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CONHARMONICALLY FLAT FIBRED RIEMANNIAN SPACE

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ABSTRACT. We prove that a conharmonically flat fibred Riemannian space with totally geodesic fibre is locally the product manifold of two spaces of constant curvature with constant scalar curvatures K and $-K$ respectively.

1. Introduction

It is well known that the angle between two vectors at a point is invariant under conformal transformations. But, in general, the harmonicity of functions, vectors and forms are not preserved by conformal transformations. Related to this fact, Y. Ishi [2] have studied conharmonic transformation which is a conformal transformation that preserves the harmonicity of a certain function. In [5], it is proved that the conharmonic curvature tensor is invariant under conharmonic transformations. In this paper, we study the fibred Riemannian spaces with vanishing conharmonic curvature tensor. It is also proved that conharmonically flat manifold M is locally the product of two spaces of constant curvature with constant scalar curvatures K and $-K$ respectively.

Key Words : conharmonically flat, fibred Riemannian space, space of constant curvature.

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Throughout this paper, the ranges of indices are as follows :

$$i, j, k, \cdots : 1, 2, \cdots, n+p = m,$$

$$a, b, c, \cdots : 1, 2, \cdots, n,$$

$$x, y, z, \cdots : n+1, \cdots, n+p$$

unless otherwise stated.

2. Conharmonic transformation

Let M be an m-dimensional Riemannian manifold with metric tensor G and ρ a positive function on M. Then the Riemannian manifold $(M, \bar{G} = e^{2\rho}G)$ is conformally diffeomorphic to (M, G) and the conformal diffeomorphism $\phi : (M, G) \to (M, \bar{G})$ is called a conformal transformation. When ρ is constant, ϕ is called a homothety. A harmonic function is defined as a function whose Laplacian vanishes. It is well known that the harmonic function is not in general transformed into a harmonic function by conformal transformations [2,5]. In this point of view, Y. Ishi [2] have studied the condition on ρ in order that the function

$$\bar{f} = e^{2\alpha\rho} f \tag{2.1}$$

may become a harmonic function with respect to the Riemannian metric $\bar{G}$ for a suitable constant α and a harmonic function f. In [2], it is proved that

Proposition 2.1. *Let f be a harmonic function on (M, G) and $\alpha = \frac{2-m}{4}$ $(m > 2)$. Then the function $\bar{f}$ defined by (2.1) is harmonic for $\bar{G}$ if and only if*

$$\nabla\rho + \frac{m-2}{2}\|d\rho\|^2 = 0. \tag{2.2}$$

Y. Ishi [2] defined a conharmonic transformation which is the conformal transformation $\phi : (M, G) \to (M, \bar{G})$ satisfying the equation (2.2). The conharmonic curvature tensor T is defined by

$$T_{kji}{}^h = R_{kji}{}^h + \frac{1}{m-2}\{S_{ik}\delta_j^h + S_j{}^h G_{ik} - S_{ij}\delta_k^h - S_k{}^h G_{ij}\} \tag{2.3}$$

and T is invariant under conharmonic transformations [1,2,5].

3. Fibred Riemannian space

Let (M, B, G, π) be a fibred Riemannian space, that is, M an m-dimensional total space with projectable Riemannian metric G, B an n-dimensional base space, and $\pi : M \to B$ a projection with maximal rank n. The fibre passing through a point $x \in M$ is denoted by $F(x)$ or generally F, which is a p-dimensional submanifold of M, where $p = m - n$.

Let $h = (h_{\gamma\beta}{}^{a})$ and $L = (L_{cb}{}^{\beta})$ be the components of the second fundamental tensor and normal connection of each fibre respectively. The geometric objects $\{\tilde{R}, \tilde{S}, \tilde{K}\}$ are the Riemannian curvature, Ricci curvature and scalar curvature tensors of M respectively. $\{R, S, K\}$ and $\{\bar{R}, \bar{S}, \bar{K}\}$ are the corresponding objects of B and F. Then the structure equations are given by [3,4]

$$\tilde{R}_{dcb}{}^{a} = R_{dcb}{}^{a} - L_{d}{}^{a}{}_{\epsilon} L_{cb}{}^{\epsilon} + L_{c}{}^{a}{}_{\epsilon} L_{db}{}^{\epsilon} + 2L_{dc}{}^{\epsilon} L_{b}{}^{a}{}_{\epsilon}, \tag{3.1}$$

$$\tilde{R}_{d\gamma b}{}^{\alpha} = -{}^{*}\nabla_{d} h_{\gamma}{}^{\alpha}{}_{b} +{}^{**} \nabla_{\gamma} L_{db}{}^{\alpha} + L_{d}{}^{e}{}_{\gamma} L_{eb}{}^{\alpha} + h_{\gamma}{}^{\epsilon}{}_{d} h_{\epsilon}{}^{\alpha}{}_{b}, \tag{3.2}$$

$$\tilde{R}_{\delta\gamma\beta}{}^{\alpha} = \bar{R}_{\delta\gamma\beta}{}^{\alpha} + h_{\delta\beta}{}^{e} h_{\gamma}{}^{\alpha}{}_{e} - h_{\gamma\beta}{}^{e} h_{\delta}{}^{\alpha}{}_{e}, \tag{3.3}$$

where we put

$$^{*}\nabla_{d} h_{\gamma\beta}{}^{a} = \partial_{d} h_{\gamma\beta}{}^{a} + \Gamma^{a}_{de} h_{\gamma\beta}{}^{e} - Q_{d\gamma}{}^{\epsilon} h_{\epsilon\beta}{}^{a} - Q_{d\beta}{}^{\epsilon} h_{\gamma\epsilon}{}^{a}, \tag{3.4}$$

$$^{**}\nabla_{\delta} L_{cb}{}^{\alpha} = \partial_{\delta} L_{cb}{}^{\alpha} + \bar{\Gamma}^{\alpha}_{\delta\epsilon} L_{cb}{}^{\epsilon} - L_{c}{}^{e}{}_{\delta} L_{eb}{}^{\alpha} - L_{b}{}^{e}{}_{\delta} L_{ce}{}^{\alpha}, \tag{3.5}$$

$$Q_{c\beta}{}^{\alpha} = P_{c\beta}{}^{\alpha} - h_{\beta}{}^{\alpha}{}_{c}, \tag{3.6}$$

and $P_{c\beta}{}^{\alpha}$ are local functions defined by

$$\mathcal{L}_{C_\beta} C^{\alpha} = P_{d\beta}{}^{\alpha} E^{d},$$

where $\{E^{a}, C^{\alpha}\}$ are dual to the local frame $\{E_b, C_\beta\}$ of M and $\mathcal{L}_{C_\beta}$ is the Lie derivative with respect to C_β. Let g and $\bar{g}$ be the Riemannian metrics on B and F respectively.

The Ricci curvature and scalar curvature are given by

$$\tilde{S}_{cb} = S_{cb} - 2L_{ce}{}^{\epsilon} L_{b}{}^{e}{}_{\epsilon} - h_{\beta\alpha c} h^{\beta\alpha}{}_{b} + \frac{1}{2}({}^{*}\nabla_c h_{\epsilon}{}^{\epsilon}{}_{b} + {}^{*}\nabla_b h_{\epsilon}{}^{\epsilon}{}_{c}), \tag{3.7}$$

$$\tilde{S}_{\gamma b} = {}^{**}\nabla_{\gamma} h_{\epsilon}{}^{\epsilon}{}_{b} - {}^{**}\nabla_{\epsilon} h_{\gamma}{}^{\epsilon}{}_{b} + {}^{*}\nabla_{e} L_{b}{}^{e}{}_{\gamma} - 2h_{\gamma}{}^{\epsilon}{}_{e} L_{b}{}^{e}{}_{\epsilon}, \tag{3.8}$$

$$\tilde{S}_{\gamma\beta} = \bar{S}_{\gamma\beta} - h_{\gamma\beta}{}^{e} h_{\epsilon}{}^{\epsilon}{}_{e} + {}^{*}\nabla_{e} h_{\gamma\beta}{}^{e} + L_{ae\gamma} L^{ae}{}_{\beta}, \tag{3.9}$$

$$\tilde{K} = K + \bar{K} - ||L_{cb}{}^{\alpha}||^2 - ||h_{\gamma\beta}{}^{a}||^2 - h_{\gamma}{}^{\gamma}{}_{e} h_{\beta}{}^{\beta e} + 2^{*}\nabla_{e} h_{\epsilon}{}^{\epsilon e}, \tag{3.10}$$

where we put

$$^{**}\nabla_{\delta} h_{\gamma\beta}{}^{a} = \partial_{\delta} h_{\gamma\beta}{}^{a} - \bar{\Gamma}^{\epsilon}_{\delta\gamma} h_{\epsilon\beta}{}^{a} - \bar{\Gamma}^{\epsilon}_{\delta\beta} h_{\gamma\epsilon}{}^{a} + L_{e}{}^{a}{}_{\delta} h_{\gamma\beta}{}^{e}, \tag{3.11}$$

$$^{*}\nabla_{d} L_{cb}{}^{\alpha} = \partial_{d} L_{cb}{}^{\alpha} - \Gamma^{e}{}_{dc} L_{eb}{}^{\alpha} - \Gamma^{e}{}_{db} L_{ce}{}^{\alpha} + Q_{d\epsilon}{}^{\alpha} L_{cb}{}^{\epsilon}. \tag{3.12}$$

The following lemma is well known [3].

Lemma 3.1. *If the components $L = (L_{cb}{}^{\alpha})$ and $h = (h_{\gamma\beta}{}^{a})$ vanish identically in a fibred Riemannian space, then the fibred space is locally the Riemannian product of the base space and a fibre.*

4. Conharmonically flat fibred Riemannian space

Let M be a conharmonically flat fibred Riemannian space with totally geodesic fibre and $m = \dim M > 2$. Then, by use of (2.3) and (3.2), we have

$$(4.1)\quad (m-2)({}^{**}\nabla_{\gamma} L_{db}{}^{\alpha} + L_{d}{}^{e}{}_{\gamma} L_{eb}{}^{\alpha}) = -(S_{db} - 2L_{be\epsilon} L_{d}{}^{e\epsilon})\delta_{\gamma}^{\alpha} - (\bar{S}_{\gamma}{}^{\alpha} + L_{ae\gamma} L^{ae\alpha}) g_{db}.$$

If we contract (4.1) with respect to b and d, then we get

$$(4.2)\qquad (m-2) L_{de\gamma} L^{de\alpha} = (K - 2\|L\|^{2})\delta_{\gamma}^{\alpha} + n(\bar{S}_{\gamma}{}^{\alpha} + L_{ae\gamma} L^{ae\alpha}),$$

and that

$$(4.3)\qquad (m-2)\|L\|^{2} = p(K - 2\|L\|^{2}) + n(\bar{K} + \|L\|^{2}),$$

that is

$$(4.4)\qquad (3p-2)\|L\|^{2} = pK + n\bar{K}.$$

Since the scalar curvature $\tilde{K}$ vanishes in the conharmonically flat space, we obtain

$$(4.5)\qquad K + \bar{K} = \|L\|^{2}$$

from (3.10). If we consider (4.4) and (4.5), then we have

$$(4.6)\qquad 2(p-1)K + (3p-2-n)\bar{K} = 0.$$

From (2.3), (3.1) and (3.7), it follows

$$(4.7)\quad R_{dcb}{}^{a} - L_{d}{}^{a}{}_{\epsilon}L_{cb}{}^{\epsilon} + L_{c}{}^{a}{}_{\epsilon}L_{db}{}^{\epsilon} + 2L_{dc}{}^{\epsilon}L_{b}{}^{a}{}_{\epsilon} = \frac{-1}{m-2}\{S_{bd}\delta_{c}^{a} + S_{c}{}^{a}g_{db} - S_{cb}\delta_{d}^{a}$$

$$-S_{d}{}^{a}g_{cb} - 2(L_{de}{}^{\epsilon}L_{b}{}^{e}{}_{\epsilon}\delta_{c}^{a} + L_{ce}{}^{\epsilon}L^{ae}{}_{\epsilon}g_{db} - L_{ce}{}^{\epsilon}L_{b}{}^{e}{}_{\epsilon}\delta_{d}^{a} - L_{de\epsilon}L^{ae\epsilon}g_{cb})\}.$$

Contracting (4.7) with respect to a and d, we have

$$(4.8)\qquad pS_{cb} = Kg_{cb} - (n+3p-2)L_{ce\epsilon}L_{b}{}^{e\epsilon} - 2\|L\|^{2}g_{cb},$$

that is

$$(4.9)\qquad (p-n)K = (2-3m)\|L\|^{2}.$$

If we assume that $n = p$, then $L = 0$. This fact and Lemma 3.1 give

Theorem 4.1. *Let M be a conharmonically flat fibred Riemannian space with totally geodesic fibre. Then M is locally the Riemannian product of the base space and a fibre when $n = p$.*

Hence, we see that

$$(4.10)\qquad S_{cb} = \frac{K}{n}g_{cb}$$

from (4.8), that is B becomes an Einstein space. If we consider $L = 0$ and (4.10) in (4.7), then we get

$$(4.11)\qquad R_{dcb}{}^{a} = \frac{K}{n(n-1)}(g_{cb}\delta_{d}^{a} - g_{bd}\delta_{c}^{a}),$$

that is B is a space of constant curvature.

Theorem 4.2. *Let M be a conharmonically flat fibred Riemannian space with totally geodesic fibre. Then the base space is a space of constant curvature when $n = p$.*

On the other hand, by use of $L = 0$, (2.3), (3.3) and (3.9), we see that

$$\bar{R}_{\delta\gamma\beta}{}^{\alpha} = \frac{-1}{m-2}\{\bar{S}_{\beta\delta}\delta_{\gamma}^{\alpha} + \bar{S}_{\gamma}{}^{\alpha}\bar{g}_{\delta\beta} - \bar{S}_{\gamma\beta}\delta_{\delta}^{\alpha} - \bar{S}_{\delta}{}^{\alpha}\bar{g}_{\gamma\beta}\}. \tag{4.12}$$

If we contract (4.12) with respect to α and δ, then we get

$$\bar{S}_{\gamma\beta} = \frac{\bar{K}}{p}\bar{g}_{\gamma\beta}. \tag{4.13}$$

Hence if we substitute (4.13) into (4.12), we obtain

$$\bar{R}_{\delta\gamma\beta}{}^{\alpha} = \frac{\bar{K}}{p(p-1)}(\bar{g}_{\gamma\beta}\delta_{\delta}^{\alpha} - \bar{g}_{\beta\delta}\delta_{\gamma}^{\alpha}).$$

Thus we have

Theorem 4.3. *Let M be a conharmonically flat fibred Riemannian space with totally geodesic fibre. Then each fibre is a space of constant curvature when $n = p$.*

Finally, if we consider (4.5) and Theorems 4.1, 4.2 and 4.3, then we have

Theorem 4.4. *Let M be a conharmonically flat fibred Riemannian space with totally geodesic fibre and $n = p$. Then M is locally the Riemannian product of two spaces of constant curvature with constant scalar curvatures K and $-K$ respectively.*

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