オーリッツ関数のリースポテンシャルの連続性*

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Continuity properties of Riesz potentials of functions in Orlicz classes

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要　　旨

In this thesis we study continuity properties of Riesz potentials of order α, $0 < \alpha < n$, of a nonnegative measurable function f on R^n, which is defined by

$$U_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)dy.$$

Here it is natural to assume that $U_\alpha f \not\equiv \infty$, which is equivalent to

$$\int_{R^n} (1+|y|)^{\alpha-n} f(y)dy < \infty. \tag{1}$$

We are concerned with the behavior of Riesz potentials $U_\alpha f$ near a given point, which may be assumed, without loss of generality, to be the origin.

To obtain general results, we treat functions f satisfying an Orlicz condition with weight of the form

$$\int_{R^n} \Phi_p(f(y))\omega(|y|)dy < \infty.$$

Here $\Phi_p(r)$ and $\omega(r)$ are positive monotone functions on the interval $(0,\infty)$ with the following properties:

(φ1) $\Phi_p(r)$ is of the form $r^p\varphi(r)$, where $1 \le p < \infty$ and φ is a positive monotone function on the interval $(0,\infty)$; set $\varphi(0) = \lim_{r\to 0} \varphi(r)$.

(φ2) φ is of logarithmic type, that is, there exists $A_1 > 0$ such that

$$A_1^{-1}\varphi(r) \le \varphi(r^2) \le A_1\varphi(r) \qquad \text{whenever } r > 0.$$

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$(\omega 1)$ ω satisfies the doubling condition; that is, there exists $A_2 > 0$ such that

$$A_2^{-1}\omega(r) \le \omega(2r) \le A_2\omega(r) \qquad \text{whenever } r > 0.$$

Riesz potentials may not, in general, be continuous at any point of R^n. But, it is known that if $p > 1$ and

$$\int_0^1 \left[r^{n-\alpha p}\varphi(r^{-1})\right]^{-1/(p-1)} r^{-1}dr < \infty, \tag{2}$$

then $U_\alpha f$ is continuous everywhere on R^n; in case $\alpha p > n$, (2) holds by condition $(\varphi 2)$ and the continuity also follows from well-known Sobolev's theorem.

For simplicity, let $\omega(r) = r^\beta$, where $-n < \beta \le \alpha p - n$, and ℓ be the nonnegative integer such that $\ell \le \alpha - (n+\beta)/p < \ell + 1$. In this case, we treat functions f satisfying

$$\int_{R^n} \Phi_p(f(y))|y|^\beta dy < \infty. \tag{3}$$

In Chapter 2, we shall show that if (2) holds, then there exists a polynomial P_ℓ such that

$$\lim_{x\to 0}\, [K(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0 \tag{4}$$

for any function f satisfying (1) and (3), where

$$K(r) = \begin{cases} [r^{n-\alpha p+\beta}\varphi(r^{-1})]^{-1/p} & \text{in case } \ell < \alpha - (n+\beta)/p < \ell + 1 \\ & \quad \text{and } n - \alpha p < 0, \\ r^{-\beta/p}\left(\int_0^r [\varphi(t^{-1})]^{-p'/p}t^{-1}dt\right)^{1/p'} & \text{in case } \ell < \alpha - (n+\beta)/p < \ell + 1 \\ & \quad \text{and } n - \alpha p = 0, \\ r^\ell\left(\int_0^r [\varphi(t^{-1})]^{-p'/p}t^{-1}dt\right)^{1/p'} & \text{in case } \ell = \alpha - (n+\beta)/p. \end{cases}$$

Since $\lim_{r\to 0} r^{-\ell}K(r) = 0$, (4) implies that $U_\alpha f$ is ℓ times differentiable at the origin.

Let $R_\alpha(x) = |x|^{\alpha-n}$ and consider the remainder term of Taylor's expansion:

$$R_{\alpha,\ell}(x,y) = R_\alpha(x-y) - \sum_{|\mu|\le\ell}\frac{x^\mu}{\mu!}[(D^\mu R_\alpha)(-y)].$$

Then $U_\alpha f(x) - P_\ell(x)$ will be written as

$$U_{\alpha,\ell}f(x) = \int_{R^n} R_{\alpha,\ell}(x,y)f(y)dy,$$

provided

$$\int_{B(0,1)} |y|^{\alpha-n-\ell}f(y)dy < \infty,$$

where $B(x, r)$ denotes the open ball centered at x with radius $r > 0$.

In Chapter 3, we study the Hölder continuity of Riesz potentials of functions f satisfying

(5) $$\int_{R^n} \Phi_p(f(y))dy < \infty,$$

by applying the results in Chapter 2. Recently Edmunds and Krbec studied almost Lipschitz continuity for Bessel potentials of order $n/p + 1$ of functions f satisfying

$$\int_{R^n} f(y)^p[\log(e + f(y))]^{-\sigma}dy < \infty$$

for some $\sigma > 0$. In this chapter, we deal with Riesz potentials of order α, $n/p \le \alpha \le n/p + 1$, and give extensions of those results. Our aim in this direction is to find κ such that

$$U_\alpha f(x) - U_\alpha f(0) = o(\kappa(|x|)) \qquad \text{as } x \to 0$$

when f satisfies (5), and our result implies the above mentioned result by Edmunds and Krbec.

If (2) does not hold, then the potential may not be continuous anywhere, and we study the fine limits of $U_\alpha f$ with respect to the relative Orlicz capacity

$$C_{k,\Phi_p}(E; G) = \inf_g \int_G \Phi_p(g(y))dy, \qquad E \subset G,$$

where k is a nonnegative Borel measurable function on R^n, G is an open set in R^n and the infimum is taken over all nonnegative measurable functions g on G such that

$$\int_{R^n} k(|x - y|)g(y)dy \ge 1 \qquad \text{for every } x \in E.$$

In case $k(r) = r^{\alpha-n}$, we write C_{α,Φ_p} for C_{k,Φ_p}. For simplicity, we write $C_{k,\Phi_p}(E) = 0$ if $C_{k,\Phi_p}(E \cap G; G) = 0$ for every bounded open set G. If a property holds except for a set E with $C_{k,\Phi_p}(E) = 0$, then we say that the property holds C_{k,Φ_p}-quasi everywhere. In Chapter 4, we show that if f satisfies (1) and (3), then there exist a set $E \subset R^n$ and a polynomial P_ℓ such that

$$\lim_{x \to 0, x \in R^n - E} [\kappa(|x|)]^{-1}[U_\alpha f(x) - P_\ell(x)] = 0$$

and

$$\sum_{j=1}^{\infty} 2^{j(n-\alpha p)}[\varphi(2^j)]^{-1}C_{\alpha,\Phi_p}(E_j; B_j) < \infty,$$

where $E_j = \{x \in E : 2^{-j} \le |x| < 2^{-j+1}\}$, $B_j = \{x : 2^{-j-1} < |x| < 2^{-j+2}\}$ and

$$\kappa(r) = r^\ell \left(\int_0^r [t^{n-\alpha p+\beta+\ell p}\varphi(t^{-1})]^{-1/(p-1)}t^{-1}dt\right)^{1-1/p}$$

If in addition (2) holds, then the exceptional set E is empty and the above fine limit is seen to be replaced by the usual limit similar to (4).

In Chapter 5, we are concerned with the existence of radial limits. We shall show that if f satisfies (1) and (3), then there exist a set $E^* \subset \partial B(0,1)$ and a polynomial P_ℓ such that $C_{\alpha,\Phi_p}(E^*) = 0$ and

$$\lim_{r\to 0} r^{(n-\alpha p+\beta)/p}[U_\alpha f(r\xi) - P_\ell(r\xi)] = 0 \qquad \text{for any } \xi \in \partial B(0,1) - E^*.$$

In Chapter 6, we deal with L^q-mean limits for Taylor's expansion of Riesz potentials $U_\alpha f$

$$\lim_{r\to 0} \kappa(r)^{-1} \left(r^{-n} \int_{B(x_0,r)} |U_\alpha f(x) - P_{x_0}(x)|^q \, dx \right)^{1/q} = 0 \tag{6}$$

for functions f satisfying (5) and for $0 < q < \infty$ satisfying $1/q \geq 1/p - \alpha/n$; if $1/q = 1/p - \alpha/n$, then q is called the Sobolev exponent. If (2) holds, then we know that (4) holds and hence (6) trivially holds for $\kappa(r) = K(r)$. Thus we are mainly concerned with the case where (2) does not necessarily hold. In Section 3 of Chapter 6, we shall show that (6) holds for every x_0 except that in a set of C_{k,Φ_p}-capacity zero, where $k(r) = r^{\alpha-n}\kappa(r)^{-1}$. In view of the behavior at the origin of Bessel kernels, our results can be considered as generalizations of the results by Meyers concerning Bessel potentials of functions in $L^p(R^n)$.

If (6) holds for $\kappa(r) = r^\ell$, then $U_\alpha f$ is said to be L^q-differentiable of order ℓ at x_0, where ℓ is a positive integer such that $\ell \leq \alpha$. In the final section we discuss quasi every L^q-differentiability as a consequence of the proceeding results in case $\ell < \alpha$ and in fact show that $U_\alpha f$ is L^q-differentiable of order ℓ $C_{\alpha-\ell,\Phi_p}$-quasi everywhere. In case $\alpha = \ell$, $U_\ell f$ is shown to be L^q-differentiable of order ℓ almost everywhere. If (2) holds, then $U_\ell f$ is known to be ℓ times differentiable almost everywhere.