

Strict Solution Method for Linear Programming Problem with Ellipsoidal Distributions under Fuzziness

Takashi Hasuike, Hideki Katagiri

Graduate School of Information Science and Technology, Osaka University, Japan
2-1 Yamadaoka, Suita, Osaka 565-0871, Japan
email:thasuike@osaka-u.ac.jp

Graduate School of Engineering, Hiroshima University, Japan
1-4-1 Kagamiyama, Higashi-Hiroshima, Hiroshima, Japan
email:katagiri-h@hiroshima-u.ac.jp

Abstract — This paper considers a linear programming problem with ellipsoidal distributions including fuzziness. Since this problem is not well-defined due to randomness and fuzziness, it is hard to solve it directly. Therefore, introducing chance constraints, fuzzy goals and possibility measures, the proposed model is transformed into the deterministic equivalent problems. Furthermore, since it is difficult to solve the main problem analytically and efficiently due to nonlinear programming, the solution method is constructed introducing an appropriate parameter and performing the equivalent transformations.

1. GENERAL INTRODUCTIONS

In real-world decision making, one often needs to make an optimal decision under uncertainty. Stochastic programming (for example, Beale [1], Charnes and Cooper [6], Dantzig [7]) and fuzzy programming (For example, Dubois and Prade [8], Inuiguchi and Tanino [14]) have been developed as useful tools for decision makers to determine an optimal solution. Furthermore, decision makers are faced with environments including both randomness and fuzziness. In order to construct a framework of decision making models under such stochastic and fuzzy environments, fuzzy random variables (Kwakernaak [17], Puri and Ralescu [22]) and random fuzzy variable (Liu [19, 20]) have been brought to the attention of researchers.

In many previous researches, values of parameters such as costs, returns, times, etc. are assumed to be known, and in these cases, main problems are deterministic mathematical programming problems. Therefore, these optimal solutions are analytically obtained using the deterministic mathematical programming. However, decision makers may receive a lot of information and data in the real world. Then, it is almost impossible to estimate strict values of parameters and to determine their random distribution. These distributions may be statistically determined as a confidence interval involving some error. Therefore, using these statistical distributions, it is more important to consider that decision makers optimize the problem in the worst case; i.e. Robust optimization

problem. Recently, the robust optimization problem becomes a more active area of research, and there exist various studies (For example, Ben-Tal and Nemirovski [2, 3], Goldfarb and Iyengar [9]).

On the other hand, it is most important to undertake appropriate risk management such as the reduction of uncertainty and the improvement of satisfaction of decision makers. Therefore, the role of portfolio selection problems, which is mainly focused on the risk aversion, is important. As for the research history on mathematical approach, Markowitz [21] proposed the mean-variance model and it has been central to research activity in the real financial field. Then, there are some basic researches under various uncertainty conditions with respect to portfolio selection problems (Bilbao-Terol et al. [4], Carlsson et al. [5], Guo and Tanaka [10], Huang [11, 12], Inuiguchi et al. [13, 14], Katagiri et al. [15, 16], Tanaka et al. [23, 24], Watada [25]). Furthermore, there are some studies of robust portfolio selection problems determining optimal investment strategy using the robust approach (For example, [9]).

Therefore, by extending risk management methods used the portfolio theory to the general mathematical programming problem, we propose a new and versatile robust programming problem. Until now, there are few models of mathematical programming problems considering both uncertainty and ambiguity, simultaneously. Furthermore, there are no researches which are analytically extended and solved these types of robust programming problems based on the portfolio theory. Particularly, we focus on the probability maximization model. Since our proposal model is not well-defined, in this paper, we transform the main problem into the deterministic equivalent problems and construct the analytical solution method for the fuzzy robust programming problem.

This paper is organized as follows. In Section 2, we introduce and formulate a basic linear programming problem based on the robust programming problem with uncertainty considering the portfolio theory. In Section 3, introducing

fuzzy numbers to uncertainty sets of parameters, we propose fuzzy extension models of robust linear programming problems and construct the analytical solution method. Finally, in Section 4, we conclude this paper and discuss future research problems.

2. FORMULATION OF ROBUST OPTIMIZATION PROBLEMS WITH ELLIPSOIDAL DISTRIBUTIONS

In this section, we consider a basic linear programming problem and their robust models with ellipsoidal distributions. First of all, we introduce the following linear programming problem:

$$\begin{aligned} & \text{Maximize } \mathbf{r}'\mathbf{x} \\ & \text{subject to } \mathbf{x} \in X \triangleq \{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \end{aligned} \quad (1)$$

where notations mean as follows:

\mathbf{r} : n -dimensional column vector

\mathbf{A} : $m \times n$ coefficient matrix

\mathbf{b} : m -dimensional column vector

\mathbf{x} : n -dimensional column vector for decision variables

In the case that all coefficients are constant, this problem is easily and efficiently solved by using basic linear programming approaches such as Simplex method and Interior point method.

However, in real world decision making, it is hard to receive all information and data with respect to future returns and determine the distributions of their random variables. Therefore, in this paper, we consider that parameter \mathbf{r} has some uncertainty and each parameter is included in an uncertainty set. In this case, problem (1) is not the linear programming problem due to uncertainty. Therefore, we need to construct the solution procedure to solve them. In this paper, we formulate the robust portfolio selection problem Ben-tal and Nemirovski [2] have proposed. We formulate the robust problem as follows:

$$\begin{aligned} & \text{Maximize } \min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} \\ & \text{subject to } \mathbf{x} \in X \end{aligned} \quad (2)$$

where $M_d \subset \mathbb{R}^n$ is the uncertainty set. This problem is not well defined without defining uncertainty sets. Therefore, we first assume the uncertainty set of \mathbf{r} to be the following ellipsoidal set:

$$M_d \triangleq \left\{ \mathbf{r} \mid (\mathbf{r} - \mathbf{r}_0)' G (\mathbf{r} - \mathbf{r}_0) \leq d^2 \right\} \quad (3)$$

where \mathbf{r}_0 is the n -dimensional column vector for the center value of ellipsoidal set and $G \in \mathbb{R}^{n \times n}$ is the symmetric positive definite matrix. Then, d is the constant positive value decided by the decision maker. In this case that constant value of parameter d is larger, the region of ellipsoidal set or ellipsoidal distribution is also wide. Furthermore, even if d is much large, it is an useful and robust decision making that

$\min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x}$ is larger than the target value f , i.e.

$\min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} \geq f$. Therefore, we transform problem (2) and consider the following problem similar to probability maximization model:

$$\begin{aligned} & \text{Maximize } d \\ & \text{subject to } \min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} \geq f, \\ & \mathbf{x} \in X \end{aligned} \quad (4)$$

Subsequently, ellipsoidal set (3) is equivalently transformed into the following form:

$$M \triangleq \left\{ \mathbf{r} \mid \left[\frac{1}{d}(\mathbf{r} - \mathbf{r}_0) \right]' G \left[\frac{1}{d}(\mathbf{r} - \mathbf{r}_0) \right] \leq 1 \right\} \quad (5)$$

Then, using Cholesky decomposition to G , we obtain an upper triangular matrix $G^{1/2}$ satisfying $G = (G^{1/2})' G^{1/2}$

where $(G^{1/2})'$ is the transposed matrix of $G^{1/2}$. Therefore, in problem (4), constraint $\min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} \geq f$ is transformed into

the following form by introducing parameters $\hat{\mathbf{r}}$ and \mathbf{z} :

$$\begin{aligned} \min_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} &= \inf_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} \\ &= \inf_{\|G^{1/2}\hat{\mathbf{r}}\| \leq 1} (\mathbf{r}_0 + d\hat{\mathbf{r}})' \mathbf{x} = \mathbf{r}_0'\mathbf{x} + d \left(\inf_{\|\mathbf{z}\| \leq 1} \mathbf{z}' G^{-1/2} \mathbf{x} \right) \end{aligned} \quad (6)$$

where $\|G^{1/2}\hat{\mathbf{r}}\| = \sqrt{\hat{\mathbf{r}}' G \hat{\mathbf{r}}}$, $\mathbf{z} = G^{1/2}\hat{\mathbf{r}}$, and $G^{-1/2}$ is defined as the inverse matrix of $G^{1/2}$. Therefore, by solving $\inf_{\|\mathbf{z}\| \leq 1} \mathbf{z}' G^{-1/2} \mathbf{x}$ with respect to \mathbf{z} , we easily obtain the

following optimal solution:

$$\mathbf{z}^* = - \frac{G^{-1/2} \mathbf{x}}{\|G^{-1/2} \mathbf{x}\|} \quad (7)$$

Using this optimal solution \mathbf{z}^* , the expression (6) is transformed into the following form:

$$\begin{aligned} \inf_{\{\mathbf{r} \in M_d\}} \mathbf{r}'\mathbf{x} &= \mathbf{r}_0'\mathbf{x} + d \left(\left(- \frac{G^{-1/2} \mathbf{x}}{\|G^{-1/2} \mathbf{x}\|} \right)' (G^{-1/2} \mathbf{x}) \right) \\ &= \mathbf{r}_0'\mathbf{x} - d \|G^{-1/2} \mathbf{x}\| \end{aligned} \quad (8)$$

Consequently, main problem (4) is equivalently transformed into the following problem:

$$\begin{aligned} & \text{Maximize } d \\ & \text{subject to } \mathbf{r}_0'\mathbf{x} - d \|G^{-1/2} \mathbf{x}\| \geq f, \\ & \mathbf{x} \in X \end{aligned} \quad (9)$$

In this problem, constraint $\mathbf{r}_0\mathbf{x} - d\|G^{-1/2}\mathbf{x}\| \geq f$ is transformed

into $\frac{\mathbf{r}_0\mathbf{x} - f}{\|G^{-1/2}\mathbf{x}\|} \geq d$, and so problem (9) is equivalently

transformed into the following problem:

$$\begin{aligned} & \text{Maximize } \frac{\mathbf{r}_0\mathbf{x} - f}{\|G^{-1/2}\mathbf{x}\|} \\ & \text{subject to } \mathbf{x} \in X \end{aligned} \quad (10)$$

These problems are convex programming problems similar to the probability maximization model in the case $\mathbf{r}_0\mathbf{x} - f > 0$, and so we obtain each optimal solution using the convex programming approach.

3. FUZZY EXTENSION OF ROBUST MEAN VARIANCE OPTIMIZATION PROBLEMS

In Section 2, we consider that each parameter in the ellipsoidal set is fixed value. However, in real world decision making, there exist various types of effective and ineffective information, and each investor has an institution with respect to the real world. These factors include ambiguity and so we need to consider a robust portfolio selection problem including ambiguity. In this paper, we assume \mathbf{r}_0 to include ambiguity and to be a fuzzy number. Therefore, uncertainty set (7) is redefined into the following form:

$$\tilde{M}_d \triangleq \left\{ \mathbf{r} \mid (\mathbf{r} - \tilde{\mathbf{r}}_0)' G (\mathbf{r} - \tilde{\mathbf{r}}_0) \leq d^2 \right\} \quad (11)$$

Then, in this paper, the fuzzy number $\tilde{\mathbf{r}}_0$ is assumed to be a following L-shape fuzzy number:

$$\mu_{\tilde{r}_{0j}}(\omega) = \begin{cases} \max \left\{ 0, L \left(\frac{\bar{r}_{0j} - \omega}{\alpha_j} \right) \right\}, & (\omega \leq \bar{r}_{0j}) \\ \max \left\{ 0, L \left(\frac{\omega - \bar{r}_{0j}}{\alpha_j} \right) \right\}, & (\bar{r}_{0j} \leq \omega) \end{cases} \quad (12)$$

where $L(x)$ is the reference function and continuously decreasing, and $L(0) = 1, L(1) = 0$. The uncertainty set $\tilde{U}_r = (\mathbf{r} - \tilde{\mathbf{r}}_0)' G (\mathbf{r} - \tilde{\mathbf{r}}_0)$ includes fuzzy numbers vector $\tilde{\mathbf{r}}_0$ and so \tilde{U}_r is a fuzzy number. Therefore, the membership function of \tilde{U}_r is as follows:

$$\mu_{\tilde{U}_r}(\bar{u}_r) = \sup_{\gamma_0} \left\{ \min_{1 \leq j \leq n} \mu_{\tilde{r}_{0j}}(\gamma_{0j}) \mid \bar{u}_r = (\mathbf{r} - \gamma_0)' G (\mathbf{r} - \gamma_0) \right\} \quad (13)$$

Consequently, uncertainty set \tilde{M}_d is represented as a fuzzy set characterized by the following membership function:

$$\begin{aligned} \mu_{\tilde{M}_d}(d) &= \sup_{\bar{u}_r} \left\{ \mu_{\tilde{U}_r}(\bar{u}_r) \mid \bar{u}_r \leq d^2 \right\} \\ &= \sup_{\gamma_0} \min_{1 \leq j \leq n} \left\{ \mu_{\tilde{r}_{0j}}(\gamma_{0j}) \mid (\mathbf{r} - \gamma_0)' G (\mathbf{r} - \gamma_0) \leq d^2 \right\} \end{aligned} \quad (14)$$

Furthermore, taking account of the vagueness of human judgment and flexibility for the execution of a plan, we give a fuzzy goal to the target probability as the fuzzy set characterized by a membership function. In this subsection, we consider the fuzzy goal of target level d for probability which is represented by,

$$\mu_{\tilde{G}}(\omega) = \begin{cases} 1 & (d_U \leq \omega) \\ g_d(\omega) & (d_L \leq \omega < d_U) \\ 0 & (\omega < d_L) \end{cases} \quad (15)$$

where $g_d(\omega)$ is a strictly increasing continuous function, and d_L and d_U are lower and upper constant values set by the decision maker, respectively. Then, using a concept of possibility measure, we introduce the degree of possibility as follows:

$$\Pi_{\tilde{M}_d}(\tilde{G}) = \sup_d \min \left(\mu_{\tilde{M}_d}(d), \mu_{\tilde{G}}(d) \right) \quad (16)$$

Therefore, by introducing a parameter of satisfaction level h , uncertainty set (11) is transformed into the following form using the h -cut:

$$M_d(h) \triangleq \left\{ \mathbf{r} \mid \Pi_{\tilde{M}_d}(\tilde{G}) \geq h \right\} \quad (17)$$

Consequently, the main problem (4) is reformulated the following possibility maximization model:

$$\begin{aligned} & \text{Maximize } h \\ & \text{subject to } \min_{\mathbf{r} \in M_d(h)} \mathbf{r}' \mathbf{x} \geq f, \\ & \mathbf{x} \in X \end{aligned} \quad (18)$$

Subsequently, we equivalently transform $\mu_{\tilde{M}_d}(d) \geq h$ and obtain the following inequality:

$$\begin{aligned} & \mu_{\tilde{M}_d}(d) \geq h \\ & \Leftrightarrow \exists d : \sup_{\gamma_0} \min_{1 \leq j \leq n} \left\{ \mu_{\tilde{r}_{0j}}(\gamma_{0j}) \mid (\mathbf{r} - \gamma_0)' G (\mathbf{r} - \gamma_0) \leq d^2 \right\} \geq h, \\ & \Leftrightarrow \exists d : \mathbf{r}' G \mathbf{r} - 2\mathbf{r}' G (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}) \\ & \quad + (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha})' G (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}) \leq d^2 \\ & \Leftrightarrow \left(\frac{1}{d} \mathbf{r} \right)' G \left(\frac{1}{d} \mathbf{r} \right) - 2 \left(\frac{1}{d} \mathbf{r} \right)' G \left(\frac{1}{d} (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}) \right) \\ & \quad + \left(\frac{1}{d} (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}) \right)' G \left(\frac{1}{d} (\bar{\mathbf{r}}_0 - L^*(h)\boldsymbol{\alpha}) \right) \leq 1 \end{aligned} \quad (19)$$

where $L^*(h)$ is the pseudo inverse function of $L(\omega)$. Using this inequality, the expression (6) is transformed into the following expression:

$$\begin{aligned}
\min_{\{r \in M_d\}} r' \mathbf{x} &= \inf_{\{r \in M_d\}} r' \mathbf{x} \\
&= \inf_{\|G^{-1/2} \mathbf{r}\| \leq 1} \left((\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha}) + d \hat{\mathbf{r}} \right)' \mathbf{x} \\
&= (\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} + d \left(\inf_{\|z\| \leq 1} z' G^{-1/2} \mathbf{x} \right)
\end{aligned} \tag{20}$$

Then, $\inf_{\|z\| \leq 1} z' G^{-1/2} \mathbf{x}$ in expression (20) is equal to that in expression (6), and from the optimal value of (7), this expression is equal to the following form:

$$\inf_{\{r \in M_d\}} r' \mathbf{x} = (\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - d \|G^{-1/2} \mathbf{x}\| \tag{21}$$

Consequently, in the case that we consider the possibility measure constraint $\Pi_{\tilde{M}_d}(\tilde{G}) \geq h$, this constraint is transformed into the following inequality:

$$\begin{aligned}
\Pi_{\tilde{M}_d}(\tilde{G}) &\geq h, \\
\Leftrightarrow \sup_d \min(\mu_{\tilde{M}_d}(d), \mu_{\tilde{G}}(d)) \\
\Leftrightarrow (\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - d \|G^{-1/2} \mathbf{x}\| &\geq f, \quad d \geq g_d^{-1}(h) \\
\Leftrightarrow \frac{(\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - f}{\|G^{-1/2} \mathbf{x}\|} &\geq d, \quad d \geq g_d^{-1}(h) \\
\Leftrightarrow \frac{(\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - f}{\|G^{-1/2} \mathbf{x}\|} &\geq g_d^{-1}(h)
\end{aligned} \tag{22}$$

Subsequently, we assume that $(\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - f$ is positive on satisfaction level h where $0 \leq h \leq 1$. Then, using this transformation, the proposed fuzzy robust programming problem (18) is equivalently transformed into the following problem:

$$\begin{aligned}
&\text{Maximize } h \\
&\text{subject to } \frac{(\bar{\mathbf{r}}_0 - L^*(h) \boldsymbol{\alpha})' \mathbf{x} - f}{\|G^{-1/2} \mathbf{x}\|} \geq g_d^{-1}(h), \\
&\mathbf{x} \in X
\end{aligned} \tag{23}$$

It should be noted here that problem (23) is a nonconvex programming problem due to nonlinear functions $L^*(h)$ and $g_d^{-1}(h)$, and so it cannot be solved by any linear programming techniques or convex programming techniques. However, if we fix decision variable h as $h = q$ and introduce the following auxiliary problem;

$$\begin{aligned}
&\text{Maximize } \frac{(\bar{\mathbf{r}}_0 - L^*(q) \boldsymbol{\alpha})' \mathbf{x} - f}{\|G^{-1/2} \mathbf{x}\|} \\
&\text{subject to } \mathbf{x} \in X
\end{aligned} \tag{24}$$

This problem is equivalent to previous problem (10). Then, with respect to the relation between problem (23) and the auxiliary problem (24), the following theorem holds based on the previous study (e.g. [15]).

Theorem 1

Let $\mathbf{x}(q)$ and $Z(q)$ be an optimal solution of problem (24) and its optimal value, respectively. Then, for q satisfying $0 < q < 1$, $Z(q)$ is a strictly increasing function of q .

Theorem 2

Let \hat{q} denote q satisfying $Z(\hat{q}) = g^{-1}(\hat{q})$ and the optimal solutions of main problem (23) be (\mathbf{x}^*, h^*) . Then $(\mathbf{x}(\hat{q}), \hat{q})$ is equal to (\mathbf{x}^*, h^*) in $0 < \hat{q} < 1$.

From these theorems, by using bisection algorithm for parameter q and comparing objective function $Z(q)$ with $g_d^{-1}(q)$, we repeatedly solve problem (24) for each q using branch-and-bound method, and finally obtain the optimal solution. Consequently, we develop the following strict solution method.

Solution method

STEP1: Elicit the membership function of a fuzzy goal with respect to the total expected return and variance.

STEP2: Set $h \leftarrow 1$ and solve problem (24). If a feasible solution \mathbf{x} exists, then terminate. In this case, the obtained current solution is an optimal solution of main problem.

STEP3: Set $h \leftarrow 0$ and solve problem (24). If a feasible solution \mathbf{x} does not exist, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal with respect to the total expected return and variance.

STEP4: Set $h_L \leftarrow 0$ and $h_U \leftarrow 1$.

STEP5: Set $h \leftarrow \frac{h_L + h_U}{2}$

STEP6: Solve problem (24) and find the optimal solution $\mathbf{x}(h)$. Then, if $|h_U - h_L| < \varepsilon$ holds with respect to a sufficiently small number ε , $\mathbf{x}(h)$ is the optimal

solution of main problem (18), and terminate this algorithm. If not, go to Step 7.

STEP7: If an optimal solution exists, then set $h_L \leftarrow h$ and return to Step 5. If not, then set $h_U \leftarrow h$ and return to Step 5.

It is surely possible that we find an optimal solution of problem (24) for each value of parameter h . Furthermore, in the special case the positive definite matrix G^{-1} is assumed to be a variance-covariance matrix \mathbf{V} , we obtain the optimal solution more efficiently.

Subsequently, as an approximate function for $\|G^{-1/2}\mathbf{x}\| = \sqrt{\mathbf{x}'G^{-1}\mathbf{x}} = \sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}}$, we introduce the following mean absolute deviation:

$$\begin{aligned} W(\mathbf{x}) &= E \left\| \sum_{j=1}^n r_j^{(g)} x_j - \sum_{j=1}^n \bar{r}_j^{(g)} x_j \right\| \\ &= \sum_{t=1}^T p_t \left\| \sum_{j=1}^n (r_{ij}^{(g)} - \bar{r}_{ij}^{(g)}) x_j \right\| \end{aligned} \quad (25)$$

where $\mathbf{r}_t^{(g)} = \{r_{t1}^{(g)}, r_{t2}^{(g)}, \dots, r_{tm}^{(g)}\}$, ($t=1, 2, \dots, T$) is the discrete distribution to random variable \mathbf{r} based on the uncertainty set (3), and $\bar{r}_j^{(g)}$ is the arithmetic mean. Then, p_t is each occurrence probability of $\mathbf{r}_t^{(g)}$. In the case that \mathbf{V} is a variance-covariance matrix derived from a normal distribution, it was shown that $\mathbf{x}'\mathbf{V}\mathbf{x} = \frac{\pi}{2} \{W(\mathbf{x})\}^2$ by the previous study [18]. Therefore, absolute deviation $W(\mathbf{x})$ is considered to be an approximate function to the quadratic function. Using this mean absolute deviation, problem (24) is approximately transformed into the following problem;

$$\begin{aligned} &\text{Maximize } \frac{(\bar{r}_0 - L^*(q)\boldsymbol{\alpha})' \mathbf{x} - f}{\sqrt{\frac{\pi}{2} W(\mathbf{x})}} \\ &\text{subject to } \mathbf{x} \in X \end{aligned} \quad (26)$$

Furthermore, by introducing the parameter ξ_t , problem (26) is equivalently transformed into the following problem based on the study of Konno [18];

$$\begin{aligned} &\text{Maximize } \frac{(\bar{r}_0 - L^*(q)\boldsymbol{\alpha})' \mathbf{x} - f}{\sum_{t=1}^T p_t \xi_t} \\ &\text{subject to } \xi_t \pm (r_{ij}^{(g)} - \bar{r}_{ij}^{(g)}) x_j \geq 0, \quad (t=1, 2, \dots, T) \\ &\quad \mathbf{x} \in X \end{aligned} \quad (27)$$

Problem (27) is also a basic fractional linear programming

problem and it can be equivalently transformed into the following linear programming problem by introducing

$$\text{parameter } \eta = \frac{1}{\sum_{t=1}^T p_t \xi_t}, \quad \mathbf{x}' = \eta \mathbf{x}, \quad \xi_t' = \eta \xi_t :$$

$$\begin{aligned} &\text{Maximize } (\bar{r}_0 - L^*(q)\boldsymbol{\alpha})' \mathbf{x}' - f \eta \\ &\text{subject to } \sum_{t=1}^T p_t \xi_t' = 1, \\ &\quad \xi_t' \pm (r_{ij}^{(g)} - \bar{r}_{ij}^{(g)}) x_j' \geq 0, \quad (t=1, 2, \dots, T) \\ &\quad \mathbf{x}' \in X' \triangleq \{\mathbf{x}' \mid \mathbf{A}\mathbf{x}' \leq \mathbf{b}\eta, \mathbf{x}' \geq \mathbf{0}\} \end{aligned} \quad (28)$$

Therefore, we obtain an optimal portfolio more efficiently than the proposed standard approach. Consequently, using a bisection algorithm with respect to h , we construct the following solution method.

Efficient Solution method to the special case

STEP0: Set a discrete distribution $\mathbf{r}_t^{(g)}$, ($t=1, 2, \dots, T$) to random variable \mathbf{r} and the occurrence probability p_t .

STEP1: Elicit the membership function of a fuzzy goal with respect to the total expected return and variance.

STEP2: Set $h \leftarrow 1$ and solve problem (28). If a feasible solution \mathbf{x} exists, then terminate. In this case, the obtained current solution is an optimal solution of main problem.

STEP3: Set $h \leftarrow 0$ and solve problem (28). If a feasible solution \mathbf{x} does not exist, then terminate. In this case, there is no feasible solution and it is necessary to reset a fuzzy goal with respect to the total expected return and variance.

STEP4: Set $h_L \leftarrow 0$ and $h_U \leftarrow 1$.

STEP5: Set $h \leftarrow \frac{h_L + h_U}{2}$

STEP6: Solve problem (28), and find the optimal solution $\mathbf{x}(h)$. Then, if $|h_U - h_L| < \varepsilon$ holds with respect to a sufficiently small number ε , $\mathbf{x}(h)$ is the optimal solution of main problem (17), and terminate this algorithm. If not, go to Step 7.

STEP7: If an optimal solution exists, then set $h_L \leftarrow h$ and return to Step 5. If not, then set $h_U \leftarrow h$ and return to Step 5.

4. CONCLUSION

In this paper, we have proposed an extension model of robust linear programming problems considering uncertainty

conditions with ellipsoidal distribution and fuzziness, particularly, the probability maximization model. Since this problem is not well defined due to fuzzy numbers, we have introduced the degree of possibility and transformed the main problem into the deterministic equivalent problem. Furthermore, to solve the special case with variance-covariance matrix efficiently, we have constructed the efficient solution method by using the mean-absolute deviation. Our proposed models include the other robust practical problems and so we may apply our models to the more flexible and complex problems in real world decision making than the previous models.

As the future studies, we are now attacking the cases that optimal solutions are restricted to be integers and multi-period models.

REFERENCES

- [1] E. M. L. Beale, "On optimizing a convex function subject to linear inequalities", *Journal of the Royal Statistical Society* **17**, pp.173-184, 1955.
- [2] A. Ben-Tal and A. Nemirovski, "Robust solutions of uncertain linear programs", *Operations Research Letters* **25**(1), pp.1-13, 1999.
- [3] A. Ben-Tal and A. Nemirovski, "Lectures on modern convex optimization", *SIAM*, Philadelphia, PA, 2001.
- [4] A. Bilbao-Terol, B. Perez-Gladish, M. Arenas-Parra and M.V. Rodriguez-Uria, "Fuzzy compromise programming for portfolio selection", *Applied Mathematics and Computation* **173**, pp.251-264, 2006.
- [5] C. Carlsson, R. Fuller and P. Majlender, "A possibilistic approach to selecting portfolios with highest utility score", *Fuzzy Sets and Systems* **131**, pp.12-21, 2002.
- [6] A. Charnes and W. W. Cooper, "Deterministic equivalents for optimizing and satisfying under chance constraints", *Operations Research* **11**, pp.18-39, 1955.
- [7] G. B. Dantzig, "Linear programming under uncertainty", *Management Science* **1**, pp.197-206, 1955.
- [8] D. Dubois and H. Prade, *Fuzzy Sets and Systems*, New York: Academic Press, 1980.
- [9] D. Goldfarb and G. Iyengar, "Robust portfolio selection problems", *Mathematics of Operations Research* **28**, pp.1-38, 2003.
- [10] P. Guo and H. Tanaka, "Possibility data analysis and its application to portfolio selection problems", *Fuzzy Economic Rev.* **3**, pp.3-23, 1998.
- [11] X. Huang, "Fuzzy chance-constrained portfolio selection", *Applied Mathematics and Computation* **177**, pp.500-507, 2006.
- [12] X. Hung, "Two new models for portfolio selection with stochastic returns taking fuzzy information", *European Journal of Operational Research* **180**, pp.396-405, 2007.
- [13] M. Inuiguchi, J. Ramik, "Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem", *Fuzzy Sets and Systems* **111**, pp.3-28, 2000.
- [14] M. Inuiguchi and T. Tanino, "Portfolio selection under independent possibilistic information", *Fuzzy Sets and Systems* **115**, pp.83-92, 2000.
- [15] H. Katagiri, H. Ishii and M. Sakawa, "On fuzzy random linear knapsack problems", *Central European Journal of Operations Research* **12**(1), pp.59-70, 2004.
- [16] H. Katagiri, M. Sakawa and H. Ishii, "A study on fuzzy random portfolio selection problems using possibility and necessity measures", *Scientiae Mathematicae Japonicae* **65**(2), pp.361-369, 2005.
- [17] H. Kwakernaak, "Fuzzy random variable-1", *Information Sciences* **15**, pp.1-29, 1978.
- [18] H. Konno, "Piecewise linear risk functions and portfolio optimization", *Journal of Operations Research Society of Japan* **33**, pp.139-156, 1990.
- [19] B. Liu, *Theory and Practice of Uncertain Programming*, Physica Verlag, 2002.
- [20] B. Liu, *Uncertainty theory*, Physica Verlag, 2004.
- [21] H. Markowitz, *Portfolio Selection*, New York: Wiley, 1952.
- [22] M. L. Puri and D. A. Ralescu, "Fuzzy random variables", *Journal of Mathematical Analysis and Applications* **14**, pp.409-422, 1986.
- [23] H. Tanaka and P. Guo, "Portfolio selection based on upper and lower exponential possibility distributions", *European Journal of Operational Researches* **114**, pp.115-126, 1999.
- [24] H. Tanaka, P. Guo and I. B. Turksen, "Portfolio selection based on fuzzy probabilities and possibility distributions", *Fuzzy Sets and Systems* **111**, pp.387-397, 2000.
- [25] J. Watada, "Fuzzy portfolio selection and its applications to decision making", *Tatra Mountains Math. Pub.* **13**, pp.219-248, 1997.