Crosscap numbers of 2-bridge knots

Mikami Hirasawa [1]

Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan

Masakazu Teragaito [2]

Department of Mathematics and Mathematics Education, Hiroshima University, 1-1-1 Kagamiyama, Higashi-hiroshima 739-8524, Japan

Abstract

We present a practical algorithm to determine the minimal genus of non-orientable spanning surfaces for 2-bridge knots, called the crosscap numbers. We will exhibit a table of crosscap numbers of 2-bridge knots up to 12 crossings (all 362 of them).

Key words: crosscap number, 2-bridge knot, non-orientable spanning surface
1991 MSC: 57M25

1 Introduction

For a knot K in the 3-sphere S^3, there is a connected compact embedded surface F in S^3 whose boundary is K. In particular, F can be chosen to be orientable, and then it is called a Seifert surface for K. The genus $g(K)$ of K is the minimal number of genera of all Seifert surfaces for K. Thus the unknot is the only knot of genus zero.

On the other hand, we can choose the above F to be non-orientable, for example, by adding a half-twisted band to a Seifert surface. In this paper, such F is referred to as a *non-orientable spanning surface* for K. We define the *crosscap number* $\gamma(K)$ of a non-trivial knot K as the minimal number of the first betti numbers β_1 of all non-orientable spanning surfaces for K,

[1] E-mail:hirasawa@math.gakushuin.ac.jp. Partially supported by MEXT, Grant-in-Aid for Young Scientists (B) 14740048.
[2] Tel.:+81-824-247079, Fax:+81-824-247084, E-mail:teragai@hiroshima-u.ac.jp

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and set $\gamma(\text{unknot}) = 0$ for convenience. We call $\gamma(K)$ the crosscap number because it counts the number of 'crosscap summands' in the closed surface obtained by capping off a non-orientable spanning surface with a disk, which is well known to be a connected sum of projective planes. In the literature, a crosscap number is also called a non-orientable genus [9]. For a non-trivial knot K, if a non-orientable spanning surface F satisfies $\beta_1(F) = \gamma(K)$, then F is called a *minimal genus non-orientable spanning surface* for K.

In general, it is very hard to determine the crosscap number for a given knot. Any minimal genus Seifert surface becomes a non-orientable spanning surface for the same knot if we attach a small half-twisted band as above, and hence we have an obvious inequality $\gamma(K) \leq 2g(K)+1$. There are only a few results about crosscap numbers of knots. Clark [4] introduced the notion of crosscap number and pointed out that $\gamma(K) = 1$ if and only if K is a 2-cabled knot. He also asked about the existence of a knot satisfying the equality $\gamma(K) = 2g(K) + 1$, and Murakami and Yasuhara [8] came up with the first example, showing $\gamma(7_4) = 3$ algebraically. In [11], the crosscap numbers of torus knots are completely determined.

The purpose of this paper is to determine the crosscap numbers of 2-bridge knots, which form a special but important class of knots. For 2-bridge knots, Hatcher and Thurston [7] constructed all incompressible, boundary-incompressible orientable or non-orientable spanning surfaces. However, for the 2-bridge knot 7_4, a minimal genus non-orientable spanning surface can be realized only by a boundary-compressible surface. Then Bessho [1] proved that any incompressible, boundary-compressible spanning surface for a 2-bridge knot becomes an incompressible, boundary-incompressible surface after several boundary-compressions. Therefore, theoretically, we can obtain $\gamma(K)$ as follows:

> For a 2-bridge knot K, generate all incompressible, boundary-incompressible spanning surfaces according to [7]. Let n be the minimal first betti number of them. Then if n is realized by a non-orientable spanning surface, then $\gamma(K) = n$, and otherwise $\gamma(K) = n + 1$. Here, n equals the minimal length of all continued fraction expansions for K.

However, an effective algorithm to determine n was missing, and one could not tell, for example, for which 2-bridge knots, the equality $\gamma(K) = 2g(K)+1$ holds.

In the following, we present a practical algorithm to find a shortest continued fraction expansion for all rational numbers representing a 2-bridge knot K. This enables us to determine the crosscap number from any continued fraction expansion for K. The main tool is so-called the modular diagram, whose vertices correspond to rational numbers, on which we introduce the notion of depth. In Section 6, we exhibit a table of crosscap numbers of 2-bridge knots up to 12 crossings (all 362 of them).

2 Statement of results

Let K be a 2-bridge knot $S(q,p)$ in Schubert's notation. Here, p and q are coprime integers, and q is odd. As is well-known, $S(q,p)$ and $S(q',p')$ are equivalent if and only if $q'=q$ and $p' \equiv p^{\pm 1} \pmod{q}$, and $S(q,-p)$ gives the mirror image of $S(q,p)$.

Consider a *subtractive* continued fraction expansion of p/q (see [7])

$$\frac{p}{q} = r + [b_1, b_2, \ldots, b_n] = r + \cfrac{1}{b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots - \cfrac{1}{b_n}}}}}$$

where $r, b_i \in \mathbb{Z}$ and $b_i \neq 0$. The *length* of this expansion is n. Then K is the boundary of the surface obtained by plumbing n bands in a row, the ith band having b_i half-twists (right-handed if $b_i > 0$ and left-handed if $b_i < 0$). If some b_i is odd, then the expansion is said to be of *odd type*. Otherwise, it is of *even type*. Any fraction has expansions of odd type and even type, e.g., $1/3 = 1 - 2/3 = 1 + [-2,-2] = 1 + [-1,2]$. In this paper, an expansion always means a subtractive one. We remark the following equality:

$$r + [a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1}a_n] = r + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}.$$

The crosscap number of a 2-bridge knot K can be described in terms of the length of expansion corresponding to K. The first theorem is due to Bessho, but we will give its proof for reader's convenience in Section 3.

Theorem 1 (Bessho [1]) *Let K be a 2-bridge knot.*

(1) The crosscap number $\gamma(K)$ equals the minimal length of all expansions of odd type of all fractions corresponding to K.

(2) If a minimal genus non-orientable spanning surface F for K is boundary-compressible, then F is obtained from a minimal genus Seifert surface for K by attaching a Möbius band as in Figure 1.

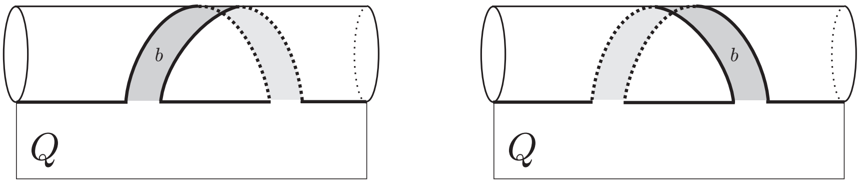

Fig. 1. Möbius bands attached to Seifert surfaces

We present a practical algorithm to obtain a shortest expansion from any one of p/q.

Theorem 2 *Let $p/q = r + [b_1, b_2, \ldots, b_n]$ be an expansion obtained from an arbitrary expansion of p/q by fully reducing the length by a repetition of the following three reductions. Then n is the minimal length of all expansions of p/q.*

(1) Removal of coefficient 0.

$$[\ldots, a, 0, b, \ldots] = [\ldots, a+b, \ldots],$$
$$[\ldots, a, b, 0] = [\ldots, a],$$
$$[0, a, b, \ldots] = -a + [b, \ldots].$$

(2) Removal of coefficient $\varepsilon = \pm 1$.

$$[\ldots, a, \varepsilon, b, \ldots] = [\ldots, a-\varepsilon, b-\varepsilon, \ldots],$$
$$[\ldots, a, \varepsilon] = [\ldots, a-\varepsilon],$$
$$r + [\varepsilon, a, \ldots] = (r+\varepsilon) + [a-\varepsilon, \ldots].$$

(3) Removal of a subsequence $2\varepsilon, 2\varepsilon$ or $2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon$.

$$[\ldots, a, \underbrace{2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon}_{m}, b, \ldots,] = [\ldots, a-\varepsilon, \underbrace{-3\varepsilon, -3\varepsilon, \ldots, -3\varepsilon}_{m-1}, b-\varepsilon, \ldots],$$
$$[\ldots, a, \underbrace{2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon}_{m}] = [\ldots, a-\varepsilon, \underbrace{-3\varepsilon, -3\varepsilon, \ldots, -3\varepsilon}_{m-1}],$$
$$r + [\underbrace{2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon}_{m}, a, \ldots] = (r+\varepsilon) + [\underbrace{-3\varepsilon, -3\varepsilon, \ldots, -3\varepsilon}_{m-1}, a-\varepsilon, \ldots],$$
$$r + [\underbrace{2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon}_{m}] = (r+\varepsilon) + [\underbrace{-3\varepsilon, -3\varepsilon, \ldots, -3\varepsilon}_{m-1}].$$

(Here, $\varepsilon = \pm 1$, and possibly $m = 2$.)

In Theorem 2, we fix a fraction p/q. Although there are infinitely many fractions corresponding to a 2-bridge knot, the next theorem guarantees that we can start from any fraction.

Theorem 3 *Let K be a 2-bridge knot. If the reduction in Theorem 2 yields a length n expansion, then n is the minimal length of all expansions of all fractions corresponding to K.*

The next theorem is the key to determine whether a fraction p/q admits a shortest expansion of odd type.

Theorem 4 *Two shortest expansions for p/q are deformed to each other by a finite repetition of the following, where $\varepsilon = \pm 1$:*

$$[\dots, a, 2\varepsilon, b, \dots] = [\dots, a-\varepsilon, -2\varepsilon, b-\varepsilon, \dots],$$
$$[\dots, a, 2\varepsilon] = [\dots, a-\varepsilon, -2\varepsilon],$$
$$r + [2\varepsilon, a, \dots] = (r+\varepsilon) + [-2\varepsilon, a-\varepsilon, \dots].$$

Theorem 5 *Let $K = S(q,p)$ be a 2-bridge knot. If a shortest expansion of p/q obtained by Theorem 2 contains an odd coefficient or ± 2, then $\gamma(K) = n$, otherwise $\gamma(K) = n+1$, where n is the length of the expansion.*

Remark that if there is a coefficient ± 2 in an expansion with only even coefficients, then we can apply Theorem 4 to obtain an expansion of odd type.

Example 6 Let $K = 6_1$ in the knot table (see [10]). It is the 2-bridge knot $S(9,2)$. Then $2/9 = [5,2]$. Thus $\gamma(K) = 2$ by Theorem 5.

It is known that any 2-bridge knot K has a unique expansion of even type modulo integer parts, and the length of which equals $2g(K)$. As a direct corollary to Theorem 5, we can completely characterize those 2-bridge knots satisfying the equality $\gamma(K) = 2g(K)+1$.

Corollary 7 *For a 2-bridge knot K, the equality $\gamma(K) = 2g(K)+1$ holds if and only if there is no coefficient ± 2 in the (unique) expansion for K containing only even coefficients.*

Example 8 Let $K = 7_4 = S(15,4)$. Note that K has genus one. Since $4/15 = [4,4]$, $\gamma(K) = 2g(K)+1 = 3$ by Corollary 7. More examples will be given in Section 5.

Some minimal genus non-orientable surfaces for 2-bridge knots are boundary-incompressible, but others boundary-compressible, and some 2-bridge knots have several such surfaces. This makes a strong contrast to the case of torus knots, where minimal genus non-orientable spanning surfaces are boundary-incompressible and even unique [11].

By the theorems above, we can characterize 2-bridge knots with boundary-compressible minimal genus non-orientable spanning surfaces. It is unknown whether Corollary 10 below generalizes to all knots.

Theorem 9 *Let $K = S(q,p)$ be a 2-bridge knot, and $\mathcal{C}$ the set of shortest expansions for p/q. Then we have:*

(1) $\mathcal{C}$ contains an expansion of odd type if and only if any minimal genus non-orientable spanning surface for K is boundary-incompressible.
(2) $\mathcal{C}$ contains no expansion of odd type if and only if any minimal genus non-orientable spanning surface for K is boundary-compressible.

The following is immediate from Theorem 9.

Corollary 10 *A 2-bridge knot never has two minimal genus non-orientable spanning surfaces such that one is boundary-incompressible and the other is boundary-compressible.*

In Section 4, we give an algorithm to visualize a minimal genus non-orientable spanning surface for 2-bridge knots.

Theorem 11 *Any 2-bridge knot K has a Conway diagram D such that a minimal genus non-orientable spanning surface for K is obtained as a checker-board surface on D.*

Example 12 We note that such diagrams are not unique and not shortest in general. Let $K = S(15,4)$ as in Example 8 with $\gamma(K) = 3$. Then the Conway diagrams $[2,1,5,-1,3]$ and $[4,1,1,1,4]$ representing K respectively yield a desired surface as a checker-board surface. It is interesting to confirm Theorem 1(2) for these surfaces.

3 Proof of Theorem 1

Let K be a 2-bridge knot with a minimal genus non-orientable spanning surface F. Let $E(K) = S^3 - \mathrm{Int}\, N(K)$ be its exterior. Then $F \cap N(K)$ can be assumed to be a collar neighborhood of ∂F in F, and hence we will use the same notation F for $F \cap E(K)$.

Lemma 13 *F is incompressible in $E(K)$.*

PROOF. Assume not. Let D be a compressing disk for F. Then ∂D is an orientation-preserving loop on F. Let F' be the resulting surface from F by compressing along D. Then $\chi(F') = \chi(F) + 2$. If F' is disconnected, then it consists of a closed orientable component F_1 and a non-orientable component F_2 with $\partial F_2 \neq \emptyset$. Since $\beta_1(F_1) + \beta_1(F_2) = \beta_1(F)$ and $\beta_1(F_1) > 0$, we have $\beta_1(F_2) < \beta_1(F)$. This contradicts the minimality of $\beta_1(F)$. If F' is connected and non-orientable, then $\beta_1(F') = \beta_1(F) - 2$, a contradiction. Hence F' is connected and orientable. This means that F' is a Seifert surface for K. Then adding a small half-twisted band to F' gives a non-orientable spanning surface R for K with $\beta_1(R) = \beta_1(F') + 1 = \beta_1(F) - 1$, a contradiction. □

Proof of Theorem 1 (1) Let $p/q = r + [b_1, b_2, \dots, b_n]$ be an expansion of odd type of some fraction p/q for K. We assume that the length n is minimal among all expansions of odd type of all fractions for K. The surface obtained by plumbing n bands corresponding to this expansion in the usual way gives a non-orientable spanning surface for K with the first betti number n. Thus $\gamma(K) \leq n$.

The argument to show $n \leq \gamma(K)$ is divided into two cases according to the boundary-incompressibility of a minimal genus non-orientable spanning surface F.

First assume that K has a minimal genus non-orientable spanning surface F which is boundary-incompressible. Then it is isotopic to one of the surfaces obtained by plumbing k bands corresponding to some expansion $s + [a_1, a_2, \dots, a_k]$ of some fraction for K with $s \in \mathbb{Z}$ and $|a_i| \geq 2$ for each i by [7, Theorem 1(b)]. Hence $\gamma(K) = k$. Since F is non-orientable, this expansion $s + [a_1, a_2, \dots, a_k]$ must be of odd type. Thus $n \leq k$ by the minimality of n, and hence we have $n \leq \gamma(K)$.

Next, assume that any minimal genus non-orientable spanning surface F for K is boundary-compressible. Let D be a boundary-compressing disk such that $\partial D = \alpha \cup \beta$, where $D \cap F = \alpha$ is a properly embedded essential arc in F and $D \cap \partial E(K) = \beta$. Then β intersects ∂F in two points. If these two points have distinct signs (after orienting β and ∂F suitably), then β and a subarc of ∂F bound a disk δ in $\partial E(K)$. Thus $D \cup \delta$, pushed away from $\partial E(K)$ slightly, gives a compressing disk for F, which contradicts Lemma 13. Hence β intersects ∂F twice in the same direction. Let F_1 be the surface obtained by boundary-compressing F along D. From the above observation, F_1 is a connected surface with connected boundary. Also, we see $\beta_1(F_1) = \beta_1(F) - 1$.

Claim 14 *F_1 is incompressible in $E(K)$.*

Proof of Claim 14 Let $N(D) = D \times [-1, 1]$ be a product neighborhood of D such that $N(D) \cap F = \partial N(D) \cap F = \alpha \times [-1, 1]$. Then $F_1 = (F - N(D) \cap F) \cup (D \times \{-1, 1\})$. If F_1 is compressible, then it has a compressing disk E disjoint from $N(D)$. Since F is incompressible, ∂E bounds a disk E' in F. We can choose E' disjoint from the disk $\alpha \times [-1, 1]$. Thus ∂E bounds a disk in F_1, a contradiction. □

If F_1 is orientable, then it is boundary-incompressible. (Any orientable incompressible surface in $E(K)$ is boundary-incompressible if it has a connected boundary.) If F_1 is non-orientable and boundary-compressible, we continue a boundary-compression. Thus for some $\ell > 0$ we have a sequence of incompressible surfaces $F = F_0 \to F_1 \to F_2 \to \cdots \to F_\ell$ where $\beta_1(F_i) = \beta_1(F_{i-1}) - 1$ for $i = 1, 2, \ldots, \ell$ and F_ℓ is boundary-incompressible. By [7, Proposition 2], ∂F_ℓ runs once longitudinally on $\partial E(K)$.

If F_ℓ is orientable, then it has minimal genus [7, Corollary]. Note that $\beta_1(F_\ell) = \beta_1(F) - \ell = \gamma(K) - \ell$ and that F_ℓ corresponds to the unique expansion $r' + [b'_1, b'_2, \ldots, b'_m]$ with each b'_i even. Then $\beta_1(F_\ell) = m$. Since K admits an odd type expansion of length $m+1$, $n \leq m+1 = \beta_1(F_\ell)+1 = \gamma(K)-\ell+1 \leq \gamma(K)$. Thus we have $n \leq \gamma(K)$, and so $\gamma(K) = n$ and $\ell = 1$.

If F_ℓ is non-orientable, then F_ℓ is isotopic to some surface obtained by plumbing k bands and $n \leq k$ as before. Since $\beta_1(F_\ell) = k$, $n \leq k = \gamma(K) - \ell < \gamma(K) \leq n$, a contradiction. Thus such a case never occurs.

(2) If a minimal genus non-orientable spanning surface F is boundary-compressible, then the above argument shows that boundary-compressing F gives a minimal genus Seifert surface Q for K. Then F is obtained from $Q \subset E(K)$ by attaching a band $b = [0, 1] \times [0, 1] \subset \partial E(K)$ such that $b \cap Q = [0, 1] \times \{0, 1\}$. In fact, since ∂F runs once longitudinally on $\partial E(K)$, there are only two possibilities for b as shown in Figure 1. Thus F is obtained from Q by adding a Möbius band locally as desired. □

4 Calculation by the modular diagram

We use the modular diagram $\mathcal{D}$ as shown in Figure 2 to compute the crosscap numbers of 2-bridge knots. This diagram comes from the action of $PSL(2, \mathbb{Z})$ on the hyperbolic plane. (But $\mathcal{D}$ is distorted to space the vertices evenly along the circle.)

The vertices are labelled with $\mathbb{Q} \cup \{1/0\}$, inductively: Start with $1/0$ and $0/1$ at the ends of the 'horizontal' edge. If two vertices of a triangle are already labelled with a/b and c/d, then the remaining vertex of the triangle is labelled $(a+c)/(b+d)$. (This is the rule to label the vertices on the upper circle only, and for those on the lower circle, regard $1/0$ and $0/1$ as $-1/0$ and $-0/1$.)

We call the third vertex the *child* of the first two vertices, which themselves are called the *parents*, and call the edge connecting the parents the *longest side* of a triangle. Note that two vertices a/b and c/d are connected by an edge if and only if $|ad - bc| = 1$.

We will identify a vertex with the corresponding label for convenience. In fact, all rational numbers appear on the circle with the usual order. That is, if the vertices u and v correspond to a/b and c/d, respectively, and if $a/b < c/d$ in $\mathbb{Q}$, then u and v lie on the circle with the counterclockwise orientation in the order u, v.

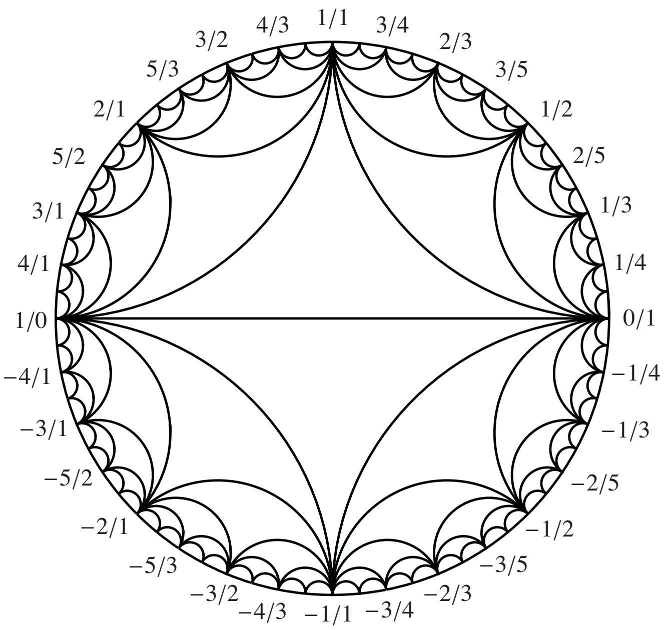

Fig. 2. The modular diagram $\mathcal{D}$

An edge-path from $1/0$ to p/q in $\mathcal{D}$ corresponds uniquely to an expansion $p/q = r + [b_1, b_2, \ldots, b_n]$, where the partial sums $p_i/q_i = r + [b_1, b_2, \ldots, b_i]$ ($p_0/q_0 = r$) are the successive vertices of the edge-path. At the vertex p_{i-1}/q_{i-1} the path turns left or right across $|b_i|$ triangles, left if $b_i > 0$ and right if $b_i < 0$. See Figure 3. If an edge-path corresponds to an expansion of odd type, then the path is also said to be of odd type.

We assign the *depth* $d(v)$ to each vertex v on $\mathcal{D}$. First, set $d(v) = 0$ for $v \in \mathbb{Z} \cup \{1/0\}$. If a triangle in $\mathcal{D}$ has vertices a/b, c/d and $(a+c)/(b+d)$, then define the depth of the child from those of parents by setting

$$d((a+c)/(b+d)) = \min\{d(a/b), d(c/d)\} + 1.$$

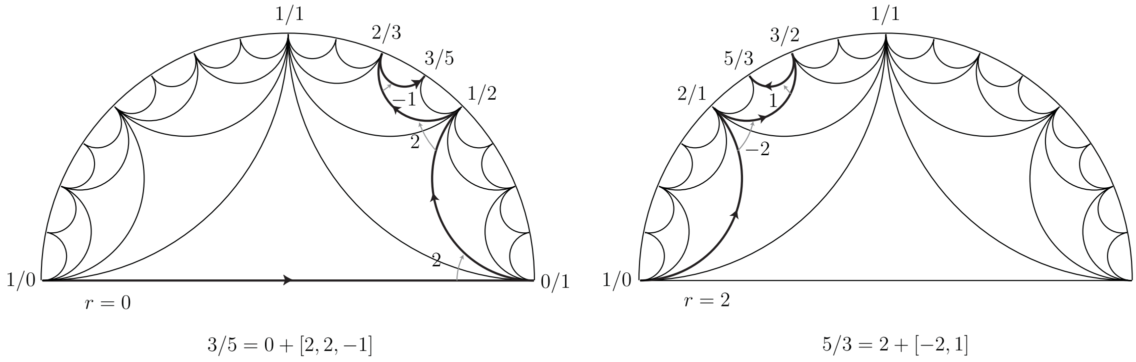

Fig. 3. Edge-paths from $1/0$ to p/q

Thus all vertices can be assigned the depths. Notice that if two vertices u and v are connected by an edge in $\mathcal{D}$, then $|d(u) - d(v)| \leq 1$. Also there are only three kinds of triangles in $\mathcal{D}$ as shown in Figure 4, where depths of vertices are indicated, except triangles with vertices $\{1/0, n, n+1\}$ of depth 0 where $n \in \mathbb{Z}$.

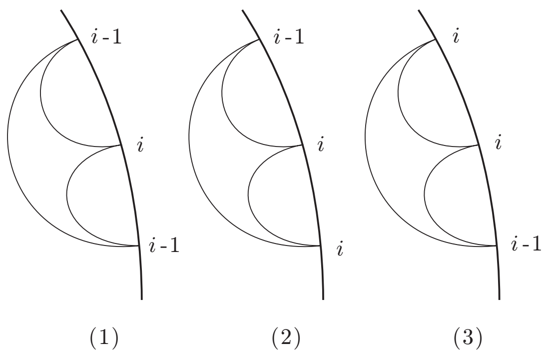

Fig. 4. Triangles

Lemma 15 *Let ℓ be the length of a shortest edge-path from $1/0$ to $p/q \neq 1/0$. Then $d(p/q) = \ell - 1$.*

PROOF. By the definition of depth, there is an edge-path from $1/0$ to p/q of length $d(p/q) + 1$. Thus $\ell \leq d(p/q) + 1$. Conversely, let ξ be a shortest edge-path from $1/0$ to p/q. Recall that $|d(u) - d(v)| \leq 1$ for any consecutive vertices u, v on ξ. In particular, $d(1/0) = d(v_0) = 0$, where v_0 is the second vertex on ξ, and hence $d(p/q) \leq \ell - 1$. Thus we have $d(p/q) = \ell - 1$. □

The next theorem gives a criterion for a shortest edge-path in terms of depths.

Theorem 16 *An edge-path $\xi : 1/0 \to v_0 \to v_1 \to v_2 \to \cdots \to v_n = p/q \neq 1/0$ is shortest if and only if $d(v_i) = i$ for $i = 0, 1, 2, \ldots, n$.*

PROOF. Assume that ξ is shortest. Then $d(v_n) = n$ by Lemma 15. Since the depth can increase by at most one along ξ and $d(v_0) = 0$, $d(v_i) = i$ for each i.

Conversely, since $d(v_n) = n$, any edge-path from $1/0$ to p/q has length at least $n + 1$ by Lemma 15. Thus we can conclude that ξ is shortest. $\square$

Proof of Theorem 2 Let $p/q = r + [b_1, b_2, \ldots, b_n]$ be an expansion fully reduced by the reductions in the statement of Theorem 2. Let $\xi : 1/0 \to v_0 \to v_1 \to \cdots \to v_n = p/q$ be the edge-path corresponding to the expansion. Suppose that the expansion is not of minimal length, that is, ξ is not shortest. Then the sequence $\mathcal{S}$ of depths $d(v_0), d(v_1), d(v_2), \ldots, d(v_n)$ is not strictly increasing by Theorem 16. Notice that $d(v_0) = 0$.

First, suppose that $\mathcal{S}$ contains $i-1, i, i-1$ as a subsequence. Then we can see that there is a triangle of Figure 4(1) such that ξ runs along the shorter two edges. This means that some coefficient b_j is ± 1, a contradiction.

If $\mathcal{S}$ contains $0, 0$, then $b_1 = \pm 1$, a contradiction. Thus $\mathcal{S}$ contains $i-1, i, i, \ldots, i$ (i is repeated k (≥ 2) times) for some $i \geq 1$. We choose i minimal among such subsequences of $\mathcal{S}$.

Let u_1 and u_2 be the depth i vertices on ξ, appearing in the order u_2, u_1. We can suppose that the vertex before u_2 on ξ has depth $i-1$. There is the unique triangle T_1 which contains the edge between u_1 and u_2 as one of two shorter edges. Without loss of generality, we can assume that T_1 has the form of Figure 4(3). Let w_1 be the remaining vertex of T_1. If u_2 is the child of $\{u_1, w_1\}$, then ξ contains the edges $w_1 \to u_2 \to u_1$. Then some $b_j = -1$, a contradiction. Hence u_1 is the child of $\{u_2, w_1\}$. Let T_2 be the (unique) triangle sharing the edge between u_2 and w_1 with T_1, and let u_3 be the remaining vertex of T_2. See Figure 5(1). (Since $d(w_1) = i-1$ and $d(u_2) = i$, u_3 is located in this position.) Then $d(u_3) = i$ or $i-1$. If $d(u_3) = i$, then ξ contains $w_1 \to u_2 \to u_1$, so some coefficient is -1. Thus $d(u_3) = i-1$. If ξ contains $w_1 \to u_2 \to u_1$, then some coefficient is -1 again. Hence ξ contains $u_3 \to u_2 \to u_1$.

If $i = 1$, then $d(u_3) = d(w_1) = 0$. By the minimality of i, $u_3 = v_0$. Thus ξ contains $1/0 \to u_3 \to u_2 \to u_1$, so $b_1 = b_2 = -2$, a contradiction. (If T_1 has the form of Figure 4(2), then we encounter 1 or $2, 2$.)

Suppose $i \geq 2$. Let T_3 be the triangle sharing the edge between u_3 and w_1 with T_2, and let w_2 be the remaining vertex of T_3. If u_3 is the child of $\{w_1, w_2\}$, then $d(w_2) = i-2$, and ξ contains $w_2 \to u_3 \to u_2 \to u_1$. Then we have $-2, -2$ in the coefficients, a contradiction. Hence w_1 is the child of $\{u_3, w_2\}$, and $d(w_2) = i-2$. See Figure 5(2). Let T_4 be the triangle sharing the edge between u_3 and w_2 with T_3, and let u_4 be the remaining vertex of T_4. Then u_3 is the child of $\{u_4, w_2\}$, and so $d(u_4) = i-1$ or $i-2$. If $d(u_4) = i-1$,

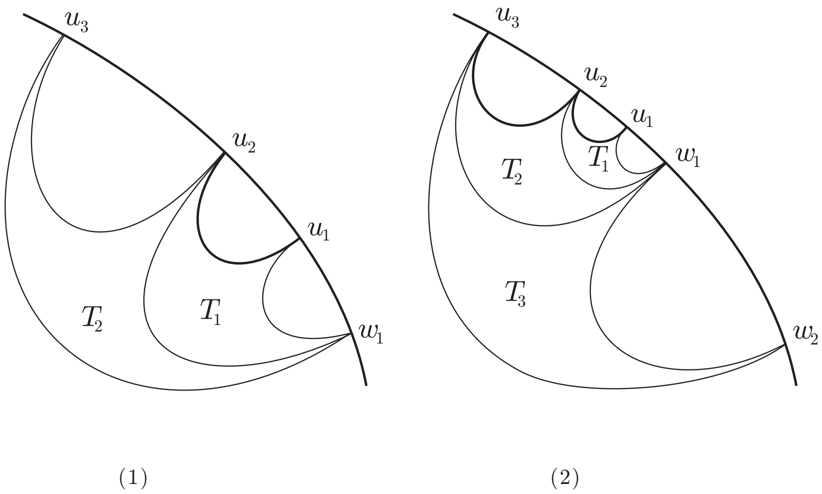

Fig. 5. Paths $u_2 \to u_1$ and $u_3 \to u_2 \to u_1$

then ξ contains $w_2 \to u_3 \to u_2 \to u_1$. Then we have $-2, -2$ in the coefficients. Hence $d(u_4) = i - 2$. If ξ contains $w_2 \to u_3 \to u_2 \to u_1$, then we have $-2, -2$ again. Thus ξ contains $u_4 \to u_3 \to u_2 \to u_1$. See Figure 6(1). If $i = 2$, then $d(u_4) = d(w_2) = 0$ and hence $u_4 = v_0\,(\neq 1/0)$. Thus ξ contains $1/0 \to u_4 \to u_3 \to u_2 \to u_1$, so $b_1 = -2$, $b_2 = -3$ and $b_3 = -2$, a contradiction. Thus $i \geq 3$.

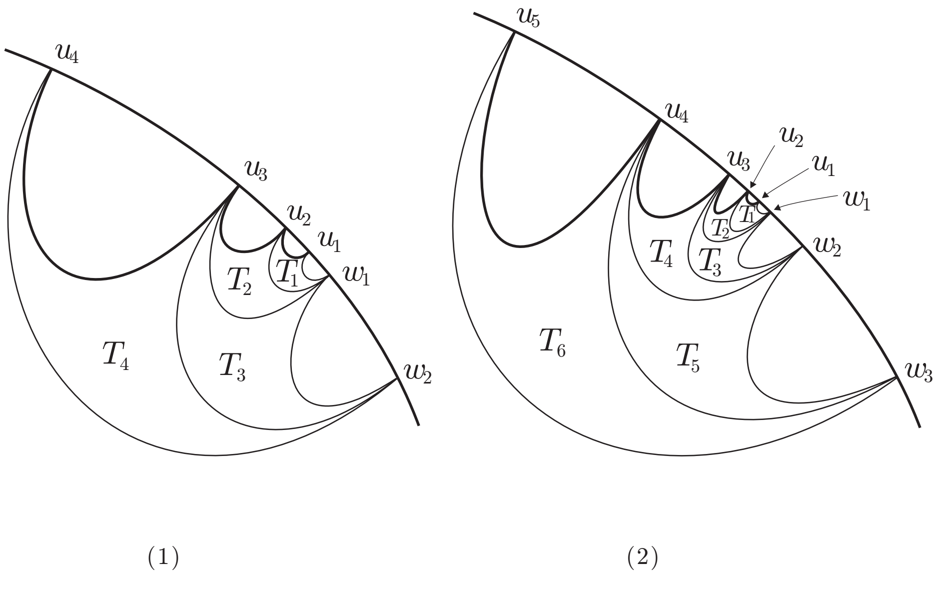

Fig. 6. Paths $u_4 \to u_3 \to u_2 \to u_1$ and $u_5 \to u_4 \to u_3 \to u_2 \to u_1$

Continuing this process, we obtain the triangles $T_1, T_2, \ldots, T_{2i}$, where T_{2m-1} is the form of Figure 4(3), and T_{2m} is of Figure 4(1). Also, T_1 has vertices $\{u_1, u_2, w_1\}$, T_{2m-1} $(m \geq 2)$ has vertices $\{w_{m-1}, w_m, u_{m+1}\}$, and T_{2m} $(m \geq 1)$ has vertices $\{u_{m+1}, u_{m+2}, w_m\}$, and $d(u_1) = i$, $d(u_j) = i - j + 2$ for $j \geq 2$, $d(w_j) = i - j$ for $j \geq 1$. The path ξ contains the edges $u_{i+2} \to u_{i+1} \to \cdots \to u_2 \to u_1$. Since $d(u_{i+2}) = 0$, $u_{i+2} = v_0$. Then $-2, -3, \ldots, -3, -2$ appears in the coefficients, a contradiction. (If T_1 has the form of Figure 4(2), then we encounter 1 or $2, 3, 3, \ldots, 3, 2$ in the coefficients.) $\square$

Lemma 17 *Let p/q and p'/q' be two fractions for a 2-bridge knot K. Then there exists a one-to-one correspondence between the set of all expansions for p/q and that of p'/q', such that the correspondence preserves both length and type of expansions.*

PROOF. Since both p/q and p'/q' represent the same knot, $q = q'$ and (i) $p \equiv p' \pmod q$ or (ii) $pp' \equiv 1 \pmod q$. Suppose $p/q = p'/q + s$, where $s \in \mathbb{Z}$. Then for any expansion $r + [a_1, a_2, \ldots, a_n]$ of p/q, we can associate $r - s + [a_1, a_2, \ldots, a_n]$ of p'/q. Therefore, it suffices to establish a one-to-one correspondence only for p/q and p'/q lying between 0 and 1. Under such a restriction, suppose that $pp' \equiv 1 \pmod q$. Then it is well known that if $[a_1, a_2, \ldots, a_n] = p/q$ then $[a_n, \ldots, a_2, a_1] = p'/q$. Thus we can establish a required one-to-one correspondence. $\square$

Proof of Theorem 3 Let p/q be a fraction corresponding to K. Let n be the minimal length of an expansion of p/q obtained by the reduction of Theorem 2. If another fraction for K admits an expansion of length shorter than n, then p/q has an expansion of the same length by Lemma 17. This contradicts the minimality of n. $\square$

Next, we consider when there exists a shortest edge-path from $1/0$ to p/q, which is of odd type. A *rectangle* in $\mathcal{D}$ is the union of two triangles sharing one edge.

Lemma 18 *Assume that a shortest edge-path ξ from $1/0$ to p/q is not of odd type. If there is a rectangle in $\mathcal{D}$ containing two successive edges of ξ, then there is a shortest edge-path of odd type from $1/0$ to p/q.*

PROOF. Without loss of generality, we can assume that the successive two edges on ξ, $v_{i-2} \to v_{i-1}$ and $v_{i-1} \to v_i$ lie on a rectangle which is the union of two triangles whose vertices $\{v_{i-2}, v_{i-1}, v'_{i-1}\}$ and $\{v_{i-1}, v'_{i-1}, v_i\}$. Then we replace these two edges by $v_{i-2} \to v'_{i-1}$ and $v'_{i-1} \to v_i$. This will change ξ to a new shortest edge-path of odd type. For, if $v_{i-2} \neq 1/0$, then the expansion corresponding to ξ changes from $r + [\ldots, a, \pm 2, b, \ldots]$ to $r + [\ldots, a \mp 1, \mp 2, b \mp 1, \ldots]$, or $r + [\ldots, a, \pm 2]$ to $r + [\ldots, a \mp 1, \mp 2]$. If $v_{i-2} = 1/0$, then $v_{i-1} \in \mathbb{Z}$. Thus ξ corresponds to an expansion $p/q = v_{i-1} + [\pm 2, a, \ldots]$. Then the new edge-path corresponds to $v_{i-1} \pm 1 + [\mp 2, a \mp 1, \ldots]$. $\square$

The deformation used in the proof of Lemma 18 is referred to as a *rectangle move*.

Proof of Theorem 4 Let

$$\xi : 1/0 \to v_0 \to v_1 \to v_2 \to \cdots \to v_n = p/q,$$

$$\xi' : 1/0 \to v'_0 \to v'_1 \to v'_2 \to \cdots \to v'_n = p/q$$

be two shortest edge-paths from $1/0$ to p/q. Recall that each of v_i and v'_i has depth i for any i by Theorem 16. In particular, $v_0, v'_0 \in \mathbb{Z}$.

Suppose $v_0 \neq v'_0$. We may assume that $v_0 < v'_0$. Since $p/q \in (v_0 - 1, v_0 + 1)$ and $p/q \in (v'_0 - 1, v'_0 + 1)$, $v'_0 - v_0 = 1$. In fact, v_1 and v'_1 lie in the interval (v_0, v'_0). Let u be the child of $\{v_0, v'_0\}$. Then $d(u) = 1$. If neither of v_1 nor v'_1 is u, then we would have $p/q < u < p/q$, a contradiction. Hence we may assume $v'_1 = u$. Then a single application of rectangle move on ξ' changes the edges $1/0 \to v'_0 \to v'_1 = u$ to $1/0 \to v_0 \to u$.

Suppose that $v_i = v'_i$ for $0 \leq i \leq k$ and $v_{k+1} \neq v'_{k+1}$ for some $k \geq 0$. If $v_{k+1} < v_k = v'_k < v'_{k+1}$ or $v_{k+1} > v_k = v'_k > v'_{k+1}$, then we would have $p/q < v_k = v'_k < p/q$, a contradiction. Hence we consider only the case where $v_{k+1}, v'_{k+1} > v_k = v'_k$. The case where $v_{k+1}, v'_{k+1} < v_k = v'_k$ is similar. Without loss of generality, assume $v'_{k+1} < v_{k+1}$. Then p/q lies in the interval (v'_{k+1}, v_{k+1}). If there is a vertex u with depth $k+1$ inside the interval, then we have $p/q < u < p/q$, a contradiction. Hence there are a triangle Δ_1 whose vertices are v_k, v_{k+1} and v'_{k+1}, and a triangle Δ_2 whose vertices are v_{k+1}, v'_{k+1} and w. Here, v_{k+1}, v'_{k+1} are the parents of w. If $w = p/q$, then $v_{k+2} = v'_{k+2}$. Then ξ can be changed to ξ' by the rectangle move on $\Delta_1 \cup \Delta_2$. Otherwise p/q lies in (v'_{k+1}, w) or (w, v_{k+1}). Then $w = v_{k+2}$ in the former case, and $w = v'_{k+2}$ in the latter. After the rectangle move on $\Delta_1 \cup \Delta_2$ to ξ or ξ', we have $v_{k+1} = v'_{k+1}$. Thus, ξ can be changed to ξ' gradually. $\square$

Proof of Theorem 5 If a shortest expansion of p/q obtained by Theorem 2 contains an odd coefficient, then $\gamma(K) = n$ by Theorems 1 and 3. Let ξ be the corresponding edge-path and assume that ξ is of even type. If the expansion contains a coefficient ± 2, then there is a rectangle in $\mathcal{D}$ containing two successive edges of ξ. Hence a rectangle move creates another shortest edge-path of odd type by Lemma 18. Then $\gamma(K) = n$ as above. Otherwise ξ is the unique shortest edge-path from $1/0$ to p/q by Theorem 4.

If another fraction for K admits an expansion of odd type of length n, then p/q also admits an expansion of odd type of length n by Lemma 17. Thus there is no expansion of odd type of length n, and so $\gamma(K) = n + 1$. $\square$

Proof of Corollary 7 It is well known that even type expansions for a 2-bridge knot are unique modulo integer parts, and the length equals $2g(K)$. If the expansion does not contain ± 2, then it is shortest by Theorem 2. Hence $\gamma(K) = 2g(K) + 1$ by Theorem 5. If the expansion contains ± 2, then there is another expansion of odd type with the same length by Lemma 18. This means that $\gamma(K) \leq 2g(K)$. $\square$

Proof of Theorem 9 Let F be a minimal genus non-orientable spanning surface for K. By Lemma 13, F is incompressible.

First, assume that the minimal length n of expansions of all fractions for K is realized by an expansion of odd type.

If F is boundary-compressible, then boundary-compression yields a minimal genus Seifert surface S for K with $\beta_1(S) = \beta_1(F) - 1$ as in the proof of Theorem 1. Note that S is isotopic to a plumbing surface corresponding to the unique expansion with only even coefficients. In particular, such expansion has length $n - 1$. This contradicts the minimality of n. Hence F is boundary-incompressible. (In this case F is isotopic to a plumbing surface by [7].)

Next, assume that only expansions of even type realize the minimal length n. If F is boundary-incompressible, then F is isotopic to a plumbing surface which corresponds to some expansion of odd type, which has length $k \geq n + 1$. Indeed, $k = n + 1$ by the minimality of $\beta_1(F)$. If F corresponds to an expansion $r + [b_1, b_2, \ldots, b_{n+1}]$, then $|b_i| \geq 2$ for each i by [7]. This expansion is not shortest, and hence the sequence $b_1, b_2, \ldots, b_{n+1}$ contains a subsequence $2\varepsilon, 3\varepsilon, \ldots, 3\varepsilon, 2\varepsilon$, $\varepsilon = \pm 1$, where the number of 3ε may be zero, by Theorem 2. But such expansion can be reduced as shown in Theorem 2. In particular, we have a shorter expansion of odd type, a contradiction. Thus F must be boundary-compressible. $\square$

Proof of Theorem 11 Let $K = S(q,p)$ with $\gamma(K) = \gamma$. We omit the trivial case $\gamma = 1$. Let $C = [a_1, b_1, a_2, b_2, \cdots]$ be a shortest expansion for p/q among those of odd type. To be precise take $[a_1, b_1, \cdots, a_{(\gamma+1)/2}]$ if γ is odd and if otherwise, take $[a_1, b_1, \cdots, b_{\gamma/2}]$. Figure 7(1) represents a Conway diagram for $K = S(q,p)$ of length γ. Deform it to Figure 7(3) through (2) corresponding to the expansion $C' = [a_1 - 1, -1, b_1, 1, a_2, -1, b_2, 1, \cdots]$, which ends with $a_{(\gamma+1)/2} + 1$ (resp. 1) if γ is odd (resp. even). Then we have a desired Conway diagram with a checkerboard surface F with $\beta(F) = \gamma$. We remark it is not preferable if $a_1 - 1 = 0$ or $a_{(\gamma+1)/2} + 1 = 0$. In that case, apply the following: Take the mirror image of K and thus change all the signs of C, apply the deformation in Figure 7, and the take the mirror image again. Then we obtain a desired Conway diagram for K. (This modification works since C has at most one coefficient ± 1 because of the minimality of γ.) $\square$

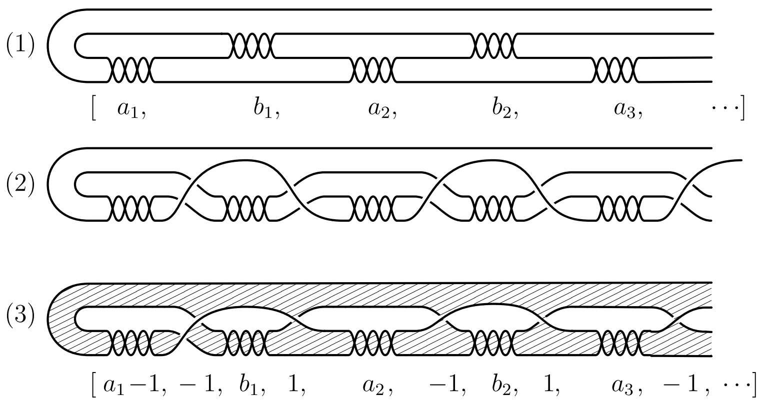

Fig. 7. Deformation of Conway diagram

5 Examples

In [12], it is proved that any positive integer can appear as the crosscap number of some pretzel knot. We can show that such examples can be found among 2-bridge knots.

As seen in Section 6, among 2-bridge knots up to 12 crossings, exactly 7_4, 8_3, 9_5, 10_3, $11a_{343}$, $11a_{363}$, $12a_{1166}$ and $12a_{1287}$ do not have a shortest expansion of odd type and hence satisfy the equality $\gamma(K) = 2g(K) + 1$. The next example also gives an infinite series of such 2-bridge knots, as a generalization of [8].

Example 19 Let $K_{m,n}$ be the 2-bridge knot corresponding to $[m, 4, 4, \ldots, 4]$ of length n for any $n \geq 1$. When $n = 1$, $K_{m,1} = S(m, 1)$.

If m is odd and $m \geq 3$, then this expansion is shortest by Theorem 2. Thus $\gamma(K_{m,n}) = n$ by Theorem 5. Also, if $m \neq m'$, then $K_{m,n}$ and $K_{m',n}$ have distinct denominators, and hence they are not equivalent. Thus we have infinitely many 2-bridge knots $K_{m,n}$ with $\gamma(K_{m,n}) = n$ for any $n \geq 1$.

If m and n are even and $m \geq 4$, $n \geq 2$, then $g(K_{m,n}) = n/2$. By Corollary 7, $\gamma(K_{m,n}) = n+1$. Also, distinct m's give distinct knots. Thus we have infinitely many 2-bridge knots $K_{m,n}$ satisfying the equality $\gamma(K_{m,n}) = 2g(K_{m,n}) + 1$.

The Murasugi sum of two minimal genus Seifert surfaces gives a minimal genus Seifert surface [6]. Finally, we give the examples of 2-bridge knots, showing that an analogous statement does not hold in non-orientable case. This is a generalization of Bessho's example [1].

Example 20 For any odd integer $m \geq 3$, let K_m be the 2-bridge knot corresponding to $[m, 2]$. Then $\gamma(K) = 2$ and its minimal genus non-orientable spanning surface F is obtained by plumbing two bands with m and 2 half-twists respectively. Let R be the Murasugi sum of two copies F_1 and F_2 of F.

Here, the plumbing disks are chosen to lie in the band with 2 half-twists of F_1 and in the band with 2 half-twists of F_2. Then $\beta_1(R) = 4$. But ∂R is the 2-bridge knot corresponding to $[m, 2, 2, m]$. Since $[m, 2, 2, m] = [m-1, -3, m-1]$, it has crosscap number 3. Also, distinct m's give inequivalent knots. Thus the Murasugi sum of two minimal genus non-orientable spanning surfaces is not necessarily minimal genus.

6 Table

Here is the table of crosscap numbers of 362 2-bridge knots up to 12 crossings. The numbering of knots with 10 or less crossings follows that of [10]. For 11, 12 crossings knots, we have used Dowker-Thistlethwaite notation. The last column gives a minimal length subtractive continued fraction expansion of p/q. We chose them to be of odd type except for the ones (indicated by $*$) where the shortest expansion is unique and of even type. We referred to [2] for 2-bridge knots up to 10 crossings, to [3] for those of 11 and 12 crossings, for which we also used a table compiled by David De Wit [5].

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
3_1	1/3	1	$[3]$	4_1	2/5	2	$[3, 2]$
5_1	1/5	1	$[5]$	5_2	3/7	2	$[2, -3]$
6_1	2/9	2	$[5, 2]$	6_2	4/11	2	$[3, 4]$
6_3	5/13	3	$[3, 2, -2]$	7_1	1/7	1	$[7]$
7_2	5/11	2	$[2, -5]$	7_3	4/13	2	$[3, -4]$
7_4	4/15	3	$[4, 4]^*$	7_5	7/17	3	$[2, -2, 3]$
7_6	7/19	3	$[3, 3, -2]$	7_7	8/21	3	$[3, 3, 3]$
8_1	2/13	2	$[7, 2]$	8_2	6/17	2	$[3, 6]$
8_3	4/17	3	$[4, -4]^*$	8_4	5/19	2	$[4, 5]$
8_6	10/23	3	$[2, -3, 3]$	8_7	9/23	3	$[3, 2, -4]$
8_8	9/25	3	$[3, 4, -2]$	8_9	7/25	3	$[4, 2, -3]$
8_{11}	10/27	3	$[3, 3, -3]$	8_{12}	12/29	4	$[3, 2, 4, 2]$
8_{13}	11/29	3	$[3, 3, 4]$	8_{14}	12/31	4	$[3, 2, -2, 2]$
9_1	1/9	1	$[9]$	9_2	7/15	2	$[2, -7]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
9_3	$6/19$	2	$[3,-6]$	9_4	$5/21$	2	$[4,-5]$
9_5	$6/23$	3	$[4,6]^*$	9_6	$5/27$	3	$[5,-2,2]$
9_7	$13/29$	3	$[2,-4,3]$	9_8	$11/31$	3	$[3,5,-2]$
9_9	$9/31$	3	$[3,-2,4]$	9_{10}	$10/33$	3	$[3,-3,3]$
9_{11}	$14/33$	3	$[2,-3,-5]$	9_{12}	$13/35$	3	$[3,3,-4]$
9_{13}	$10/37$	3	$[4,3,-3]$	9_{14}	$14/37$	3	$[3,3,5]$
9_{15}	$16/39$	4	$[2,-2,3,-2]$	9_{17}	$14/39$	3	$[3,5,3]$
9_{18}	$17/41$	4	$[2,-2,2,-3]$	9_{19}	$16/41$	4	$[3,2,-3,2]$
9_{20}	$15/41$	3	$[3,4,4]$	9_{21}	$18/43$	4	$[2,-3,-2,3]$
9_{23}	$19/45$	4	$[2,-3,-3,2]$	9_{26}	$18/47$	4	$[3,3,2,-3]$
9_{27}	$19/49$	4	$[3,2,-3,-3]$	9_{31}	$21/55$	4	$[3,3,3,3]$
10_1	$2/17$	2	$[9,2]$	10_2	$8/23$	2	$[3,8]$
10_3	$6/25$	3	$[4,-6]^*$	10_4	$7/27$	2	$[4,7]$
10_5	$13/33$	3	$[3,2,-6]$	10_6	$16/37$	3	$[2,-3,5]$
10_7	$16/43$	3	$[3,3,-5]$	10_8	$6/29$	2	$[5,6]$
10_9	$11/39$	3	$[4,2,-5]$	10_{10}	$17/45$	3	$[3,3,6]$
10_{11}	$13/43$	3	$[3,-3,4]$	10_{12}	$17/47$	3	$[3,4,-4]$
10_{13}	$22/53$	4	$[3,2,3,-4]$	10_{14}	$22/57$	4	$[3,2,-2,4]$
10_{15}	$19/43$	3	$[2,-4,-5]$	10_{16}	$14/47$	3	$[3,-3,-5]$
10_{17}	$9/41$	3	$[5,2,-4]$	10_{18}	$23/55$	4	$[2,-3,-2,4]$
10_{19}	$14/51$	3	$[4,3,5]$	10_{20}	$16/35$	3	$[2,-5,3]$
10_{21}	$16/45$	3	$[3,5,-3]$	10_{22}	$13/49$	3	$[4,4,-3]$
10_{23}	$23/59$	4	$[3,2,-3,3]$	10_{24}	$24/55$	4	$[2,-3,2,-3]$
10_{25}	$24/65$	4	$[3,3,-2,3]$	10_{26}	$17/61$	4	$[4,2,-2,3]$
10_{27}	$27/71$	4	$[3,3,3,-3]$	10_{28}	$19/53$	3	$[3,5,4]$
10_{29}	$26/63$	4	$[2,-2,3,4]$	10_{30}	$26/67$	4	$[3,2,-3,-4]$
10_{31}	$25/57$	4	$[2,-4,-2,3]$	10_{32}	$29/69$	4	$[2,-3,-3,-4]$
10_{33}	$18/65$	4	$[4,3,2,-3]$	10_{34}	$13/37$	3	$[3,6,-2]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
10_{35}	$20/49$	4	$[3, 2, 6, 2]$	10_{36}	$20/51$	4	$[3, 2, -4, 2]$
10_{37}	$23/53$	4	$[2, -3, 3, -2]$	10_{38}	$25/59$	4	$[2, -3, -4, 2]$
10_{39}	$22/61$	4	$[3, 4, -2, 2]$	10_{40}	$29/75$	5	$[3, 2, -2, 2, -2]$
10_{41}	$26/71$	4	$[3, 4, 3, -2]$	10_{42}	$31/81$	5	$[3, 3, 2, -2, 2]$
10_{43}	$27/73$	4	$[3, 3, -3, -3]$	10_{44}	$30/79$	4	$[3, 3, 4, 3]$
10_{45}	$34/89$	5	$[3, 3, 3, 2, -2]$	$11a_{13}$	$28/61$	4	$[2, -6, -3, -2]$
$11a_{59}$	$20/43$	3	$[2, -7, -3]$	$11a_{65}$	$27/59$	4	$[2, -5, 2, -2]$
$11a_{75}$	$36/83$	4	$[2, -3, 4, 3]$	$11a_{77}$	$55/131$	5	$[2, -3, -3, -3, -3]$
$11a_{84}$	$44/101$	5	$[2, -3, 3, 2, -2]$	$11a_{85}$	$47/107$	5	$[2, -4, -2, 2, 3]$
$11a_{89}$	$44/119$	5	$[3, 3, -3, -2, 2]$	$11a_{90}$	$23/87$	4	$[4, 4, -2, -3]$
$11a_{91}$	$50/129$	5	$[3, 2, -3, -3, -3]$	$11a_{93}$	$41/93$	4	$[2, -4, -5, -3]$
$11a_{95}$	$33/73$	4	$[2, -5, -3, 2]$	$11a_{96}$	$50/121$	5	$[2, -2, 3, 3, 3]$
$11a_{98}$	$18/77$	4	$[4, -4, -3, -2]$	$11a_{110}$	$35/97$	4	$[3, 4, -3, -3]$
$11a_{111}$	$37/103$	4	$[3, 5, 3, 3]$	$11a_{117}$	$49/117$	5	$[2, -3, -2, 3, 3]$
$11a_{119}$	$34/77$	4	$[2, -4, -5, -2]$	$11a_{120}$	$45/109$	5	$[2, -2, 3, 3, -2]$
$11a_{121}$	$50/119$	5	$[2, -3, -3, -3, 2]$	$11a_{140}$	$17/65$	3	$[4, 6, 3]$
$11a_{144}$	$17/73$	4	$[4, -3, 2, -2]$	$11a_{145}$	$22/83$	4	$[4, 4, -3, -2]$
$11a_{154}$	$30/67$	4	$[2, -4, 3, -2]$	$11a_{159}$	$46/111$	5	$[2, -2, 2, -3, -3]$
$11a_{166}$	$14/59$	3	$[4, -5, -3]$	$11a_{174}$	$28/79$	4	$[3, 5, -2, -3]$
$11a_{175}$	$41/105$	5	$[3, 2, -4, -2, 2]$	$11a_{176}$	$31/111$	5	$[4, 2, -3, -2, 2]$
$11a_{177}$	$21/97$	4	$[5, 3, 3, 3]$	$11a_{178}$	$34/123$	5	$[4, 2, -2, -3, -3]$
$11a_{179}$	$20/57$	3	$[3, 7, 3]$	$11a_{180}$	$25/89$	4	$[4, 2, -4, -3]$
$11a_{182}$	$13/73$	4	$[6, 2, -2, -3]$	$11a_{183}$	$34/115$	5	$[3, -3, -2, 2, 3]$
$11a_{184}$	$19/87$	4	$[5, 2, -3, -3]$	$11a_{185}$	$30/109$	4	$[4, 3, 4, 3]$
$11a_{186}$	$39/95$	5	$[2, -2, 3, -2, 2]$	$11a_{188}$	$14/67$	3	$[5, 5, 3]$
$11a_{190}$	$18/85$	4	$[5, 3, -2, -3]$	$11a_{191}$	$19/83$	4	$[4, -3, -3, 2]$
$11a_{192}$	$26/97$	4	$[4, 4, 3, -2]$	$11a_{193}$	$29/95$	4	$[3, -4, -3, -3]$
$11a_{195}$	$8/53$	3	$[7, 3, 3]$	$11a_{203}$	$11/63$	3	$[6, 4, 3]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
$11a_{204}$	30/101	4	$[3,-3,-4,-3]$	$11a_{205}$	25/91	4	$[4,3,-4,-2]$
$11a_{206}$	7/47	3	$[7,3,-2]$	$11a_{207}$	26/85	4	$[3,-4,-3,2]$
$11a_{208}$	31/105	5	$[3,-3,-2,2,-2]$	$11a_{210}$	16/73	4	$[5,2,-3,2]$
$11a_{211}$	12/67	4	$[6,2,-3,-2]$	$11a_{220}$	23/85	4	$[4,3,-3,2]$
$11a_{224}$	27/89	4	$[3,-3,3,3]$	$11a_{225}$	11/53	3	$[5,5,-2]$
$11a_{226}$	20/71	4	$[4,2,-5,-2]$	$11a_{229}$	16/71	4	$[4,-2,3,-2]$
$11a_{230}$	8/51	3	$[6,-3,-3]$	$11a_{234}$	5/37	3	$[7,-2,2]$
$11a_{235}$	22/71	4	$[3,-4,2,-2]$	$11a_{236}$	29/99	5	$[3,-2,2,-2,2]$
$11a_{238}$	12/65	4	$[5,-2,2,-2]$	$11a_{242}$	9/47	3	$[5,-4,2]$
$11a_{243}$	20/69	4	$[3,-2,4,-2]$	$11a_{246}$	13/41	3	$[3,-6,2]$
$11a_{247}$	2/19	2	$[9,-2]$	$11a_{306}$	29/105	4	$[4,3,3,4]$
$11a_{307}$	18/83	4	$[5,2,-2,-4]$	$11a_{308}$	15/71	3	$[5,4,4]$
$11a_{309}$	25/93	4	$[4,3,-2,-4]$	$11a_{310}$	14/61	3	$[4,-3,-5]$
$11a_{311}$	18/79	4	$[4,-3,-2,3]$	$11a_{333}$	14/65	3	$[5,3,5]$
$11a_{334}$	9/49	3	$[5,-2,4]$	$11a_{335}$	17/75	4	$[4,-2,2,-3]$
$11a_{336}$	11/59	3	$[5,-3,-4]$	$11a_{337}$	26/89	4	$[3,-2,3,4]$
$11a_{339}$	13/55	3	$[4,-4,3]$	$11a_{341}$	19/61	3	$[3,-5,-4]$
$11a_{342}$	4/29	2	$[7,-4]$	$11a_{343}$	4/31	3	$[8,4]^*$
$11a_{355}$	7/45	3	$[6,-2,3]$	$11a_{356}$	24/79	4	$[3,-3,2,-3]$
$11a_{357}$	27/91	4	$[3,-3,-3,3]$	$11a_{358}$	5/31	2	$[6,-5]$
$11a_{359}$	10/53	3	$[5,-3,3]$	$11a_{360}$	10/57	3	$[6,3,-3]$
$11a_{363}$	6/35	3	$[6,6]^*$	$11a_{364}$	3/25	2	$[8,-3]$
$11a_{365}$	16/51	3	$[3,-5,3]$	$11a_{367}$	1/11	1	$[11]$
$12a_{38}$	33/71	4	$[2,-6,2,3]$	$12a_{169}$	23/49	3	$[2,-8,-3]$
$12a_{197}$	32/69	4	$[2,-6,3,2]$	$12a_{204}$	76/173	5	$[2,-4,-3,-3,-3]$
$12a_{206}$	47/105	4	$[2,-4,4,3]$	$12a_{221}$	66/169	5	$[3,2,-4,-3,-3]$
$12a_{226}$	75/181	6	$[2,-2,2,-2,2,3]$	$12a_{239}$	40/87	4	$[2,-6,-3,2]$
$12a_{241}$	57/127	5	$[2,-4,3,2,-2]$	$12a_{243}$	60/133	5	$[2,-4,2,3,3]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
$12a_{247}$	71/163	5	$[2, -3, 3, 3, 3]$	$12a_{251}$	59/159	5	$[3, 3, -3, 2, 3]$
$12a_{254}$	23/97	4	$[4, -4, 2, 3]$	$12a_{255}$	28/107	4	$[4, 5, -2, -3]$
$12a_{257}$	80/191	6	$[2, -2, 2, 3, -2, -3]$	$12a_{259}$	52/115	4	$[2, -5, -4, -3]$
$12a_{300}$	68/155	5	$[2, -3, 2, 4, 3]$	$12a_{302}$	61/147	5	$[2, -2, 2, -4, -3]$
$12a_{303}$	64/153	5	$[2, -2, 2, 5, 3]$	$12a_{306}$	64/147	5	$[2, -3, 3, 3, -2]$
$12a_{307}$	69/157	5	$[2, -4, -3, -3, 2]$	$12a_{330}$	43/95	4	$[2, -5, -4, 2]$
$12a_{378}$	45/127	4	$[3, 6, 3, 3]$	$12a_{379}$	17/71	3	$[4, -6, -3]$
$12a_{380}$	20/77	3	$[4, 7, 3]$	$12a_{384}$	62/151	5	$[2, -2, 3, -3, -3]$
$12a_{385}$	66/161	5	$[2, -2, 4, 3, 3]$	$12a_{406}$	74/179	6	$[2, -2, 3, 2, -2, 2]$
$12a_{425}$	37/81	4	$[2, -5, 3, -2]$	$12a_{437}$	65/149	5	$[2, -3, 2, -3, -3]$
$12a_{447}$	43/121	4	$[3, 5, -3, -3]$	$12a_{454}$	27/103	4	$[4, 5, -2, 2]$
$12a_{471}$	38/85	4	$[2, -4, 5, 2]$	$12a_{477}$	70/169	6	$[2, -2, 2, -2, 3, 2]$
$12a_{482}$	22/93	4	$[4, -4, 3, 2]$	$12a_{497}$	81/209	6	$[3, 2, -3, -2, 2, 3]$
$12a_{498}$	76/207	5	$[3, 4, 3, 3, 3]$	$12a_{499}$	89/233	6	$[3, 3, 3, 3, 2, -2]$
$12a_{500}$	60/167	5	$[3, 4, -2, -3, -3]$	$12a_{501}$	55/199	5	$[4, 3, 3, 3, 3]$
$12a_{502}$	37/91	4	$[2, -2, 6, 3]$	$12a_{506}$	68/185	5	$[3, 3, -2, -4, -3]$
$12a_{508}$	56/129	5	$[2, -3, 3, -2, 2]$	$12a_{510}$	81/193	6	$[2, -2, 2, 3, 3, -2]$
$12a_{511}$	51/125	5	$[2, -2, 5, 2, -2]$	$12a_{512}$	64/151	5	$[2, -3, -4, 2, 3]$
$12a_{514}$	79/187	5	$[2, -3, -4, -3, -3]$	$12a_{517}$	52/145	4	$[3, 5, 4, 3]$
$12a_{518}$	34/157	5	$[5, 2, -2, -3, -3]$	$12a_{519}$	25/111	4	$[4, -2, 4, 3]$
$12a_{520}$	36/133	4	$[4, 3, -4, -3]$	$12a_{521}$	48/113	4	$[2, -3, -6, -3]$
$12a_{522}$	73/173	5	$[2, -3, -3, 3, 3]$	$12a_{528}$	67/183	5	$[3, 4, 4, 2, -2]$
$12a_{532}$	33/125	4	$[4, 5, 3, -2]$	$12a_{533}$	31/137	5	$[4, -2, 3, 2, -2]$
$12a_{534}$	44/163	5	$[4, 3, -3, -2, 2]$	$12a_{535}$	47/175	5	$[4, 3, -2, -3, -3]$
$12a_{536}$	29/137	4	$[5, 4, 3, 3]$	$12a_{537}$	50/179	5	$[4, 2, -3, -3, -3]$
$12a_{538}$	13/83	4	$[6, -2, 2, 3]$	$12a_{539}$	44/145	5	$[3, -3, 3, 2, -2]$
$12a_{540}$	49/165	5	$[3, -3, -3, 2, 3]$	$12a_{541}$	41/153	4	$[4, 4, 4, 3]$
$12a_{545}$	63/143	5	$[2, -4, -3, 2, -2]$	$12a_{549}$	26/111	4	$[4, -4, -3, 2]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
$12a_{550}$	$34/149$	5	$[4,-2,2,3,3]$	$12a_{551}$	$18/103$	4	$[6,3,-2,-3]$
$12a_{552}$	$30/131$	4	$[4,-3,-4,-3]$	$12a_{579}$	$49/177$	5	$[4,2,-2,-4,-3]$
$12a_{580}$	$11/69$	3	$[6,-4,-3]$	$12a_{581}$	$36/119$	4	$[3,-3,4,3]$
$12a_{582}$	$39/131$	4	$[3,-3,-5,-3]$	$12a_{583}$	$45/161$	5	$[4,2,-3,-3,2]$
$12a_{584}$	$31/143$	5	$[5,2,-2,-3,2]$	$12a_{585}$	$50/181$	5	$[4,3,3,3,-2]$
$12a_{595}$	$30/139$	4	$[5,3,4,3]$	$12a_{596}$	$14/81$	3	$[6,5,3]$
$12a_{597}$	$26/123$	4	$[5,4,3,-2]$	$12a_{600}$	$25/109$	4	$[4,-3,-4,2]$
$12a_{601}$	$34/127$	4	$[4,4,5,2]$	$12a_{643}$	$23/99$	4	$[4,-3,3,-2]$
$12a_{644}$	$30/113$	4	$[4,4,-3,2]$	$12a_{649}$	$27/127$	4	$[5,3,-3,-3]$
$12a_{650}$	$46/165$	5	$[4,2,-2,3,3]$	$12a_{651}$	$17/97$	4	$[6,3,-2,2]$
$12a_{652}$	$46/155$	5	$[3,-3,-3,2,-2]$	$12a_{682}$	$29/107$	4	$[4,3,-4,2]$
$12a_{684}$	$41/135$	5	$[3,-3,2,-2,2]$	$12a_{690}$	$20/89$	4	$[4,-2,5,2]$
$12a_{691}$	$12/77$	4	$[6,-2,3,2]$	$12a_{713}$	$39/139$	5	$[4,2,-3,2,-2]$
$12a_{714}$	$19/107$	4	$[6,3,3,-2]$	$12a_{715}$	$50/169$	5	$[3,-3,-3,-3,2]$
$12a_{716}$	$5/43$	3	$[9,2,-2]$	$12a_{717}$	$28/89$	4	$[3,-6,-2,2]$
$12a_{718}$	$41/141$	5	$[3,-2,4,2,-2]$	$12a_{720}$	$21/113$	4	$[5,-3,-3,-3]$
$12a_{721}$	$50/171$	5	$[3,-2,3,3,3]$	$12a_{722}$	$3/29$	2	$[10,3]$
$12a_{723}$	$20/63$	3	$[3,-7,-3]$	$12a_{724}$	$31/107$	4	$[3,-2,5,3]$
$12a_{726}$	$19/103$	4	$[5,-2,3,3]$	$12a_{727}$	$46/157$	5	$[3,-2,2,-3,-3]$
$12a_{728}$	$29/133$	5	$[5,2,-2,2,-2]$	$12a_{729}$	$46/167$	5	$[4,3,3,-2,2]$
$12a_{731}$	$22/105$	4	$[5,4,-2,2]$	$12a_{732}$	$18/95$	4	$[5,-3,2,3]$
$12a_{733}$	$14/73$	3	$[5,-5,-3]$	$12a_{736}$	$43/141$	5	$[3,-3,2,3,-2]$
$12a_{738}$	$37/119$	4	$[3,-5,-3,-3]$	$12a_{740}$	$35/113$	4	$[3,-4,3,3]$
$12a_{743}$	$12/79$	4	$[7,2,-2,2]$	$12a_{744}$	$8/61$	3	$[8,3,3]$
$12a_{745}$	$8/59$	3	$[7,-3,-3]$	$12a_{758}$	$31/113$	4	$[4,3,5,-2]$
$12a_{759}$	$9/61$	3	$[7,4,-2]$	$12a_{760}$	$34/111$	4	$[3,-4,-4,2]$
$12a_{761}$	$41/139$	5	$[3,-2,2,4,-2]$	$12a_{762}$	$7/51$	3	$[7,-3,2]$
$12a_{763}$	$30/97$	4	$[3,-4,3,-2]$	$12a_{764}$	$39/133$	5	$[3,-2,2,-3,2]$

knot	p/q	γ	continued fraction	knot	p/q	γ	continued fraction
$12a_{773}$	20/91	4	$[4,-2,-5,2]$	$12a_{774}$	16/89	4	$[5,-2,-4,2]$
$12a_{775}$	16/87	4	$[5,-2,3,-2]$	$12a_{791}$	13/63	3	$[5,6,-2]$
$12a_{792}$	24/85	4	$[4,2,-5,2]$	$12a_{796}$	11/57	3	$[5,-5,2]$
$12a_{797}$	24/83	4	$[3,-2,5,-2]$	$12a_{802}$	15/47	3	$[3,-7,2]$
$12a_{803}$	2/21	2	$[11,2]$	$12a_{1023}$	29/127	4	$[4,-3,-3,-4]$
$12a_{1024}$	40/149	4	$[4,4,3,4]$	$12a_{1029}$	19/81	3	$[4,-4,-5]$
$12a_{1030}$	19/91	3	$[5,5,4]$	$12a_{1033}$	25/107	4	$[4,-3,2,4]$
$12a_{1034}$	32/121	4	$[4,5,2,-3]$	$12a_{1039}$	37/137	4	$[4,3,-3,-4]$
$12a_{1040}$	26/115	4	$[4,-2,3,4]$	$12a_{1125}$	23/101	4	$[4,-2,2,5]$
$12a_{1126}$	26/119	4	$[5,2,-3,-4]$	$12a_{1127}$	22/97	4	$[5,2,3,-4]$
$12a_{1128}$	9/59	3	$[7,2,-4]$	$12a_{1129}$	23/105	4	$[4,-2,-4,3]$
$12a_{1130}$	27/125	4	$[5,3,3,-3]$	$12a_{1131}$	11/73	3	$[7,3,4]$
$12a_{1132}$	40/131	4	$[3,-4,-3,-4]$	$12a_{1133}$	47/159	5	$[3,-2,2,3,4]$
$12a_{1134}$	7/53	3	$[8,2,-3]$	$12a_{1135}$	32/103	4	$[3,-4,2,4]$
$12a_{1136}$	43/147	5	$[3,-2,3,2,-3]$	$12a_{1138}$	14/79	3	$[6,3,5]$
$12a_{1139}$	18/101	4	$[6,3,2,-3]$	$12a_{1140}$	18/97	4	$[5,-2,2,4]$
$12a_{1145}$	15/79	3	$[5,-4,-4]$	$12a_{1146}$	34/117	4	$[3,-2,4,4]$
$12a_{1148}$	23/73	3	$[3,-6,-4]$	$12a_{1149}$	4/35	2	$[9,4]$
$12a_{1157}$	5/39	2	$[8,5]$	$12a_{1158}$	16/77	3	$[5,5,-3]$
$12a_{1159}$	24/113	4	$[5,3,-2,3]$	$12a_{1161}$	14/75	3	$[5,-3,-5]$
$12a_{1162}$	13/69	3	$[5,-3,4]$	$12a_{1163}$	24/103	4	$[4,-3,2,-3]$
$12a_{1165}$	16/67	3	$[4,-5,3]$	$12a_{1166}$	4/33	3	$[8,-4]^*$
$12a_{1273}$	11/61	3	$[6,2,-5]$	$12a_{1274}$	17/95	4	$[6,2,-2,3]$
$12a_{1275}$	44/149	5	$[3,-3,-2,2,-3]$	$12a_{1276}$	13/75	3	$[6,4,-3]$
$12a_{1277}$	37/121	4	$[3,-4,-3,3]$	$12a_{1278}$	6/41	2	$[7,6]$
$12a_{1279}$	10/67	3	$[7,3,-3]$	$12a_{1281}$	33/109	4	$[3,-3,3,-3]$
$12a_{1282}$	10/63	3	$[6,-3,3]$	$12a_{1287}$	6/37	3	$[6,-6]^*$

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