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Dualisation with Respect to Restricted s-Tuples

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Dualisation with respect to restricted s-tuples for constructions of partially balanced incomplete block designs is proposed. As a by-product, balanced incomplete block designs are also obtained.

Keywords: Balanced incomplete block (BIB) design, partially balanced incomplete block (PBIB) design, t-design, quasi-symmetric, dual.

1. Introduction

Bose and Nair (1939) first introduced the concept of "dualisation" in the field of design of experiments. They derived a new class of block designs by interchanging the role of treatments and blocks in a block design $\mathcal{D}$. This is called a dual of $\mathcal{D}$ and denoted by $\mathcal{D}_1^*$. This dualisation, namely, writing the block numbers of blocks in which a treatment occurs in the original design, is extended to another concept as writing the block numbers of blocks in which a pair of treatments occurs in the original design. This is named as "dualisation with respect to pairs", denoted by $\mathcal{D}_2^*$ for a given block design $\mathcal{D}$, and is dealt with in Vanstone (1975) and Mohan and Kageyama (1983).

Kageyama and Mohan (1984) generalized the concept of "dualisation with respect to pairs" to "dualisation with respect to s-tuples" for any $s \geq 1$. Kageyama et al. (1995) and Philip et al. (1997) used the concept of "restricted dualisation" to construct some nested balanced incomplete block (BIB) designs and partially balanced incomplete block (PBIB) designs. In this paper, we introduce the concept of "dualisation with respect to restricted s-tuples" for any $s \geq 1$ to construct PBIB designs. This dual design is here denoted by $\mathcal{D}_s^{**}$. Hence this paper forms a companion of Kageyama and Mohan (1984).

A BIB design is an arrangement of v treatments into b blocks such that

(1) each block contains $k(< v)$ distinct treatments,
(2) each treatment appears in r different blocks,
(3) every pair of distinct treatments appears together in exactly λ different blocks.

Here, the parameters v, b, r, k, λ are related by identities $vr = bk$ and $\lambda(v-1) = r(k-1)$.

A t-(v, k, λ_t) design (or simply t-design) is a system with v treatments and b blocks containing k distinct treatments, each treatment contained in r different blocks and every t distinct treatments are contained in exactly λ_t different blocks. For a t-design $\lambda_t\binom{v}{t} = b\binom{k}{t}$ and for each $0 \leq s \leq t$, every t-(v, k, λ_t) design is an s-(v, k, λ_s) design with

$$\lambda_s = \lambda_t\binom{v-s}{t-s}\Big/\binom{k-s}{t-s},$$

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where $\binom{n}{r}$ is a binomial coefficient and $\binom{n}{r} = 0$ if $n < r$. Here, $\lambda_0 = b$ and $\lambda_1 = r$. A 2-design coincides with a BIB design with parameters $v, b, r, k, \lambda(= \lambda_2)$.

For $t \geq 2$, a t-design is said to be quasi-symmetric if any of its two blocks have either x or y common treatments, where $x \neq y$, namely, the number of treatments incident with two blocks takes just two distinct values.

For the technical terms like association schemes and PBIB designs, we refer the reader to Raghavarao (1971).

Lemma 1.1. If $\mathcal{D}$ is a quasi-symmetric BIB design with parameters v, b, r, k, λ, then among the r blocks which contain a particular treatment, say θ, each block has t_1 blocks having $x - 1$ treatments in common with it other than θ and t_2 blocks having $y - 1$ treatments in common with it other than θ, where

$$t_1 = \frac{(y-1)(r-1) - (k-1)(\lambda-1)}{y-x}, \tag{1.1}$$

$$t_2 = \frac{(k-1)(\lambda-1) - (x-1)(r-1)}{y-x}. \tag{1.2}$$

Proof. Let the r blocks of $\mathcal{D}$ which contain θ be denoted by $B_1, B_2, ..., B_r$. Without loss of generality, suppose that B_1 has t_1 blocks having $x - 1$ treatments in common with it other than θ and t_2 blocks having $y - 1$ treatments in common with it other than θ. Then among B_i's it clearly holds that

$$t_1 + t_2 = r - 1, \tag{1.3}$$

$$(x-1)t_1 + (y-1)t_2 = (k-1)(\lambda-1). \tag{1.4}$$

Solving (1.3) and (1.4) for t_1 and t_2 we can obtain their values as in (1.1) and (1.2), respectively. □

2. Method

We consider here an equireplicate and proper block design $\mathcal{D}$ in which the number of treatments (with the replication number r) is v and the number of blocks of size k each is b. The present method is described as follows. Number the r blocks of a given block design $\mathcal{D}$. Now, in $\mathcal{D}_s^{**}$ if the ith block of $\mathcal{D}$ includes an s-tuple containing a fixed treatment, say θ, then the corresponding block of $\mathcal{D}_s^{**}$ will have the ith treatment of $\mathcal{D}_s^{**}$.

For a given block design $\mathcal{D}$ with parameters v, b, r, k, it is obvious that its dual design $\mathcal{D}_s^{**}$ with respect to restricted s-tuples for $s < k$ is characterized by the parameters in the following form

$v^{**} = r$, $b^{**} = \binom{v-1}{s-1}$, $r^{**} = \binom{k-1}{s-1}$,

k^{**} = the number of times s-tuples of treatments occur in the original design $\mathcal{D}$,

$\lambda^{**} = \binom{\mu_s - 1}{s-1}$, where μ_s denotes the number of treatments common to any two blocks in $\mathcal{D}$ where the treatment θ occurs.

Note that if the number of times s-tuples of treatments occur and the number of treatments common to any two blocks of $\mathcal{D}$ are not constant, then the values of k^{**} and λ^{**} are varying. By noting this fact, in this paper, constructions of PBIB designs are discussed to show some advantage of the present approach.

Theorem 2.1. For positive integers x, y $(x \neq y)$, the existence of a quasi-symmetric t-(v, k, λ_t) design with any of its two blocks having either x or y common treatments implies the existence of a 2-associate PBIB design with parameters

$$v^{**} = r,\ b^{**} = \binom{v-1}{s-1},\ r^{**} = \binom{k-1}{s-1},\ k^{**} = \lambda_s,\ \lambda_1^{**} = \binom{x-1}{s-1},\ \lambda_2^{**} = \binom{y-1}{s-1},$$

$$n_1^{**} = \frac{(y-1)(r-1)-(k-1)(\lambda_2-1)}{y-x},\ n_2^{**} = \frac{(k-1)(\lambda_2-1)-(x-1)(r-1)}{y-x},$$

for $1 \leq s \leq t$.

Proof. Consider a quasi-symmetric t-(v, k, λ_t) design with any of its two blocks having either x or y common treatments. Dualise this quasi-symmetric t-design with respect to restricted s-tuples for $1 \leq s \leq t$. Then, the parameters $v^{**}, b^{**}, r^{**}, k^{**}$, λ_1^{**}, λ_2^{**} are obvious from the definition of the dualisation with respect to restricted s-tuples. Also, the values of n_1^{**} and n_2^{**} follow from Lemma 1.1. The proof is complete. □

Example 2.1. Consider a quasi-symmetric 5-(7,5,1) design with parameters $b = 21, r = 15, \lambda_2 = 10, x = 3, y = 4$, whose blocks are given by (1, 2, 3, 4, 5), (1, 2, 3, 4, 6), (1, 2, 3, 4, 7), (1, 2, 3, 5, 6), (1, 2, 3, 5, 7), (1, 2, 3, 6, 7), (1, 2, 4, 5, 6), (1, 2, 4, 5, 7), (1, 2, 4, 6, 7), (1, 2, 5, 6, 7), (1, 3, 4, 5, 6), (1, 3, 4, 5, 7), (1, 3, 4, 6, 7), (1, 3, 5, 6, 7), (1, 4, 5, 6, 7), (2, 3, 4, 5, 6), (2, 3, 4, 5, 7), (2, 3, 4, 6, 7), (2, 3, 5, 6, 7), (2, 4, 5, 6, 7), (3, 4, 5, 6, 7). Then Theorem 2.1 yields a 2-associate PBIB design $\mathcal{D}_2^{**}$ with parameters $v^{**} = 15, b^{**} = 6, r^{**} = 4, k^{**} = 10, \lambda_1^{**} = 2, \lambda_2^{**} = 3, n_1^{**} = 6, n_2^{**} = 8$, whose blocks are as follows.

(1,2,3,4,5,6,7,8,9,10), (1,2,3,4,5,6,11,12,13,14),
(1,2,3,7,8,9,11,12,13,15), (1,4,5,7,8,10,11,12,14,15),
(2,4,6,7,9,10,11,13,14,15), (3,5,6,8,9,10,12,13,14,15).

Note that the complement of this resulting design is a 2-associate PBIB design with parameters $v = 15, b = 6, r = 2, k = 5, \lambda_1 = 0, \lambda_2 = 1, n_1 = 6, n_2 = 8$.

Example 2.2. Consider a quasi-symmetric 5-(7,5,1) design with parameters $b = 21, r = 15, \lambda_2 = 10, \lambda_3 = 6, x = 3, y = 4$, having blocks given in Example 2.1. Then Theorem 2.1 yields a 2-associate PBIB design $\mathcal{D}_3^{**}$ with parameters $v^{**} = 15, b^{**} = 15, r^{**} = 6, k^{**} = 6, \lambda_1^{**} = 1, \lambda_2^{**} = 3, n_1^{**} = 6, n_2^{**} = 8$, whose blocks are as follows.

(1,2,3,4,5,6), (1,2,3,7,8,9), (1,4,5,7,8,10), (2,4,6,7,9,10),
(3,5,6,8,9,10), (1,2,3,11,12,13), (1,4,5,11,12,14), (2,4,6,11,13,14),
(3,5,6,12,13,14), (1,7,8,11,12,15), (2,7,9,11,13,15), (3,8,9,12,13,15),
(4,7,10,11,14,15), (5,8,10,12,14,15), (6,9,10,13,14,15).

Example 2.3. There exists a quasi-symmetric 4-(23,7,1) design with $r = 77, \lambda_2 = 21, \lambda_3 = 5, x = 1$ and $y = 3$ (see, e.g., Hedayat and Kageyama, 1980; Kageyama and Hedayat, 1983). Then Theorem 2.1 yields 2-associate PBIB designs for $n_1^{**} = 16$ and $n_2^{**} = 60$:

$\mathcal{D}_2^{**}$: $v^{**} = 77, b^{**} = 22, r^{**} = 6, k^{**} = 21, \lambda_1^{**} = 0, \lambda_2^{**} = 2$;
$\mathcal{D}_3^{**}$: $v^{**} = 77, b^{**} = 231, r^{**} = 15, k^{**} = 5, \lambda_1^{**} = 0, \lambda_2^{**} = 1$.

Remark 2.1. In Theorem 2.1, if the starting design is affine resolvable, then $b = v + r - 1$, $x = 0$ and $y = k^2/v$ (see, e.g.,Hedayat and Kageyama, 1980; Kageyama and Hedayat, 1983).

Hence an idea of the present procedure yields a BIB design. This observation implies that we can utilize an affine resolvable t-design as a starting quasi-symmetric block design. In fact, an affine resolvable block design has $x = 0$. Available affine resolvable t-designs yield some new BIB designs. For example, Kimberley (1971) showed that all resolvable 3-(v, k, λ_3) designs are affine resolvable if and only if they are 3-$(4\lambda_3 + 4, 2\lambda_3 + 2, \lambda_3)$ designs. In this case, they have other parameters as $b = 2(4\lambda_3 + 3), r = 4\lambda_3 + 3, \lambda_2 = 2\lambda_3 + 1, x = 0$ and $y = \lambda_3 + 1$. Hence the present procedure yields two BIB designs, $\mathcal{D}_2^{**}$ and $\mathcal{D}_3^{**}$, with parameters

$$v^{**} = 4\lambda_3 + 3,\ b^{**} = \binom{4\lambda_3 + 3}{s-1},\ r^{**} = \binom{2\lambda_3 + 1}{s-1},\ k^{**} = \lambda_s,\ \lambda^{**} = \binom{\lambda_3}{s-1}$$

for $s = 2, 3$.

As another example that is not affine resolvable, we can consider a quasi-symmetric 3-(22, 6, 1) design with $r = 21, \lambda_2 = 5, x = 0$ and $y = 2$ (see, e.g., Kageyama and Hedayat, 1983). In this case the present procedure yields a BIB design, $\mathcal{D}_2^{**}$, with parameters $v^{**} = 21, b^{**} = 21, r^{**} = 5, k^{**} = 5, \lambda^{**} = 1$.

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