in (\mathbb{C}^*)^3$ be a triple of a root of the following polynomial equation:

$$y_1^2 = \frac{4b_0}{b_1}, \quad z^2 = \frac{4b_2}{b_1}, \quad y_2^2 = \frac{1}{b_2b_0}.$$

Replacing $y_2$ by $-y_2$ if necessary, the triple $(y_1, z, y_2) \in (\mathbb{C}^*)^3$ satisfies

$$y_1^2 + z^2 + 4 = y_1 y_2$$

and $y_2$ is not equal to 0. Hence the triple $(y_1, z, y_2) \in (\mathbb{C}^*)^3$ is a pseudo-Markoff triple. Set $y_{12} = y_1y_2 - z$. By direct calculation, we can see that the triple $(y_1, y_{12}, y_2) \in (\mathbb{C}^*)^3$ is also a pseudo-Markoff triple, namely, the triple satisfies (Eq1) and (Eq2). Hence, for the triple $(y_1, y_{12}, y_2)$, we have an element $\rho_2 \in \Omega(N_{2,1})$ which is as in Lemma 4.4(2.2). By the formula in Lemma 4.4(2.2), we have the following (cf. Figure 7):

- $c(\rho_2(Q_0)) - c(\rho_2(Q_2Q_0)) = b_0$. 


\begin{itemize}
  \item $c(p_2(Q_0)) - c(p_2(Q_0)) = b_1$.
  \item $c(p_2(Q_0)) - c(p_2(Q_2)) = b_2$.
  \item $c(p_2(Q_0)) = \frac{1}{2}(c(p_2(Q_2)) + c(p_2(Q_0)))$.
  \item $p_2(Q_0)$ is the $\pi$-rotation about the geodesic with endpoints $c(p_2(Q_0)) \pm \sqrt{b_0(-b_1)}$.
  \item $p_2(Q_2)$ is the $\pi$-rotation about the geodesic with endpoints $c(p_2(Q_2)) \pm \sqrt{b_1(-b_2)}$.
  \item $p_2(Q_0Q_2)$ is the composition of the $\pi$-rotation about the geodesic with endpoints $c(p_2(Q_0Q_2)) \pm \sqrt{b_2(-b_0)}$ and the translation $z \mapsto z - 1$.
  \item $p_2(Q_1)$ is the $\pi$-rotation about the geodesic with endpoints $c(p_2(Q_0))$ and $c(p_2(Q_2))$.
\end{itemize}

Recall that the sequence of elliptic generators \{Q_j\} satisfies the following:

$$Q_{K_0}^j = Q_{-1}, \quad Q_{K_2}^j = Q_{-j+5}, \quad \text{where } K_0 = Q_{1}^{Q_0}, \quad K_2 = Q_{1}^{Q_2}.$$  

Note that the isometry $\rho_2(K_0)$ (resp. $\rho_2(K_2)$) is the $\pi$-rotation about the vertical geodesic above 0 (resp. 1). Here a vertical geodesic above a point $z \in \mathbb{C}$ means the geodesic \{z\} $\times \mathbb{R}$ in $\mathbb{H}^2 = \mathbb{C} \times \mathbb{R}_+$. Hence, by Lemma 4.7, we have $I(\gamma_{p_2(K_0)}) = \rho_2(K_0)(I(\gamma))$ and $I(\gamma_{p_2(K_2)}) = \rho_2(K_2)(I(\gamma))$ for any $\gamma \in \text{PSL}(2, \mathbb{C})$ such that $\gamma(\infty) \neq \infty$. Thus we obtain the desired result.

(2) Let $p_2$ be an element of $\Omega(N_{2,1})$. Since $p_2$ is normalized, the representation $p_2$ is conjugate to a representation as in Lemma 4.4(2.2) by some Euclidean translation. Since the properties in Proposition 4.11(1) are invariant by Euclidean translations, we have the desired result by the above proof.

**Notation 4.12.** Under Convention 4.5, let $p_2$ be an element of $\Omega(N_{2,1})$ and let \{Q_j\} be a sequence of elliptic generators of $\pi_1(O_{N_{2,1}})$. Let $\sigma$ be the automorphism of $\pi_1(O_{N_{2,1}})$ given by Proposition 3.7(1.2), namely,

$$\langle \sigma(Q_0), \sigma(Q_1), \sigma(Q_2) \rangle = (Q_2, Q_1^{Q_2Q_0}, Q_0^{Q_2}).$$

If the complex probability associated with \{p_2(\sigma^k(Q_j))\} is well-defined, then we denote it by $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$.

The following lemma can be verified by simple calculation.

**Lemma 4.13.** Under Convention 4.5, let $p_2$ be an element of $\Omega(N_{2,1})$ and let \{Q_j\} be a sequence of elliptic generators of $\pi_1(O_{N_{2,1}})$. Suppose that the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with \{p_2(\sigma^k(Q_j))\} is well-defined for any $k \in \mathbb{Z}$. Then we have the following identities (cf. Figure 8):

\[
\begin{align*}
 b_0^{(k+1)} &= 1 - b_2^{(k)}, \quad b_1^{(k+1)} = b_1^{(k)} b_2^{(k)} \frac{b_2^{(k)}}{1 - b_2^{(k)}}, \quad b_2^{(k+1)} = b_2^{(k)} b_0^{(k)} \frac{b_0^{(k)}}{1 - b_0^{(k)}}, \\
 b_0^{(k-1)} &= b_2^{(k)} b_0^{(k)} \frac{b_0^{(k)}}{1 - b_0^{(k)}}, \quad b_1^{(k-1)} = b_1^{(k)} b_0^{(k)} \frac{b_1^{(k)}}{1 - b_1^{(k)}}, \quad b_2^{(k-1)} = b_2^{(k)} b_0^{(k)} \frac{b_2^{(k)}}{1 - b_2^{(k)}}.
\end{align*}
\]
As a consequence of Propositions 4.8, 4.11 and Lemmas 4.10, 4.13, we have the following corollary.

**Corollary 4.14.** Under Convention 4.5, let \( \rho_1 \) and \( \rho_2 \) be elements of \( \Omega(\Sigma_{1,1}) \) and \( \Omega(N_{2,1}) \), respectively. Let \( \{P_j\} \) and \( \{Q_j\} \) be sequences of elliptic generators of \( \pi_1(O_{\Sigma_{1,1}}) \) and \( \pi_1(O_{N_{2,1}}) \), respectively. Suppose that the complex probabilities \((a_0, a_1, a_2)\) and \((b_0, b_1, b_2)\) associated with \(\{\rho_1(P_j)\}\) and \(\{\rho_2(Q_j)\}\), respectively, are well-defined. Then the following hold.

1. \((a_0, a_1, a_2) = (b_0, b_1, b_2)\) and \(c(\rho_1(P)) = c(\rho_2(Q))\) if and only if \(\rho_1(P_{0j+2}) = \rho_2(Q_{0j+2})\) for some \(j \in \mathbb{Z}\). Moreover, if these conditions hold, then the following identities hold for any \(j, k \in \mathbb{N}\):

   \[
   (a_0^{(k)}, a_1^{(k)}, a_2^{(k)}) = (b_0^{(k)}, b_1^{(k)}, b_2^{(k)}),
   \]

2. \((a_0, a_1, a_2) = (b_0, b_1, b_2)\) and \(c(\rho_1(P)) = c(\rho_2(Q))\) if and only if \(\rho_1(P_{0j+6}) = \rho_2(Q_{0j+6})\) for some \(j \in \mathbb{Z}\). Moreover, if these conditions hold, then the following identities hold for any \(j, k \in \mathbb{N}\):

   \[
   (a_0^{(k)}, a_1^{(k)}, a_2^{(k)}) = (b_2^{(k)}, b_1^{(k)}, b_0^{(k)}),
   \]

At the end of this section, we prove the following lemma.

**Lemma 4.15.** Let \( \rho_1 \) and \( \rho_2 \) be type-preserving \(\text{PSL}(2, \mathbb{C})\)-representations of \(\pi_1(\Sigma_{1,1})\) and \(\pi_1(N_{2,1})\), respectively. Let \(\tilde{\rho}_1\) and \(\tilde{\rho}_2\), respectively, be the unique extensions of \( \rho_1 \) and \( \rho_2 \) given by Proposition 4.1. Then \( \rho_1 \) and \( \rho_2 \) are commensurable if and only if \(\tilde{\rho}_1\) and \(\tilde{\rho}_2\) are commensurable.

**Proof.** We first show the if part. Suppose that \(\tilde{\rho}_1\) and \(\tilde{\rho}_2\) are commensurable, i.e., there exist double coverings \( \rho_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) and \( \rho_2 : \Sigma_{1,2} \to \Sigma_{1,1} \) such that \(\rho_1 \circ (\rho_1)_* = \rho_2 \circ (\rho_2)_*\). By the correspondence between double coverings described in Section 2 (see Figure 2), there exist double coverings \(\tilde{\rho}_1 : \Sigma_{1,2} \to \Sigma_{1,1}\) and \(\tilde{\rho}_2 : \Sigma_{1,2} \to N_{2,1}\) such that \(p_{\Sigma_{1,1}} \circ \tilde{\rho}_1 = p_1 \circ p_{\Sigma_{1,2}}\) and \(p_{\Sigma_{1,1}} \circ \tilde{\rho}_2 = p_2 \circ p_{\Sigma_{1,2}}\).
One solution. Hence we have the following identity:

\[ \rho_1 \circ (\tilde{p}_1)_* = \rho_1 \circ (\tilde{p}_1)_* \]

\[ = \tilde{p}_1 \circ (\rho_1)_* \]

\[ = \rho_2 \circ (\rho_1)_* \]

\[ = \rho_2 \circ (\tilde{p}_2)_* = \rho_2 \circ (\tilde{p}_2)_* . \]

Next we show the only if part. Suppose that \( \rho_1 \) and \( \rho_2 \) are commensurable, namely there exist double coverings \( \tilde{p}_1 : \Sigma_{1,2} \to \Sigma_{1,1} \) and \( p_2 : \Sigma_{1,2} \to N_{2,1} \) such that \( \rho_1 \circ (\tilde{p}_1)_* = \rho_2 \circ (\tilde{p}_2)_* . \) By the correspondence between double coverings described in Section 2 (see Figure 2), we have double coverings \( \rho_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}} \) and \( p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}} \) such that \( \rho_1 \circ p_1 = \tilde{p}_1 \circ p_2 \) and \( \rho_2 \circ p_2 = \rho_2 \circ p_\Sigma_{1,2} . \) Hence we have the following identity (see Figure 9):

\[ \tilde{p}_1 \circ (\tilde{p}_1)_* \circ (\rho_1)_* = \rho_1 \circ (\tilde{p}_1)_* = \rho_2 \circ (\tilde{p}_2)_* = \rho_2 \circ (\rho_1)_* . \]

This means that both \( \tilde{p}_1 \circ (\tilde{p}_1)_* \) and \( \tilde{p}_2 \circ (\tilde{p}_2)_* \) are extensions of \( \rho := \rho_1 \circ (\tilde{p}_1)_* = \rho_2 \circ (\tilde{p}_2)_* \) of \( \pi_1(\Sigma_{1,2}) \) to \( \pi_1(\Sigma_{1,1}) \). Note that \( \pi_1(\mathcal{O}_{\Sigma_{1,2}}) \) is generated by \( \pi_1(\Sigma_{1,2}) = (Z_1, Z_2, Z_3) \) and the element \( R_1 \) and that the generators satisfy the following identities (see Section 2):

\[ R_1 Z_j R_1^{-1} = Z_j^{-1} \text{ for } j = 1, 2, 3. \]

Hence both \( \tilde{p}_1 \circ (\tilde{p}_1)_*(R_1) \) and \( \tilde{p}_2 \circ (\tilde{p}_2)_*(R_1) \) are solutions of the following system of equation in \( \text{PSL}(2, \mathbb{C}) \).

\[ g \rho(Z_j) g^{-1} = \rho(Z_j)^{-1} \text{ for } j = 1, 2, 3. \]

On the other hand, since \( \rho \) is irreducible, the system of equations have at most one solution. Hence we have \( \tilde{p}_1 \circ (\tilde{p}_1)_*(R_1) = \tilde{p}_2 \circ (\tilde{p}_2)_*(R_1) \), and therefore we have \( \tilde{p}_1 \circ (\tilde{p}_1)_* = \tilde{p}_2 \circ (\tilde{p}_2)_* . \)

\[ \text{Figure 9} \]
Remark 4.16. Let \( \rho_1, \rho_2, \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) be as in Lemma 4.15 and assume that \( \rho_1 \) and \( \rho_2 \) (and so \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \)) are commensurable. Then we can easily see, as in the proof of Proposition 4.1, that if one of the representations \( \rho_1, \rho_2, \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) is faithful, then all of them are faithful.

5. Main Theorem

In this section, we give a partial answer to Problem 2.3. By Lemma 4.15, we may only consider the problem for the quotient orbifolds. Our partial answer to the commensurability problem for representations of the fundamental groups of the orbifolds \( O_{1,1} \) and \( O_{N,1} \) is given as follows.

Theorem 5.1. Under Convention 4.5, the following hold:

1. Let \( \rho_2 \) be an element of \( \Omega(N_{2,1}) \). Suppose that \( \rho_2 \) is faithful. Then the following conditions are equivalent.
   
   (i) There exists a faithful representation \( \rho_1 \in \Omega(\Sigma_{1,1}) \) which is commensurable with \( \rho_2 \).
   
   (ii) There exist a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(O_{N,1}) \) and an integer \( k_0 \) such that the complex probability \( (b_0, b_1, b_2) \) associated with \( \{\rho_2(Q_j)\} \) satisfies the following identity under Notation 4.12 (cf. Figure 10):
   
   \[
   (b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).
   \]

   (iii) There exists a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(O_{N,1}) \) such that the complex probability \( (b_0, b_1, b_2) \) associated with \( \{\rho_2(Q_j)\} \) satisfies one of the following identities:
   
   (a) \( (b_0^{(0)}, b_1^{(0)}, b_2^{(0)}) = (b_2, b_1, b_0) \),
   
   (b) \( (b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0) \).

2. If the conditions in (1) hold, the representation \( \rho_1 \) is unique up to precomposition by an automorphism of \( \pi_1(O_{1,1}) \) preserving \( K \).

3. Moreover, the following hold:

   (a) \( \rho_2 \) extends to a type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representation of \( \pi_1(O_n) \) if and only if \( \rho_2 \) satisfies the condition (iii)-(a). Moreover, if these conditions are satisfied, the extension is unique.

   (b) \( \rho_2 \) extends to a type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representation of \( \pi_1(O_3) \) if and only if \( \rho_2 \) satisfies the condition (iii)-(b). Moreover, if these conditions are satisfied, the extension is unique.

Remark 5.2. By using this theorem, we can prove the “converse” condition, namely, we can give a condition for a faithful type-preserving \( \text{PSL}(2, \mathbb{C}) \)-representation of \( \pi_1(\Sigma_{1,1}) \) to be commensurable with that of \( \pi_1(N_{2,1}) \) (see [4]).

Proof. We prove (1) by proving the implications (iii) \( \Rightarrow \) (ii), (ii) \( \Rightarrow \) (iii), (ii) \( \Rightarrow \) (i) and (i) \( \Rightarrow \) (ii).

(iii) \( \Rightarrow \) (ii). This is obvious.
Pick a sequence of elliptic generators \( f \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \). Then by Proposition 4.8(1), there exists a (normalized) type-preserving representation \( \rho_1 \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) such that the complex probability associated with \( \{\rho_1(P_j)\} \) is equal to \( \{b_0, b_1, b_2\} \). After taking conjugate of \( \rho_1 \) by a parallel translation, we may

(ii) \( \Rightarrow \) (iii). Suppose that there exist a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) and an integer \( k_0 \) such that the complex probability \( \{b_0, b_1, b_2\} \) associated with \( \{\rho_2(Q_j)\} \) satisfies the following identity:

\[
(b_{0}^{(k_0)}, b_{1}^{(k_0)}, b_{2}^{(k_0)}) = (b_2, b_1, b_0).
\]

Recall that the triple \( (b_{0}^{(k_0)}, b_{1}^{(k_0)}, b_{2}^{(k_0)}) \) is the complex probability associated with \( \{\rho_2(\sigma^{k_0}(Q_j))\} \), where \( \sigma \) is the automorphism of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) given as in Notation 4.12. Since \( \rho_2 \) is faithful, we have \( \text{tr}(\rho_2(\gamma)) \neq 0 \) for any \( \gamma \in \pi_1(N_{2,1}) \).

Hence the complex probability \( (b_{0}^{(k)}, b_{1}^{(k)}, b_{2}^{(k)}) \) associated with \( \{\rho_2(\sigma^{k}(Q_j))\} \) is well-defined for any \( k \in \mathbb{Z} \). By the assumption \( (b_{0}^{(k_0)}, b_{1}^{(k_0)}, b_{2}^{(k_0)}) = (b_2, b_1, b_0) \) and Lemma 4.13, we have

\[
(b_{0}^{(k_0 \pm l)}, b_{1}^{(k_0 \pm l)}, b_{2}^{(k_0 \pm l)}) = (b_{l}^{(k_0 \pm l)}, b_{1}^{(k_0 \pm l)}, b_{2}^{(k_0 \pm l)})
\]

for any \( l \in \mathbb{Z} \). In particular we have

\[
(b_{0}^{(k_0 - \frac{k_0}{2})}, b_{1}^{(k_0 - \frac{k_0}{2})}, b_{2}^{(k_0 - \frac{k_0}{2})}) = (b_{0}^{(k_0)}, b_{1}^{(k_0)}, b_{2}^{(k_0)}) \quad \text{if } k_0 \text{ is even},
\]

\[
(b_{0}^{(k_0 - \frac{k_0 - 1}{2})}, b_{1}^{(k_0 - \frac{k_0 - 1}{2})}, b_{2}^{(k_0 - \frac{k_0 - 1}{2})}) = (b_{0}^{(k_0 - 1)}, b_{1}^{(k_0 - 1)}, b_{2}^{(k_0 - 1)}) \quad \text{if } k_0 \text{ is odd}.
\]

Hence, by replacing \( \{Q_j\} \) with \( \{\sigma^{k_0 - \frac{k_0}{2}}(Q_j)\} \) or \( \{\sigma^{k_0 - \frac{k_0 - 1}{2}}(Q_j)\} \) according to whether \( k_0 \) is even or odd, we obtain the desired result.

(ii) \( \Rightarrow \) (i). Suppose that there exist a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) and an integer \( k_0 \) such that the complex probability \( \{b_0, b_1, b_2\} \) associated with \( \{\rho_2(Q_j)\} \) satisfies the following identity:

\[
(b_{0}^{(k_0)}, b_{1}^{(k_0)}, b_{2}^{(k_0)}) = (b_2, b_1, b_0).
\]
assume that \( c(\rho_1(P_1)) = c(\rho_2(Q_2Q_0)) \). Then, by Corollary 4.14(1), we see that
\[
(\rho_1(P_2), \rho_1(P_3)) = (\rho_2(Q_0), \rho_2(Q_2)).
\]
By Lemmas 4.10 and 4.13, we see that the complex probability associated with \( \{\rho_1(\xi^{k_0}(P_j))\} \) is equal to the complex probability \( \{b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}\} \) associated with \( \{\rho_2(\sigma^{k_0}(Q_j))\} \), where \( \xi \) and \( \sigma \) are, respectively, the automorphisms of \( \pi_1(O_{\Sigma_{1,1}}) \) and \( \pi_1(O_{N_{2,1}}) \) given by Notations 4.9 and 4.12. Hence, by the assumption \( (b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0) \) and Corollary 4.14(2), we have
\[
(\rho_1(\xi^{k_0}(P_5)), \rho_1(\xi^{k_0}(P_6))) = (\rho_2(Q_2^{K_2}), \rho_2(Q_0^{K_2})).
\]
Hence we have
\[
(\rho_1(P_2), \rho_1(P_3), \rho_1(\xi^{k_0}(P_5)), \rho_1(\xi^{k_0}(P_6))) = (\rho_2(Q_0), \rho_2(Q_2), \rho_2(Q_2^{K_2}), \rho_2(Q_0^{K_2})).
\]

**Claim 5.3.** Let \( (R_0, R_1, R_2, R_3) \) be the generator system of \( \pi_1(O_{\Sigma_{1,2}}) \) given in Section 2.

1. There is a double covering \( p_1 : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}} \) such that
\[
(\rho_1(P_2), \rho_1(P_3), \rho_1(P_5), \rho_1(P_6)) = (P_2, P_3, \xi^{k_0}(P_5), \xi^{k_0}(P_6)).
\]

2. There is a double covering \( p_2 : O_{\Sigma_{1,2}} \to O_{N_{2,1}} \) such that
\[
(\rho_2(Q_0), \rho_2(Q_2), \rho_2(Q_2^{K_2}, Q_0^{K_2})).
\]

**Proof.** (2) can be seen by choosing \( p_2 \) to be the covering corresponding to the epimorphism \( \phi_2 : \pi_1(O_{N_{2,1}}) \to \mathbb{Z}/2\mathbb{Z} \) defined by the following formula (see Figure 2):
\[
\phi_2(Q_j) = \begin{cases} 0 & \text{if } j = 0 \text{ or } 2, \\ 1 & \text{if } j = 1. \end{cases}
\]
To prove (1), let \( q_1 : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}} \) be the double covering such that the following holds (see Figure 2):
\[
(q_1)_*(R_0), (q_1)_*(R_1), (q_1)_*(R_2), (q_1)_*(R_3)) = (P_0, P_1, P_0^K, P_1^K).
\]
Let \( f \) be a self-homeomorphism of \( O_{\Sigma_{1,1}} \) such that \( f_* \) maps \( (P_0, P_1, P_2) \) to \( (P_2, P_3, P_1^K) \), and consider the double covering \( p_1^{(0)} := f \circ q_1 : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}} \). Then we have
\[
((p_1^{(0)})_* (R_0), (p_1^{(0)})_* (R_1), (p_1^{(0)})_* (R_2), (p_1^{(0)})_* (R_3)) = (P_2, P_3, P_0, P_0).
\]
Let \( \xi \) be the self-homeomorphism of \( O_{\Sigma_{1,2}} \) such that
\[
((\tilde{\xi})_* (R_0), (\tilde{\xi})_* (R_1), (\tilde{\xi})_* (R_2), (\tilde{\xi})_* (R_3)) = (R_0, R_1, R_3, R_2^{R_1}).
\]
Then the double covering \( p_1 := p_1^{(0)} \circ \tilde{\xi} : O_{\Sigma_{1,2}} \to O_{\Sigma_{1,1}} \) satisfies the desired condition.
By Claim 5.3, there are double coverings $p_1 : \mathcal{O}_{\Sigma_{1.2}} \to \mathcal{O}_{\Sigma_{1.1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1.2}} \to \mathcal{O}_{\Sigma_{1.1}}$, satisfying the following identity:

$$(p_1 \circ (p_1)_*(R_0), p_1 \circ (p_1)_*(R_1), p_1 \circ (p_1)_*(R_2), p_1 \circ (p_1)_*(R_3))$$

$$= (p_1(P_2), p_1(P_3), p_1(\xi^K_0(P_3)), p_1(\xi^K_0(P_0)))$$

$$= (p_2(Q_0), p_2(Q_2), p_2(Q^K_2), p_2(Q^K_0))$$

$$= (p_2 \circ (p_2)_*(R_0), p_2 \circ (p_2)_*(R_1), p_2 \circ (p_2)_*(R_2), p_2 \circ (p_2)_*(R_3)).$$

Hence $p_1 \circ (p_1)_* = p_2 \circ (p_2)_*$, namely, the representation $p_2$ is commensurable with $p_1$. By Remark 4.16, $p_1$ is faithful. Thus we obtain the desired representation $p_1$.

(i) $\Rightarrow$ (ii). Suppose that there exists a faithful (normalized) type-preserving $\text{PSL}(2, \mathbb{C})$-representation $p_1$ of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$ which is commensurable with $p_2$, i.e., there exist double coverings $p_1 : \mathcal{O}_{\Sigma_{1.2}} \to \mathcal{O}_{\Sigma_{1.1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1.2}} \to \mathcal{O}_{\Sigma_{1.1}}$, such that $p_1 \circ (p_1)_* = p_2 \circ (p_2)_*$. Recall that $(p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1.2}}))$ is equal to the kernel of the epimorphism $\phi_2 : \pi_1(\mathcal{O}_{\Sigma_{1.2}}) \to \mathbb{Z}/2\mathbb{Z}$ and that the kernel of $\phi_2$ is equal to the subgroup of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$ generated by the quadruple $(Q_0, Q_2, Q^K_2, Q^K_0)$. Hence we have $Q_0, Q_2 \in (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1.2}}))$. Set $P^{(0)} = (p_1 \circ p_2^{-1})_*(Q_0)$ and $P^{(1)} = (p_1 \circ p_2^{-1})_*(Q_2)$.

**Claim 5.4.** The ordered triple $(K^{-1}P^{(1)}P^{(0)}P^{(0)}P^{(1)})$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$.

**Proof.** Note that $P^{(0)}$ and $P^{(1)}$ have order 2, because

1. $(p_1 \circ p_2^{-1})_*(Q_0), (p_1 \circ p_2^{-1})_*(Q_2)$ are represented by $P^{(0)}$, $P^{(1)}$.
2. $Q_0, Q_2 = (p_1 \circ p_2^{-1})_*(Q_0), (p_1 \circ p_2^{-1})_*(Q_2)$.

By using the third assertion of Proposition 4.11(1.2) and the fact that $p_1(K)$ is the horizontal translation $z \mapsto z + 1$, we see that $p_1(K^{-1})p_2(Q_2)$ has order 2. By the definition of $P^{(0)}$ and $P^{(1)}$ and by the identity $p_1 \circ (p_1)_* = p_2 \circ (p_2)_*$, we have $p_1(P^{(1)}P^{(0)}) = p_2(Q_2Q_0)$. Hence $p_1(K^{-1}P^{(1)}P^{(0)})$ has order 2. Since $p_1$ is faithful, this implies that $K^{-1}P^{(1)}P^{(0)}$ has order 2. Hence, by Remark 3.2, the triple $(K^{-1}P^{(1)}P^{(0)}P^{(0)}P^{(1)})$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$. $\square$

By Claim 5.4 and Proposition 3.3(1.1), $(P^{(1)}P^{(0)}P^{(1)}P^{(0)}P^{(1)})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$. Let $\{P_j\}$ be the sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$ associated with this triple. Then $p_1(P_j) = p_1(P^{(0)}) = p_2(Q_0)$ and $p_1(P_j) = p_1(P^{(1)}) = p_2(Q_2)$. This implies, together with Corollary 4.14(1), that the complex probability associated with $\{p_1(P_j)\}$ is equal to $(b_0, b_1, b_2, b_3)$. Set $P^{(0)} = (p_1 \circ p_2^{-1})_*(Q^K_2), (p_1 \circ p_2^{-1})_*(Q^K_0)$. Then, by a parallel argument, the triple $(P^{(3)}K^{-2}, (K^{-1}P^{(3)}P^{(2)})K^{-1}, (P^{(2)}K^{-1}))$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$. Let $\{P'_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1.1}})$ associated with this triple. Then $p_1(P'_j) = p_1(P^{(2)}) = p_2(Q^K_2)$ and $p_1(P'_j) = p_1(P^{(3)}) = p_2(Q^K_0)$. This implies, together with Corollary 4.14(2), that the complex probability associated with $\{p_1(P'_j)\}$ is equal to $(b_2, b_1, b_0)$. 


Since $\rho_1(K)\rho_2(Q_0Q_2) = \rho_1(K^{-1})\rho_2(Q_0^KQ_2^K)$ by Proposition 4.11(1.2), we have
\[
\rho_1(P_4) = \rho_1(KP_2P_3) = \rho_1(K)p_2(Q_0Q_2)
\]
\[
= \rho_1(K^{-1})\rho_2(Q_0^KQ_2^K) = \rho_1(K^{-1}P_6^0P_5^0) = \rho_1(P_4')
\]
Since $\rho_1$ is faithful, this implies $P_4 = P_4'$. Hence, by Proposition 3.3, there is an integer $k_0$ such that $P_j^0 = \xi^{k_0}(P_j)$. By Lemmas 4.10 and 4.13, the complex probability associated with $\{ \rho_1(\xi^k(P_j)) \}$ is equal to the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with $\{ \rho_2(\sigma^k(Q_j)) \}$ for any $k \in \mathbb{Z}$. Hence we have
\[
(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).
\]
This the proof of the assertion (1) is complete.

Next we prove the assertion (2). Let $\rho_1$ and $\rho_1'$ be type-preserving PSL$(2, \mathbb{C})$-representations of $\pi_1(O_{1,1})$ such that they are commensurable with $\rho_2$. Then there exist coverings $\rho_1, \rho_1' : O_{1,1} \to O_{1,1}$, and $\rho_2 : O_{1,1} \to O_{1,1}$, such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1' \circ (p_1')_* = \rho_2 \circ (p_2)_*$. By Claim 5.4, the following triples are elliptic generator triples of $\pi_1(O_{1,1})$:
\[
(P_0, P_1, P_2) := (K^{-1}(p_1 \circ p_2^{-1})_*, (Q_2Q_0), (p_1 \circ p_2^{-1})_* (Q_0), (p_1 \circ p_2^{-1})_* (Q_2)),
\]
\[
(P_0', P_1', P_2') := (K^{-1}(p_1' \circ p_2^{-1})_*, (Q_2Q_0), (p_1' \circ p_2^{-1})_* (Q_0), (p_1' \circ p_2^{-1})_* (Q_2)).
\]
Since $\rho_1$ and $\rho_1'$ are commensurable with $\rho_2$, we have the following identity
\[
(\rho_1(P_0), \rho_1(P_1), \rho_1(P_2)) = (\rho_1(K^{-1})\rho_2(Q_2Q_0), \rho_2(Q_0), \rho_2(Q_2))
\]
\[= (\rho_1'(K^{-1})\rho_2(Q_2Q_0), \rho_2(Q_0), \rho_2(Q_2))
\]
\[= (\rho_1'(P_0'), \rho_1'(P_1'), \rho_1'(P_2')).
\]
By Proposition 3.3(2), there is an automorphism $f$ of $\pi_1(O_{1,1})$ preserving $K$ which maps $(P_0, P_1, P_2)$ to $(P_0', P_1', P_2')$. Hence we have $\rho_1 = \rho_1' \circ f$.

Finally we prove the assertion (3).

The if part of (a). Suppose that there exists a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(O_{1,1})$ such that the complex probability $(b_0, b_1, b_2)$ associated with $\{ \rho_2(Q_j) \}$ satisfies the following identity:
\[
(b_0^{(0)}, b_1^{(0)}, b_2^{(0)}) = (b_2, b_1, b_0), \text{ namely, } b_0 = b_2.
\]
Let $K$ be the horizontal translation $z \mapsto z + 1$. For simplicity of notation, we write $(g_0, g_1, g_2)$ instead of $(\rho_2(Q_2), \rho_2(Q_1), \tilde{K}\rho_2(K_0))$. We first show that there is a representation $\rho_2^* = \pi_1(O_0) = (S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = 1, (S_1S_2)^2 = 1)$ to PSL$(2, \mathbb{C})$ sending $(S_0, S_1, S_2)$ to $(g_0, g_1, g_2)$. Since $g_0 = \rho_2(Q_2)$ and $g_1 = \rho_2(Q_1)$, we have $g_0^2 = g_1^2 = 1$. Thus the existence of the representation $\rho_2^*$ is guaranteed by the following claim.

CLAIM 5.5. (1) $g_2$ is the $\pi$-rotation about the axis which is the image of the vertical geodesic Axis$(\rho_2(K_0))$ by the translation $z \mapsto z + {1 \over 2}$, where Axis$(A)$ denotes the axis of $A \in$ PSL$(2, \mathbb{C})$. In particular, $g_2^2 = 1$. 

(2) The axes of the $\pi$-rotations $g_1$ and $g_2$ intersect orthogonally and hence $g_1g_2$ is also a $\pi$-rotation. In particular, $(g_1g_2)^2 = 1$.

PROOF. (1) Since $K(z) = z + 1$ and since $\rho_2(K_0)$ is the $\pi$-rotation about the vertical geodesic $\text{Axis}(\rho_2(K_0))$, the isometry $g_2 = K\rho_2(K_0)$ is also the $\pi$-rotation about the vertical geodesic, which is the image of $\text{Axis}(\rho_2(K_0))$ by the translation $z \mapsto z + \frac{1}{2}$.

(2) Note that $\rho_2(K_0)$ is the $\pi$-rotation about the vertical geodesic above $c(\rho_2(Q_2Q_0))$, because we have the following identity by Lemma 4.7:

$$\rho_2(K_0) I(\rho_2(Q_2Q_0)) = I(\rho_2(Q_2Q_0K_0)) = I(\rho_2(K_2Q_2Q_0)) = I(\rho_2(Q_2Q_0)).$$

Thus the axis of $g_2$ is the vertical geodesic above $c(\rho_2(Q_2Q_0)) + \frac{1}{2}$ by Claim 5.5(1).

Moreover, we have the following identity:

$$\begin{align*}
c(\rho_2(Q_2Q_0)) + \frac{1}{2} &= c(\rho_2(Q_2Q_0)) + \frac{1}{2}(c(\rho_2(Q_0Q_2)) - c(\rho_2(Q_2Q_0))) & \text{by Proposition 4.11(1.1)} \\
&= \frac{1}{2}c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2)) \\
&= \frac{1}{2}(c(\rho_2(Q_2Q_0)) + b_0 - b_2 + c(\rho_2(Q_0Q_2))) & \text{by the assumption } b_0 = b_2 \\
&= \frac{1}{2}(c(\rho_2(Q_0)) + c(\rho_2(Q_2))) & \text{by Proposition 4.11(1.1)} \\
&= c(\rho_2(Q_1)). & \text{by Proposition 4.11(1.1).}
\end{align*}$$

Hence $g_2$ is the $\pi$-rotation about the vertical geodesic above $c(\rho_2(Q_1)) = c(g_1)$ and hence the axes of $g_1$ and $g_2$ intersect orthogonally. 

Recall that $\pi_1(\mathcal{O}_{N_2})$ is identified with a subgroup of $\pi_1(\mathcal{O}_n)$ and their generators satisfy the following identities:

$$Q_0 = S_0^{S^2}, \quad Q_1 = S_1, \quad Q_2 = S_0.$$

Since $g_2$ is the $\pi$-rotation about the vertical geodesic above $c(\rho_2(Q_1))$, we have $c(\rho_2(Q_0)) = c(\rho_2(Q_2)) = c(\rho_2(S_0^{S^2})).$ This together with the assumption $b_0 = b_2$ implies that $\rho_2(Q_0) = \rho_2(Q_2)^{g_2} = \rho_2(S_0^{S^2})$ by Proposition 4.11. Hence the restriction of $\rho_2$ to $\pi_1(\mathcal{O}_{N_2})$ is equal to the original representation $\rho_2$.

The only if part of (a). Suppose that $\rho_2$ extends to a type-preserving representation $\tilde{\rho}_2$ of $\pi_1(\mathcal{O}_n)$. Pick a sequence of elliptic generators $\{Q_i\}_i$ of $\pi_1(\mathcal{O}_{N_2})$. Since $\rho_2$ is faithful, we have $\text{tr}(\rho_2(Y_1))\text{tr}(\rho_2(Y_2))\text{tr}(\rho_2(Y_1Y_2^{-1})) \neq 0$, where $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Thus the complex probability $(b_0,b_1,b_2)$ associated with $\{\rho_2(Q_j)\}_j$ is well-defined. Since $\pi_1(\mathcal{O}_{N_2})$ is identified with a subgroup of $\pi_1(\mathcal{O}_n)$, the isometry $\tilde{\rho}_2(S_2)$ satisfies the following identities:

$$\tilde{\rho}_2(S_2)^2 = 1, \quad \tilde{\rho}_2(Q_0^{S^2}) = \rho_2(Q_2), \quad (\tilde{\rho}_2(Q_1S_2))^2 = 1.$$
CLAIM 5.6. The isometry \( \hat{\rho}_2(S_2) \) is the \( \pi \)-rotation about the vertical geodesic above \( \frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_1)) \).

PROOF. Since \( (\hat{\rho}_2(S_2))^2 = 1 \), \( \hat{\rho}_2(S_2) \) is either the identity or a \( \pi \)-rotation. If \( \hat{\rho}_2(S_2) = 1 \), then \( \rho_2(K_{N_{L_1}}) = \hat{\rho}_2(S_0(S_0S_2)) ) = \hat{\rho}_2(S_0(S_0S_2)) = 1 \), a contradiction. Hence \( \hat{\rho}_2(S_2) \) is a \( \pi \)-rotation. By \( \hat{\rho}_2(Q_0S_2) = \rho_2(Q_2) \) and \( (\hat{\rho}_2(Q_1S_2))^2 = 1 \), we have \( \hat{\rho}_2(K_0S_2) = \rho_2(K_2) \). Hence \( \hat{\rho}_2(S_2) \) maps \( \Fix(\rho_2(K_0)) \) to \( \Fix(\rho_2(K_2)) \), \( \Fix(\rho_2(Q_0Q_2)) \), and hence \( \hat{\rho}_2(Y_2) = 0 \), a contradiction to (Eq2). Hence we have \( \Fix(\hat{\rho}_2(S_2)) = \{ \frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) \} \). By the faithfulness of \( \rho_2 \), the isometry \( \rho_2(Q_1) \) does not fix \( \infty \). In fact, if \( \rho_2(Q_1) \) fixes \( \infty \), then \( \tr(\rho_2(Y_1))\tr(\rho_2(Y_2)) = 0 \) by Lemma 4.4(2.2). Since \( \hat{\rho}_2(S_2) \) fixes \( \infty \), the axes of \( \rho_2(Q_1) \) and \( \hat{\rho}_2(S_2) \) intersect orthogonally by \( (\hat{\rho}_2(Q_1S_2))^2 = 1 \). Hence the isometry \( \hat{\rho}_2(S_2) \) is the \( \pi \)-rotation about the vertical geodesic above \( \frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_1)) \).

Hence we have

\[ b_0 = c(\rho_2(Q_0)) - c(\rho_2(Q_2Q_0)) \quad \text{by Proposition 4.11(1.1)} \]
\[ = -c(\rho_2(Q_2)) + 2c(\rho_2(Q_1)) - c(\rho_2(Q_2Q_0)) \quad \text{by Proposition 4.11(1.1)} \]
\[ = -c(\rho_2(Q_2)) + c(\rho_2(Q_0Q_2)) \quad \text{by Claim 5.6} \]
\[ = b_2 \quad \text{by Proposition 4.11(1.1).} \]

To show the uniqueness of the extensions of \( \rho_2 \), let \( \hat{\rho}_2 \) and \( \hat{\rho}_2 \) be extensions of \( \rho_2 \) to \( \pi_1(\mathcal{O}_0) \). Then we have the following identity:

\[ (\hat{\rho}_2(S_0S_2), \hat{\rho}_2(S_1), \hat{\rho}_2(S_0)) = (\hat{\rho}_2(Q_0), \hat{\rho}_2(Q_1), \hat{\rho}_2(Q_2)) = (\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2)) = (\rho_2(S_0S_2), \rho_2(S_1), \rho_2(S_0)). \]

By Claim 5.6, we have \( \hat{\rho}_2(S_2) = \hat{\rho}_2(S_2) \). Hence we have \( \hat{\rho}_2 = \hat{\rho}_2 \).

The part of (\( \beta \)). Suppose that there exists a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(\mathcal{O}_{N_1}) \) such that the complex probability \( (b_0, b_1, b_2) \) associated with \( \{\rho_2(Q_j)\} \) satisfies the following identity:

\[ (b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0). \]

Let \( \tilde{K} \) be the horizontal translation \( z \mapsto z + 1 \). For simplicity of notation, we write \( (g_0, g_1, g_2, g_3) \) instead of \( (\tilde{K}\rho_2(K_0), g_0^{-1}\rho_2(Q_2), g_1^{-1}\rho_2(Q_0), \rho_2(K_0)) \). We first show that there is a representation \( \rho_2 \) from \( \pi_1(\mathcal{O}_3) = (T_0, T_1, T_2, T_3) \) to \( \text{PSL}(2, \mathbb{C}) \) sending \( (T_0, T_1, T_2, T_3) \) to \( (g_0, g_1, g_2, g_3) \). Since \( g_3 = \rho_2(K_0), g_0g_1 = \rho_2(Q_2) \) and \( g_1g_2 = \rho_2(Q_0) \), we have \( g_3^2 = (g_0g_1)^2 = (g_1g_2)^2 = 1 \). By Convention 4.5, \( g_0 = \tilde{K}\rho_2(K_0) \)
are \(\pi\)-rotations and hence \(g_2^2 = 1\). Thus the existence of the representation \(\rho_2^*\) is guaranteed by the following claim.

**Claim 5.7.** (1) \(g_0\) is a \(\pi\)-rotation satisfying \(\rho_2(Q_2Q_0)^{g_0} = \rho_2(Q_0Q_2)\). In particular, \(g_2 = g_0\rho_2(Q_2Q_0)\) has order 2, and hence \(g_2^2 = 1\).

(2) The axes of \(g_0\) and \(\rho_2(Q_2)\) intersect orthogonally and hence \(g_0^{-1}\rho_2(Q_2) = g_1\) is also a \(\pi\)-rotation. In particular, \(g_1^2 = 1\).

(3) \(g_2g_3\) is a \(\pi\)-rotation and hence \((g_2g_3)^2 = 1\).

**Proof.** (1) By the proof of Claim 5.5(1), the isometry \(g_0 = \rho_2(K_2)\tilde{K}\) is the \(\pi\)-rotation about the vertical geodesic above \(\frac{1}{2}(c(\rho_2(Q_0Q_2)) + c(\rho_2(Q_2Q_0)))\). This together with Proposition 4.11 implies that \(\rho_2(Q_2Q_0)^{g_0} = \rho_2(Q_0Q_2)\).

(2) By Lemma 4.13, we have

\[
 b_0^{(1)} = 1 - b_2, \quad b_1^{(1)} = \frac{b_1b_2}{1-b_2}, \quad b_2^{(1)} = \frac{b_2b_0}{1-b_2}.
\]

This together with the assumption \((b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0)\) implies that \(b_2 = b_0^{(1)} = 1/2\). In particular, \(\frac{1}{2}(c(\rho_2(Q_0Q_2)) + c(\rho_2(Q_2Q_0))) = c(\rho_2(Q_2))\) by Proposition 4.11. Hence \(g_0\) is the \(\pi\)-rotation about the vertical geodesic above \(c(\rho_2(Q_2))\) and hence \(g_0\) and \(\rho_2(Q_2)\) intersect orthogonally.

(3) Since \(g_0g_3 = \tilde{K}\), we have \(g_0g_3(z) = z + 1\). By Proposition 4.11(1.2), the isometry \(g_0g_2 = g_3g_0\rho_2(Q_2Q_0)\) is a \(\pi\)-rotation. \(\square\)

Recall that \(\pi_1(\mathcal{O}_{N_{\xi,i}})\) is identified with a subgroup of \(\pi_1(\mathcal{O}_\beta)\) and their generators satisfy the following identities:

\[
 Q_0 = T_1T_2, \quad Q_1 = T_3^{T_1}, \quad Q_2 = T_0T_1.
\]

Since

\[
 (\rho_2(T_0), \rho_2(T_1), \rho_2(T_2), \rho_2^*(T_3)) = (g_0, g_1, g_2, g_3) = (K\rho_2(K_0), g_0^{-1}\rho_2(Q_2), g_1^{-1}\rho_2(Q_0), \rho_2(K_0)),
\]

we have

\[
 (\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2)) = (\rho_2^*(T_1T_2), \rho_2^*(T_3^{T_1T_2}), \rho_2^*(T_0T_1)) = (\rho_2^*(T_1T_2), \rho_2^*(T_3^{T_1}), \rho_2^*(T_0T_1)).
\]

Thus the restriction of \(\rho_2^*\) to \(\pi_1(\mathcal{O}_{N_{\xi,i}})\) is equal to the original representation \(\rho_2\).

The only if part of (\(\beta\)). Suppose that \(\rho_2\) extends to a type-preserving representation \(\tilde{\rho}_2\) of \(\pi_1(\mathcal{O}_\beta)\). Pick a sequence of elliptic generators \(\{Q_1\}\) of \(\pi_1(\mathcal{O}_{N_{\xi,i}})\). Since \(\rho_2\) is faithful, we have \(\text{tr}(\rho_2(Y_1))\text{tr}(\rho_2(Y_2))\text{tr}(\rho_2(Y_1Y_2^{-1})) \neq 0\) and \(\text{tr}(\rho_2(Y_1))\text{tr}(\rho_2(Y_2))\text{tr}(\rho_2(Y_1Y_2)) \neq 0\), where \(Y_1 = Q_0Q_1\) and \(Y_2 = Q_0Q_2\). Thus the complex probability \((b_0, b_1, b_2)\) associated with \(\{\rho_2(Q_1)\}\) and the complex probability \((b_0^{(1)}, b_1^{(1)}, b_2^{(1)})\) associated with \(\{\rho_2(\sigma(Q_1))\}\) are well-defined. Since \(\pi_1(\mathcal{O}_{N_{\xi,i}})\) is identified with a subgroup of \(\pi_1(\mathcal{O}_\beta)\), the isometry \(\tilde{\rho}_2(T_0)\) satisfies the following identities:

\[
 (\tilde{\rho}_2(T_0))^2 = 1, \quad (\tilde{\rho}_2(T_0Q_2))^2 = 1, \quad (\tilde{\rho}_2(T_0Q_2Q_0))^2 = 1, \quad (\tilde{\rho}_2(T_0Q_2Q_0K_0))^2 = 1.
\]
CLAIM 5.8. The isometry $\tilde{\rho}_2(T_0)$ is the $\pi$-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_2))$. Moreover, we have $\tilde{\rho}_2(T_0)(c(\rho_2(Q_0))) = c(\rho_2(Q_0^{Q_2}))$.

PROOF. Since $(\tilde{\rho}_2(T_0))^2 = 1$, $\tilde{\rho}_2(T_0)$ is either the identity or a $\pi$-rotation. If $\tilde{\rho}_2(T_0) = 1$, then $\rho_2(K_0) = \tilde{\rho}_2(T_0)T_0^2 = \tilde{\rho}_2(T_0^2) = 1$, a contradiction. Hence $\tilde{\rho}_2(T_0)$ is a $\pi$-rotation. By $(\tilde{\rho}_2(T_0Q_2Q_0))^2 = 1$ and $(\tilde{\rho}_2(T_0Q_2Q_0K_0))^2 = 1$, we have $\tilde{\rho}_2(K_0^{Q_2}) = \rho_2(K_0)$. Hence $\tilde{\rho}_2(T_0)$ maps $\text{Fix}(\rho_2(K_2)) = \{c(\rho_2(Q_0Q_2)), \infty\}$ to $\text{Fix}(\rho_2(K_0)) = \{c(\rho_2(Q_2Q_0)), \infty\}$. Since $\tilde{\rho}_2(T_0)$ has order 2, the isometry $\tilde{\rho}_2(T_0)$ must fix $\infty$. (Otherwise $c(\rho_2(Y_1^{-1})) = c(\rho_2(Q_0Q_2)) = c(\rho_2(Q_0Q_2)) = c(\rho_2(Q_2))$ and hence $\tilde{\rho}_2(T_0)$ is a $\pi$-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_2))$.

By $\lambda_2(\tilde{\rho}_2(T_0Q_2)^2) = 1$ and $(\tilde{\rho}_2(T_0Q_2Q_0)^2 = 1$, we have $\tilde{\rho}_2(Q_0^{Q_2}) = c(\rho_2(Q_0^{Q_2}))$. By the above argument, the isometry $\tilde{\rho}_2(T_0)$ is a Euclidean isometry preserving $\infty$. Hence, by Lemma 4.7, we have $\tilde{\rho}_2(T_0)(c(\rho_2(Q_0))) = c(\rho_2(Q_0^{Q_2}))$. □

Then we have

\[ b_0 = c(\rho_2(Q_0)) - c(\rho_2(Q_0Q_2)) \] by Proposition 4.11(1.1)

\[ = c(\rho_2(Q_0Q_2)) - c(\rho_2(Q_0^{Q_2})) \] by Claim 5.8

\[ = b_2^{(1)} \] by Proposition 4.11(1.1) and Notation 4.12,

\[ b_1 = c(\rho_2(Q_0)) - c(\rho_2(Q_0Q_2)) \] by Proposition 4.11(1.1)

\[ = c(\rho_2(Q_0^{Q_2})) - c(\rho_2(Q_0)) \] by Claim 5.8

\[ = b_1^{(1)} \] by Proposition 4.11(1.1) and Notation 4.12,

\[ b_2 = c(\rho_2(Q_0Q_2)) - c(\rho_2(Q_0)) \] by Proposition 4.11(1.1)

\[ = c(\rho_2(Q_0^{Q_2})) - c(\rho_2(Q_0Q_2)) \] by Claim 5.8

\[ = b_0^{(1)} \] by Proposition 4.11(1.1) and Notation 4.12.

To show the uniqueness of the extension of $\rho_2$, let $\tilde{\rho}_2$ and $\tilde{\rho}_2'$ be extensions of $\rho_2$ to $\tau_1(O_B)$. Then we have the following identity:

\[ (\tilde{\rho}_2(T_1T_2), \tilde{\rho}_2(T_3^{T_2}), \tilde{\rho}_2(T_0T_1)) = (\tilde{\rho}_2(Q_0), \tilde{\rho}_2(Q_1), \tilde{\rho}_2(Q_2)) \]

\[ = (\tilde{\rho}_2'(Q_0), \tilde{\rho}_2'(Q_1), \tilde{\rho}_2'(Q_2)) \]

\[ = (\tilde{\rho}_2(T_1T_2), \tilde{\rho}_2'(T_3^{T_2}), \tilde{\rho}_2'(T_0T_1)). \]

By Claim 5.8, we have $\tilde{\rho}_2(T_0) = \tilde{\rho}_2'(T_0)$. Hence we have $\tilde{\rho}_2 = \tilde{\rho}_2'$. □
In the remainder of this section, we study what happens if we drop the faithfulness condition in Theorem 5.1.

**Proposition 5.9.** Under Convention 4.5, the following hold for every \( \rho_2 \in \Omega(N_{2,1}) 

1. For the conditions \((i)\) and \((iii)\) in Theorem 5.1(1) and the condition \((i)\)' defined below, the implication \((iii) \Rightarrow (ii) \Rightarrow (i)\)' holds.

\((i)\)' There exists a (possibly non-faithful) representation \( \rho_1 \in \Omega(\Sigma_{1,1}) \) which is commensurable with \( \rho_2 \).

2. The assertion \((\alpha)\) in Theorem 5.1(3) holds.

3. The if part of \((\beta)\) in Theorem 5.1(3) holds.

**Proof.** (1) In the proof of the implication \((iii) \Rightarrow (ii)\) in Theorem 5.1(1), we do not use the faithfulness of \( \rho_2 \). In the proof of the implication \((ii) \Rightarrow (i)\) in Theorem 5.1(1), we do not use the faithfulness of \( \rho_2 \) to show the existence of the representation \( \rho_1 \in \Omega(\Sigma_{1,1}) \) which is commensurable with \( \rho_2 \). Hence we have the desired results.

(2) The proof of the if part of \((\alpha)\) in Theorem 5.1(3) does not use the faithfulness. In the proof of the only if part of \((\alpha)\) in Theorem 5.1(3), we use the faithfulness of \( \rho_2 \) only to guarantee the existence of a sequence of elliptic generators \( \{Q_j\} \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) such that

- \( \text{tr}(\rho_2(Y_1))\text{tr}(\rho_2(Y_2))\text{tr}(\rho_2(Y_1Y_2^{-1})) \neq 0 \) with \( Y_1 = Q_0Q_1 \) and \( Y_2 = Q_0Q_2 \) and
- \( \rho_2(Q_1) \) does not fix \( \infty \).

On the other hand, Lemma 4.4(2.2) implies that the above two conditions are equivalent. Hence, we have only to show that \( \rho_2(Q_1) \) does not fix \( \infty \) without the faithfulness of \( \rho_2 \).

Let \( \tilde{\rho}_2 \) be the extension of \( \rho_2 \) to \( \pi_1(\mathcal{O}_a) \). Since \( \pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 | Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle \) is identified with a subgroup of \( \pi_1(\mathcal{O}_a) = \langle S_0, S_1, S_2 | S_0^2 = S_1^2 = S_2^2 = 1, (S_1S_2)^2 = 1 \rangle \), we have \( (\tilde{\rho}_2(Q_1S_2))^2 = (\tilde{\rho}_2(S_1S_2))^2 = 1 \). By the proof of Claim 5.6, we have \( \text{Fix}(\tilde{\rho}_2(S_2)) = \{ \frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) \} \). Suppose to the contrary that \( \rho_2(Q_1) \) fixes \( \infty \). Then \( \rho_2(Q_1) \) is equal to \( \rho_2(K_0) \) or \( \rho_2(K_2) \) by Lemma 4.4(2.2). Hence \( \text{Fix}(\rho_2(Q_1)) = \{ c(\rho_2(Q_2Q_0)) \} \) or \( \{ c(\rho_2(Q_0Q_2)) \} \). Since \( (\tilde{\rho}_2(Q_1S_2))^2 = 1 \), we have \( \rho_2(Q_1) = \tilde{\rho}_2(S_2) \). Hence \( c(\rho_2(Q_2Q_0)) = c(\rho_2(Q_0Q_2)) \), and therefore \( \text{tr}(\rho_2(Q_0Q_2)) = \text{tr}(\rho_2(Y_2)) = 0 \). This contradicts \((\text{Eq}2)\). Hence \( \rho_2(Q_1) \) does not fix \( \infty \).

(3) The proof of the if part of \((\beta)\) in Theorem 5.1(3) does not use the faithfulness. \( \square \)

**Definition 5.10.** An element \( \rho_2 \in \Omega(N_{2,1}) \) is strongly non-faithful if there exists an elliptic generator \( Q_j \) of \( \pi_1(\mathcal{O}_{N_{2,1}}) \) with \( j \not\equiv 1 \pmod{3} \) such that \( \text{tr}(\rho_2(K_0Q_j)) = 0 \).

**Proposition 5.11.** Under Convention 4.5, let \( \rho_2 \) be an element of \( \Omega(N_{2,1}) \). Then the following conditions are equivalent.
(1) \( \rho_2 \) is strongly non-faithful.

(2) The conditions (i)' in Proposition 5.9(1) and (ii) in Theorem 5.1(1) hold, but the condition (iii) in Theorem 5.1(1) does not hold.

(3) \( \rho_2 \) extends to a type-preserving representation \( \tilde{\rho}_2 \) of \( \pi_1(O_\beta) \) such that \( \tilde{\rho}_2(T_1) = 1 \), where \( T_1 \) is the generator of \( \pi_1(O_\beta) \) as in Figure 3.

PROOF. We prove this proposition by proving the implications (1) \(\Rightarrow\) (2), (2) \(\Rightarrow\) (1), (1) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (1).

(1) \(\Rightarrow\) (2). Suppose that \( \rho_2 \) is strongly non-faithful, i.e., there is an elliptic generator \( Q_j \) of \( \pi_1(O_{N_2,1}) \) with \( j \neq 1 \) (mod 3) such that \( \text{tr}(\rho_2(K_0Q_j)) = 0 \). We may assume without losing generality that \( j = 0 \). Then the pseudo-Markoff triple associated with \( \{\rho_2(Q_j)\} \) is equal to \((0, 2i, r)\) for some \( r \in \mathbb{C}^* \). In particular, the complex probability associated with \( \{\rho_2(Q_j)\} \) is not defined. Hence \( \rho_2 \) does not satisfy the condition (iii) in Theorem 5.1(1). However, the complex probability \((b_0^{(2)}, b_1^{(2)}, b_2^{(2)})\) associated with \( \{\rho_2(\sigma(Q_j))\} \) is equal to \((1, -1/r^2, 1/r^2)\), and the complex probability \((b_0^{(-1)}, b_1^{(-1)}, b_2^{(-1)})\) associated with \( \{\rho_2(\sigma^{-1}(Q_j))\} \) is equal to \((1/r^2, -1/r^2, 1)\). Thus we have \((b_0^{(2)}, b_1^{(2)}, b_2^{(2)}) = (b_0^{(-1)}, b_1^{(-1)}, b_2^{(-1)})\).

By replacing \( \{Q_j\} \) with \( \{\sigma(Q_j)\} \), the representation \( \rho_2 \) (together with \( \{Q_j\} \)) satisfies the condition (ii) in Theorem 5.1(1), and hence \( \rho_2 \) satisfies the condition (i)' in Proposition 5.9(1) by Proposition 5.9(1).

(2) \(\Rightarrow\) (1) Suppose that the conditions (i)' in Proposition 5.9(1) and (ii) in Theorem 5.1(1) hold, but the condition (iii) in Theorem 5.1(1) does not hold. Then, by the proof of (ii) \(\Rightarrow\) (iii) in Theorem 5.1(1), for some sequence of elliptic generators \( \{Q_j\} \) and some integer \( k \), the complex probability associated with \( \{\rho_2(\sigma^k(Q_j))\} \) is not defined. This implies \( y_1^{(k)}y_2^{(k)}y_1^{(k)}y_2^{(k)} = 0 \) for the pseudo-Markoff triple \((y_1^{(k)}, y_2^{(k)}, y_2^{(k)})\) associated with \( \{\rho_2(\sigma^k(Q_j))\} \). Hence \( \text{tr}(\rho_2(K_0\sigma^k(Q_0))) \) or \( \text{tr}(\rho_2(K_0\sigma^{-k}(Q_0))) \) is equal to 0, and therefore \( \rho_2 \) is strongly non-faithful.

(1) \(\Rightarrow\) (3) Suppose that \( \rho_2 \) is strongly non-faithful, namely, there is an elliptic generator \( Q_j \) of \( \pi_1(O_{N_2,1}) \) with \( j \neq 1 \) (mod 3) such that \( \text{tr}(\rho_2(K_0Q_j)) = 0 \). We may assume \( j = 0 \) without losing generality. Then there is a representation \( \rho_2' \) from \( \pi_1(O_\beta) = \langle T_0, T_1, T_2, T_3 \mid T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, (T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1 \rangle \) to \( \text{PSL}(2, \mathbb{C}) \) sending \((T_0, T_1, T_2, T_3) \) to \((g_0, g_1, g_2, g_3) := (K_0\rho_2(K_0), g_0^{-1}\rho_2(Q_0), g_1^{-1}\rho_2(Q_0), \rho_2(K_0)) \), where \( K_0 \) is the horizontal translation \( z \mapsto z + 1 \) and \( \rho_2' \) is an extension of \( \rho_2 \) to \( \pi_1(O_\beta) \). This can be seen as follows. By the proof of the if part of (\( \beta \)) in Theorem 5.1(3), we have \( g_3^2 = (g_0g_1)^2 = (g_1g_2)^2 = g_0^2 = 1 \). Hence we have only to show that \( g_1^2 = 1 \), \( g_2^2 = 1 \) and \( (g_0g_3)^2 = 1 \). Since \( g_1 = \text{tr}(\rho_2(T_1)) = \text{tr}(\rho_2(K_0Q_0)) = 0 \), we have \( c(\rho_2(Q_0)) = 0 \in \text{Fix}(\rho_2(K_0)) \) and \( \rho_2(Q_2) = \tilde{K}_0\rho_2(K_0) \) by Lemma 4.4(2). By \( \rho_2(Q_2) = \tilde{K}_0\rho_2(K_0) \), we have \( g_1 = g_0^{-1}\rho_2(Q_2) = 1 \), and hence \( g_1^2 = 1 \). Since \( g_2 = g_1^{-1}\rho_2(Q_0) = \rho_2(Q_0) \), we have \( g_2^2 = 1 \). By \( c(\rho_2(Q_0)) \in \text{Fix}(\rho_2(K_0)) \), the axes of \( \rho_2(Q_0) \) and \( \rho_2(K_0) \) intersect orthogonally, and hence \( g_2g_3 = g_1^{-1}\rho_2(Q_0)\rho_2(K_0) = 1 \) as required.
\(\rho_2(Q_0)\rho_2(K_0)\) is a \(\pi\)-rotation. In particular, \((g_2g_3)^2 = 1\). Moreover, we have \(\rho_2^*(T_1) = g_1 = 1\). Thus we obtain the desired representation \(\rho_2^*\).

(3) Suppose that the representation \(\rho_2\) extends to a type-preserving \(\text{PSL}(2,\mathbb{C})\)-representation \(\tilde{\rho}_2\) of \(\pi_1(O_3)\) such that \(\tilde{\rho}_2(T_1) = 1\). Note that \(\pi_1(O_{N_2,1})\) is identified with a subgroup of \(\pi_1(O_3)\) and \(T_1 = T_0Q_2\). By the proof of Claim 5.8, the isometry \(\tilde{\rho}_2(T_0)\) fixes \(\infty\). Since \(\tilde{\rho}_2(T_1) = \tilde{\rho}_2(T_0Q_2) = 1\), the isometry \(\rho_2(Q_2)\) fixes \(\infty\). By Lemma 4.4(2.2), we have \(\text{tr}(\rho_2(Y_1))\) equals to 0. Hence \(\rho_2\) is strongly non-faithful.

6. Application to Ford domains

In this section, we give an application to the study of the Ford domains.

**Definition 6.1.** Let \(\Gamma\) be a non-elementary Kleinian group such that the stabilizer \(\Gamma_\infty\) of \(\infty\) contains parabolic transformations. Then the Ford domain \(P(\Gamma)\) of \(\Gamma\) is the polyhedron in \(H^3\) defined below:

\[
P(\Gamma) := \bigcap \{E(\gamma) \mid \gamma \in \Gamma - \Gamma_\infty\}.
\]

**Lemma 6.2.** Under Convention 4.5, the following hold:

1. Let \(\rho_1\) be an element of \(\Omega(\Sigma_{1,1})\). Suppose that \(\rho_1\) is discrete. Then \(P(\rho_1(\pi_1(O_{\Sigma_{1,1}}))) = P(\rho_1(\pi_1(\Sigma_{1,1})))\).

2. Let \(\rho_2\) be an element of \(\Omega(N_{2,1})\). Suppose that \(\rho_2\) is discrete. Then \(P(\rho_2(\pi_1(O_{N_{2,1}}))) = P(\rho_2(\pi_1(N_{2,1})))\).

**Proof.** The assertion (1) is well-known (see [1, Proposition 2.2.8]). The assertion (2) follows from the fact that \(\pi_1(O_{N_{2,1}}) = \langle \pi_1(N_{2,1}), K_2 \rangle\) and that \(\rho_2(K_2)\) is a Euclidean transformation preserving \(\infty\) by Convention 4.5.

**Proposition 6.3.** Under Convention 4.5, let \(\rho_1\) and \(\rho_2\) be elements of \(\Omega(\Sigma_{1,1})\) and \(\Omega(N_{2,1})\), respectively. Suppose that they are discrete and commensurable. Then \(P(\rho_1(\pi_1(\Sigma_{1,1}))) = P(\rho_1(\pi_1(O_{\Sigma_{1,1}})))\) equals to \(P(\rho_2(\pi_1(O_{N_{2,1}})))\).

**Proof.** We prove only the second equality, because the remaining equalities can be proved by Lemma 6.2.

Since \(\rho_1\) and \(\rho_2\) are commensurable, there exist a double covering \(p_1 : \Sigma_{1,2} \to \Sigma_{1,1}\) and a double covering \(p_2 : \Sigma_{2,2} \to \Sigma_{2,1}\) such that \(\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*\). Then we can easily observe that

\[
\pi_1(O_{\Sigma_{1,1}}) = \langle (p_1)_*(\pi_1(O_{\Sigma_{1,2}})), K_1 \rangle,
\]

\[
\pi_1(O_{N_{2,1}}) = \langle (p_2)_*(\pi_1(O_{\Sigma_{2,2}})), K_2 \rangle.
\]

Since \(\rho_1(K_1)\) and \(\rho_2(K_2)\) are Euclidean transformations preserving \(\infty\), we see

\[
P(\rho_1(\pi_1(O_{\Sigma_{1,1}}))) = P(\rho_1((p_1)_*(\pi_1(O_{\Sigma_{1,2}})))) = P(\rho_2((p_2)_*(\pi_1(O_{\Sigma_{2,2}})))) = P(\rho_2(\pi_1(O_{N_{2,1}})))).
\]

\(\Box\)
Example 6.4. Jorgensen and Marden [7] constructed complete hyperbolic structures of the punctured torus bundles over the circle with monodromy matrices \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \) by explicitly constructing the fiber groups \( G_1 \) and \( G_2 \) and their Ford domains. The groups \( G_1 \) and \( G_2 \), respectively, are the images of (faithful) representations \( \rho_1 \) and \( \rho'_1 \) in \( \Omega(\Sigma_{1,1}) \) constructed by Proposition 4.8(1) from the following triples:

\[
(a_0, a_1, a_2) = \left( \frac{1}{\sqrt{3}} e^{\frac{\pi}{3} i}, \frac{1}{\sqrt{3}} e^{-\frac{\pi}{3} i}, \frac{1}{\sqrt{3}} e^{\frac{\pi}{3} i} \right),
\]

\[
(a'_0, a'_1, a'_2) = \left( -\frac{1}{2}, -\frac{1}{2} e^{\frac{\pi}{3} i}, \frac{1}{2} \right).
\]

Let \( \rho_2 \) and \( \rho'_2 \), respectively, be elements of \( \Omega(N_{2,1}) \) constructed by Proposition 4.11(1) from the above triples. Then \( \rho_2 \) and \( \rho'_2 \), respectively, satisfy the conditions (iii)-(a) and (iii)-(b) in Theorem 5.1(1). Hence, by Proposition 5.9(1), for each of \( \rho_2 \) and \( \rho'_2 \), there is an element of \( \Omega(\Sigma_{1,1}) \) which is commensurable with it. In fact, we can easily check that \( \rho_1 \) (resp. \( \rho'_1 \)) is commensurable with \( \rho_2 \) (resp. \( \rho'_2 \)). Hence \( \rho_2 \) and \( \rho'_2 \) are faithful by Remark 4.16, and \( P(\rho_1(\pi_1(\mathcal{O}_{N_{2,1}}))) = P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}}))) \) and \( P(\rho'_1(\pi_1(\mathcal{O}_{N_{2,1}}))) = P(\rho'_2(\pi_1(\mathcal{O}_{N_{2,1}}))) \) by Proposition 6.3. The Ford domain \( P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}}))) \) (resp. \( P(\rho'_2(\pi_1(\mathcal{O}_{N_{2,1}}))) \)) is illustrated in the left (resp. right) of Figure 11 (compare with [7, FIG. 1 and 2]).

Figure 11. Left: The Ford domain of \( \rho_2(\pi_1(\mathcal{O}_{N_{2,1}})) \). Right: The Ford domain of \( \rho'_2(\pi_1(\mathcal{O}_{N_{2,1}})) \).

Acknowledgements

I wish to express my deepest thanks to my supervisor, Professor Makoto Sakuma for his valuable comments and suggestions. I would also like to thank Professor Hirotaka Akiyoshi for inspiring conversation and for pointing out mistakes in the first draft. I also wish to express my gratitude to Professor Yasushi Yamashita and Professor Masaaki Wada for teaching me Jorgensen’s theory and
computer programming. I would like to thank Mr. Naoki Sakata for his encouragement and helpful comments. Finally, I would like to thank the referee for his very careful reading.

References


Mikio Furokawa

Department of Mathematics

Graduate School of Science

Hiroshima University

Higashi-Hiroshima 739-8526 JAPAN

E-mail: mfurokawa@hiroshima-u.ac.jp