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On Lyapunov Theorem for Descriptor Systems

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Abstract. : In this note, we present a different version of Lyapunov theorem for descriptor systems by introducing several new definitions.

Key words : Descriptor systems, generalized Lyapunov theorem.

1. Introduction

Consider a linear time-invarant descriptor system

$$E\dot{x}(t)=Ax(t), \tag{1a}$$
$$y(t)=Cx(t), \tag{1b}$$

where $x\in R^n$ is the state, $y\in R^{\ell}$ is the output. E is a square matrix of rank $r\leq n$. We assume that the system (1) is regular, that is, $det(sE-A)$ is not identically zero. We also assume that the response of the system (1) is impulse-free, that is, $deg\ [det(sE-A)]=rank\ E$.

It is well known that there exist two nonsingular matrices M, H such that system (1) can take the form

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_1(t) \end{bmatrix}, \tag{2a}$$

and

$$y=[C_1\ \ C_2]\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, \tag{2b}$$

where $[z_1^T(t)\ \ z_2^T(t)]=x^T(t)H^{-T}$ and

$$MEH=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},\ MAH=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ CH=[C_1\ \ C_2]. \tag{3}$$

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In a recent paper [1], a new generalized Lyapunov theorem for descriptor systems is proposed by extending the theorem due to Lewis [2].

In this note, we present a different version of the Lyapunov theorem for descriptor systems by introducing several new definitions and a generalized Lyapunov equation.

2. Stability Analysis of Descriptor Systems

Definition 1. [3] The system (1) is called asymptotically stable if

$$rank\ [sE-A]=n, \text{ for all } Re(s)\geq 0. \tag{4}$$

Definition 2. A matrix A is called a generalized stable matrix corresponding to E, simply, the generalized stable matrix, if the condition (4) is satisfied.

Remark 1. It is different from the general definition of stable matrix that a generalized stable matrix A does not necessarily satisfy the condition of $Re(\lambda_i(A)) < 0$, for all eigenvalues $\lambda_i(A)$ of A.

Example 1.

$$E=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A=\begin{bmatrix} -2 & -3 \\ 4/3 & 3 \end{bmatrix}$$

Since $\lambda_{1,2}(A)=\{-1, 2\}$, A is not a stable matrix. However, since

$$rank\ \left(\begin{bmatrix} s+2 & 3 \\ -4/3 & -3 \end{bmatrix}\right)=rank\ \left(\begin{bmatrix} s+2/3 & 0 \\ -4/3 & -3 \end{bmatrix}\right)=2, \text{ except for } s=-2/3,$$

A is really a generalized stable matrix.

Definition 3. [3] The system (1) is called finite dynamic detectable if

$$rank\ [sE^T-A^T,\ C^T]=n, \text{ for all } Re(s)\geq 0. \tag{5}$$

Lemma 1. The system (1) is asymptotically stable if and only if

$$rank\ [sI_r-A_0]=r, \text{ for all } Re(s)\geq 0, \tag{6}$$

where $A_0=A_{11}-A_{12}A_{22}^{-1}A_{21}$.

Note that A_{22}^{-1} exists asebecu the response of the system (1) is assumed to be impulse-free.

Proof. Since

$$M(sE-A)H=\begin{bmatrix} sI_r-A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}, \tag{7}$$

and

$$\begin{bmatrix} I_r & -A_{12}A_{22}^{-1} \\ 0 & -A_{22}^{-1} \end{bmatrix}\begin{bmatrix} sI_r-A_{11} & -A_{12} \\ -A_{21} & -A_{22} \end{bmatrix}\begin{bmatrix} I_r & 0 \\ -A_{22}^{-1}A_{21} & I_{n-r} \end{bmatrix}=\begin{bmatrix} sI_r-A_0 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{8}$$

the proof follows from Definition 1 directly.

Lemma 2. The system (1) is finite dynamic detectable if and only if

$$rank\ [sI_r-A_0^T,\ C_0^T]=r, \text{ for all } Re(s)\geq 0, \tag{9}$$

where $A_0=A_{11}-A_{12}A_{22}^{-1}A_{21}$, $C_0=C_1-C_2A_{22}^{-1}A_{21}$.

Proof. Since

$$[H^T(sE^T-A^T)M^T,\ H^TC^T]=\begin{bmatrix} sI_r-A_{11}^T & -A_{21}^T & C_1^T \\ -A_{12}^T & -A_{22}^T & C_2^T \end{bmatrix}, \tag{10}$$

and

$$\begin{bmatrix} I_r & -A_{21}^TA_{22}^{-T} \\ 0 & -A_{22}^{-T} \end{bmatrix}\begin{bmatrix} sI_r-A_{11}^T & -A_{21}^T & C_1^T \\ -A_{12}^T & -A_{22}^T & C_2^T \end{bmatrix}\begin{bmatrix} I_r & 0 & 0 \\ -A_{22}^{-T}A_{12}^T & I_{n-r} & A_{22}^{-T}C_2^T \\ 0 & 0 & I_e \end{bmatrix}$$

$$=\begin{bmatrix} sI_r-A_0^T & 0 & C_0^T \\ 0 & I_{n-r} & 0 \end{bmatrix}, \tag{11}$$

the proof follows from Definition 3 directly.

Definition 4. A matrix P is called a generalized positive semidefinite (definite) matrix, denoted by $P\underline{>}0$ $(P>0)$, if $x^TE^TPx\underline{>}0$ (>0), for all x such that $Ex\neq 0$.

Let us introduce a generalized Lyapunov equation as follows.

$$i)\ A^TP+P^TA+C^TC=0, \tag{12a}$$

$$ii)\ E^TP=P^TE. \tag{12b}$$

Defining

$$M^{-T}PH=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{13}$$

we can obtain some properties on the generalized positive semidefinite (definite) matrix P from *ii)*, that is, P is a lower-triangular block matrix of $P_{12}\equiv 0$ and $P_{11}^T=P_{11}\geq 0\ (>0)$ after the transformation (13).

Theorem 1. Suppose that the descriptor system (1) is regular and impulse-free. Then,

(i) if A is a generalized stable matrix then the generalized Lyapunov equation (12) admits a generalized positive semidefinite solution $P\geq 0$,

(ii) if there is a generalized positive semidefinite matrix P satisfying (12) and the system (1) is finite dynamic detectable, then A is a generalized stable matrix.

Proof. Making transformation to the generalized Lyapunov equation (12a) by using the relations (3), (13), we have

$$\begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\begin{bmatrix} P_{11} & 0 \\ P_{21} & P_{22} \end{bmatrix}+\begin{bmatrix} P_{11} & P_{21}^T \\ 0 & P_{22}^T \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}+\begin{bmatrix} C_1^TC_1 & C_1^TC_2 \\ C_2^TC_1 & C_2^TC_2 \end{bmatrix}=0. \tag{14}$$

(14) can be partitioned into

$$P_{11}A_{11}+P_{21}^TA_{21}+A_{11}^TP_{11}+A_{21}^TP_{21}+C_1^TC_1=0, \tag{15a}$$

$$P_{22}^TA_{21}+A_{12}^TP_{11}+A_{22}^TP_{21}+C_2^TC_1=0, \tag{15b}$$

$$P_{22}^TA_{22}+A_{22}^TP_{22}+C_2^TC_2=0. \tag{15c}$$

(15c) is a Lyapunov-like equation and has a solution P_{22}. Furthermore,

$$P_{21}=-A_{22}^{-T}[A_{12}^TP_{11}+P_{22}^TA_{21}+C_2^TC_1] \tag{16}$$

according to (15b). Substituting (16) into (15a) and making some manipulations, we obtain

$$A_0^TP_{11}+P_{11}A_0+C_0^TC_0=0, \tag{17}$$

where $A_0=A_{11}-A_{12}A_{22}^{-1}A_{21}$, $C_0=C_1-C_2A_{22}^{-1}A_{21}$. Therefore,

(i) if A is a generalized stable matrix, Lemma 1 and standard Lyapunov theorem ensure that (17) has a unique solution $P_{11} \geq 0$. Hence, there exists a generalized positive semidefinite solution of (12).

(ii) That there is a generalized positive semidefinite matrix P satisfying (12) implies that (17) has a solution$P_{11} \geq 0$. Furthermore, the system (1) is finite dynamic detectable means that $(A_0\ C_0)$ is a detectable matrix pair from Lemma 2. Therefore, A_0 is stable, which means that A is generalized stable matrix according to Lemma 1 and Definition 2.
Remark 2. Different from the standard Lyapunov theorem for state space systems, here the generalized positive semidefinite matrix P satisfying (12) may not be unique since the solution of (15c) may not be unique.

3. Conclusion

By introducing the concepts of the generalized stable matrix, the generalized positive semidefinite (definite) matrix and the generalized Lyapunov equation, we have introduced a different version of the Lyapunov theorem for descriptor systems.

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