Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser. IV, Vol. 27 129-133, Dec. 2001

ON CONFORMAL TRANSFORMATIONS AND FIBRED RIEMANNIAN SPACES

YOSHIO AGAOKA* AND BYUNG HAK KIM**

** Department of Mathematics, Faculty of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739-8521, Japan*
*** Department of Mathematics and Institute of Natural Sciences, Kyung Hee University, Suwon 449-701, Korea*

Abstract

We prove that a conformal transformation $\phi:(M,g^*)\to(M,g)$ with $Ric_{g^*}=Ric_g$ preserves Riemannian curvature tensors. Moreover, in a fibred Riemannian space, if any horizontal mapping covering is a Ricci-invariant conformal transformation and the total space is Einstein, then each fibre is a totally geodesic submanifold of the total space.

1. Introduction

Let M be an m-dimensional Riemannian manifold with metric tensor g. A diffeomorphism $\phi: M\to M$ is called a conformal transformation if there is a positive function ρ on M such that $\phi^* g=\rho^{-2}g$. In this case, we express ϕ as $\phi:(M,g^*)\to(M,g)$, where $g^*=\rho^{-2}g$. If ρ is constant, then ϕ is called a homothety [3,4,6]. A classical theorem of Liouville determines all possible conformal transformations between the Euclidean space. As a generalization, we call a conformal transformation $\phi:(M,g^*)\to(M,g)$ a Liouville transformation if $Ric_{g^*}=Ric_g$ [3], where Ric_g is the Ricci curvature with respect to g. In [2,5], they proved that a globally defined Liouville transformation of a complete Riemannian manifold is a homothety.

In this paper, we study the local properties of Liouville transformations by use of adapted coordinates and fibred Riemannian spaces. We prove that

Theorem 1. *Let $\phi:(M,g^*)\to(M,g)$ $(g^*=\rho^{-2}g)$ be a conformal transformation with $Ric_{g^*}=Ric_g$. Then ϕ preserves Riemannian curvature tensors. That is $R_{g^*}=R_g$, where R_g is the Riemannian curvature tensor with respect to g.*

On the other hand, in a fibred Riemannian space (M,B,G,π), the horizontal mapping covering is a transformation between fibres. If any local horizontal mapping covering in a fibred Riemannian space is an isometry (resp. conformal mapping), then we call it a fibred Riemannian space with isometric fibre (resp. conformal fibre). It is well known that [2] a necessary and sufficient condition for a fibred Riemannian space have isometric fibre (resp. conformal fibre) is

---

Key words: Conformal transformation, fibred Riemannian space, Lie derivative.
** This work was supported by Korea Research Foundation Grant KRF-2000-042-D00007.
Received October 1 2001; Accepted November 1 2001

$$\left(\mathcal{L}_{X^L} G^V\right)^V = 0 \quad \left(resp.\ \rho G^V\right)$$

for any vector field X in B, ρ a function on M and X^L is a lift of X. In a fibred Riemannian space with isometric fibre (resp. conformal fibre), each fibre is a totally geodesic (resp. totally umbilical) submanifold of the total space and vice versa [2]. From these facts, we have

Theorem 2. *In a fibred Riemannian space, if any local horizontal mapping covering is a Ricci-invariant conformal transformation and the total space is a non-Euclidean Einstein space, then each fibre is a totally geodesic submanifold of the total space.*

Throughout this paper, the ranges of indices are as follows:

$$i, j, k, \cdots : 1, 2, \cdots, n + p = m,$$
$$a, b, c, \cdots : 1, 2, \cdots, n,$$
$$\alpha, \beta, \gamma, \cdots : n + 1, \cdots, m,$$

unless otherwise stated.

2. Fibred Riemannian space

Let (M, B, G, π) be a fibred Riemannian space, where (M, G) is the total space with a projectable Riemannian metric G, B the base space and π the projection $M \to B$ of maximum rank everywhere. We suppose that the dimensions of M and B are m and n respectively. The fibre at any point Q in M is of dimension $p = m - n$ and denoted by $F(Q)$ or simply by F. We suppose that each fibre is connected. The horizontal and vertical parts of a tensor field T are denoted by T^H and T^V respectively. Let $h = \left(h_{\gamma\beta}{}^a\right)$ and $L = \left(L_{cb}{}^\beta\right)$ be the components of the second fundamental tensor and the normal connection of each fibre respectively. For a basis $\left\{E_b, C_\beta\right\}$ of the tangent space of M and $\left\{E^a, C^\alpha\right\}$ dual to $\left\{E_b, C_\beta\right\}$, it is well known that [2,5]

$$\tilde{\nabla}_j E^h{}_b = \Gamma_{cb}{}^a E_j{}^c E^h{}_a - L_{cb}{}^\alpha E_j{}^c C^h{}_\alpha + L_b{}^a{}_\gamma C_j{}^\gamma E^h{}_a - h_\gamma{}^\alpha{}_b C_j{}^\gamma C^h{}_\alpha, \tag{2.1}$$

$$\mathcal{L}_c E^a = 0, \quad \mathcal{L}_c C^\alpha = 2L_{cb}{}^\alpha E^b - P_{c\beta}{}^\alpha C^\beta, \tag{2.2}$$

where Γ is the Christoffel symbol of the Riemannian metric of B, $\mathcal{L}_c$ the Lie derivation with respect to E_c and $P_{c\beta}{}^\alpha$ is defined by

$$\left[E_c, C_\beta\right] = P_{c\beta}{}^\alpha C_\alpha$$

for the bracket operatoin [,].

3. Proof of Theorem 1

Let $\phi : \left(M, g^*\right) \to (M, g)\ \left(g^* = \rho^{-2} g\right)$ be a conformal transformation. The geometric objects $\{R, S, K, \Gamma\}$

are the Riemannian curvature, Ricci curvature, scalar curvature, and the Christoffel symbol of (M, g) respectively. $\{R^*, S^*, K^*, \Gamma^*\}$ are the corresponding objects of (M, g^*). Then we have [1]

$$(3.1)\qquad R^*{}_{kji}{}^h = R_{kji}{}^h + \rho^{-1}\left(\delta_k^h \nabla_j \rho_i - \delta_j^h \nabla_k \rho_i + g_{ji}\nabla_k \rho^h - g_{ki}\nabla_j \rho^h\right) - \rho^{-2}\rho_l \rho^l \left(\delta_k^h g_{ji} - \delta_j^h g_{ki}\right),$$

$$(3.2)\qquad S^*{}_{ji} = S_{ji} + (m-2)\rho^{-1}\nabla_j \rho^i + \rho^{-1} g_{ji}\nabla_l \rho^l - (m-1)\rho^{-2}\rho_l \rho^l g_{ji},$$

$$(3.3)\qquad K^* = \rho^2 K + 2m^{-1}\rho \nabla_l \rho^l - \rho_l \rho^l.$$

If we assume that $S^*{}_{ji} = S_{ji}$, then we get

$$(3.4)\qquad \nabla_j \rho_i = \frac{1}{2\rho}\rho_l \rho^l g_{ji}.$$

For an arc length u of a ρ-curve, if we take an adapted coordinate system (u, u^h), then the metric ds^2 of M is given by

$$(3.5)\qquad ds^2 = du^2 + \{\rho'(u)\}^2 \bar{d}s^2$$

where $\bar{d}s^2 = f_{ji}du^j du^i$ is the metric form of the ρ-hypersurface $\overline{M}$ of M. Along the ρ-curve, from (3.4), we get

$$(3.6)\qquad 2\rho\rho'' = (\rho')^2.$$

The general solution of (3.6) is given by

$$(3.7)\qquad \rho = (Au + B)^2,$$

where A and B are constants. If M is complete, then $\rho = B^2$, that is ϕ is a homothety. Hence it has a meaning only in the case of local version. For an adapted coordinate, the metric of M is given by

$$(3.8)\qquad ds^2 = du^2 + 4A^2(Au + B)^2 \bar{d}s^2.$$

Then the Christoffel symbols of M are given by

$$(3.9)\qquad \begin{aligned} &\Gamma_{1\,1}^{1} = \Gamma_{1\,i}^{1} = \Gamma_{1\,1}^{h} = 0, \\ &\Gamma_{ji}{}^{1} = -4A^3(Au + B)f_{ji}, \\ &\Gamma_{1\,i}^{h} = \frac{A}{Au + B}\delta_i^h, \\ &\Gamma_{ji}{}^{h} = \bar{\Gamma}_{ji}{}^{h}. \end{aligned}$$

Hence, the non-zero component of the curvature tensor R of M is

$$R_{kji}{}^h = \bar{R}_{kji}{}^h - 4A^4\left(\delta_k^h f_{ji} - \delta_j^h f_{ki}\right).$$

The Riemannian metric of (M, g^*) is given by

$$(3.10)\qquad ds^{*2} = \rho^{-2}ds^2 = \frac{1}{(Au + B)^4}du^2 + \frac{4A^2}{(Au + B)^2}\bar{d}s^2,$$

that is

$$g_{11}{}^* = \frac{1}{(Au + B)^4}, \qquad g_{ji}{}^* = \frac{4A^2}{(Au + B)^2}f_{ji}.$$

So, the Christoffel symbols of ds^{*2} are given by

$$\Gamma^{*}{}_{1}{}^{1}{}_{i} = \Gamma^{*}{}_{1}{}^{h}{}_{1} = 0, \qquad \Gamma^{*}{}_{1}{}^{1}{}_{1} = -\frac{2A}{Au+B},$$

$$(3.11)\quad \Gamma^{*}{}_{ji}{}^{1} = 4A^{3}(Au+B)f_{ji}, \qquad \Gamma^{*}{}_{1}{}^{h}{}_{i} = -\frac{A}{Au+B}\delta_{i}^{h},$$

$$\Gamma^{*}{}_{j}{}^{h}{}_{i} = \bar{\Gamma}_{j}{}^{h}{}_{i}.$$

Therefore, we can calculate the non-zero component of the curvature tensor of (M, g^{*}) as

$$(3.12)\quad R^{*}{}_{kji}{}^{h} = \bar{R}_{kji}{}^{h} + \Gamma^{*}{}_{k}{}^{h}{}_{1}\Gamma^{*}{}_{j}{}^{1}{}_{i} - \Gamma^{*}{}_{j}{}^{h}{}_{1}\Gamma^{*}{}_{k}{}^{1}{}_{i} = \bar{R}_{kji}{}^{h} - 4A^{4}\left(\delta_{k}^{h}f_{ji} - \delta_{j}^{h}f_{ki}\right).$$

Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

From (2.1), we have

$$(4.1)\quad \nabla_{h}X^{l} = \nabla_{h}\left(X^{b}E^{l}{}_{b}\right) = \left(\partial_{h}X^{b}\right)E^{l}{}_{b} + X^{b}\left\{\Gamma_{cb}{}^{a}E_{h}{}^{c}E^{l}{}_{a} - L_{cb}{}^{\alpha}E_{h}{}^{c}C^{l}{}_{\alpha} + L_{b}{}^{a}{}_{\gamma}C_{h}{}^{\gamma}E^{l}{}_{a} - h_{\gamma}{}^{\alpha}{}_{b}C_{h}{}^{\gamma}C^{l}{}_{\alpha}\right\}.$$

Let $\tilde{S}$ be the Ricci tensor of (M, G). By use of (2.2) and (4.1), we obtain

$$(4.2)\quad \left(\pounds_{X^{L}}\tilde{S}^{V}\right)^{V} = \left(\pounds_{X^{L}}\tilde{S}\right)^{V} = \left(\pounds_{X^{j}\partial_{j}}\tilde{S}_{hk}\right)C^{h}\otimes C^{k}.$$

Since

$$(4.3)\quad \pounds_{X^{j}\partial_{j}}\tilde{S}_{hk} = X^{j}\nabla_{j}\tilde{S}_{hk} + \tilde{S}_{lk}\nabla_{h}X^{l} + \tilde{S}_{hl}\nabla_{k}X^{l},$$

we have

$$(4.4)\quad \left(\pounds_{X^{L}}\tilde{S}_{hk}\right)C^{h}{}_{\delta}C^{k}{}_{\epsilon} = \left(X^{j}\nabla_{j}\tilde{S}_{hk}\right)C^{h}{}_{\delta}\otimes C^{k}{}_{\epsilon} + \tilde{S}_{a\delta}L_{b}{}^{a}{}_{\epsilon}X^{b} - \tilde{S}_{\alpha\delta}h_{\epsilon}{}^{\alpha}{}_{b}X^{b} + \tilde{S}_{a\epsilon}L_{b}{}^{a}{}_{\delta}X^{b} - \tilde{S}_{\alpha\epsilon}h_{\delta}{}^{\alpha}{}_{b}X^{b}$$

by substituting (4.1) into (4.3) and taking account of (4.2). In a fibred Riemannian space, if any local horizontal mapping covering is a Ricci-invariant conformal transformation, then $\left(\pounds_{X^{L}}\tilde{S}^{V}\right)^{V} = 0$. Therefore the condition that the metric G on M is Einstein and the equations (4.2) and (4.4) imply Theorem 2.

References

1. A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin (1987).
2. S. Ishihara and M. Konishi, *Differential geometry of fibred spaces*, Publ. Study Group of Geometry, Kyoto Univ. **8** (1973).
3. W. Kühnel and H.B.Rademacher, *Conformal diffeomorphisms preserving the Ricci tensor*, Proc. Amer. Math. Soc., **123** (1995), 2841-2848.
4. Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Amer. Math. Soc., **117** (1965), 251-275.
5. Y. Tashiro and B.H. Kim, *Almost complex and almost contact structures in fibred Riemannian spaces*, Hiroshima Math. J. **18** (1988), 161-188.
6. Xingwang Xu, *Prescribing a Ricci tensor in a conformal class of Riemannian metrics*, Proc. Amer. Math. Soc., **115** (1992), 455-459, **118** (1993), 333.