広島大学学位請求論文

Two-point homogeneous quandles with cardinality of prime power

(素数冪位数の二点等質カンドル)

2015年

広島大学大学院理学研究科
数学専攻

和田　幸史朗

目　次

主論文

TWO-POINT HOMOGENEOUS QUANDLES WITH CARDINALITY OF PRIME POWER

KOSHIRO WADA

Abstract. The main result of this paper classifies two-point homogeneous quandles with cardinality of prime power. More precisely, such quandles are isomorphic to Alexander quandles defined by primitive roots over finite fields. This result classifies all two-point homogeneous finite quandles, by combining with the recent result of Vendramin.

1. Introduction

Quandles were introduced to study knots by Joyce ([7]). Let X be a set, and assume that there exists a map $s : X \to \mathrm{Map}(X, X) : x \mapsto s_x$. Here $\mathrm{Map}(X, X)$ denotes the set of all maps from X to X. Then a pair (X, s) is called a *quandle* if s satisfies the conditions corresponding to Reidemeister moves of classical knots (see Definition 2.1). In knot theory, quandles provide a complete algebraic framework, and provide several invariants of knots (see [2, 4] and references therein). Among others, Carter, Jelsovsky, Kamada, Langford and Saito ([3]) gave strong invariants, called quandle cocycle invariants, defined by quandle cocycles. They apply it to prove the non-invertibility of the 2-twist spun trefoil by using a 3-cocycle of the dihedral quandle R_3 with cardinality 3. In [9], Mochizuki gave a systematic method for calculating some quandle cocycles of dihedral quandles. In addition, Nosaka ([10]) applied the method of Mochizuki, and provided quandle cocycles of some Alexander quandles. However, in general, calculation of quandle cocycles is difficult, even in the case of low cardinality. Therefore, it is of importance to study special classes of quandles, whose quandle structures are helpful to induce algebraic properties of quandle cohomologies.

From this point of view, we study two-point homogeneous quandles and quandles of cyclic type. A quandle (X, s) with $\#X \geq 3$ is said to be *two-point homogeneous* if for any $(x_1, x_2), (y_1, y_2) \in X \times X$ satisfying $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists an inner automorphism f of (X, s) such that $(f(x_1), f(x_2)) = (y_1, y_2)$. On the other hand, a quandle (X, s) with finite cardinality $n \geq 3$ is said to be of *cyclic type* if s_x are cyclic permutations of order $n - 1$ for any $x \in X$. These quandles have been studied in [5, 6, 8, 11, 12, 13]. In particular, all two-point homogeneous quandles with prime cardinality were classified in [12]. In addition, [12] proved that all quandles of cyclic type are two-point homogeneous, and gave

Key words and phrases. Finite quandles, two-point homogeneous quandles, quandles of cyclic type.

the following conjecture. Note that this conjecture is true when the cardinalities are prime numbers ([12]).

Conjecture 1.1. *All two-point homogeneous quandles with finite cardinality are of cyclic type.*

Recently, Vendramin ([13]) proved that the cardinalities of two-point homogeneous quandles must be prime power.

In this paper, we classify two-point homogeneous quandles (X, s) with cardinality of prime power. A key of the proof is that (X, s) are simple crossed sets (Proposition 3.7). By using a classification of simple crossed sets with cardinality of prime power in [1], we have the following.

Main theorem (Theorem 4.3)**.** Let q be a prime power and X be a quandle with cardinality q. Then the following conditions are mutually equivalent:
(1) (X, s) is two-point homogeneous,
(2) (X, s) is isomorphic to the Alexander quandle $(\mathbb{F}_q, \omega)$, where ω is a primitive root over the finite field $\mathbb{F}_q$,
(3) (X, s) is of cyclic type.

This result is an extension of the result of [12]. Moreover, by applying the result of Vendramin ([13]), we obtain a classification of all two-point homogeneous finite quandles (Corollary 4.5). In particular, Corollary 4.5 shows that Conjecture 1.1 is true.

This paper is organized as follows. In Section 2, we recall some notions of quandles. In Section 3, some properties of two-point homogeneous quandles and quandles of cyclic type are summarized. In Section 4, we prove the main result.

2. Preliminaries for quandles

In this section we recall some notions on quandles.

Definition 2.1. Let X be a set, and assume that there exists a map $s : X \to \mathrm{Map}(X, X) : x \mapsto s_x$. Then a pair (X, s) is called a *quandle* if s satisfies the following conditions:

(S1) $\forall x \in X$, $s_x(x) = x$,
(S2) $\forall x \in X$, s_x is bijective, and
(S3) $\forall x, y \in X$, $s_x \circ s_y = s_{s_x(y)} \circ s_x$.

We denote by $\#X$ the cardinality of X.

Example 2.2. The following (X, s) are quandles:
(1) Let X be any set and $s_x := \mathrm{id}_X$ for every $x \in X$. Then the pair (X, s) is called the *trivial quandle*.
(2) Let $X := \{1, \ldots, n\}$ and $s_i(j) := 2i - j \pmod n$ for any $i, j \in X$. Then the pair (X, s) is called the *dihedral quandle* with cardinality n.

(3) Let q be a prime power and $\mathbb{F}_q$ be the finite field of order q. If $\omega \in \mathbb{F}_q$ ($\omega \neq 0, 1$), then the pair $(\mathbb{F}_q, \omega)$ with the following operator is called the *Alexander quandle* of order q:

$$s_x(y) := \omega y + (1 - \omega)x. \tag{2.1}$$

Definition 2.3. Let (X, s^X) and (Y, s^Y) be quandles, and $f : X \to Y$ be a map.
(1) f is called a *homomorphism* if for every $x \in X$, $f \circ s_x^X = s_{f(x)}^Y \circ f$ holds.
(2) f is called an *isomorphism* if f is a bijective homomorphism.

An isomorphism from a quandle (X, s) onto itself is called an *automorphism*. The set of automorphisms of (X, s) forms a group, which is called the *automorphism group* and denoted by $\mathrm{Aut}(X, s)$.

Note that s_x ($x \in X$) is an automorphism of (X, s). The subgroup of $\mathrm{Aut}(X, s)$ generated by $\{s_x \mid x \in X\}$ is called the *inner automorphism group* of (X, s) and denoted by $\mathrm{Inn}(X, s)$. A quandle (X, s) is said to be *connected* if $\mathrm{Inn}(X, s)$ acts on (X, s) transitively. Let $\mathrm{Inn}(X, s)_x$ be the stabilizer subgroup of $\mathrm{Inn}(X, s)$ at $x \in X$.

Definition 2.4. A quandle (X, s) is called a *crossed set* if $s_x(y) = y$ whenever $s_y(x) = x$.

Definition 2.5. Let (X, s^X) and (Y, s^Y) be finite quandles. A surjective homomorphism $f : X \to Y$ is called *trivial* if $\#Y$ is equal to either $\#X$ or 1.

Definition 2.6. A quandle (X, s) is *simple* if it is not a trivial quandle and any surjective homomorphism $f : X \to Y$ is trivial.

On classification of simple crossed sets with cardinality of prime power, Andruskiewitsch and Grana ([1]) give the following theorem.

Theorem 2.7 (Corollary 3.10 in [1])**.** *Let p be a prime number and $l \in \mathbb{N}$. If a quandle (X, s) with cardinality p^l is a simple crossed set, then (X, s) isomorphic to an Alexander quandle $(\mathbb{F}_{p^l}, \omega)$, where ω generates $\mathbb{F}_{p^l}$ over $\mathbb{F}_p$.*

Recall that $\omega \in \mathbb{F}_{p^l}$ is said to *generate* $\mathbb{F}_{p^l}$ if $\{1, \omega, \ldots, \omega^{l-1}\}$ is a basis of $\mathbb{F}_{p^l}$ over $\mathbb{F}_p$.

3. Two-point homogeneous quandles and quandles of cyclic type

In this section, we recall the definitions and some properties of two-point homogeneous quandles and quandles of cyclic type, which are given in [12].

Definition 3.1. A finite quandle (X, s) with $\#X = n \geq 3$ is said to be of *cyclic type* if for every $x \in X$, s_x acts on $X - \{x\}$ as a cyclic permutation of order $n - 1$.

This notion is closely related to the notion of two-point homogeneous quandles.

Definition 3.2. A quandle (X, s) with $\#X \geq 3$ is said to be *two-point homogeneous* if for any $(x_1, x_2), (y_1, y_2) \in X \times X$ satisfying $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists $f \in \mathrm{Inn}(X, s)$ such that $(f(x_1), f(x_2)) = (y_1, y_2)$.

Note that quandles of cyclic type have finite cardinalities. On the other hand, two-point homogeneous quandles are not necessarily finite.

Proposition 3.3 ([12])**.** *Every quandle of cyclic type is two-point homogeneous.*

The following are characterizations of quandles of cyclic type and two-point homogeneous quandles.

Proposition 3.4 ([12])**.** *Let (X, s) be a finite quandle with $\#X \geq 3$. Then the following conditions are equivalent*:
(1) *(X, s) is of cyclic type,*
(2) *(X, s) is connected, and there exists $x \in X$ such that s_x acts on $X - \{x\}$ as a cyclic permutation of order $\#X - 1$.*

Proposition 3.5 ([12])**.** *Let (X, s) be a quandle with $\#X \geq 3$. Then the following conditions are equivalent*:
(1) *(X, s) is two-point homogeneous,*
(2) *for every $x \in X$, the action of $\mathrm{Inn}(X, s)_x$ on $X - \{x\}$ is transitive,*
(3) *(X, s) is connected, and there exists $x \in X$ such that the action of $\mathrm{Inn}(X, s)_x$ on $X - \{x\}$ is transitive.*

The following lemma will be used to prove that every two-point homogeneous finite quandles is a simple crossed set.

Lemma 3.6. *Let (X, s^X) and (Y, s^Y) be quandles, and $f : X \to Y$ be a homomorphism. Then for any $g \in \mathrm{Inn}(X, s^X)$, there exists $h \in \mathrm{Inn}(Y, s^Y)$ satisfying*

$$f \circ g = h \circ f. \tag{3.1}$$

Proof. The inner automorphism g can be written as

$$g = \prod_{i=1}^{k} (s^X_{x_i})^{\epsilon_i} \tag{3.2}$$

for some $x_i \in X, \epsilon_i \in \mathbb{Z}$. By the assumption on f, we have

$$f \circ (s^X_x)^{\epsilon} = (s^Y_{f(x)})^{\epsilon} \circ f \tag{3.3}$$

for any $\epsilon \in \mathbb{Z}$. It follows that

$$h := (\prod_{i=1}^{k} (s^Y_{f(x_i)})^{\epsilon_i}) \in \mathrm{Inn}(Y, s^Y) \tag{3.4}$$

satisfies the required condition. □

This lemma gives the following proposition for two-point homogeneous quandles.

Proposition 3.7. *Every two-point homogeneous quandle with finite cardinality is a simple crossed set.*

Proof. Let (X, s) be a two-point homogeneous quandle with finite cardinality. We prove that (X, s) is crossed. Suppose that

$$s_x(y) = y \ (x, y \in X). \tag{3.5}$$

It is enough to consider the case $x \neq y$. By the definition of two-point homogeneous quandles, there exists $f \in \mathrm{Inn}(X, s)$ such that

$$f(x) = y, \ f(y) = x. \tag{3.6}$$

Hence one has

$$s_y(x) = s_{f(x)} \circ f(y) = f \circ s_x(y) = f(y) = x. \tag{3.7}$$

Next, we prove that (X, s) is simple. Take any quandle (Y, s') and any surjective homomorphism $f : X \to Y$. Assume that f is not injective, and we prove $\#Y = 1$. By the assumption, there exist $x, y \in X$ $(x \neq y)$ with $f(x) = f(y)$. Take any $z \in X - \{x, y\}$. Then there exists $g \in \mathrm{Inn}(X, s)$ satisfying

$$g(y) = z, \ g(x) = x. \tag{3.8}$$

Lemma 3.6 yields that there exists $h \in \mathrm{Inn}(Y, s')$ satisfying

$$f \circ g = h \circ f. \tag{3.9}$$

Hence we have

$$f(z) = f \circ g(y) = h \circ f(y) = h \circ f(x) = f \circ g(x) = f(x). \tag{3.10}$$

This shows $\#f(X) = 1$. Since f is surjective, we have $\#Y = 1$. $\square$

4. Main theorem

Let $q \geq 3$ be a prime power. In this section, we classify two-point homogeneous quandles with cardinality q. We also show that all two-point homogeneous quandles with cardinality q are of cyclic type. Note that two-point homogeneous quandles with prime cardinality are already classified in [12]. The main theorem is an extension of the result of [12].

4.1. The inner automorphism group of $(\mathbb{F}_q, \omega)$.

In this subsection, we determine the inner automorphism group of the Alexander quandle $(\mathbb{F}_q, \omega)$ of order q with $\omega \in \mathbb{F}_q - \{0, 1\}$. Recall that the map $s : \mathbb{F}_q \to \mathrm{Map}(\mathbb{F}_q, \mathbb{F}_q)$ is given by assigning

$$s_x : \mathbb{F}_q \to \mathbb{F}_q : y \mapsto \omega y + (1 - \omega)x$$

to each $x \in \mathbb{F}_q$.

For each $x \in \mathbb{F}_q$, we define a map ψ_x as follows:

$$\psi_x : \mathbb{F}_q \to \mathbb{F}_q : y \mapsto y + x. \tag{4.1}$$

Proposition 4.1. *The inner automorphism group of* $(\mathbb{F}_q, \omega)$ *satisfies*

$$\mathrm{Inn}(\mathbb{F}_q, \omega) = \{(s_x)^k \mid x \in \mathbb{F}_q, k \in \mathbb{Z}\} \cup \{\psi_y \mid y \in \mathbb{F}_q\}. \tag{4.2}$$

Proof. Let us denote by G the right side of (4.2). First of all, we prove $G \subset \mathrm{Inn}(\mathbb{F}_q, \omega)$. It is clear that $(s_x)^k \in \mathrm{Inn}(\mathbb{F}_q, \omega)$ for any $k \in \mathbb{Z}$ and $x \in \mathbb{F}_q$. Hence we have only to prove $\psi_x \in \mathrm{Inn}(\mathbb{F}_q, \omega)$ for each $x \in \mathbb{F}_q$. Note that there exists the inverse $(1-\omega)^{-1}$. For $x, y \in \mathbb{F}_q$, one has

$$\begin{aligned} (s_{x(1-\omega)^{-1}})(s_0)^{q-2}(y) &= s_{x(1-\omega)^{-1}}(\omega^{q-2} y) \\ &= \omega^{q-1} y + x(1-\omega)(1-\omega)^{-1} \\ &= y + x \\ &= \psi_x(y). \end{aligned} \tag{4.3}$$

This shows $\psi_x = (s_{x(1-\omega)^{-1}})(s_0)^{q-2} \in \mathrm{Inn}(\mathbb{F}_q, \omega)$, and hence $G \subset \mathrm{Inn}(\mathbb{F}_q, \omega)$.

Next, we prove $G \supset \mathrm{Inn}(\mathbb{F}_q, \omega)$. Since G contains generators $\{s_x \mid x \in \mathbb{F}_q\}$ of $\mathrm{Inn}(\mathbb{F}_q, \omega)$, it is enough to prove that G is a group. For any $a, b \in \mathbb{Z}$, we show

$$(s_x)^a (s_y)^b, (s_x)^a \psi_y, \psi_y (s_x)^a \in G. \tag{4.4}$$

Case (1): We show $(s_x)^a (s_y)^b \in G$. For any $z \in \mathbb{F}_q$, we have

$$\begin{aligned} (s_x)^a (s_y)^b (z) &= (s_x)^a (\omega^b z + (1-\omega^b) y) \\ &= \omega^a (\omega^b z + (1-\omega^b) y) + (1-\omega^a) x \\ &= \omega^{(a+b)} z + (1-\omega^b)\omega^a y + (1-\omega^a) x. \end{aligned} \tag{4.5}$$

Let $\alpha := (1-\omega^b)\omega^a y + (1-\omega^a) x$.

Subcase (1)-i: We consider the case of $1 - \omega^{a+b} = 0$. By (4.5), we have

$$(s_x)^a (s_y)^b (z) = z + \alpha = \psi_\alpha(z) \tag{4.6}$$

for any $z \in \mathbb{F}_q$. This yields that $(s_x)^a (s_y)^b = \psi_\alpha \in G$.

Subcase (1)-ii: We deal with the case of $1 - \omega^{a+b} \neq 0$. In this case, there exists the inverse $(1-\omega^{a+b})^{-1}$. Therefore (4.5) yields

$$\begin{aligned} (s_x)^a (s_y)^b (z) &= \omega^{(a+b)} z + (1-\omega^{a+b})(1-\omega^{a+b})^{-1} \alpha \\ &= (s_{(1-\omega^{a+b})^{-1}\alpha})^{(a+b)}(z). \end{aligned} \tag{4.7}$$

This yields $(s_x)^a (s_y)^b = (s_{(1-\omega^{a+b})^{-1}\alpha})^{(a+b)} \in G$.

Case (2): We show $(s_x)^a \psi_y \in G$. Let $z \in \mathbb{F}_q$.

Subcase (2)-i: If $1 - \omega^a = 0$, then one has

$$(s_x)^a (z) = \omega^a z + (1-\omega^a) x = z. \tag{4.8}$$

This yields

$$(s_x)^a \psi_y = \psi_y \in G. \tag{4.9}$$

Subcase (2)-ii: Suppose that $1-\omega^a \neq 0$. Note that there exists the inverse $(1-\omega^a)^{-1}$. For any $z \in \mathbb{F}_q$, we have

$$
\begin{aligned}
(s_x)^a \psi_y(z) &= \omega^a(z+y) + (1-\omega^a)x \\
&= \omega^a z + (1-\omega^a)(1-\omega^a)^{-1}\omega^a y + (1-\omega^a)x \\
&= (s_{(1-\omega^a)^{-1}\omega^a y + x})^a(z).
\end{aligned} \tag{4.10}
$$

This yields $(s_x)^a \psi_y = (s_{(1-\omega^a)^{-1}\omega^a y+x})^a \in G$.

Case (3): We show $\psi_y(s_x)^a \in G$. For any $z \in \mathbb{F}_q$, we have

$$\psi_y(s_x)^a(z) = \omega^a z + (1-\omega^a)x + y. \tag{4.11}$$

By considering two cases as in Case (2), we have $\psi_y(s_x)^a \in G$. □

Let us concern the stabilizer subgroup of $\mathrm{Inn}(\mathbb{F}_q, \omega)$ at 0,

$$\mathrm{Inn}(\mathbb{F}_q, \omega)_0 := \{f \in \mathrm{Inn}(\mathbb{F}_q, \omega) \mid f(0) = 0\}. \tag{4.12}$$

Corollary 4.2. *The group* $\mathrm{Inn}(\mathbb{F}_q, \omega)_0$ *is generated by* s_0.

Proof. Since $s_0 \in \mathrm{Inn}(\mathbb{F}_q, \omega)_0$, we have

$$\mathrm{Inn}(\mathbb{F}_q, \omega)_0 \supset \langle s_0 \rangle. \tag{4.13}$$

Hence, we have only to prove

$$\mathrm{Inn}(\mathbb{F}_q, \omega)_0 \subset \langle s_0 \rangle. \tag{4.14}$$

Take any $g \in \mathrm{Inn}(\mathbb{F}_q, \omega)_0$. In view of Proposition 4.1, we have only to consider the following two cases.

Case (1): We deal with the case $g = (s_x)^a$ for $x \in \mathbb{F}_q$ and $a \in \mathbb{Z}$. Since $(s_x)^a(0) = 0$, one has

$$0 = (s_x)^a(0) = (1-\omega^a)x. \tag{4.15}$$

Since $\mathbb{F}_q$ is a field, one has $1-\omega^a = 0$ or $x = 0$. If $1-\omega^a = 0$, then $(s_x)^a(y) = (1-\omega^a)x + \omega^a y$ yields

$$g = (s_x)^a = \mathrm{id} \in \langle s_0 \rangle. \tag{4.16}$$

If $x = 0$, then we have

$$g = (s_0)^a \in \langle s_0 \rangle. \tag{4.17}$$

Case (2): We deal with the case $g = \psi_x$. Since $\psi_x(0) = 0$, we clear have $x = 0$ and $\psi_x = \mathrm{id} \in \mathrm{Inn}(\mathbb{F}_q, \omega)_0$. Therefore (4.14) holds. □

4.2. **Proof of the main theorem.** In this subsection, we prove the main theorem. Let q be a prime power and $\mathbb{F}_q$ be a finite field of order q. Recall that $\omega \in \mathbb{F}_q$ is called a *primitive root modulo* q if $\langle \omega \rangle := \{1, \omega, \ldots, \omega^{q-2}\} = \mathbb{F}_q - \{0\}$.

Theorem 4.3. *Let q be a prime power and X be a quandle with cardinality q. Then the following conditions are mutually equivalent*:

(1) *(X, s) is two-point homogeneous,*
(2) *(X, s) is isomorphic to the Alexander quandle $(\mathbb{F}_q, \omega)$, where ω is a primitive root over the finite field $\mathbb{F}_q$,*
(3) *(X, s) is of cyclic type.*

Proof. First of all, we deal with (1) ⇒ (2). By Proposition 3.7, X is simple and crossed. Thus, by Theorem 2.7, there exists $\omega \in \mathbb{F}_q$ satisfying $(X, s) \cong (\mathbb{F}_q, \omega)$. It is enough to show that ω is a primitive root modulo q. Note that $(\mathbb{F}_q, \omega)$ is two-point homogeneous. Hence, by Proposition 3.5, $\mathrm{Inn}(\mathbb{F}_q, \omega)_0$ acts on $\mathbb{F}_q - \{0\}$ transitively. Thus Corollary 4.2 yields

$$\mathbb{F}_q - \{0\} = \mathrm{Inn}(\mathbb{F}_q, \omega)_0 \cdot \{1\} = \langle s_0 \rangle \cdot \{1\} = \langle \omega \rangle. \tag{4.18}$$

Hence ω is a primitive root modulo q.

Next, we prove (2) ⇒ (3). Note that $(\mathbb{F}_q, \omega)$ is connected from Proposition 4.1. Hence it is enough to prove that s_0 is a cyclic permutation of order $q - 1$ from Proposition 3.4. One knows $(s_0)^t(x) = \omega^t x$ for any $x \in \mathbb{F}_q - \{0\}$ and $t \in \mathbb{Z}$. Since ω is a primitive root, we have

$$\langle s_0 \rangle \cdot \{1\} = \{1, \omega, \omega^2, \ldots, \omega^{q-2}\} = \mathbb{F}_q - \{0\}. \tag{4.19}$$

This means that s_0 is a cyclic permutation of order $q - 1$.

The implication (3) ⇒ (1) follows from Proposition 3.3. □

This theorem yields the existence of two-point homogeneous quandles with cardinality of prime power.

Corollary 4.4. *For any prime power $q \geq 3$, there exists a quandle of cyclic type, and hence a two-point homogeneous quandle, with cardinality q.*

In addition, by combining the result of [13] with Theorem 4.3, we obtain the following corollary.

Corollary 4.5. *All two-point homogeneous quandles with finite cardinality are isomorphic to Alexander quandles defined by primitive roots over finite fields. In particular, they are of cyclic type.*

Acknowledgement

The author would like to thank T. Hashinaga, S. Kamada, A. Kubo, Y. Taketomi and H. Tamaru for helpful comments. He also wishes to acknowledge valuable discussions with L. Vendramin.

References

[1] N. Andruskiewitsch, M. Grana, *From racks to pointed Hopf algebras*, Adv. Math., **178** (2003), no. 2, 177–243.

[2] J. S. Carter, *A Survey of Quandle Ideas*, in: Introductory Lectures on Knot Theory, Ser. Knots Everything, **46** (2012), 22–53.

[3] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc., **355** (2003), no. 10, 3947–3989.

[4] R. Fenn, C. Rourke, *Racks and links in codimension two*, J. Knot Theory Ramifications, **1** (1992), 343–406.

[5] C. Hayashi, *Canonical forms for operation tables of finite connected quandles*, Comm. Algebra, **41** (2013), 3340–3349.

[6] S. Iwanaga, *On the classification of two-point homogeneous quandles of cardinality* p^2, Master's thesis (in Japanese), Tokyo University of Science, (2013).

[7] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra, **23** (1982), 37–65.

[8] S. Kamada, H. Tamaru, K. Wada, *On classification of quandles of cyclic type*, arXiv:1312.6917.

[9] T. Mochizuki, *Some calculations of cohomology groups of finite Alexander quandles*, J. Pure Appl. Algebra, **179** (2003), 287–330.

[10] T. Nosaka, *On quandle homology groups of Alexander quandles of prime order*, Trans. Amer. Math. Soc., **365** (2013), 3413–3436.

[11] P. Lopes, D. Roseman, *On finite racks and quandles*, Comm. Algebra, **34** (2006), no. 1, 371–406.

[12] H. Tamaru, *Two-point homogeneous quandles with prime cardinally*, J. Math. Soc. Japan, **65** (2013), no. 4, 1117–1134.

[13] L. Vendramin, *Doubly transitive groups and cyclic quandles*, J. Math. Soc. Japan, in press.

参考論文

On classification of two-point homogeneous quandles of cyclic type
Koshiro Wada.
In: 研究集会「結び目の数学IV」報告集, 144-148, 2012.

On classification of two-point homogeneous quandles of cyclic type

和田　幸史朗 (広島大学 大学院理学研究科数学専攻 博士課程前期 2 年)

概要

本研究は Riemann 多様体論からのカンドルへのアプローチを研究することを目的としている. Riemann 多様体論における 2 点等質という概念のアナロジーとして, 2 点等質カンドルと呼ばれるカンドルが定義される. またその特別なケースである巡回型と呼ばれるカンドルが存在する. ここでは, まず 2 点等質カンドルと巡回型カンドルとの関係をいくつか述べ, 巡回型カンドルの分類定理と位数 12 までの分類結果を紹介する. 尚, 本研究は広島大学の田丸博士先生との共同研究である.

1　背景と定義

Riemann 多様体とは可微分多様体 M に Riemann 計量と呼ばれるある計量 g をいれた多様体 (M,g) のことである (ここでは Riemann 多様体は連結であることを仮定する). この計量によって距離関数 d が与えられ, Riemann 対称空間という多様体が定義される. また別に 2 点等質 Riemann 多様体と呼ばれるものを定義する. 以下 (M,g) を Riemann 多様体, $\mathrm{Isom}(M,g)$ を等長変換の成す群とする.

定義 1.1. 任意の $x \in M$ に対し次を満たす等長変換 s_x が存在するとき, (M,g) は Riemann 対照空間であるという.
(i) x は s_x の孤立固定点である.
(ii) $s_x^2 = \mathrm{id}_M$

定義 1.2. Riemann 多様体 (M,g) に対し, 次が成り立つ時それは 2 点等質であるという.
$\forall (x_1,x_2),(y_1,y_2) \in M \times M \ (d(x_1,x_2) = d(y_1,y_2)),\ \exists f \in \mathrm{Isom}(M,g)$ s.t. $f(x_i) = y_i \ (i = 1,2)$

2 点等質 Riemann 多様体は Riemann 対称空間であることが知られている. Joyce は [1] において Riemann 対称空間がカンドルであることを述べており, 我々は 2 点等質に相当する概念をカンドルにおいて定義する.

定義 1.3. 集合 X と二項演算 $* : X \to X \mid (x,y) \mapsto x * y$ の組 $(X,*)$ が次を満たす時 $(X,*)$ はカンドルであるという.
$\forall x \in X, x * x = x$
$\forall x,y \in X, \exists! z \in X$ s.t. $z * x = y$
$\forall x,y,z \in X, (x*y)*z = (x*z)*(y*z)$

Riemann 対称空間 (M,g) に対し二項演算を $* : M \times M | (x,y) \mapsto x * y := s_y(x)$ で定義する. この時 $(M,*)$ はカンドルとなり満たすべき等式は $s_x \circ s_y = s_{s_x(y)} \circ sx$ である. 以下カンドルを $(X,*)$ として, 2 点等質カンドルを定義する. まず各 $x \in X$

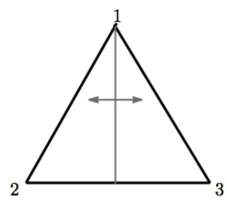
に対し, $*x : X \to X \mid y \mapsto y * x$ とし, $\mathrm{Inn}(X,*) := \langle *x (x \in X) \rangle$ とする (積は $(*x)\cdot(*y) := *x \circ *y$ とする). また任意の $x \in X$ に対して, $\mathrm{Inn}(X,*)$ の部分群 G_x を $\{f \in \mathrm{Inn}(X,*) \mid f(x) = x\}$ とする.

定義 1.4. 次が $(X,*)$ に対して成立するとき $(X,*)$ は 2 点等質カンドルであるという.

$$\forall (x_1,x_2),(y_1,y_2) \in X \times X \ (x_1 \neq x_2, y_1 \neq y_2), \exists f \in \mathrm{Inn}(X,*) \text{ s.t. } f(x_i) = y_i \ (i = 1,2)$$

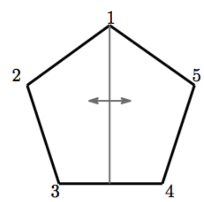
この例として, 位数 3 の二面体カンドルが挙げられる (位数が 4 以上の二面体カンドルの場合は 2 点等質カンドルにはならない). 位数 n の二面体カンドルとは各要素を正 n 角形の各頂点に対応させ, $*x$ を x に対し対称に変換させる作用として二項演算 $*$ を定めたものである (下図は $*1$ の作用).

$n = 3$



$*1 = (2,3)$
$*2 = (1,3)$
$*3 = (1,2)$

$n = 5$



$*1 = (2,5)(3,4)$
$*2 = (1,3)(4,5)$
$*3 = (1,5)(2,4)$
$*4 = (1,2)(3,5)$
$*5 = (1,4)(2,3)$

まとめると以下のようになる.

{ Riemann 対称空間 } $\supset$ { 2 点等質 Riemann 多様体 }
$\cap$
{ カンドル } $\supset$ { 2 点等質カンドル }

注意 1.1. 2 点等質 Riemann 多様体において f は等長変換全体からとるため, 2 点等質 Riemann 多様体は必ずしも 2 点等質カンドルになるとは限らない.

以下カンドルに関するいくつかの定義を述べる. ここで述べる以外のカンドルに関する基礎的な部分については [3, 4, 5] を参照されたい.

定義 1.5. $(X,*), (X',*')$ をカンドルとする. 写像 $f : X \to X'$ が任意の $x, y \in X$ に対し, $f(x * y) = f(x) *' f(y)$ を満たす時, **カンドル準同型**であるという. また全単射なカンドル準同型が存在する時, 二つのカンドルは**同型**であるという.

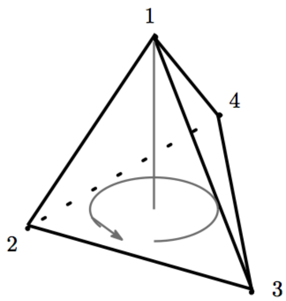
定義 1.6. カンドル $(X,*)$ に対し, 二項演算 $\bar{*} : X \times X \to X$ を $x \bar{*} y = z$ で定義する. 但し z は $x = z * y$ を満たす元である. $(X, \bar{*})$ はカンドルとなりこれを**双対カンドル**という.

定義 1.7. カンドル $(X,*)$ に対し, $\mathrm{Inn}(X,*)$ が X に推移的に作用する時 X は**連結**であるという.

定義 1.8. 位数 $n \geq 3$ のカンドル $(X,*)$ の任意の元 $x \in X$ に対し, $*x$ の $X - \{x\}$ への作用が位数 $n-1$ の巡回置換として作用するとき, $(X,*)$ は**巡回型**であるという.

四面体カンドルは巡回型である. これは正四面体の各頂点に元を割り当て, 各々の作用 $*x$ が他の頂点を反時計周りに置換するものとして二項演算を与えたカンドルである (下

図は $*1$ の作用).



$$*1 = (2,3,4)$$
$$*2 = (1,4,3)$$
$$*3 = (1,2,4)$$
$$*4 = (1,3,2)$$

2　2 点等質カンドルと巡回型カンドル

ここでは 2 点等質カンドルと巡回型カンドルとの関係を述べる. 定義から直ちに次のことが導かれる.

命題 2.1. *2 点等質カンドルは連結である.*

命題 2.2. *カンドル $(X,*)$ に対し, 2 点等質カンドルであることと任意の $x \in X$ に対して, G_x が $X - \{x\}$ に推移的に作用することは同値である.*

命題 2.2 から次の命題が成立する.

命題 2.3. *巡回型カンドルは 2 点等質カンドルである.*

逆が成立するかどうかはまだ一般には知られていないが, ある仮定を付け加えると成立する. それを以下で述べる.

命題 2.4. *位数 $p+1$ の 2 点等質カンドルは巡回型カンドルである. ここで p は素数とする.*

証明　$x \in X \subset \mathrm{Inn}(X,*)$ に対し, $\langle *x \rangle$ による $X - \{x\}$ の軌道分解を施すと,

$$X - \{x\} = \coprod_{i=1}^{m} \langle *x \rangle \cdot y_i$$

となる. よって $m = 1$ を示せばよい.
まず $f \in G_x$ に対して, $f \circ (*x) = (*x) \circ f$ が成立する. よって各軌道 $\langle *x \rangle \cdot y_i$ と $f \in G_x$ について $f(\langle *x \rangle \cdot y_i) = \langle *x \rangle \cdot f(y_i)$ となる. また命題 2.2 から G_x は $X - \{x\}$ に推移的に作用するのため, 各軌道は G_x によって移り合い軌道の位数は全て一致する. よって位数に関して $\#(X - \{x\}) = m\#(\langle *x \rangle \cdot y_1) = p$ が成立する. 一般の連結なカンドルに対し各 $*x$ は恒等写像とならないことが知られているため, 命題 2.1 より $*x \neq \mathrm{id}$ となり, $\#(\langle *x \rangle \cdot y_1) \neq 1$ が成り立つ. 今 p は素数なのでこれらのことから $m = 1$ となる. □

これらの命題から次の問題が考えられる.

問題: 全ての 2 点等質カンドルは巡回型か?

この問題は [2] において位数が素数の場合に解決されている. その場合 2 点等質カンドル

は必ず存在し, 全て巡回型となる (ある種の線型アレクサンダーカンドルで具体的に与えられる).

3　巡回型カンドルの分類

この章では, 巡回型カンドルの分類定理と位数が 12 までの分類結果を紹介する. A_n を位数 n の巡回型カンドルの同型類の集合とする. S_n を n 次対称群, また s_1 を $(2,\ldots,n) \in S_n$ なる $n-1$ 次巡回置換とする. 次の条件を満たす $n-1$ 次巡回置換 s の集合を D_n と置く.

1. $s(2)=2$
2. $\{s^m s_1 s^{-m} \mid m=1,2,...,n-2\} = \{s_1^m s s_1^{-m} \mid m=1,2,...,n-2\}$
3. s は $n-1$ 次巡回置換である.

定理 3.1. (**Tamaru**)
A_n と D_n は一対一に対応する.

また D_n から対応する巡回型カンドルを簡単に構成することができる.
以下に巡回型カンドルの D_n による分類表を与える.

n	D_n
3	$\{(1,3)\}$
4	$\{(1,4,3)\}$
5	$\{(1,3,5,4),(1,4,3,5)\}$
6	$\emptyset$
7	$\{(1,7,4,6,5,3),(1,7,5,4,6,3)\}$
8	$\{(1,5,8,3,7,6,4),(1,7,5,4,8,3,6)\}$
9	$\{(1,4,3,8,6,9,5,7),(1,5,7,3,6,4,9,8)\}$
10	$\emptyset$
11	$\{(1,3,6,8,4,11,5,10,9,7),(1,4,3,7,10,5,11,9,6,8),$ $(1,6,8,5,3,9,4,7,11,10),(1,7,5,4,9,3,10,6,8,11)\}$
12	$\emptyset$

巡回型の定義からすぐに分かるように, それらの双対もまた巡回型になる. この表においては $n=3,4$ の場合を除いて, 自己双対となる巡回型カンドルは存在しないことが分かっている.

参考文献

[1] D. Joyce, *A classification invariant of knots, the knot quandle,* J. Pure Appl. Algebra **23** (1982) 37–65.
[2] H. Tamaru, *Two-point homogeneous quandles with prime cardinally,* preprint.
[3] R. Fenn, C. Rourke, *Racks and links in codimention two,* J. Knot Theory Ramifications, **10** (1992) 343–406.

[4] S. Kamada, K. Oshiro, *Homology groups of symmetric quandles and cocycle invariants of links and surface-links,* Amer. Math. Soc. **362** (2010), 5501–5527.

[5] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc., **355** (2003), no. 10, 3947–3989.