広島大学学位請求論文

The values of the generalized matrix functions of 3×3 matrices

（3×3 行列の一般化行列関数の値）

２０１５年

広島大学大学院理学研究科
数学専攻

田端　亮

目　次

主論文

The values of the generalized matrix functions of 3×3 matrices

Ryo TABATA

Abstract. When A is a 3×3 positive semi-definite Hermitian matrix, Schur's inequality and the permanental dominance conjecture are known to hold. In [5], we determined the possible positions of the normalized generalized matrix functions relative to the determinant and the permanent except in the case that the order of the subgroup is 2. The purpose of this paper is to determine the possible positions in the last open case.

1 Introduction

Let $M_n(\mathbb{C})$ be the set of $n \times n$ complex matrices. *The generalized matrix function* on $M_n(\mathbb{C})$ associated to G and χ is defined to be

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where G is a subgroup of the symmetric group $\mathfrak{S}_n$, and χ a character of G. When $G = \mathfrak{S}_n$ and χ is an irreducible character, d_χ^G is called an *immanant*. The determinant and permanent, which are well-known functions on matrices, are examples of immanants: These are the special cases where χ are the alternating character and the trivial character of $\mathfrak{S}_n$.

If the domain is restricted to positive semi-definite Hermitian matrices, then each d_χ^G is a real-valued function. We also define the normalized generalized matrix function as $\overline{d}_\chi^G = d_\chi^G / \chi(\mathrm{id})$, where id is the identity element of G, hence $\chi(\mathrm{id})$ is the dimension of the corresponding representation. In 1918, Schur[4] proved an interesting inequality on generalized matrix functions.

Theorem 1 (Schur [4]). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\overline{d}_\chi^G(A) \geq \det A.$$

Namely, the determinant is the smallest normalized generalized matrix function. On the other hand, the permanent is conjectured to be the largest normalized generalized matrix function:

Conjecture (Lieb [2]). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\operatorname{per} A \geq \overline{d}_{\chi}^{G}(A).$$

The conjecture holds for immanants with $n \leq 13$ ([3]), and for all generalized matrix functions with $n = 3$ ([1]). Hence one can write

$$\overline{d}_{\chi}^{G}(A) = t \operatorname{per} A + (1-t) \det A$$

for some $t \in [0, 1]$. In [5], the possible values of t are determined when $G = \mathfrak{S}_3, \{\mathrm{id}\}$ or $\mathfrak{A}_3$. Let $R(G, \chi)$ denote the set of all possible values $t \in [0, 1]$ such that

$$\overline{d}_{\chi}^{G}(A) = t \operatorname{per} A + (1-t) \det A$$

for some 3×3 positive semi-definite Hermitian matrices A with $\operatorname{per} A \neq \det A$.

Theorem 2 ([5]). *Let χ_λ be the character of $\mathfrak{S}_3$ corresponding to the partition λ of* 3, triv *the trivial character, and ω a non-trivial irreducible character of $\mathfrak{A}_3$ with $\overline{\omega}$ its conjugate. Then*

1. $R(\mathfrak{S}_3, \chi_{(3)}) = \{1\}$.
2. $R(\mathfrak{S}_3, \chi_{(1,1,1)}) = \{0\}$.
3. $R(\mathfrak{S}_3, \chi_{(2,1)}) = \left[0, \frac{3}{4}\right]$.
4. $R(\{(\mathrm{id})\}, \mathrm{triv}) = \left[\frac{1}{6}, \frac{2}{3}\right]$.
5. $R(\mathfrak{A}_3, \mathrm{triv}) = \left\{\frac{1}{2}\right\}$.
6. $R(\mathfrak{A}_3, \omega) = R(\mathfrak{A}_3, \overline{\omega}) = \left[0, \frac{1}{\sqrt[3]{2}}\right]$.

The goal of this paper is to complete this table:

Theorem 3 (Main Theorem). *Let $G \subset \mathfrak{S}_3$ be a subgroup of order* 2, *and $\chi_+ : G \to \mathbb{C}^*$ be the trivial character and $\chi_- : G \to \mathbb{C}^*$ the non-trivial irreducible character of G. Then*

$$R(G, \chi_+) = \left[\frac{1}{3}, 1\right] \quad \textit{and} \quad R(G, \chi_-) = \left[0, \frac{1}{\sqrt{3}}\right].$$

2 Proof of Main Theorem

In this section, we work with the 3×3 positive semi-definite Hermitian matrix

$$A = \begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{pmatrix}.$$

The variables a, b, c, d, e, and f always refer to the entries of A.

If $\det A = \operatorname{per} A$, then we have $\overline{d}_\chi^G(A) = \det A = \operatorname{per} A$ for any subgroup G and its character χ, and nothing interesting happens. Throughout this paper we assume $\det A < \operatorname{per} A$, namely $(\operatorname{per} A - \det A)/2 = a|e|^2 + d|c|^2 + f|b|^2 > 0$. We denote the set of 3×3 positive semi-definite Hermitian matrices with $a|e|^2 + d|c|^2 + f|b|^2 > 0$ by $\mathcal{H}_3^+(\mathbb{C})$. For such A, following [5], we define the function T as follows.

Definition 1. *For $A \in \mathcal{H}_3^+(\mathbb{C})$, define*

$$T(A) = \frac{b\bar{c}e}{a|e|^2 + d|c|^2 + f|b|^2}.$$

The value of $T(A)$ determines the value of $t \in [0, 1]$ such that $\overline{d}_\chi^G(A) = t \operatorname{per} A + (1 - t) \det A$ for all $G \subset \mathfrak{S}_3$ and χ, except in the case of $|G| = 2$ (See [5]).

Proposition 1 ([5], Lemma 2)**.** *Writing $T(A) = x + yi$ $(x, y \in \mathbb{R})$, the possible values of $T(A)$ are given by*

$$\begin{cases} 54x(x^2 + y^2) - 27(x^2 + y^2) + 1 \geq 0, \\ -\dfrac{1}{6} \leq x \leq \dfrac{1}{3}. \end{cases}$$

In [5], Theorem 2 was deduced from Proposition 1.

Definition 2. *Define real-valued functions X, u_1, u_2 and u_3 on $\mathcal{H}_3^+(\mathbb{C})$ by*

$$X = X(A) = a|e|^2 + d|c|^2 + f|b|^2,$$

$$u_1 = u_1(A) = \frac{a|e|^2}{X}, u_2 = u_2(A) = \frac{d|c|^2}{X}, \quad \textit{and} \quad u_3 = u_3(A) = \frac{f|b|^2}{X}.$$

Hence $u_i \geq 0$ and $u_1 + u_2 + u_3 = 1$. Also, define $K(A) = u_1 u_2 u_3$.

Proposition 2. *The possible values of $K(A)$ are $0 \leq K(A) \leq 1/27$. More precisely, for $\lambda \in [0, 1/27]$ and $x + yi \in \mathbb{C}$ $(x, y \in \mathbb{R})$, there exists a positive semi-definite Hermitian matrix A such that*

$$T(A) = x + yi \qquad \textit{and} \qquad K(A) = u_1 u_2 u_3 = \lambda$$
$$u_1, u_2, u_3 \geq 0, \qquad u_1 + u_2 + u_3 = 1$$

if and only if

$$(*)\begin{cases}\lambda \geq -2x(x^2+y^2)+(x^2+y^2),\\ x \leq \dfrac{1}{3}.\end{cases}$$

The region of $x, y \in \mathbb{R}$ satisfying $(*)$ for a few values of λ are shown in Figure 1.

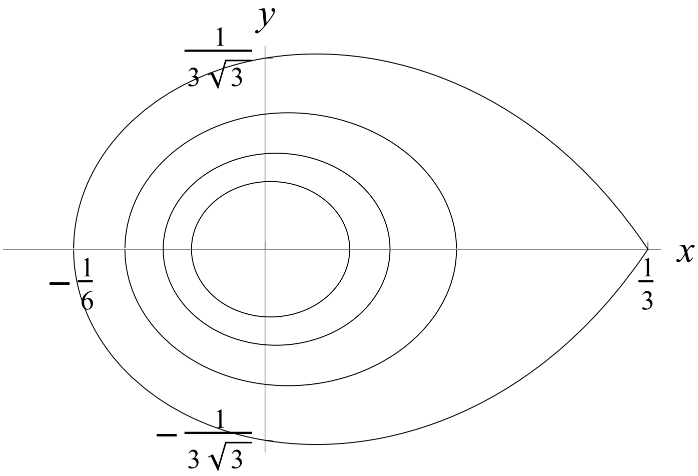

Fig. 1. (Contour plot of $2x(x^2+y^2)-(x^2+y^2)+\lambda = 0$ for $\lambda = 1/216, 1/108, 1/54, 1/27$)

Proof. $K(A) = \left(\sqrt[3]{u_1u_2u_3}\right)^3 \leq ((u_1+u_2+u_3)/3)^3 = 1/27$, hence $0 \leq K(A) \leq 1/27$. Also since $0 \leq \det A = adf + 2\operatorname{Re}(b\bar{c}e) - (a|e|^2 + d|c|^2 + f|b|^2)$,

$$\begin{aligned} K(A) &= \frac{adf\,|bce|^2}{X^3}\\ &\geq \frac{\left(-2\operatorname{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2\right)|bce|^2}{X^3}\\ &= \frac{-2\operatorname{Re}(b\bar{c}e)\,|bce|^2}{X^3} + \frac{|bce|^2}{X^2}. \end{aligned}$$

When $T(A) = x+yi = (b\bar{c}e)/X$ and $K(A) = \lambda$, since $|b\bar{c}e|^2/X^2 = |T(A)|^2 = x^2+y^2$ and $\operatorname{Re}(b\bar{c}e)/X = x$, we have

$$\lambda \geq -2x(x^2+y^2)+(x^2+y^2).$$

By Proposition 1, $x \leq 1/3$ follows.

Conversely, for $x+yi$ and λ as above, if $\lambda = 0$, which implies $x = y = 0$, we can find $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$. If $\lambda > 0$, let A be the matrix

$$\begin{pmatrix} 1 & \frac{1}{\sqrt[3]{\lambda}}(x+yi) & \frac{1}{\sqrt[3]{\lambda}}(x+yi) \\ \frac{1}{\sqrt[3]{\lambda}}(x-yi) & 1 & \frac{1}{\sqrt[3]{\lambda}}(x+yi) \\ \frac{1}{\sqrt[3]{\lambda}}(x-yi) & \frac{1}{\sqrt[3]{\lambda}}(x-yi) & 1 \end{pmatrix}.$$

Then $(x^2+y^2) \leq \lambda/(1-2x)$, therefore $1-(x^2+y^2)/\sqrt[3]{\lambda^2} \geq 1-\sqrt[3]{\lambda}/(1-2x) \geq 1 - \left(1/27^{\frac{1}{3}}\right)/(1-2/3) = 0$, which means the 2×2 principal minor of A is non-negative. Combining with the fact that $\det A = 1 + 2x(x^2+y^2)/\lambda - (x^2+y^2)/\sqrt[3]{\lambda^2} = \left(\lambda + 2x(x^2+y^2) - 3\sqrt[3]{\lambda}(x^2+y^2)\right) \geq 0$, we can conclude that A is a positive semi-definite Hermitian matrix satisfying $T(A) = x + yi$ and $K(A) = \lambda$. $\square$

Now let us consider the case of $G = \{(1), (12)\} \subset \mathfrak{S}_3$. Let χ_+ be the trivial character and χ_- the other irreducible character of G. The character table of G is the following:

G	(1)	(12)
χ_+	1	1
χ_-	1	-1

Writing

$$\overline{d}^G_{\chi_\pm}(A) = F_\pm(A) \operatorname{per} A + (1 - F_\pm(A)) \det A,$$

we have

$$\begin{aligned} F_+(A) &= \frac{1}{2}u_3 - \operatorname{Re} T(A) + \frac{1}{2}, \\ F_-(A) &= -\frac{1}{2}u_3 - \operatorname{Re} T(A) + \frac{1}{2}. \end{aligned}$$

Note that the values of $F_\pm(A)$ depend on the values of u_3 and $x = \operatorname{Re} T(A)$. If we fix u_3, the possible values of $K(A) = u_1u_2u_3$ are $0 \leq K(A) \leq u_3(1-u_3)^2/4$. Proposition 2 says that the possible values of x are

$$\begin{cases} 2x^3 - x^2 \geq -K(A) + y^2(1-2x), \\ x \leq \frac{1}{3}. \end{cases}$$

Thus, $K(A) = u_3(1-u_3)^2/4$ and $y = 0$ give the largest range for the values of x. Hence it is enough to give the possible values for $F_\pm(A)$ under the assumption

$$\begin{cases} 2x^3 - x^2 + \dfrac{u_3(1-u_3)^2}{4} = (2x+u_3-1)\left(x^2 - \dfrac{u_3}{2}x + \dfrac{1}{4}(u_3^2 - u_3)\right) \geq 0, \\ x \leq \dfrac{1}{3}, \end{cases}$$

whose region is in Figure 2.

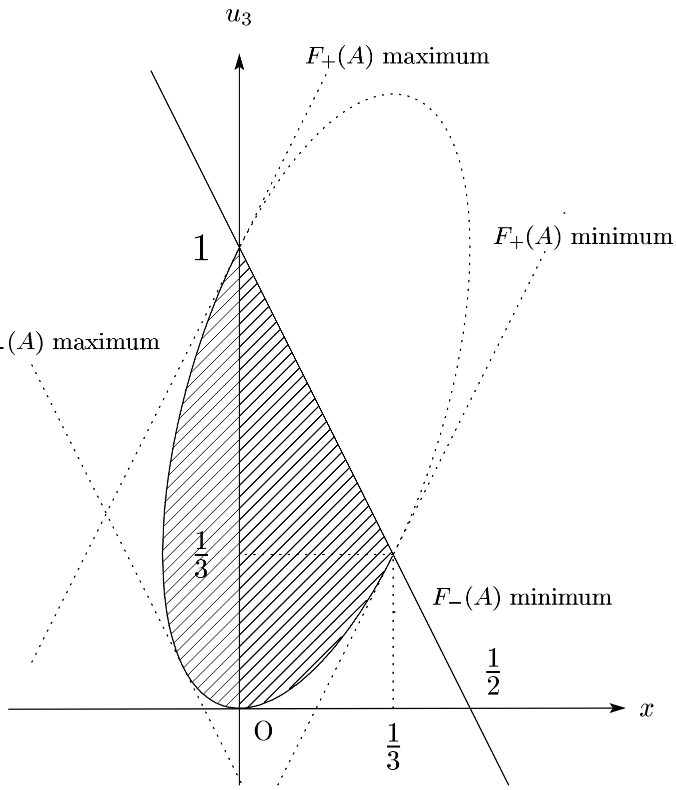

Fig. 2

We state a theorem that implies Theorem 3 below.

Theorem 4. *If A is a 3×3 positive semi-definite Hermitian matrix, and χ_+ and χ_- are the trivial and non-trivial irreducible characters of a subgroup G of $\mathfrak{S}_3$ with order 2, then*

$$\frac{1}{3}\operatorname{per} A + \frac{2}{3}\det A \leq \overline{d}^G_{\chi_+}(A) \leq \operatorname{per} A.$$

$$\det A \leq \overline{d}^G_{\chi_-}(A) \leq \frac{1}{\sqrt{3}}\operatorname{per} A + \left(1 - \frac{1}{\sqrt{3}}\right)\det A.$$

Proof. All we need to calculate is the possible values of

$$F_+(A) = \frac{1}{2}u_3 - x + \frac{1}{2} \quad \text{and} \quad F_-(A) = -\frac{1}{2}u_3 - x + \frac{1}{2}.$$

in Figure 2. One easily sees that

$$F_+(A) \text{ is maximum at } (x,u_3)=(0,1) \text{ with } F_+(A)=1,$$
$$\text{and minimum at } (x,u_3)=\left(\frac{1}{3},\frac{1}{3}\right) \text{ with } F_+(A)=\frac{1}{3}.$$

Also,

$$F_-(A) \text{ is maximum at } (x,u_3)=\left(\frac{1-\sqrt{3}}{6},\frac{2-\sqrt{3}}{3}\right) \text{ with } F_-(A)=\frac{1}{\sqrt{3}},$$
$$\text{and minimum on the line segment } (0,1)\text{-}\left(\frac{1}{3},\frac{1}{3}\right) \text{ with } F_-(A)=0.$$

□

Remark 1. *Let* $A_1=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$, $A_2=\begin{pmatrix}1&1&0\\1&1&0\\0&0&1\end{pmatrix}$, *and*

$$A_3=\begin{pmatrix}1 & -2+\sqrt{3} & \sqrt{\frac{\sqrt{3}-1}{2}}\\ -2+\sqrt{3} & 1 & \sqrt{\frac{\sqrt{3}-1}{2}}\\ \sqrt{\frac{\sqrt{3}-1}{2}} & \sqrt{\frac{\sqrt{3}-1}{2}} & 1\end{pmatrix}.$$

Then

$$F_+(A_1)=\frac{1}{3}, \qquad F_+(A_2)=1,$$
$$F_-(A_1)=F_-(A_2)=0, \qquad F_-(A_3)=\frac{1}{\sqrt{3}}.$$

Acknowledgements. I thank Professor Shun-ichi Kimura for his great help on this work.

References

[1] P. Heyfron, Positive functions defined on Hermitian positive semi-definite matrices. Ph.D. thesis, University of London (1989).

[2] E. H. Lieb, Proofs of some conjectures on permanents, J. Math. and Mech. 16 (1966) 127-134.

[3] T. H. Pate, Row appending maps, Ψ functions, and immanant inequalities for Hermitian positive semi-definite matrices, Proc. London Math. Soc. (3) 76(2) (1998) 307-358.

[4] I. Schur, Über endliche Gruppen und Hermitische Formen, Math. Z. 1 (1918) 184-207.

[5] R. Tabata, Sharp inequalities for the permanental dominance conjecture, Hiroshima Math. J. 40(2) (2010) 205-213.

参考論文

Sharp inequalities for the permanental dominance conjecture
Ryo Tabata
Hiroshima Mathematical Journal, 40(2) (2010) 205-213.

Hiroshima Math. J.
40 (2010), 205–213

Sharp inequalities for the permanental dominance conjecture

Ryo Tabata

(Received July 10, 2009)
(Revised October 13, 2009)

Abstract. For the normalized generalized matrix function $\bar{d}_\chi^G(A)$ for 3×3 positive semi-definite Hermitian matrices A, the permanental dominance conjecture per $A \geq \bar{d}_\chi^G(A)$ is known to hold. In this paper, we show that this inequality is not sharp, and give a sharper bound.

1. Introduction

The normalized generalized matrix function is a complex valued function on $n \times n$ square matrices $M_n(\mathbf{C})$, defined by

$$\bar{d}_\chi^G(A) := \frac{1}{\chi(\mathrm{id})} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)},$$

where G is a subgroup of the symmetric group $\mathfrak{S}_n$ and χ a character of G. In particular, it is called *immanant* if $G = \mathfrak{S}_n$ and χ is an irreducible character. For example, when $\chi(\sigma) = \operatorname{sgn} \sigma$, the immanant is the determinant, and when $\chi(\sigma) \equiv 1$, the immanant $\bar{d}_1^{\mathfrak{S}_n}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod a_{i\sigma(i)}$ is called *the permanent* of A, denoted by per A.

Note that if its domain is restricted to the (positive semi-definite) Hermitian matrices, the generalized matrix function takes real values. These values have been studied for a long time.

Theorem 1 (Hadamard [3] 1893). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\det A \leq a_{11} \dots a_{nn}.$$

Theorem 2 (Fisher [2] 1907). *If* $\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ *is an $n \times n$ positive semi-definite Hermitian matrix with A_{11} and A_{22} square matrices, then*

$$\det\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \leq (\det A_{11})(\det A_{22}).$$

2000 *Mathematics Subject Classification*. 15A15, 20A30.
Key words and phrases. Symmetric Group, Permanent, Generalized Matrix Function.

These inequalities were generalized by Schur's theorem for the generalized matrix functions.

THEOREM 3 (Schur [11] 1918). *If A is an $n \times n$ positive semi-definite Hermitian matrix, G a subgroup of $\mathfrak{S}_n$ and χ a character of G, then*

$$\det A \leq \bar{d}_\chi^G(A).$$

Indeed, Theorems 1, 2 are the special cases, the right-hand sides of which are $\bar{d}_1^{\{\mathrm{id}\}}$, $\bar{d}_{\mathrm{sgn}}^{\mathfrak{S}_k \times \mathfrak{S}_\ell}$, respectively. Schur's theorem says that the determinant is the smallest normalized generalized matrix function. Analogously, the following theorems for the permanent are known.

THEOREM 4 (Marcus [9] 1964). *If A is an $n \times n$ positive semi-definite Hermitian matrix, then*

$$\operatorname{per} A \geq a_{11} \dots a_{nn}.$$

THEOREM 5 (Lieb [8] 1966). *If $\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$ is an $n \times n$ positive semi-definite Hermitian matrix with A_{11} and A_{22} square matrices, then*

$$\operatorname{per}\left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right) \geq (\operatorname{per} A_{11})(\operatorname{per} A_{22}).$$

From these results, it is natural to expect

CONJECTURE (Lieb [8] 1966). *(Permanental Dominance Conjecture) If A is an $n \times n$ positive semi-definite Hermitian matrix, G a subgroup of $\mathfrak{S}_n$ and χ a character of G, then*

$$\operatorname{per} A \geq \bar{d}_\chi^G(A).$$

It is known that the permanental dominance conjecture holds for immanants with $n \leq 13$ (see [10]), and for all subgroups of $\mathfrak{S}_n$ and all characters when $n \leq 3$ ([5], [7]).

A stronger result is known for single hook immanants (see [4]), namely

$$\operatorname{per} A = \bar{d}^{\mathfrak{S}_n}_{\square\cdots\square}(A) \geq \bar{d}^{\mathfrak{S}_n}_{\substack{\square\cdots\square \\ \square}}(A) \geq \cdots \geq \bar{d}^{\mathfrak{S}_n}_{\substack{\square\square \\ \vdots \\ \square}}(A) \geq \bar{d}^{\mathfrak{S}_n}_{\substack{\square \\ \vdots \\ \square}}(A) = \det A.$$

Hence the permanental dominance conjecture for $n = 3$ is already settled. However, in this paper we will show that the inequality for the conjecture is not sharp, and represent sharper bounds by the internally dividing points between the determinant and the permanent. In particular we prove the following theorem for the alternating group $\mathfrak{A}_3$.

Main Theorem. *If A is a 3×3 positive semi-definite Hermitian matrix and ω is a non-trivial irreducible character of $\mathfrak{A}_3$, then*

$$\bar{d}_{\omega}^{\mathfrak{A}_3}(A) \leq 2^{-1/3} \operatorname{per} A + (1 - 2^{-1/3}) \det A.$$

This is an improvement of the inequality in the conjecture for this special case.

2. Proof of Main Theorem

Let G be a subgroup of the symmetric group $\mathfrak{S}_n$ and χ a character of G. We define *the generalized matrix function* by

$$d_{\chi}^{G}(A) := \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}.$$

In particular, when $G = \mathfrak{S}_n$ and χ is an irreducible character, $d_{\chi}^{G}(A)$ is called *immanant*. Moreover, *the normalized generalized matrix function* $\bar{d}_{\chi}^{G}$ is defined by

$$\bar{d}_{\chi}^{G}(A) := \frac{1}{\chi(\mathrm{id})} d_{\chi}^{G}(A).$$

For the rest of the paper, we suppose $n = 3$ unless otherwise stated. It is known that the permanental dominance conjecture is true for $n = 3$ ([5], [7]).

In this section, we always write A as $\begin{pmatrix} a & b & c \\ \bar{b} & d & e \\ \bar{c} & \bar{e} & f \end{pmatrix}$. We will display the values of the three normalized immanants for A.

$$\operatorname{per} A = \bar{d}_{(3)}^{\mathfrak{S}_3}(A) = adf + (b\bar{c}e + \bar{b}c\bar{e}) + (a|e|^2 + d|c|^2 + f|b|^2),$$

$$\det A = \bar{d}_{(1,1,1)}^{\mathfrak{S}_3}(A) = adf + (b\bar{c}e + \bar{b}c\bar{e}) - (a|e|^2 + d|c|^2 + f|b|^2),$$

$$\bar{d}_{\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}}^{\mathfrak{S}_3}(A) = \bar{d}_{(2,1)}^{\mathfrak{S}_3}(A) = adf - \frac{1}{2}(b\bar{c}e + \bar{b}c\bar{e}).$$

Remark 1. *If A is a positive semi-definite Hermitian matrix, then $a|e|^2 + d|c|^2 + f|b|^2 \geq 0$. Moreover when the equality holds, the values of all the normalized generalized matrix functions coincide.*

The following function plays the key role in this paper.

Definition 1. Define a complex valued function T for 3×3 semi-definite Hermitian matrices A with $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$ by

$$T(A) := \frac{b\bar{c}e}{a|e|^2 + d|c|^2 + f|b|^2}.$$

Proposition 1. *Let A be a positive semi-difinite Hermitian matrix. If $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$, then*

$$\operatorname{Re} T(A) \leq \frac{1}{3}.$$

Proof. It is a restatement of the inequality $\bar{d}_{(2,1)}(A) - \det A \geq 0$, which is a special case of Schur's theorem.

$$\begin{aligned}
0 &\leq \bar{d}_{(2,1)}(A) - \det A \\
&= \left(adf - \frac{1}{2}(b\bar{c}e + \bar{b}c\bar{e})\right) - (adf + (b\bar{c}e + \bar{b}c\bar{e}) - (a|e|^2 + d|c|^2 + f|b|^2)) \\
&= -3 \operatorname{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2.
\end{aligned}$$

Divide both sides by $a|e|^2 + d|c|^2 + f|b|^2$ to obtain $-3 \operatorname{Re} T(A) + 1 \geq 0$. □

Remark 2. *Conversely, we obtain $\bar{d}_{(2,1)}(A) \geq \det A$ from $\operatorname{Re} T(A) \leq 1/3$. The equality holds for the positive semi-definite Hermitian matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The eigenvalues of this matrix are $3, 0, 0$. More generally, if A has 0 as the eigenvalue with multiplicity 2, then the equality holds.*

Similarly, the inequality $\operatorname{Re} T(A) \geq -1/3$ is equivalent to $\operatorname{per} A \geq \bar{d}_{(2,1)}(A)$, a special case of the permanental dominance conjecture. Conversely, we obtain $\operatorname{per} A \geq \bar{d}_{(2,1)}(A)$ from $\operatorname{Re} T(A) \geq -1/3$. However, this inequality is not sharp:

Lemma 1. *Let A be a positive semi-difinite Hermitian matrix. If $a|e|^2 + d|c|^2 + f|b|^2 \neq 0$, then*

$$\operatorname{Re} T(A) \geq -\frac{1}{6}.$$

This result was already shown implicitly in [6] and explicitly in [1, 12]. We include a proof for the reader's convenience. We will prove our main theorem by a similar argument.

Proof (Proof of Lemma 1). As the arithmetic mean is larger than or equal to the geometric mean, we have

$$a|e|^2 + d|c|^2 + f|b|^2 \geq 3\sqrt[3]{adf|bce|^2}.$$

Also as $\det A \geq 0$, we have

$$adf \geq -2 \operatorname{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2.$$

Combining them, we obtain

$$a|e|^2 + d|c|^2 + f|b|^2 \geq 3\sqrt[3]{adf|bce|^2}$$

$$\geq 3\sqrt[3]{(-2\,\mathrm{Re}(b\bar{c}e) + a|e|^2 + d|c|^2 + f|b|^2)(\mathrm{Re}(b\bar{c}e))^2}.$$

For simplicity, we write X for $a|e|^2 + d|c|^2 + f|b|^2$ and Z for $\mathrm{Re}(b\bar{c}e)$ so that $\mathrm{Re}\,T(A) = Z/X$. Then we have

$$X \geq 3\sqrt[3]{(-2Z + X)Z^2},$$

$$X^3 \geq -54Z^3 + 27XZ^2,$$

$$54Z^3 - 27XZ^2 + X^3 \geq 0,$$

$$54\left(\frac{Z}{X}\right)^3 - 27\left(\frac{Z}{X}\right)^2 + 1 \geq 0.$$

The graph of the function $Y = 54(Z/X)^3 - 27(Z/X)^2 + 1$ looks like the following.

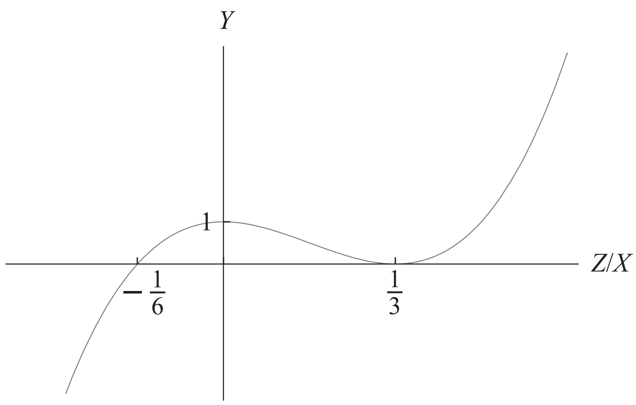

Fig. 1. $Y = 54(\frac{Z}{X})^3 - 27(\frac{Z}{X})^2 + 1$

From the graph, we can conclude $\mathrm{Re}\,T(A) = Z/X \geq -1/6$. □

The equality holds for the positive semi-definite Hermitian matrices

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & -1 \\ \frac{1}{\sqrt{2}} & 2 & \sqrt{2} \\ -1 & \sqrt{2} & 4 \end{pmatrix}.$$

The eigenvalues of these matrices are $3, 3, 0$ and $(7 \pm \sqrt{7})/2, 0$, respectively.

The inequality $\operatorname{Re} T(A) \geq -1/6$ immediately implies

$$\bar{d}^{\mathfrak{S}_3}_{\boxplus}(A) \leq \frac{3}{4} \operatorname{per} A + \frac{1}{4} \det A.$$

Corollary 1. *If A is a 3×3 positive semi-definite Hermitian matrix, then*

$$\frac{1}{6} \operatorname{per} A + \frac{5}{6} \det A \leq \bar{d}^{\{\mathrm{id}\}}_1(A) \leq \frac{2}{3} \operatorname{per} A + \frac{1}{3} \det A.$$

These inequalities are sharp, and are improvements of the permanental dominance conjecture and Hadamard's theorem.

From here, we study the values of $T(A)$ in the complex plane. In particular, we obtain the sharper inequalities for $d^{\mathfrak{A}_3}_{\omega_1}$ and $d^{\mathfrak{A}_3}_{\omega_2}$ in terms of the above, where $\mathfrak{A}_3$ is the alternating group and $\omega_i : \mathfrak{A}_3 \to \mathbf{C}$ $(i = 1, 2)$ are the two non-trivial irreducible characters.

Lemma 2. *For the complex number $x + yi$ $(x, y \in \mathbf{R})$, $T(A) = x + yi$ for some positive semi-definite Hermitian matrices if and only if*

$$\begin{cases} 54x(x^2 + y^2) - 27(x^2 + y^2) + 1 \geq 0, \\ -\frac{1}{6} \leq x \leq \frac{1}{3}. \end{cases}$$

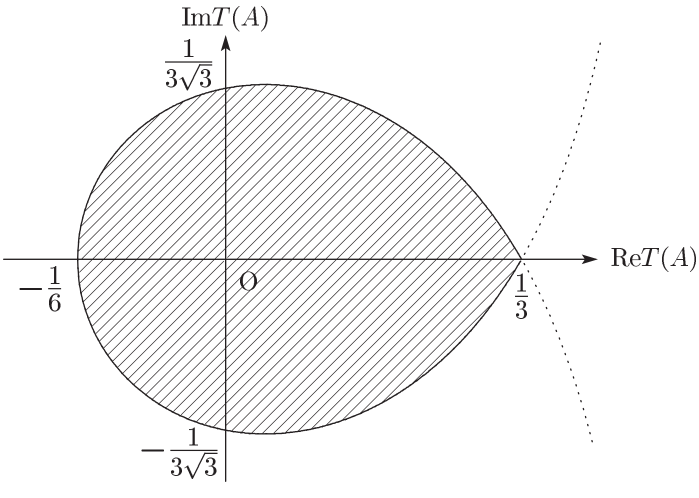

Fig. 2

Proof. We have shown $-1/6 \leq x \leq 1/3$ in Proposition 1 and Lemma 1. Suppose $X = a|e|^2 + d|c|^2 + f|b|^2, R = |bce|, S = \operatorname{Re}(b\bar{c}e)$. Calculating as in the beginning of the proof of Lemma 1, we have

$$X \geq 3(adfR^2)^{1/3} \geq 3((X - 2S)R^2)^{1/3}.$$

Note that $T(A) = x + yi, R/X = (x^2 + y^2)^{1/2}$ and $S/X = x$. An easy calculation shows that

$$54x(x^2 + y^2) - 27(x^2 + y^2) + 1 \geq 0,$$

hence the "only if" part follows.

Conversely for each $x + yi$ satisfying the inequality, the matrix

$$A = \begin{pmatrix} 1 & b & b \\ \bar{b} & 1 & b \\ \bar{b} & \bar{b} & 1 \end{pmatrix}$$

is a positive semi-definite Hermitian matrix with $T(A) = x + yi$, where $b = 3(x + yi)$. □

Proof (Proof of Main Theorem). Let $\omega_0, \omega_1, \omega_2$ denote the characters of the three irreducible representations of $\mathfrak{A}_3$. The following is the character table of $\mathfrak{A}_3$.

$\mathfrak{A}_3$	(1)	(123)	(132)
ω_0	1	1	1
ω_1	1	ω	ω^2
ω_2	1	ω^2	ω

where $\omega = (-1 + \sqrt{3}i)/2$. We prove the assertion only in the case $\omega = \omega_1$. The other case is similar. We want to find μ with $\bar{d}^{\mathfrak{A}_3}_{\omega_1}(A) \leq \mu \operatorname{per} A + (1 - \mu) \det A$. We observe

$$\bar{d}^{\mathfrak{A}_3}_{\omega_1}(A) = adf + 2\operatorname{Re}(\omega b\bar{c}e),$$

hence $\bar{d}^{\mathfrak{A}_3}_{\omega_1}(A) \leq \mu \operatorname{per} A + (1 - \mu) \det A$ if and only if

$$adf + 2\operatorname{Re}(\omega b\bar{c}e) \leq adf + 2\operatorname{Re}(b\bar{c}e) + (2\mu - 1)(a|e|^2 + d|c|^2 + f|b|^2).$$

This inequality can be rewritten as

$$\frac{1}{2} + \operatorname{Re}((\omega - 1)T(A)) \leq \mu.$$

From Lemma 2, finding the boundary point where the slope is $-\sqrt{3}$, calculation shows that

$$T(A) = \frac{1}{6}(2 - \sqrt[3]{4} - \sqrt[3]{2}) - \frac{1}{2\sqrt{3}}(\sqrt[3]{4} - \sqrt[3]{2})i$$

maximizes $\mathrm{Re}((\omega-1)T(A))$ with the value $\mu = 2^{-1/3}$. When $b = 1 - \sqrt[3]{2}\omega^2 - \sqrt[3]{4}\omega$ (hence a root of $b^3 - 3b^2 - 3b - 1 = 0$), this $T(A)$ is realized by $A = \begin{pmatrix} 1 & b & b \\ \bar{b} & 1 & b \\ \bar{b} & \bar{b} & 1 \end{pmatrix}$. □

Remark 3. *For the trivial character ω_0 of $\mathfrak{A}_3$, the following equality always holds:*

$$\bar{d}^{\mathfrak{A}_3}_{\omega_0}(A) = \frac{1}{2}\ \mathrm{per}\, A + \frac{1}{2}\ \det A.$$

Corollary 2. *There are no finitely many test matrices $\{A_1, \dots, A_r\}$ such that for a linear combination*

$$F(A) = \alpha\,\mathrm{Re}(b\bar{c}e) + \beta\,\mathrm{Im}(b\bar{c}e) + \gamma(a|e|^2 + d|c|^2 + f|b|^2),$$

$F(A) \geq 0$ for any positive semi-definite Hermitian matrix A if and only if $F(A_i) \geq 0$ for all $i = 1, 2, \dots, r$.

Proof. Each $F(A_i) \geq 0$ determines a half plane in the complex plane. From Figure 2, the shaded area cannot be written as the intersection of finitely many half planes. □

Acknowledgement

I thank Professor Shun-ichi Kimura for his help. I am also grateful to Professor Gordon James for his helpful comments.

References

[1] W. Barrett, H. Tracy Hall, R. Loewy, The cone of class function inequalities for the 4-by-4 positive semidefinite matrices, Proc. London Math. Soc. (3) 79:107–130 (1999).
[2] E. Fisher, Über den Hadamardschen Determinantensatz, Archiv d. Math. u. Phys. (3)13:32–40 (1907).
[3] J. Hadamard, Resolution d'une question relative aux determinants, Bull. Sci. Math. 2:240–246 (1893).
[4] P. Heyfron, Immanant dominance orderings for hook partitions, Linear and Multilinear Algebra 24(1):65–78 (1988).
[5] P. Heyfron, Positive functions defined on Hermitian positive semi-definite matrices. Ph.D. thesis, University of London (1989).
[6] G. D. James, Immanants, Linear and Multilinear Algebra 32:197–210 (1992).
[7] G. D. James, Private communications.
[8] E. H. Lieb, Proofs of some conjectures on permanents, I. Math. and Mech. 16:127–134 (1966).

[9] M. Marcus, The Hadamard theorem for permanents, Proc. Amer. Math. Soc. 15:967–973 (1964).

[10] T. H. Pate, Row appending maps, Ψ functions, and immanant inequalities for Hermitian positive semi-definite matrices, Proc. London Math. Soc. (3) 76(2):307–358 (1998).

[11] I. Schur, Über endliche Gruppen und Hermitische Formen, Math. Z. 1:184–207 (1918).

[12] R. Tabata, A generalization of Schur's theorem. Master's thesis, Hiroshima University (2009).

Ryo Tabata
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526, *Japan*
E-mail: tbt.ryo@gmail.com