Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser. IV, Vol. 26 95−108, Dec. 2000

On the Gauss equation in the exterior algebra

BY

YOSHIO AGAOKA

Department of Mathematics, Faculty of Integrated Arts and Sciences,
Hiroshima University, Higashi-Hiroshima 739-8521, Japan
e-mail: agaoka@mis.hiroshima-u.ac.jp

Abstract

We define the Gauss equation in the exterior algebra, and state a relation to the original Gauss equation appearing in the theory of Riemannian submanifolds. We also state several necessary (and sufficient) conditions in order that this equation admits a solution mainly in the cases codimension = 1 and 2.

Key words: Gauss equation, exterior algebra, curvature.

Introduction.

The Gauss equation is a system of quadratic equations appearing in the theory of Riemannian submanifolds, and it is expressed in the following form:

$$R(X,Y,Z,W) = \langle \alpha(X,Z), \alpha(Y,W) \rangle - \langle \alpha(X,W), \alpha(Y,Z) \rangle.$$

Here, R means a curvature like tensor on the n-dimensional real vector space V (= the tangent space of M^n at x), α is an $\mathbf{R}^r$-valued symmetric form on V which corresponds to the second fundamental form of $M^n \subset \mathbf{R}^{n+r}$, and $\langle\ ,\ \rangle$ is a positive definite inner product of $\mathbf{R}^r$.

The solvability of this equation is intimately related to the existence of local isometric imbeddings of n-dimensional Riemannian manifolds into $\mathbf{R}^{n+r}$. In fact, if this equation does not admit a solution, the Riemannian manifold possessing R as a curvature at one point cannot be isometrically immersed into the Euclidean space with codimension r.

We already obtained several types of necessary conditions on the curvature in order that the Gauss equation admits a solution for low codimensional cases. And by applying these conditions, we showed several facts on the non-existence of local isometric imbeddings of Riemannian symmetric spaces (cf. [2], [3], [4], [5], [10], [12], [13], [14], etc.). But unfortunately, for higher codimensional cases, we have a little knowledge concerning the solvability of the Gauss equation because it is a quite complicated system of quadratic equations.

Received October 1 2000; Accepted November 1 2000

In this paper, we introduce a new equation in the exterior algebra, which is a natural generalization of the usual Gauss equation stated above (Theorem 1). Since the exterior algebra has a simple algebraic structure as compared with the space of curvature like tensors, it is expected that a new obstruction to the existence of local isometric imbeddings can be found from this new formulation, though it requires some deep understanding of the space of 4-forms. Roughly speaking, the solvability of this new equation is equivalent to determine "a sort of rank" of 4-forms. We can consider a similar problem for 2-forms, and in this case, the problem is completely settled in terms of the concept "rank". But as for 4-forms, we do not have such a complete understanding as 2-forms, and we give here some partial results on the solvability of this new equation.

In this paper, after stating a relation between the usual Gauss equation and the new equation, we consider the obstructions mainly for the cases codimension = 1 and 2. In the forthcoming paper, we will attack for higher codimensional cases, by introducing a more refined formulation for 4-forms.

§ 1. Exterior Gauss equation.

We first review the usual Gauss equation in Riemannian geometry. Let K be the space of curvature like tensors on the n-dimensional real vector space V, i.e.,

$$K = \{R \in \wedge^2 V^* \otimes \wedge^2 V^* \mid \mathfrak{S}_{X,Y,Z}\, R(X,Y,Z,W) = 0\}.$$

We fix a basis $\{X_1, \cdots, X_n\}$ of V and put $R_{ijkl} = R(X_i, X_j, X_k, X_l)$. We say that R admits a solution of the Gauss equation in codimension r if R is expressed in the form

$$R(X,Y,Z,W) = \langle \alpha(X,Z), \alpha(Y,W)\rangle - \langle \alpha(X,W), \alpha(Y,Z)\rangle$$

for some $\mathbf{R}^r$-valued symmetric bilinear form α on V. Here, $\langle\ ,\ \rangle$ means a positive definite inner product of $\mathbf{R}^r$.

Now, we consider a new real vector space W with double dimension $2n$, and in terms of the curvature R, define a 4-form $\tilde{R}$ on W by

$$\tilde{R} = \sum_{i<j,\,k<l} R_{ijkl}\, e_i^* \wedge e_j^* \wedge f_k^* \wedge f_l^* \in \wedge^4 W^*,$$

where $\{e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*\}$ is a basis of the dual space of W. Then, our first main result of this paper is stated as follows.

Theorem 1. *Assume that $R \in K$ admits a solution of the Gauss equation in codimension r. Then, the above 4-form $\tilde{R}$ can be expressed in the form*

$$-\tilde{R} = \Phi_1 \wedge \Phi_1 + \cdots + \Phi_r \wedge \Phi_r$$

in terms of some 2-forms $\Phi_1, \cdots, \Phi_r \in \wedge^2 W^*$.

Proof. Assume that R admits a solution of the Gauss equation in codimension r. Then, in terms of a basis of V and $\mathbf{R}^r$, we have

$$R_{ijkl} = \sum_{p=1}^{r} (\alpha_{ik}^p \alpha_{jl}^p - \alpha_{il}^p \alpha_{jk}^p),$$

where α_{ij}^p is the component of α. In this situation, we define 2-forms $\Phi_p \in \wedge^2 W^*$ by

$$\Phi_p = \frac{1}{\sqrt{2}} \sum_{i,j} \alpha_{ij}^p \, e_i^* \wedge f_j^*,$$

for $p = 1 \sim r$. Then, by easy calculations, we immediately have

$$\begin{aligned} \Phi_1 \wedge \Phi_1 + \cdots + \Phi_r \wedge \Phi_r &= -\tfrac{1}{2} \textstyle\sum_{i,j,k,l,p} \alpha_{ij}^p \alpha_{kl}^p \, e_i^* \wedge e_k^* \wedge f_j^* \wedge f_l^* \\ &= -\textstyle\sum_{i<j,k<l,p} (\alpha_{ik}^p \alpha_{jl}^p - \alpha_{jk}^p \alpha_{il}^p) \, e_i^* \wedge e_j^* \wedge f_k^* \wedge f_l^* \\ &= -\textstyle\sum_{i<j,k<l} R_{ijkl} \, e_i^* \wedge e_j^* \wedge f_k^* \wedge f_l^* \\ &= -\tilde{R}. \end{aligned}$$

q.e.d.

In the following, we consider a more general situation purely from algebraic viewpoint. Namely, for a given 4-form Ω on an abstract m-dimensional (real or complex) vector space W, we call the equation $\Omega = \Phi_1 \wedge \Phi_1 + \cdots + \Phi_r \wedge \Phi_r$ the Gauss equation in the exterior algebra, or simply the *exterior Gauss equation.* (We drop the minus sign of the 4-form in this formulation. But in the complexified category, this gives essentially the same equation because $(i\Phi_k) \wedge (i\Phi_k) = -\Phi_k \wedge \Phi_k$.) Of course, in the actual application to local isometric imbeddings of n-dimensional Riemannian manifolds, we have to put $m = 2n$ and $\Omega = -\tilde{R}$. In this situation, if we can show that $\Omega = -\tilde{R}$ cannot be expressed in the form $\sum_{p=1}^{r} \Phi_p \wedge \Phi_p$, then from the above theorem we know that the original Gauss equation also does not admit a solution in codimension r, which indicates the non-existence of local isometric imbeddings of Riemannian manifolds possessing R as a curvature at one point.

From purely algebraic viewpoint, the solvability of the exterior Gauss equation is interesting and important. This is related to the problem of determining the "rank" of 4-forms. We say that a 4-form Ω has rank r if Ω is expressed as a sum of r (but cannot $r - 1$) decomposable 4-forms. The number corresponding to the codimension r in our formulation is smaller than or equal to this rank of Ω because decomposable 4-forms can be expressed as the exterior product of a 2-form such as

$$e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* = \frac{1}{2}(e_1^* \wedge e_2^* + e_3^* \wedge e_4^*) \wedge (e_1^* \wedge e_2^* + e_3^* \wedge e_4^*).$$

In the paper [16], a combinatorial method to obtain an upper bound of the rank of generic p-forms is explained in detail.

For the case of 2-forms, the smallest number r such that a given 2-form can be expressed as $\sum_{i=1}^{r} \alpha_i \wedge \beta_i$ ($\alpha_i, \beta_i \in W^*$) is equal to the half of the rank of 2-forms. But, as for 4-forms, we do not know such a useful concept yet. Similar problems for other algebraic situations are also considered for example in [16], [17],[18], [25], etc.

§ 2. The case $(m, r) = (6, 1)$.

Since we are considering 4-forms on W, we may assume that $\dim W \geq 4$. But in the cases $\dim W = 4$ and 5, all 4-forms on W are decomposable, and hence they always can be expressed as $\Phi \wedge \Phi$ for some $\Phi \in \wedge^2 W^*$. In this section, we consider the first non-trivial case $\dim W = 6$ and $r = 1$. We express an element of $\wedge^4 W^*$ as Ω as above, and the components of Ω as Ω_{ijkl}, i.e.,

$$\Omega = \sum_{i<j<k<l} \Omega_{ijkl}\, e_i^* \wedge e_j^* \wedge e_k^* \wedge e_l^*,$$

where $\{e_1^*, \cdots, e_m^*\}$ is a basis of W^*.

Proposition 2. *Assume* $\dim W = 6$. *Then, the map* $\wedge^2 W^* \longrightarrow \wedge^4 W^*$ *defined by* $\Phi \rightarrow \Phi \wedge \Phi$ *is a local diffeomorphism.*

Proof. First, we remark that $\dim \wedge^2 W^* = \dim \wedge^4 W^* = 15$ in the case $\dim W = 6$. We show that the rank of the differential of the above map is 15 at generic points of $\wedge^2 W^*$. For this purpose, we consider the 2-form $\Phi_0 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*$. Then, it is easy to see that the condition $\Phi_0 \wedge \Psi = 0$ ($\Psi \in \wedge^2 W^*$) implies $\Psi = 0$, which indicates that the rank of the differential of the map $\Phi \rightarrow \Phi \wedge \Phi$ is 15 at this point Φ_0. Hence, this map is a local diffeomorphism at generic points of $\wedge^2 W^*$. q.e.d.

We can give the explicit inverse formula of this map, by using a representation theoretic method stated as follows.

The general linear group $GL(W)$ naturally acts on the space of homogeneous polynomials on $\wedge^4 W^*$ with degree p ($= S^p(\wedge^4 W^*)^* = S^p(\wedge^4 W)$). The $GL(W)$-irreducible decomposition of this space is given by the decomposition of the plethysm $\{1^4\} \otimes \{p\}$. For small p, this decomposition is already known, and it is explicitly given as follows. (For the definition and some results of plethysms, see [8], [11], [21], [22], [27], etc.).

Proposition 3 (cf. [8]).

$$\begin{aligned}
p = 1 &: \{1^4\} \otimes \{1\} = \{1^4\}, \\
p = 2 &: \{1^4\} \otimes \{2\} = \{2^4\} + \{2^2 1^4\} + \{1^8\}, \\
p = 3 &: \{1^4\} \otimes \{3\} = \{3^4\} + \{3^2 2^2 1^2\} + \{3^2 1^6\} + \{3 2^3 1^3\} + \{2^6\} + \{2^4 1^4\} + \{2^3 1^6\} \\
&\qquad + \{2^2 1^8\} + \{1^{12}\}.
\end{aligned}$$

Remark. We already know the general decomposition formula of the plethysm $\{1^2\} \otimes \{p\}$ (cf. [1]). But as for the plethysm $\{1^4\} \otimes \{p\}$, such a formula is not known yet for general p.

Among the above irreducible components, the generator of $\{2^6\} \subset \{1^4\} \odot \{3\} = S^3(\wedge^4 W^*)^*$ is quite important, and is given by

$$\begin{aligned}
&\Omega_{1234}\Omega_{1256}\Omega_{3456} - \Omega_{1234}\Omega_{1356}\Omega_{2456} + \Omega_{1234}\Omega_{1456}\Omega_{2356} - \Omega_{1235}\Omega_{1246}\Omega_{3456} \\
&+\Omega_{1235}\Omega_{1346}\Omega_{2456} - \Omega_{1235}\Omega_{1456}\Omega_{2346} + \Omega_{1236}\Omega_{1245}\Omega_{3456} - \Omega_{1236}\Omega_{1345}\Omega_{2456} \\
&+\Omega_{1236}\Omega_{1456}\Omega_{2345} - \Omega_{1245}\Omega_{1346}\Omega_{2356} + \Omega_{1245}\Omega_{1356}\Omega_{2346} + \Omega_{1246}\Omega_{1345}\Omega_{2356} \\
&-\Omega_{1246}\Omega_{1356}\Omega_{2345} - \Omega_{1256}\Omega_{1345}\Omega_{2346} + \Omega_{1256}\Omega_{1346}\Omega_{2345}.
\end{aligned}$$

This expression can be obtained by the method stated in [3]. This is a relative invariant of the group $GL(W)$ in case $\dim W = 6$, and we denote this polynomial as Δ in the following. If we fix a volume form of W, then the space $\wedge^4 W^*$ is naturally isomorphic to $\wedge^2 W$, and under this identification, the invariant Δ just corresponds to the Pfaffian of $\wedge^2 W = \wedge^2 \mathbf{R}^6$.

If Ω is expressed in the form $\Phi \wedge \Phi$, we have

$$\Omega_{ijkl} = 2(\Phi_{ij}\Phi_{kl} - \Phi_{ik}\Phi_{jl} + \Phi_{il}\Phi_{jk}),$$

where $\Phi = \sum \Phi_{ij}\, e_i^* \wedge e_j^*$. By substituting this equality into the above Δ, we have $\Delta = 8\delta^2$, where

$$\begin{aligned}
\delta \;&= \Phi_{12}\Phi_{34}\Phi_{56} - \Phi_{12}\Phi_{35}\Phi_{46} + \Phi_{12}\Phi_{36}\Phi_{45} - \Phi_{13}\Phi_{24}\Phi_{56} + \Phi_{13}\Phi_{25}\Phi_{46} \\
&-\Phi_{13}\Phi_{26}\Phi_{45} + \Phi_{14}\Phi_{23}\Phi_{56} - \Phi_{14}\Phi_{25}\Phi_{36} + \Phi_{14}\Phi_{26}\Phi_{35} - \Phi_{15}\Phi_{23}\Phi_{46} \\
&+\Phi_{15}\Phi_{24}\Phi_{36} - \Phi_{15}\Phi_{26}\Phi_{34} + \Phi_{16}\Phi_{23}\Phi_{45} - \Phi_{16}\Phi_{24}\Phi_{35} + \Phi_{16}\Phi_{25}\Phi_{34},
\end{aligned}$$

which is nothing but the Pfaffian of Φ.

On the other hand, the generator of $\{2^2 1^4\} \subset \{1^4\} \otimes \{2\}$ (cf. Proposition 3) is given by

$$\Omega_{1234}\Omega_{1256} - \Omega_{1235}\Omega_{1246} + \Omega_{1236}\Omega_{1245}.$$

By substituting $\Omega = \Phi \wedge \Phi$ into this polynomial, we know that this is equal to $4\Phi_{12}\delta$. Hence, we have

$$\Phi_{12}^2 = \frac{1}{2\Delta}(\Omega_{1234}\Omega_{1256} - \Omega_{1235}\Omega_{1246} + \Omega_{1236}\Omega_{1245})^2.$$

For general indices, we have the following theorem.

Theorem 4 (The inverse formula). *Assume* $\dim W = 6$ *and* Ω *is expressed as* $\Phi \wedge \Phi$ *for some* $\Phi = \sum \Phi_{ij}\, e_i^* \wedge e_j^* \in \wedge^2 W^*$. *Then, if* Ω *is sufficiently generic, the 2-form* Φ *is uniquely determined from* Ω *(up to sign) by the formula:*

$$\Phi_{ij}^2 = \frac{1}{2\Delta}(\Omega_{ijpq}\Omega_{ijrs} - \Omega_{ijpr}\Omega_{ijqs} + \Omega_{ijps}\Omega_{ijqr})^2,$$

where $e_i^* \wedge e_j^* \wedge e_p^* \wedge e_q^* \wedge e_r^* \wedge e_s^* = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* \wedge e_5^* \wedge e_6^*$. *The ratio of two components of* Φ *is uniquely determined by*

$$\Phi_{kl} = \frac{\Omega_{klxy}\Omega_{klzw} - \Omega_{klxz}\Omega_{klyw} + \Omega_{klxw}\Omega_{klyz}}{\Omega_{ijpq}\Omega_{ijrs} - \Omega_{ijpr}\Omega_{ijqs} + \Omega_{ijps}\Omega_{ijqr}}\Phi_{ij},$$

where $e_i^* \wedge e_j^* \wedge e_p^* \wedge e_q^* \wedge e_r^* \wedge e_s^* = e_k^* \wedge e_l^* \wedge e_x^* \wedge e_y^* \wedge e_z^* \wedge e_w^* = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* \wedge e_5^* \wedge e_6^*$.

Remark. By changing the indices suitably, the above inverse formula also holds for the case $\dim W \geq 7$. In particular, the 2-form Φ is essentially uniquely determined from Ω also in the case $\dim W \geq 7$ if Ω is expressed as $\Phi \wedge \Phi$ (cf. Theorem 9).

As a corollary of this proposition, we have

Corollary 5. *Assume Ω is expressed as $\Phi \wedge \Phi$. Then, the inequality $\Delta \geq 0$ holds.*

In the actual application to Riemannian geometry, this inequality serves as an obstruction for the case $M^3 \subset \mathbf{R}^4$. In fact, for a 3-dimensional Riemannian manifold, by using the notations in §1, we have

$$\begin{aligned}\tilde{R} &= R_{1212}\, e_1^* \wedge e_2^* \wedge f_1^* \wedge f_2^* + R_{1213}\, e_1^* \wedge e_2^* \wedge f_1^* \wedge f_3^* + R_{1223}\, e_1^* \wedge e_2^* \wedge f_2^* \wedge f_3^* \\ &+ R_{1312}\, e_1^* \wedge e_3^* \wedge f_1^* \wedge f_2^* + R_{1313}\, e_1^* \wedge e_3^* \wedge f_1^* \wedge f_3^* + R_{1323}\, e_1^* \wedge e_3^* \wedge f_2^* \wedge f_3^* \\ &+ R_{2312}\, e_2^* \wedge e_3^* \wedge f_1^* \wedge f_2^* + R_{2313}\, e_2^* \wedge e_3^* \wedge f_1^* \wedge f_3^* + R_{2323}\, e_2^* \wedge e_3^* \wedge f_2^* \wedge f_3^*,\end{aligned}$$

where R_{ijkl} is the usual component of the curvature tensor R. By putting $f_1^* = e_4^*$, $f_2^* = e_5^*$, $f_3^* = e_6^*$ in this expression, we have

$$\begin{aligned}\tilde{R} &= R_{1212}\, e_1^* \wedge e_2^* \wedge e_4^* \wedge e_5^* + R_{1213}\, e_1^* \wedge e_2^* \wedge e_4^* \wedge e_6^* + R_{1223}\, e_1^* \wedge e_2^* \wedge e_5^* \wedge e_6^* \\ &+ R_{1312}\, e_1^* \wedge e_3^* \wedge e_4^* \wedge e_5^* + R_{1313}\, e_1^* \wedge e_3^* \wedge e_4^* \wedge e_6^* + R_{1323}\, e_1^* \wedge e_3^* \wedge e_5^* \wedge e_6^* \\ &+ R_{2312}\, e_2^* \wedge e_3^* \wedge e_4^* \wedge e_5^* + R_{2313}\, e_2^* \wedge e_3^* \wedge e_4^* \wedge e_6^* + R_{2323}\, e_2^* \wedge e_3^* \wedge e_5^* \wedge e_6^*.\end{aligned}$$

Then, by substituting

$$\begin{aligned}&\Omega_{1245} = R_{1212}, \quad \Omega_{1246} = R_{1213}, \quad \Omega_{1256} = R_{1223}, \quad \Omega_{1345} = R_{1213}, \\ &\Omega_{1346} = R_{1313}, \quad \Omega_{1356} = R_{1323}, \quad \Omega_{2345} = R_{1223}, \quad \Omega_{2346} = R_{1323}, \\ &\Omega_{2356} = R_{2323}\end{aligned}$$

and other $\Omega_{ijkl} = 0$ into the above invariant Δ, we have

$$\begin{aligned}\Delta &= -R_{1212}R_{1313}R_{2323} + R_{1212}R_{1323}R_{1323} + R_{1213}R_{1213}R_{2323} \\ &\quad - R_{1213}R_{1323}R_{1223} - R_{1223}R_{1213}R_{1323} + R_{1223}R_{1313}R_{1223} \\ &= - \begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1312} & R_{1313} & R_{1323} \\ R_{2312} & R_{2313} & R_{2323} \end{vmatrix}.\end{aligned}$$

Hence, if $-\tilde{R}$ is expressed as $\Phi \wedge \Phi$, then by reversing the sign of Ω, we have

$$\begin{vmatrix} R_{1212} & R_{1213} & R_{1223} \\ R_{1312} & R_{1313} & R_{1323} \\ R_{2312} & R_{2313} & R_{2323} \end{vmatrix} \geq 0,$$

which is nothing but Thomas' classical inequality for Riemannian submanifold $M^3 \subset \mathbf{R}^4$. (cf. [28]. See also [23].) By this inequality, it follows that the 3-dimensional hyperbolic space $H^3(-1)$ cannot be locally isometrically immersed into $\mathbf{R}^4$ because $R_{ijij} = -1$ and other $R_{ijkl} = 0$ for this space.

We can consider the exterior Gauss equation for the case $\dim W = 6$ from another viewpoint. By fixing a volume form of $\mathbf{R}^6$, the space $\wedge^4 W^*$ is naturally isomorphic to $\wedge^2 W$ as stated above. Since the normal forms of the elements of $\wedge^2 W$ is well-known, we can obtain the normal forms of $\wedge^4 W^*$ under the action of $GL(W)$:

$$\begin{aligned}
\Omega_0 &= 0,\\
\Omega_1 &= e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*,\\
\Omega_2 &= e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* + e_1^* \wedge e_2^* \wedge e_5^* \wedge e_6^*,\\
\Omega_3 &= e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* + e_1^* \wedge e_2^* \wedge e_5^* \wedge e_6^* + e_3^* \wedge e_4^* \wedge e_5^* \wedge e_6^*,\\
\Omega_4 &= e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* + e_1^* \wedge e_2^* \wedge e_5^* \wedge e_6^* - e_3^* \wedge e_4^* \wedge e_5^* \wedge e_6^*.
\end{aligned}$$

(If we complexify the vector space W and the group action, two forms Ω_3 and Ω_4 are mapped to each other by $GL(6, \mathbf{C})$.) We can easily show that among these normal forms, Ω_0, Ω_1 and Ω_3 can be expressed as $\Phi \wedge \Phi$, but the remaining two normal forms Ω_2 and Ω_4 cannot be expressed in this form. And thus, the solvability of the exterior Gauss equation is completely settled by using the above normal forms. In terms of the isomorphism $\wedge^4 W^* \longrightarrow \wedge^2 W$ stated above, a 4-form can be expressed as $\Phi \wedge \Phi$ if and only if the corresponding 2-vector $\in \wedge^2 W$ has rank ≤ 2, or has rank $= 6$ and satisfies the inequality on the Pfaffian corresponding to the condition $\Delta > 0$.

§ 3. The case $r = 1$ and $m \geq 7$.

To obtain the obstruction to the existence of solutions of the exterior Gauss equation that can be expressed as a polynomial relation of the components Ω_{ijkl}, we use the same method as in [3]. We first consider the case $r = 1$ and $m \geq 7$ in this section. The quadratic map $\wedge^2 W^* \longrightarrow \wedge^4 W^*$ defined by $\Phi \rightarrow \Phi \wedge \Phi$ naturally induces a dual polynomial map $\gamma_p : S^p(\wedge^4 W^*)^* \longrightarrow S^{2p}(\wedge^2 W^*)^*$ with degree $= p$. Polynomials contained in the kernel of γ_p vanish if we substitute $\Omega = \Phi \wedge \Phi$ into these polynomials. Hence, they serve as obstructions to the existence of solutions of the exterior Gauss equation. To obtain the kernel of γ_p explicitly, we calculate the plethysms $\{1^2\} \otimes \{2p\}$ $(= S^{2p}(\wedge^2 W^*)^* = S^{2p}(\wedge^2 W))$ for $p \leq 3$ at first, and compare them with the results in Proposition 3.

Proposition 6 (cf. [1], [8]).

$$\begin{aligned}
p = 1 \ &: \ \{1^2\} \otimes \{2\} = \{2^2\} + \{1^4\},\\
p = 2 \ &: \ \{1^2\} \otimes \{4\} = \{4^2\} + \{3^2 1^2\} + \{2^4\} + \{2^2 1^4\} + \{1^8\},\\
p = 3 \ &: \ \{1^2\} \otimes \{6\} = \{6^2\} + \{5^2 1^2\} + \{4^2 2^2\} + \{4^2 1^4\} + \{3^4\} + \{3^2 2^2 1^2\}\\
&\qquad + \{3^2 1^6\} + \{2^6\} + \{2^4 1^4\} + \{2^2 1^8\} + \{1^{12}\}.
\end{aligned}$$

By comparing these plethysms, we know that there exist (at least) two types of kernels $\{32^3 1^3\}$ and $\{2^3 1^6\}$ in degree $= 3$. These two kernels $\{32^3 1^3\}$ and $\{2^3 1^6\}$ serve as the actual obstruction for the cases $\dim W \geq 7$ and $\dim W \geq 9$, respectively. (Note that we can easily verify that there is no quadratic polynomial relation of Ω_{ijkl}.)

First, we consider the obstruction $\{32^3 1^3\}$. The generator corresponding to this irreducible component can be calculated by the method stated in [3]. By using a computer, we know that it is given by the polynomial

$$\begin{aligned}&\Omega_{1234}^2\Omega_{1567} - \Omega_{1234}\Omega_{1235}\Omega_{1467} + \Omega_{1234}\Omega_{1236}\Omega_{1457} - \Omega_{1234}\Omega_{1237}\Omega_{1456} + \Omega_{1234}\Omega_{1245}\Omega_{1367}\\ &-\Omega_{1234}\Omega_{1246}\Omega_{1357} + \Omega_{1234}\Omega_{1247}\Omega_{1356} - \Omega_{1234}\Omega_{1256}\Omega_{1347} + \Omega_{1234}\Omega_{1257}\Omega_{1346}\\ &-\Omega_{1234}\Omega_{1267}\Omega_{1345} + 2\Omega_{1235}\Omega_{1246}\Omega_{1347} - 2\Omega_{1235}\Omega_{1247}\Omega_{1346} - 2\Omega_{1236}\Omega_{1245}\Omega_{1347}\\ &+2\Omega_{1236}\Omega_{1247}\Omega_{1345} + 2\Omega_{1237}\Omega_{1245}\Omega_{1346} - 2\Omega_{1237}\Omega_{1246}\Omega_{1345}.\end{aligned}$$

We rewrite this polynomial by using the flag

$$\{0\} \subset W^1 \subset W^4 \subset W^7 \subset W^m,$$

as in the case of [9; p.253], where superscripts imply the dimensions of the spaces.

We put $\Theta = e_1 \rfloor \Omega \in \Lambda^3 W^*$, where $\rfloor$ implies the interior product. Then, the above cubic polynomial is just equal to the coefficient of $e_2^* \wedge \cdots \wedge e_7^*$ in

$$(e_2 \rfloor e_3 \rfloor \Theta) \wedge (e_4 \rfloor \Theta) \wedge \Theta - (e_2 \rfloor e_4 \rfloor \Theta) \wedge (e_3 \rfloor \Theta) \wedge \Theta + (e_3 \rfloor e_4 \rfloor \Theta) \wedge (e_2 \rfloor \Theta) \wedge \Theta$$

up to non-zero constant. Clearly, the above flag is defined by

$$\begin{aligned}W^1 &= \langle e_1 \rangle,\\ W^4 &= \langle e_1, e_2, e_3, e_4 \rangle,\\ W^7 &= \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle.\end{aligned}$$

Hence, combining these results, we have

Theorem 7. *If Ω is expressed as $\Phi \wedge \Phi$, then the 3-form Θ defined by $\Theta = e_1 \rfloor \Omega$ satisfies the equality*

$$(e_2 \rfloor e_3 \rfloor \Theta) \wedge (e_4 \rfloor \Theta) \wedge \Theta - (e_2 \rfloor e_4 \rfloor \Theta) \wedge (e_3 \rfloor \Theta) \wedge \Theta + (e_3 \rfloor e_4 \rfloor \Theta) \wedge (e_2 \rfloor \Theta) \wedge \Theta = 0$$

for any basis $\{e_1, \cdots, e_m\}$ of W.

We can directly prove this equality without using the results in Propositions 3 and 6. Assume that Ω is expressed as $\Phi \wedge \Phi$. Then, we have $\Theta = e_1 \rfloor \Omega = \alpha \wedge \Phi$, where $\alpha = 2e_1 \rfloor \Phi \in W^*$. In this situation, we substitute this equality into the above. Then, since

$$e_2 \rfloor e_3 \rfloor \Theta = \alpha(e_3)(e_2 \rfloor \Phi) - \alpha(e_2)(e_3 \rfloor \Phi) - \Phi(e_2, e_3)\alpha,$$

we have

$$\begin{aligned}&(e_2 \rfloor e_3 \rfloor \Theta) \wedge (e_4 \rfloor \Theta) \wedge \Theta\\ =\ &\{\alpha(e_3)(e_2 \rfloor \Phi) - \alpha(e_2)(e_3 \rfloor \Phi)\} \wedge \{\alpha(e_4)\Phi - \alpha \wedge (e_4 \rfloor \Phi)\} \wedge \alpha \wedge \Phi\\ =\ &\{\alpha(e_3)(e_2 \rfloor \Phi) - \alpha(e_2)(e_3 \rfloor \Phi)\} \wedge \alpha(e_4)\Phi \wedge \alpha \wedge \Phi\\ =\ &\{\alpha(e_3)\alpha(e_4)(e_2 \rfloor \Phi) - \alpha(e_2)\alpha(e_4)(e_3 \rfloor \Phi)\} \wedge \Phi \wedge \alpha \wedge \Phi.\end{aligned}$$

Then, it is easy to see that the cyclic sum of this expression is zero, and hence, we obtain the desired equality. q.e.d.

The condition on $\Theta \in \wedge^3 W^*$ in this theorem is nothing but the condition $C_2 \sim 0$ stated in [17; p.67 $\sim$ 72]. This condition $C_2 \sim 0$ is equivalent to the condition that the 3-form Θ can be expressed as $\alpha \wedge \Phi$ for some $\alpha \in W^*$ and $\Phi \in \wedge^2 W^*$.

Next, we consider the obstruction $\{2^3 1^6\}$ which is useful for the case $\dim W \geq 9$. To express this obstruction, we use a flag defined by

$$\begin{aligned} W^3 &= \langle e_1, e_2, e_3 \rangle, \\ W^9 &= \langle e_1, \cdots, e_9 \rangle \end{aligned}$$

in this case. Then, we have

Theorem 8. *Assume Ω is expressed as $\Phi \wedge \Phi$. Then the following equality holds on the 9-dimensional subspace W^9 of W containing $W^3 = \langle e_1, e_2, e_3 \rangle$ as a subspace:*

$$(e_1 \rfloor \Omega) \wedge (e_2 \rfloor \Omega) \wedge (e_3 \rfloor \Omega) = 0 \in \wedge^9 (W^9)^*.$$

Here, $\{e_1, \cdots, e_m\}$ is any basis of W.

Proof. The restriction of Φ to the subspace W^9 is expressed as $\Phi = a_1 \omega_1 \wedge \omega_2 + \cdots + a_4 \omega_7 \wedge \omega_8$ for some linearly independent 1-forms $\omega_1 \sim \omega_8$ on W^9, where $a_1, \cdots, a_4 = 1$ or 0 according as the rank of Φ restricted to W^9. Since $\Omega = \Phi \wedge \Phi$, we have

$$(e_1 \rfloor \Omega) \wedge (e_2 \rfloor \Omega) \wedge (e_3 \rfloor \Omega) = 8 (e_1 \rfloor \Phi) \wedge (e_2 \rfloor \Phi) \wedge (e_3 \rfloor \Phi) \wedge \Phi \wedge \Phi \wedge \Phi.$$

Restricting to W^9, this 9-form must reduce to zero because e_1, e_2, $e_3 \in W^9$ and $e_1 \rfloor \Phi$,$e_2 \rfloor \Phi$, $e_3 \rfloor \Phi$ are also expressed in terms of only eight 1-forms $\omega_1 \sim \omega_8$. q.e.d.

Clearly, the equality in Theorem 8 does not hold for generic 4-forms on W. For example, in the 9-dimensional case, we consider the 4-form

$$\Omega_0 = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^* + e_1^* \wedge e_2^* \wedge e_5^* \wedge e_6^* + e_3^* \wedge e_7^* \wedge e_8^* \wedge e_9^*.$$

Then we have

$$(e_1 \rfloor \Omega_0) \wedge (e_2 \rfloor \Omega_0) \wedge (e_3 \rfloor \Omega_0) = 2 e_1^* \wedge \cdots \wedge e_9^* \neq 0,$$

which implies that this Ω_0 cannot be expressed as $\Phi \wedge \Phi$. We remark that even if Ω is expressed as $\Phi \wedge \Phi$, the above 9-form $(e_1 \rfloor \Omega) \wedge (e_2 \rfloor \Omega) \wedge (e_3 \rfloor \Omega)$ does not in general vanish as a form on W, and we must introduce the flag $\{0\} \subset W^3 \subset W^9 \subset W$ to express the condition $\{2^3 1^6\}$.

In case Ω is expressed as $\Phi \wedge \Phi$, Ω also satisfies the additional condition $\Delta \geq 0$ as in the case of Corollary 5, by restricting Ω to any 6-dimensional subspace of W. But we do not know whether the combined conditions in Theorems 7, 8 and $\Delta \geq 0$ are sufficient for the solvability of the exterior Gauss equation in the case of $r = 1$ and $\dim W \geq 7$.

§ 4. The case $r = 2$.

Before considering the higher codimensional cases, we first state the following theorem, giving the dimension of the set consisting of 4-forms of the form $\sum_{i=1}^{r} \Phi_i \wedge \Phi_i$ for low dimensional cases.

Theorem 9. *The dimension of the image of the quadratic map* $\wedge^2 W^* \otimes \mathbf{R}^r \longrightarrow \wedge^4 W^*$ *defined by* $\Phi_1 \otimes \cdots \otimes \Phi_r \to \Phi_1 \wedge \Phi_1 + \cdots + \Phi_r \wedge \Phi_r$ *is given by*

(m,r)	$\wedge^2 W^* \otimes \mathbf{R}^r \longrightarrow \wedge^4 W^*$	dimension
$(7,1)$	$\mathbf{R}^{21} \longrightarrow \mathbf{R}^{35}$	21
$(7,2)$	$\mathbf{R}^{42} \longrightarrow \mathbf{R}^{35}$	34
$(7,3)$	$\mathbf{R}^{63} \longrightarrow \mathbf{R}^{35}$	35
$(8,1)$	$\mathbf{R}^{28} \longrightarrow \mathbf{R}^{70}$	28
$(8,2)$	$\mathbf{R}^{56} \longrightarrow \mathbf{R}^{70}$	54
$(8,3)$	$\mathbf{R}^{84} \longrightarrow \mathbf{R}^{70}$	70
$(9,1)$	$\mathbf{R}^{36} \longrightarrow \mathbf{R}^{126}$	36
$(9,2)$	$\mathbf{R}^{72} \longrightarrow \mathbf{R}^{126}$	71
$(9,3)$	$\mathbf{R}^{108} \longrightarrow \mathbf{R}^{126}$	105
$(9,4)$	$\mathbf{R}^{144} \longrightarrow \mathbf{R}^{126}$	126
$(10,1)$	$\mathbf{R}^{45} \longrightarrow \mathbf{R}^{210}$	45
$(10,2)$	$\mathbf{R}^{90} \longrightarrow \mathbf{R}^{210}$	89
$(10,3)$	$\mathbf{R}^{135} \longrightarrow \mathbf{R}^{210}$	132
$(10,4)$	$\mathbf{R}^{180} \longrightarrow \mathbf{R}^{210}$	174
$(10,5)$	$\mathbf{R}^{225} \longrightarrow \mathbf{R}^{210}$	210

We obtain these results by calculating the rank of the differential of the map $\Phi_1 \otimes \cdots \otimes \Phi_r \longrightarrow \sum \Phi_i \wedge \Phi_i$ at a generic point of $\wedge^2 W^* \otimes \mathbf{R}^r$ for each case. In actual calculations, we used a computer.

The orthogonal group $O(r)$ naturally acts on the space $\wedge^2 W^* \otimes \mathbf{R}^r$, which induces the $r(r-1)/2$-dimensional kernel of the differential of the above map because this map is $O(r)$-equivariant. (The group $O(r)$ acts on the space $\wedge^4 W^*$ trivially.) The results in Theorem 9 indicate that there exist (unexpected) additional kernels for two cases $(m,r) = (7,2)$ and $(8,2)$ (except for the surjective cases such as $m \geq 7$ and $r = 3$). (The similar curious phenomenon occurs for the original Gauss equation for the case $M^4 \subset \mathbf{R}^6$. For details, see [15].) In particular, for the case $(m,r) = (7,2)$, the Zariski closure of the image of the above map is a hypersurface of $\wedge^4 W^*$, and hence the defining equation of this image must be a relative $GL(W)$-invariant of $\wedge^4 W^*$.

To know the degree of this invariant, we first consider the character of the space of homogeneous polynomials on $\wedge^2 W^* \otimes \mathbf{R}^r$. Since the group $O(r)$ acts trivially on the space $\wedge^4 W^*$ (and hence on $S^p(\wedge^4 W^*)^*$), the image of the polynomial map $S^p(\wedge^4 W^*)^* \longrightarrow S^{2p}(\wedge^2 W^* \otimes \mathbf{R}^r)^*$ consists of $O(r)$-invariants. In the case $r = 2$, we have the decomposition

$$S^{2p}(\wedge^2 W^* \otimes \mathbf{R}^2)^* = \sum_{|\mu|=2p} S_\mu(\wedge^2 W^*)^* \otimes S_\mu(\mathbf{R}^2)^*,$$

and $S_\mu(\mathbf{R}^2)^*$ contains an $O(2)$-invariant if and only if μ is an even partition with depth ≤ 2. In this case, the multiplicity of $O(2)$-invariants is always one, and hence the image of the map $S^p(\wedge^4 W^*)^* \longrightarrow S^{2p}(\wedge^2 W^* \otimes \mathbf{R}^2)^*$ is contained in the subspace

$$\sum_{\lambda=(\lambda_1,\lambda_2),\,|\lambda|=p} S_{2\lambda}(\wedge^2 W^*)^*.$$

Then, by calculating the plethysms $\{1^2\} \otimes \{2\lambda\}$ for $p \leq 7$ by a computer, we have

Proposition 10. *The kernel of the map*

$$S^7(\wedge^4 W^*)^* \longrightarrow \sum_{\lambda=(\lambda_1,\lambda_2),\,|\lambda|=7} S_{2\lambda}(\wedge^2 W^*)^*$$

contains the spaces $\{73^7\} + \{73^5 2^2 1^2\} + \{73^4 2^3 1^3\} + \{73^3 2^6\} + \{4^7\} \subset \{1^4\} \otimes \{7\}$.

Remark. For the lower degree cases $p \leq 6$, all characters of $S^p(\wedge^4 W^*)^*$ formally appear in the right space, and we cannot decide whether there exists a non-trivial kernel or not. To determine the actual kernel of this map, it requires tremendous calculations as carried out in the paper [3], and we do not check it at present.

Among the above five obstructions, $\{4^7\}$ seems to be the most useful because it possesses the actual meaning in the case $\dim W \geq 7$, while other obstructions require higher dimensional space. The obstruction $\{4^7\}$ is a relative $GL(W)$-invariant of $\wedge^4 W^*$ in the case $\dim W = 7$. Hence, combining with Theorem 9, we have

Theorem 11. *In the case $(m,r) = (7,2)$, we complexify the space and the map in Theorem 9. Then, the Zariski closure of the image of this map is characterized by the vanishing of the generator of $\{4^7\}$, which is the relative $GL(7,\mathbf{C})$-invariant of $\wedge^4 W^*$ with degree seven.*

By considering a flag $\{0\} \subset W^7 \subset W^m$, this obstruction $\{4^7\}$ is also useful for the case $\dim W \geq 8$. But unfortunately, we do not know the explicit form of this obstruction yet. In case $\dim W = 7$, this invariant just corresponds to the invariant C_8 of $\wedge^3 \mathbf{C}^7$ stated in [17; p.72] under the identification $\wedge^4(\mathbf{C}^7)^* \simeq \wedge^3 \mathbf{C}^7$ stated before. It is also not known that in case $\dim W \geq 8$ and $r = 2$, this condition is sufficient in order that the exterior Gauss equation admits a solution even in the complex category.

Finally, as in §2, we consider the exterior Gauss equation from the viewpoint of normal forms. In the complex category, we have the following proposition.

Proposition 12. *Normal forms of $\wedge^4(W^c)^*$ ($\dim W = 7$) under the action of the* *group* $GL(7, \mathbf{C})$ *are given by the following*:

$$\begin{aligned}
\Omega_0 &= 0, \\
\Omega_1 &= e^*_{1234}, \\
\Omega_2 &= e^*_{1234} + e^*_{1256}, \\
\Omega_3 &= e^*_{1234} + e^*_{1256} + e^*_{1357}, \\
\Omega_4 &= e^*_{1234} + e^*_{1567}, \\
\Omega_5 &= e^*_{1234} + e^*_{1256} + e^*_{3456}, \\
\Omega_6 &= e^*_{1234} + e^*_{1256} + e^*_{1357} + e^*_{2367}, \\
\Omega_7 &= e^*_{1234} + e^*_{1256} + e^*_{3457}, \\
\Omega_8 &= e^*_{1234} + e^*_{1256} + e^*_{1357} + e^*_{2467}, \\
\Omega_9 &= e^*_{1234} + e^*_{1256} + e^*_{1357} + e^*_{2467} + e^*_{3456}.
\end{aligned}$$

Here, the symbol e^*_{1234} *means the exterior product* $e^*_1 \wedge e^*_2 \wedge e^*_3 \wedge e^*_4$, *etc.*

We can prove this proposition by using the normal forms of $\wedge^3 \mathbf{C}^7$ stated in [17; p.69 $\sim$ 73]. The orbit spaces of Ω_5 and Ω_8 have dimension 21 and 34, respectively, and the closures of these orbits just coincide with the closures of the images of the complexified maps in Theorem 9, corresponding to the cases $(m, r) = (7, 1)$ and $(7, 2)$.

In fact, the above normal forms have the expression

$$\begin{aligned}
\Omega_5 &= e^*_{1234} + e^*_{1256} + e^*_{3456} \\
&= \tfrac{1}{2}(e^*_{12} + e^*_{34} + e^*_{56}) \wedge (e^*_{12} + e^*_{34} + e^*_{56}), \\
\Omega_8 &= e^*_{1234} + e^*_{1256} + e^*_{1357} + e^*_{2467} \\
&= \Phi_1 \wedge \Phi_1 + \Phi_2 \wedge \Phi_2,
\end{aligned}$$

where

$$\begin{aligned}
\Phi_1 &= \tfrac{1}{2}(2e^*_{12} + 2e^*_{13} + 2e^*_{14} + 2e^*_{16} - e^*_{23} - 2e^*_{24} - e^*_{26} - 2e^*_{27} - e^*_{37} - 2e^*_{47} - 2e^*_{67}), \\
\Phi_2 &= \tfrac{1}{2}(2e^*_{12} + 2e^*_{14} + 2e^*_{15} + 2e^*_{16} - e^*_{26} - e^*_{37}).
\end{aligned}$$

For general codimension r, we can obtain a similar obstruction as in the case of the p-G-equation stated in [5] or [13]. But it is useful only in the range $2r \leq m - 3$, and we cannot improve the previously known estimates on the codimension of local isometric imbeddings by this method. To obtain a new obstruction useful for higher codimensional cases, we must find new group invariant concepts in the space of 4-forms, as stated in Introduction.

References

[1] S. Abeasis and A. Del Fra, *Young diagrams and ideals of Pfaffians*, Adv. in Math. **35** (1980), 158-178.

[2] Y. Agaoka, *Isometric immersions of* $SO(5)$, J. Math. Kyoto Univ. **24** (1984), 713-724.

[3] Y. Agaoka, *On the curvature of Riemannian submanifolds of codimension* 2, Hokkaido Math. J. **14** (1985), 107-135.

[4] Y. Agaoka, *A note on local isometric imbeddings of complex projective spaces*, J. Math. Kyoto Univ. **27** (1987), 501-505.

[5] Y. Agaoka, *Generalized Gauss equations*, Hokkaido Math. J. **20** (1991), 1-44.

[6] Y. Agaoka, *A table on the codimension of local isometric imbeddings of Riemannian symmetric spaces*, Mem. Fac. Integrated Arts Sci., Hiroshima Univ. Ser. IV **18** (1992), 1-10.

[7] Y. Agaoka, *Invariant subvarieties of the 3-tensor space* $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$, Mem. Fac. Integrated Arts Sci., Hiroshima Univ. Ser. IV **20** (1994), 1-18.

[8] Y. Agaoka, *An algorithm to calculate plethysms of Schur functions*, Mem. Fac. Integrated Arts Sci., Hiroshima Univ. Ser. IV **21** (1995), 1-17.

[9] Y. Agaoka, *First order partial differential equations on the curvature of 3-dimensional Heisenberg bundles*, Hiroshima Math. J. **27** (1997), 229-269.

[10] Y. Agaoka, *Solutions and almost solutions of the Gauss equations of* $SU(3)/SO(3)$, Mem. Fac. Integrated Arts Sci., Hiroshima Univ. Ser. IV **25** (1999), 1-10.

[11] Y. Agaoka, *An algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras*, preprint, Technical Report No.**74**, The Division of Math. & Inform. Sci., Fac. Integrated Arts & Sci., Hiroshima Univ., 2000.

[12] Y. Agaoka and E. Kaneda, *On local isometric immersions of Riemannian symmetric spaces*, Tôhoku Math. J. **36** (1984), 107-140.

[13] Y. Agaoka and E. Kaneda, *An estimate on the codimension of local isometric imbeddings of compact Lie groups*, Hiroshima Math. J. **24** (1994), 77-110.

[14] Y. Agaoka and E. Kaneda, *Local isometric imbeddings of symplectic groups*, Geom. Dedicata **71** (1998), 75-82.

[15] E. J. Berger, *The Gauss map and isometric embedding*, Ph. D. Thesis, Harvard Univ., 1981.

[16] R. Ehrenborg, *On apolarity and generic canonical forms*, J. Algebra **213** (1999), 167-194.

[17] F. D. Grosshans, G. C. Rota and J. A. Stein, *Invariant Theory and Superalgebras*, CBMS Regional Conf. Ser. in Math. **69**, (1987), Amer. Math. Soc.

[18] A. Iarrobino, *Inverse system of a symbolic power* II. *The Waring problem for forms*, J. Algebra **174** (1995), 1091-1110.

[19] E. Kaneda, *On the Gauss-Codazzi equations*, Hokkaido Math. J. **19** (1990), 189-213.

[20] A. A. Katanova, *Explicit form of certain multivector invariants*, Adv. in Soviet Math. **8** (1992), *Lie Groups, their Discrete Subgroups, and Invariant Theory* (E. B. Vinberg ed.), 87-93, Amer. Math. Soc.

[21] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups* (Second ed.), Oxford Univ. Press, London, 1950.

[22] I. G. Macdonald, *Symmetric Functions and Hall Polynomials* (Second ed.), Oxford Univ. Press, Oxford, 1995.

[23] M. Matsumoto, *Local imbedding of Riemann spaces* I, II, Sûgaku **5** (1953), 210-219; **6** (1954), 6-16, (in Japanese).

[24] A. N. Parshin and I. R. Shafarevich (ed.), *Algebraic Geometry* IV (*Linear Algebraic Groups, Invariant Theory*), Encyclopaedia of Math. Sci. **55**, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[25] B. Reznick, *Sums of Even Powers of Real Linear Forms*, Mem. Amer. Math. Soc. **463** (1992), pp.1-155, Amer. Math. Soc.

[26] H. J. Rivertz, *On Isometric and Conformal Immersions into Riemannian Manifolds*, Thesis, Univ. Oslo, Dept. of Math. (1999), pp.1-147.

[27] B. Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, Wien, New York, 1993.

[28] T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Math. **67** (1935), 169-211.