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On conformally flat twisted product manifolds

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Abstract : We study the conformally flat twisted product manifolds $M = B\times_f F$ of Riemannian manifolds and investigate the conditions of M to be a warped product space.

Key words : twisted product manifolds, conformally flat, Einstein

Introduction

The warped product manifolds were defined by Bishop and O'Neill [2] to study the manifolds with negative curvature. The properties of warped products have studied from many points of view and this notion is used to construct a lot of examples for studying geometry ([1],[2],[4],[6], etc.).

Generalizing the idea of warped products, B.Y. Chen [3] defined the twisted products to construct a family of totally umbilical submanifolds with various properties. Machida and Sato [5], Ponge and Reckziegel [7] studied twisted products and twistor spaces in the case of pseudo-Riemannian manifolds. However the mathematical properties of the twisted product manifold are not studied enough yet.

In this paper, we are to study the conformally flat twisted product spaces $M = B\times_f F$ of Riemannian manifolds and investigate the conditions to be a warped product space, and other geometrical properties of each space.

1. Preliminaries

Let (B, g) and $(F, \bar{g})$ be n-dimensional and p-dimensional Riemannian manifolds and f a positive smooth function on $B\times F$. Consider the product manifold $B\times F$ with projections

$$\pi : B\times F \to B \quad \text{and} \quad \sigma : B\times F \to F.$$

The twisted product manifold $M = B\times_f F$ is by definition the manifold $B\times F$ with the Riemannian structure given by (see [3])

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$$\|X\|^2 = \|\pi_* X\|^2 + (f(b,p))^2 \|\sigma_* X\|^2 \tag{1.1}$$

for any vector X tangent to M at (b,p). If f depends only on B, this is the warped product of B and F in the sense of Bishop and O'Neill [2]. Let G be the Riemannian metric on M and $\dim M = m = n+p$.

For a local coordinate system (u^a) of B, the metric tensor g has the components (g_{ba}). Similarly, for a local coordinate system (u^x) of F, $\bar{g}$ has the components $(\bar{g}_{yx})$. Then, with respect to the local coordinate system (u^a, u^x) of M, G has the components

$$(G_{ji}) = \begin{pmatrix} g_{ba} & 0 \\ 0 & f^2 \bar{g}_{yx} \end{pmatrix}. \tag{1.2}$$

Throughout this paper, the ranges of indices are as follows :

$$\begin{aligned} &i, j, k, \cdots\cdots : 1, 2, \cdots\cdots\cdots, n+p = m \\ &a, b, c, \cdots\cdots : 1, 2, \cdots\cdots\cdots, n \\ &x, y, z, \cdots\cdots : n+1, \cdots\cdots, n+p \end{aligned}$$

unless otherwise stated.

Let ∇_b (resp. $\bar{\nabla}_x$) be the components of the covariant derivative with respect to g (resp. $\bar{g}$) and $\left\{ {a \atop bc} \right\}$ (resp. $\left\{ {\bar{x} \atop yz} \right\}$) the Christoffel symbol of g (resp. $\bar{g}$). Then the Christoffel symbols $\left\{ {\tilde{k} \atop ji} \right\}$ of G on M are given as follows

$$\left\{ {\tilde{a} \atop bc} \right\} = \left\{ {a \atop bc} \right\}, \tag{1.3}$$

$$\left\{ {\tilde{x} \atop yz} \right\} = \left\{ {\bar{x} \atop yz} \right\} + \frac{1}{f}(f_y \delta_z^x + f_z \delta_y^x - f^x \bar{g}_{yz}), \tag{1.4}$$

$$\left\{ {\tilde{x} \atop ya} \right\} = \frac{1}{f} f_a \delta_y^x, \tag{1.5}$$

$$\left\{ {\tilde{a} \atop xy} \right\} = -f f^a \bar{g}_{xy} \tag{1.6}$$

and the others are zero, where $f_a = \dfrac{\partial f}{\partial u^a}$ and $f_y = \dfrac{\partial f}{\partial u^y}$.

Let $\tilde{R}$, R and $\bar{R}$ be the curvature tensors of M, B and F respectively, then we have

$$\tilde{R}_{dcb}{}^a = R_{dcb}{}^a, \tag{1.7}$$

$$\tilde{R}_{dxy}{}^z = \frac{1}{f}(\partial_d f_y)\delta_x^z - \frac{1}{f}(\partial_d f^z)\bar{g}_{xy} - \frac{1}{f^2} f_d f_y \delta_x^z + \frac{1}{f^2} f_d f^z \bar{g}_{xy}, \tag{1.8}$$

$$\tilde{R}_{dxb}{}^z = \frac{1}{f}(\nabla_d f_b)\delta_x^z, \quad \tilde{R}_{dxy}{}^a = -f(\nabla_d f^a)\bar{g}_{xy}, \tag{1.9}$$

$$\tilde{R}_{xyz}{}^a = -f(\partial_x f^a)\bar{g}_{yz} + f(\partial_y f^a)\bar{g}_{xz} - f^a f_y \bar{g}_{xz} + f^a f_x \bar{g}_{yz}, \tag{1.10}$$

(1.11)
$$\tilde{R}_{xyz}{}^{w} = \bar{R}_{xyz}{}^{w} + \frac{1}{f}[(\nabla_x f_z)\delta_y^w + (\nabla_y f^w)\bar{g}_{xz} - (\nabla_x f^w)\bar{g}_{yz} - (\nabla_y f_z)\delta_x^w]$$
$$- \frac{2}{f^2}\left[f_x f_z \delta_y^w + f_y f^w \bar{g}_{xz} - f_x f^w \bar{g}_{yz} - f_y f_z \delta_x^w\right]$$
$$+ \|f_e\|^2 \left[\bar{g}_{xz}\delta_y^w - \bar{g}_{yz}\delta_x^w\right] - \frac{\|f_v\|^2}{f^2}\left[\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w\right],$$

and the others are zero.

The components of Ricci tensors are given by

(1.12)
$$\tilde{S}_{ab} = S_{ab} - \frac{p}{f}(\nabla_a f_b),$$

(1.13)
$$\tilde{S}_{ax} = -(p-1)\left(\frac{1}{f}\partial_a f_x - \frac{1}{f^2} f_a f_x\right),$$

(1.14)
$$\tilde{S}_{yx} = \bar{S}_{yx} - f(\triangle f)\bar{g}_{yx} - \frac{1}{f}(\bar{\triangle} f)\bar{g}_{yx} - \frac{(p-2)}{f}\nabla_y f_x$$
$$+ \frac{2(p-2)}{f^2} f_y f_x - (p-1)\|f_e\|^2 \bar{g}_{yx} - \frac{(p-3)}{f^2}\|f_w\|^2 \bar{g}_{yx},$$

where $\triangle f = \nabla_e f^e$, $\bar{\triangle} f = \nabla_x f^x$ and $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively.

Let $\tilde{K}$, K and $\bar{K}$ be the scalar curvatures of M, B and F respectively, then we have

(1.15)
$$\tilde{K} = K + \frac{1}{f^2}\bar{K} - \frac{2p}{f}(\triangle f) - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}\|f_e\|^2 - \frac{(p-1)(p-4)}{f^4}\|f_x\|^2.$$

2. Conformally flat twisted product manifold

Now we put

(2.1)
$$\psi_{dy} = \frac{1}{f}\partial_d f_y - \frac{f_d f_y}{f^2}$$

and

(2.2)
$$\psi_{yx} = \frac{1}{f}\nabla_y f_x - \frac{2 f_y f_x}{f^2} + \frac{1}{2}\left(\|f_e\|^2 + \frac{\|f_v\|^2}{f^2}\right)\bar{g}_{yx},$$

then the identities (1.8), (1.11), (1.13), (1.14) and (1.15) are reduced to

$$\tilde{R}_{dzy}{}^{x} = \psi_{dy}\delta_z^x - \psi_d{}^x\bar{g}_{zy}, \tag{2. 3}$$

$$\tilde{R}_{wzy}{}^{x} = \bar{R}_{wzy}{}^{x} + \psi_{wy}\delta_z^x - \psi_{zy}\delta_w^x + \psi_z{}^x\bar{g}_{wy} - \psi_w{}^x\bar{g}_{zy}, \tag{2. 4}$$

$$\tilde{S}_{dy} = -(p-1)\psi_{dy}, \tag{2. 5}$$

$$\tilde{S}_{zy} = \bar{S}_{zy} - f(\triangle f)\bar{g}_{zy} - (p-2)\psi_{zy} - \psi_w{}^w\bar{g}_{zy}, \tag{2. 6}$$

$$\tilde{K} = K + \frac{1}{f^2}\bar{K} - 2p\ \frac{\triangle f}{f} - 2(p-1)\frac{\psi_w{}^w}{f^2}, \tag{2. 7}$$

where we have put

$$\psi_w{}^w = \frac{1}{f}\ \bar{\triangle} f + \frac{p}{2}\|f_e\|^2 + \frac{p-4}{2f^2}\|f_v\|^2.$$

Assume that the twisted product manifold is conformally flat, that is, $\tilde{R}_{kji}{}^h$ of $M = B\times_f F$ satisfies (see[1])

$$\tilde{R}_{kji}{}^h = \frac{1}{m-2}\left(\tilde{S}_{ji}\delta_k^h - \tilde{S}_{ki}\delta_j^h + \tilde{S}_k{}^hG_{ji} - \tilde{S}_j{}^hG_{ki}\right) - \frac{1}{(m-1)(m-2)}\tilde{K}\left(G_{ji}\delta_k^h - G_{ki}\delta_j^h\right). \tag{2. 8}$$

Using (1. 9) and (2. 8), we have

$$\frac{1}{f}(\nabla_d f_c)\delta_y^x = -\frac{1}{(m-2)}\left\{S_{dc} - \frac{p}{f}(\nabla_d f_c)\right\}\delta_y^x - \frac{1}{(m-2)}\tilde{S}_y{}^x g_{dc} + \frac{1}{(m-1)(m-2)}\tilde{K}g_{dc}\delta_y^x. \tag{2. 9}$$

If we contract (2. 9) with respect to x and y, then we get

$$\frac{2p}{f}(\triangle f) = pK + \frac{n}{f^2}\left\{\bar{K} - \frac{2(p-1)}{f}(\bar{\triangle} f) - p(p-1)\|f_e\|^2 - \frac{(p-1)(p-4)}{f^2}\|f_x\|^2\right\} - \frac{np}{(m-1)}\tilde{K}, \tag{2.10}$$

and that

$$\frac{2p(p-1)(n-1)}{(m-1)f}(\triangle f) = \frac{p(1-p)}{(m-1)}K + \frac{n(1-n)}{(m-1)f^2}\bar{K} + \frac{2n(n-1)(p-1)}{(m-1)f^3}(\bar{\triangle} f) + \frac{pn(n-1)(p-1)}{(m-1)f^2}\|f_e\|^2 + \frac{n(n-1)(p-1)(p-4)}{(m-1)f^4}\|f_x\|^2 \tag{2.11}$$

with the help of (1.15), or equivalently

$$p(p-1)K=\frac{n(1-n)}{f^2}\overline{K}-\frac{2p(p-1)(n-1)}{f}(\triangle f)+\frac{2n(n-1)(p-1)}{f^3}(\overline{\triangle}f)$$
$$+\frac{pn(n-1)(p-1)}{f^2}\|f_e\|^2+\frac{n(n-1)(p-1)(p-4)}{f^4}\|f_x\|^2.$$

If we use (1.10) and (2. 8), then we get

$$-f^2\psi_z{}^a\overline{g}_{yx}+f^2\psi_y{}^a\overline{g}_{zx}=\frac{f^2}{m-2}\left(\tilde{S}_z{}^a\overline{g}_{yx}-\tilde{S}_y{}^a\overline{g}_{zx}\right). \tag{2.12}$$

Contracting (2.12) with respect to x and z, we obtain

$$\tilde{S}_{ya}+(m-2)\psi_{ya}=0 \tag{2.13}$$

if $p\neq 1$. Using (2. 5) and (2.13),

$$\psi_{ya}=0 \tag{2.14}$$

if $n\neq 1$, and that $\tilde{S}_{ya}=0$. Since

$$\psi_{dy}=\frac{1}{f}\partial_d f_y-\frac{f_d f_y}{f^2}=\partial_d\partial_y \log f,$$

$\psi_{ya}=0$ means that f is the product of certain functions f^* on B and $\overline{f}$ on F. Therefore, if we denote the fibre with the metric $g^*_{xy}=\overline{f}^2\,\overline{g}_{xy}$ by F^*, then we have

Theorem 2. 1. *If the twisted product manifold $M=B\times_f F$ of the Riemannian manifolds B and F is conformally flat and $p\neq 1$, $n\neq 1$, then M is the warped product space $B\times_{f^*}F^*$ of B and F^*.*

If we assume that $M=B\times_f F$ is conformally flat and $n\neq 1$, then the identities (1. 7), (2. 8) and (2.11) give rise to

$$\begin{aligned}R_{dcb}{}^a=&\frac{1}{(m-2)}\left(S_{cb}\delta_d^a-S_{db}\delta_c^a+S_d{}^a g_{cb}-S_c{}^a g_{db}\right)\\&-\frac{p}{(m-2)f}\left(\delta_d^a\nabla_c f_b-\delta_c^a\nabla_d f_b+g_{cb}\nabla_d f^a-g_{db}\nabla_c f^a\right)\\&-\frac{1}{(m-1)(m-2)}\tilde{K}\left(g_{cb}\delta_d^a-g_{db}\delta_c^a\right)\end{aligned} \tag{2.15}$$

and

(2.16) $$\bar{K}=\frac{f^2}{n(1-n)}\left\{\frac{2p(p-1)(n-1)}{f}(\Delta f)+p(p-1)K-\frac{pn(n-1)(p-1)}{f^2}\|f_e\|^2\right\}.$$

If we substitute (2.16) into (1.15), then the scalar curvature $\tilde{K}$ of M is reduced to

(2.17) $$\tilde{K}=\frac{(m-1)(n-p)}{n(n-1)}K-\frac{2p(m-1)}{nf}(\Delta f).$$

So, contracting (2.15) with respect to b and c, and using (2.17), it follows that

(2.18) $$S_{cb}=\frac{1}{n}Kg_{cb}+\frac{(n-2)}{nf}(\Delta f)g_{cb}-\frac{(n-2)}{f}(\nabla_c f_b).$$

Hence, if $n=2$, then $S_{cb}=\frac{1}{n}Kg_{cb}$, that is, B becomes an Einstein space if K is constant. Since 2-dimensional Einstein space is a space of constant curvature, we have

Theorem 2. 2. *Let $M=B\times_f F$ be the conformally flat twisted product manifold. If the scalar curvature K of B is constant and* $\dim B=2$, *then B is the space of constant curvature.*

Assume that B is a space of constant curvature and $n\geq 3$. Then B becomes an Einstein space and that (2.18) is reduced to

(2.19) $$\nabla_c f_b=\frac{1}{n}(\Delta f)g_{cb}.$$

This means that f is the concircular function.

Conversely, if f is the concircular function, then the equations (2.18) and (2.19) imply that B is an Einstein space. Therefore if we substitute $S_{cb}=\frac{1}{n}Kg_{cb}$, (2.17) and (2.19) into (2.15), then we get

(2.20) $$R_{dcb}{}^a=\frac{1}{n(n-1)}K(g_{cb}\delta_d^a-g_{db}\delta_c^a),$$

because $m=n+p$. Hence we have

Theorem 2. 3. *Let B be a Riemannian manifold with dimension $n\geq 3$ and the twisted product manifold $M=B\times_f F$ is conformally flat. Then B is the space of constant curvature if and only if f is the concircular function.*

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