Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser. IV, Vol. 22, 1 −23, Dec. 1996

Tilings of the 2-dimensional sphere by congruent right triangles

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Abstract : We classify all spherical tilings consisting of congruent right triangles. There exist five sporadic types and four series of such tilings. We also exhibit the figures of these tilings.

Key words : tiling, spherical triangle, Euler's formula

Introduction

The purpose of this paper is to classify all tilings of the 2-dimensional sphere by congruent right triangles. Such classification is already studied in the previous papers Sommerville [6] and Davies [3]. But, Sommerville studied only "regular" tilings in the case of scalene triangles, and in the paper Davies [3], regrettably, detailed proof is not stated there. Hence, it is desirable to give a complete proof of the classification of spherical tilings, and we study in this paper tilings consisting of congruent "right" triangles.

The case where the isometry group acts transitively on geometric objects of tiles of the sphere is studied deeply in the paper Grünbaum and Shephard [4]. But in the present paper, we do not consider the group action on the sphere, and classify spherical tilings purely in combinatorial way, under the condition that the sphere is tiled by only one type of right triangles. As a result, we show that there exist five sporadic types and four series of such tilings, about half of which are obtained by "twisting" standard tilings in a sense. (For details, see Theorem 1.1 and the explanation following it.)

We now explain the contents of this paper. In §1, we state the main result and explain the construction of each tiling. In §2, we first classify spherical tilings by right equilateral or isosceles triangles. In §3, we treat tilings with small number of faces, and in §4, we determine the combinatorial type of vertices appearing in spherical tilings. Using these results, we classify tilings by right scalene triangles in §5 and §6.

In this paper, we always assume that no vertex of any triangle lies on the interior of an edge of any other triangle, and we identify two tilings if they are mapped to each other by a rotation or a reflection.

The main part of this paper is the first named author's master's thesis (Faculty of Science, Hiroshima University, 1996).

Received August 28, 1996; Accepted October 3, 1996

1. Main theorem

In this section, we state our main theorem (Theorem 1.1), and summarize some preliminary facts on spherical triangles which we use in this paper. We first fix our notations.

We denote by α, β, γ the angles of a spherical triangle. (In this paper, we consider only triangles with $0 < \alpha, \beta, \gamma < \pi$.) Assume that the sphere is tiled by several copies of one triangle, and express the number of vertices, edges, faces of this tiling by V, E, F, respectively. If a vertex on the sphere is surrounded by angles α, β, γ with numbers k, l, m, respectively, we say that the type of this vertex is

$$k\alpha + l\beta + m\gamma = 2\pi.$$

(We ignore the order of α, β, γ appearing in the vertex in this expression.) Of course, there may be various types of vertices in the tiling.

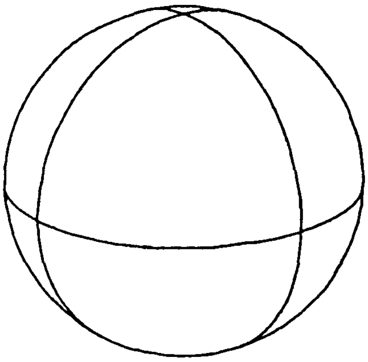
Using these notations, we state the main theorem of this paper. The meaning of the symbol in the left column and the explicit construction of each tiling are explained below.

THEOREM 1.1. *Spherical tilings by congruent triangles with $\alpha = \pi/2$ are exhausted by the following:*

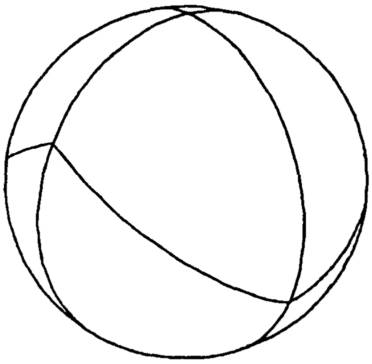
	F	E	V	β	γ	*type of vertex*
F_8	8	12	6	$\beta+\gamma=\pi$ $\left(\begin{array}{c}0<\beta,\ \gamma<\pi\\ \lvert\beta-\gamma\rvert<\pi/2\end{array}\right)$		4α, $2\beta+2\gamma$
F_{16}	16	24	10	$3\pi/8$	$3\pi/8$	4α, $\alpha+4\beta$
F_{48}	48	72	26	$\pi/4$	$\pi/3$	4α, 8β, 6γ
TF_{48}	48	72	26	$\pi/4$	$\pi/3$	4α, 8β, 6γ, $2\alpha+4\beta$
F_{120}	120	180	62	$\pi/5$	$\pi/3$	4α, 10β, 6γ
G_{2n} $(n\geqq 3)$	$2n$	$3n$	$n+2$	$2\pi/n$	$\pi/2$	4α, $n\beta$
TG_{4n} $(n\geqq 3)$	$4n$	$6n$	$2n+2$	π/n	$\pi/2$	4α, $2\alpha+n\beta$
H_{8n} $(n\geqq 3)$	$8n$	$12n$	$4n+2$	$\dfrac{(n-1)\pi}{2n}$	$\dfrac{\pi}{n}$	4α, $2n\gamma$, $4\beta+2\gamma$
TH_{16n-8} $(n\geqq 3)$	$16n-8$	$24n-12$	$8n-2$	$\dfrac{(n-1)\pi}{2n-1}$	$\dfrac{\pi}{2n-1}$	4α, $4\beta+2\gamma$, $2\beta+2n\gamma$

(Note that the tiling F_8 contains one continuous parameter.)

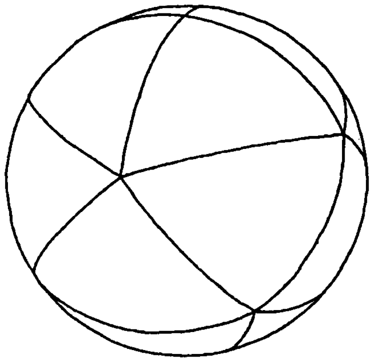
Next, we exhibit the figures of these tilings as follows: (The figures of H_{24}, F_{48}, F_{120} are also exhibited in [1; p.16].)



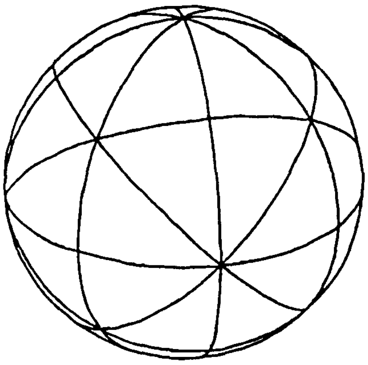
F_8 $(\beta=\gamma=\pi/2)$



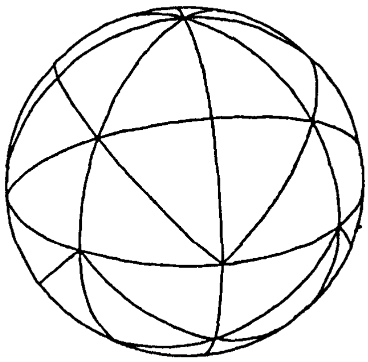
F_8



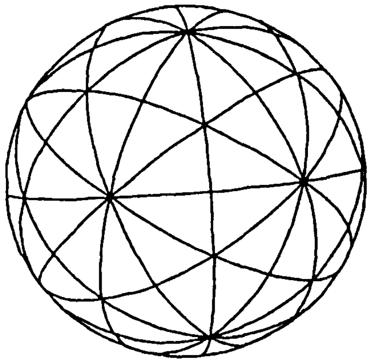
F_{16}



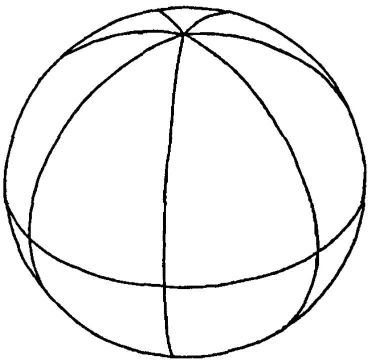
F_{48}



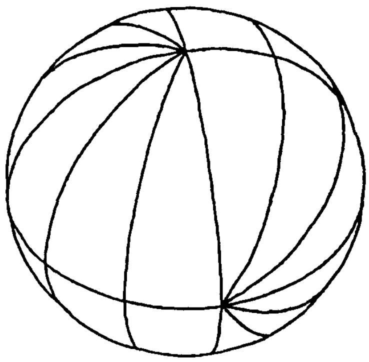
TF_{48}



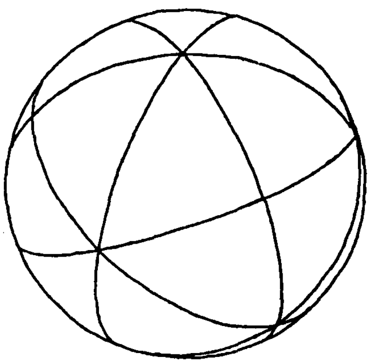
F_{120}



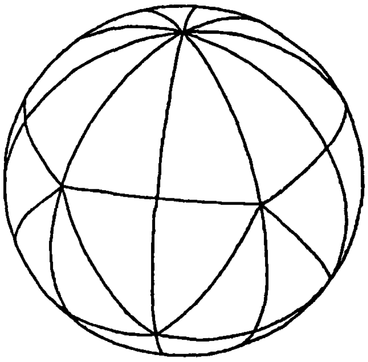
G_{14}



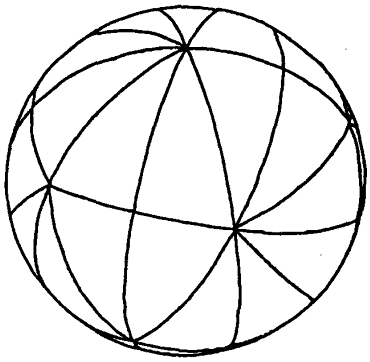
TG_{24}



H_{24}



H_{40}



TH_{40}

Figure 1

Now, we state the explicit construction of each tiling briefly.

F_8 : We first project the regular octahedron to the circumscribed sphere, and next move four vertices on the equator up and down by the same angle alternately.

F_{48}, F_{120} : These are the standard tilings obtained by projecting regular polyhedra to the circumscribed sphere and by dividing faces suitably.

TF_{48} : Rotate the south hemisphere of F_{48} along the equator with the angle $\pi/4$.

G_{2n} : This tiling is obtained by drawing the equator and n longitudes having the same angle.

TG_{4n} : Rotate a half part of G_{4n} along one longitude with the angle $\pi/2$.

H_{8n} : We first construct a tiling consisting $2n$ rhombuses with angles $2\pi/n$, $(n-1)\pi/n$, $2\pi/n$, $(n-1)\pi/n$, by attaching n angles $2\pi/n$ to the north and south poles. Next, we divide rhombuses into four triangles by drawing two diagonal lines.

TH_{16n-8} : By using the rhombuses with angles $2\pi/(2n-1)$, $2(n-1)\pi/(2n-1)$, $2\pi/(2n-1)$, $2(n-1)\pi/(2n-1)$, we construct a half tiling of the sphere, attaching n and $n-1$ angles $2\pi/(2n-1)$ to the north and south poles respectively in a consecutive way. Next, we prepare two copies of this figure and patch them together such that four poles do not meet. (See Figure 10 in the proof of Proposition 5.4.) Finally, we divide rhombuses into four triangles in the same way as H_{8n}.

F_{16} : In constructing the tiling H_{32}, delete all long diagonal lines of rhombuses.

REMARK. In the above table, the tiling G_8 is a special case of F_8 with $\beta = \gamma = \pi/2$. We may also consider the tilings TG_8, H_{16} and TH_{24} in the above construction. But, these are equal to G_8, TG_{16} and H_{24}, respectively. The tiling H_{24} can be constructed from the regular hexahedron, and by dividing the triangles in H_{24} into two parts, we obtain the tiling F_{48}. The tilings H_{40} and TH_{40} also can be constructed from regular icosahedron.

Now, we state some facts on spherical triangles for later use. The following result is well known.

PROPOSITION 1.2 (cf. [7; p.62]). *The angles* α, β, γ *of a spherical triangle satisfy the following inequalities:*

$$\pi < \alpha + \beta + \gamma,$$
$$\alpha + \beta < \pi + \gamma, \quad \beta + \gamma < \pi + \alpha, \quad \gamma + \alpha < \pi + \beta.$$

Conversely, if α, β, γ *satisfy these inequalities, then there exists a spherical triangle with angles* α, β, γ.

We note that the shape of a triangle is uniquely determined by α, β, γ. For example, the length of an edge is determined by the cosine rule.

We assume that the radius of the sphere is always 1. Then, in this situation, the area of this triangle is expressed as

$$S = \alpha + \beta + \gamma - \pi.$$

Since the sphere is tiled by this triangle, the number of faces of the tiling is equal to $F = 4\pi/S = 4\pi/(\alpha + \beta + \gamma - \pi)$. We often use this formula in this paper.

In the rest of this paper, we always assume $\alpha = \pi/2$, unless otherwise stated.

2. Spherical tilings by right equilateral or isosceles triangles

In this section, we classify spherical tilings by congruent right equilateral or isosceles triangles. In the case of right equilateral triangles, we have

THEOREM 2.1. *The tiling* F_8 *with* $\beta = \gamma = \pi/2\,(= G_8)$ *is the unique tiling of the sphere by right equilateral triangles.*

PROOF. Since the area of this triangle is equal to $S = \alpha + \beta + \gamma - \pi = 3\pi/2 - \pi = \pi/2$, we have $F = 4\pi/S = 8$. Then, by using the fact that the type of all vertices is $4\alpha = 2\pi$, we can easily show that this tiling is obtained by projecting the regular octahedron to the circumscribed sphere. q. e. d.

In the case of right isosceles triangles, we have

THEOREM 2.2. *Tilings by right isosceles (but not equilateral) triangles of the sphere are exhausted by* F_{16}, H_{24}, G_{2n} $(n=3$ *or* $n \geq 5)$ *and* TG_{4n} $(n \geq 3)$.

PROOF. We divide the proof into the following two cases according as the type of isosceles triangles:

I. The case $\beta=\gamma\,(\neq \pi/2)$,
II. The case $\alpha=\gamma=\pi/2\,(\neq\beta)$.

We first consider the case I. Assume that there exists a vertex of type $k\alpha+l\beta=2\pi$ $(0\leq k\leq 4,\ 0\leq l)$. By using the formula $S=\alpha+2\beta-\pi=2\beta-\pi/2$ and $F=4\pi/S$, we have $\beta=(F+8)\pi/4F$. Substituting this into $k\alpha+l\beta=2\pi$, we have $l=2F(4-k)/(F+8)$. Then, by using the conditions $F>0$, $0\leq k\leq 4$ and $l\geq 0$, it is easy to see that the triple of integers (k, l, F) satisfying this equality is one of the following:

	k	l	F
i	0	6	24
ii	0	7	56
iii	1	2	4
iv	1	4	16
v	1	5	40
vi	2	3	24
vii	4	0	undetermined.

(We omit the case $F=8$ because in this case β is equal to $\pi/2$.) By drawing the figures below, we know that vertices of type ii, v, vi cannot exist.

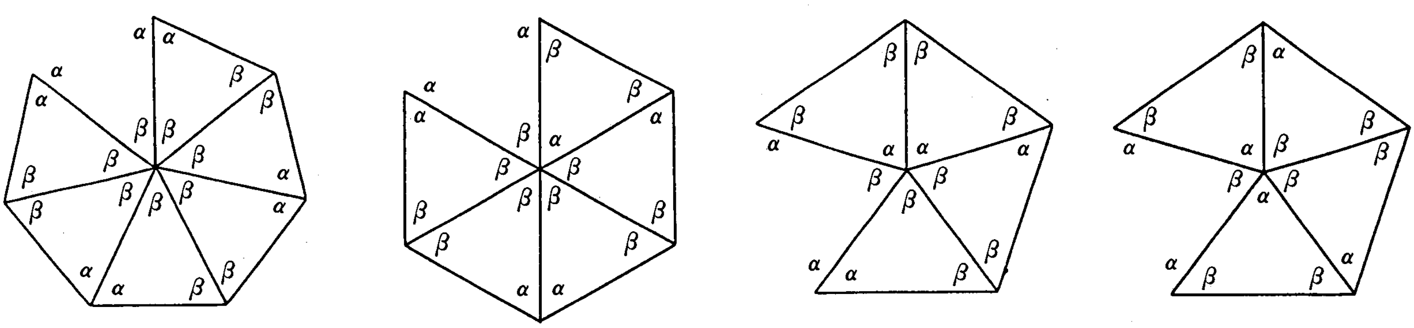


Figure 2

Next, vertices of type iii also cannot exist because the angle β does not satisfy the condition $2\beta < \pi + \alpha$ in Proposition 1.2. Hence, the remaining vertices are i , iv and vii.

If the tiling contains only one type of vertices, then the ratio of the number of α and β is 1 : 2, which implies $k : l = 1 : 2$. But this is not the case for i , iv, vii, and hence the tiling contains at least two types of vertices. Since vertices of type i and iv cannot exist simultaneously, possible combination of vertices is i and vii, or iv and vii. And by drawing development maps exhibited below, we know that for each case, there exists a unique tiling of the sphere, which corresponds to H_{24} or F_{16}.

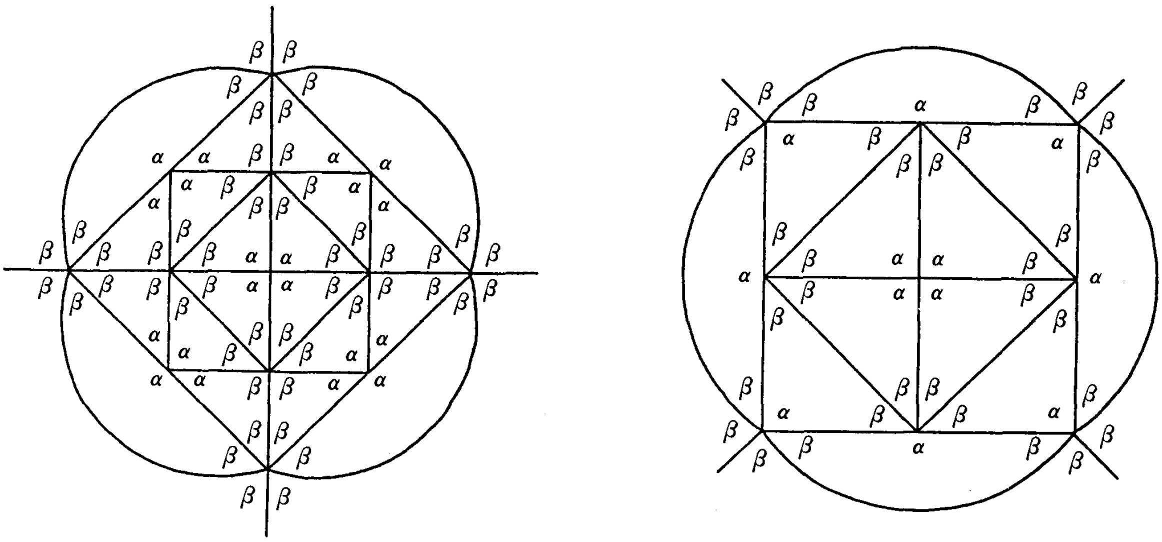


Figure 3

Next, we consider the case II. In this case, we have $\alpha = \gamma = \pi/2 \neq \beta$. Assume that there exists a vertex of type $k\alpha + l\beta = 2\pi$ $(0 \leq k \leq 4,\ 0 \leq l)$. By drawing the following figures, we know that k cannot be equal to 1 nor 3.

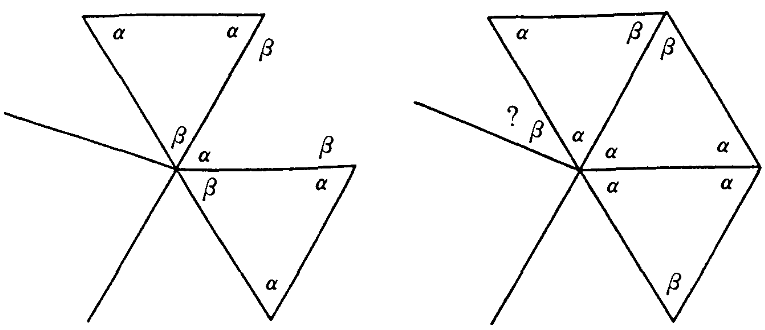


Figure 4

Hence, types of vertices must be one of the following:

$$4\alpha = 2\pi,\quad 2\alpha + p\beta = 2\pi,\quad q\beta = 2\pi \quad (p,\ q \geq 3,\ q \neq 4).$$

In addition, from the above figure, we know that two α's in the vertex of type $2\alpha+p\beta=2\pi$ must be adjacent. If the tiling contains only one type of vertices, we have $k:l=2:1$ because the ratio of the number of α and β is $2:1$. But from the above expressions, this case does not occur, and hence the tiling contains at least two types of vertices.

Now, we show that two types of vertices $2\alpha+p\beta=2\pi$ and $q\beta=2\pi$ cannot exist simultaneously. In fact, if these two types both exist, we have $2p=q$ and $F=4\pi/(2\alpha+\beta-\pi)=4p$. We denote by x and y the number of vertices of type $2\alpha+p\beta=2\pi$ and $q\beta=2\pi$ appearing in the tiling, respectively. Then, since the number of β appearing in the sphere is equal to F, we have $px+qy=F$, which implies $x+2y=4$. Therefore, we have $x=2$ and $y=1$. We draw the following development map by starting from the vertex of type $2\alpha+p\beta=2\pi$. Since the number of vertices of this type is 2, two dot points in this figure must coincide, which is a contradiction. Hence, two vertices $2\alpha+p\beta=2\pi$ and $q\beta=2\pi$ cannot exist simultaneously.

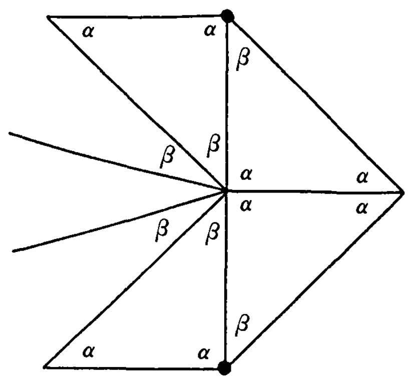


Figure 5

Therefore, remaining combinations are the following:

(i) $4\alpha=2\pi$ and $2\alpha+p\beta=2\pi$ $(p\geq 3)$,
(ii) $4\alpha=2\pi$ and $q\beta=2\pi$ $(q=3$ or $q\geq 5)$.

For each case, by drawing a development map, we can easily show that there exists uniquely a tiling of the sphere, which corresponds to TG_{4p} or G_{2q}. (For each case, the sphere is tiled by lunes consisting of two triangles.) q. e. d.

3. Spherical tilings by right scalene triangles with small number of faces

In this section, we determine spherical tilings by right scalene triangles with small number of faces. We assume that angles $\alpha(=\pi/2)$, β, γ are mutually distinct, and we often use this property in drawing development maps. The purpose of this section is to prove the following theorem, giving the minimum number of faces of such tilings.

THEOREM 3.1. *For spherical tilings by right scalene triangles, the number of faces satisfies the inequality* $F \geq 8$. *The equality* $F=8$ *holds if and only if the tiling is equal to* F_8 *for some* β, γ *satisfying* $0<\beta, \gamma<\pi$, $\beta+\gamma=\pi$ *and* $0<|\beta-\gamma|<\pi/2$.

To prove this theorem, we prepare two lemmas.

LEMMA 3.2. *There exists no 3-valent vertex in spherical tilings by right scalene triangles.*

PROOF. By drawing a development map, we can easily show that the type of 3-valent vertex is $\alpha+\beta+\gamma=2\pi$. In particular, we have $\beta+\gamma=\pi+\alpha$, which contradicts Proposition 1.2. q. e. d.

The next lemma can also be easily proved by drawing suitable development maps.

LEMMA 3.3. *If there exists a 4-valent vertex, then the type of this vertex is* $4\alpha=2\pi$ *or* $2\beta+2\gamma=2\pi$.

PROOF of THEOREM 3.1. Each face contains three edges, and each edge is contained in two faces. Hence, we have $E=3F/2$. Together with Euler's formula, we have $F=2V-4$. Next, we denote by V_k the number of k-valent vertices $(k\geq 3)$. Then, we have

$$V= V_k+V_l+\cdots\cdots+V_m \quad (k>l>\cdots\cdots>m),$$
$$F=(kV_k+lV_l+\cdots\cdots+mV_m)/3.$$

Combining these equalities, we have

$$(6-m)V\geq(6-k)V_k+(6-l)V_l+\cdots\cdots+(6-m)V_m$$
$$=6V-3F=12>0,$$

which implies $m=3, 4, 5$. But, from Lemma 3.2, we have $m\neq 3$, and hence, $m=4$ or 5. Then, from the inequality $(6-m)V\geq 12$, we have $V\geq 6$, which implies $F\geq 8$. In the case of $F=8$, we have $E=12$ and $V=6$. In particular, if $k\geq 6$, we have $V_k=0$. Hence, from the equalities

$$V=V_4+V_5=6,$$
$$F=(4V_4+5V_5)/3=8,$$

we have $V_4=6$ and $V_5=0$. Therefore, from Lemma 3.3, the vertices of this tiling are of type $4\alpha=2\pi$ and $2\beta+2\gamma=2\pi$, and both types must actually appear. Then, the uniqueness of the tiling with $F=8$ can be proved easily by drawing a development map starting from the vertex $4\alpha = 2\pi$. Clearly, we have $\beta + \gamma = \pi$, and the inequality $|\beta - \gamma| < \pi/2$ follows immediately from Proposition 1.2. q. e. d.

4. Combinatorial type of vertices

In this section, we determine the combinatorial type of vertices, appearing in spherical tilings by right scalene triangles. In the rest of this paper, we always assume $F>8$, except in Lemma 4.2. The purpose of this section is to prove the following proposition, which plays a fundamental role during the classification in §5 and §6.

PROPOSITION 4.1. *The type of vertices of spherical tilings by right scalene triangles with $F>8$ is one of the following:*

$$
\begin{array}{lll}
[A] & 4\alpha = 2\pi, & \\
[B] & 2\alpha + 2p\beta = 2\pi & (p \geq 2), \\
[B'] & 2\alpha + 2q\gamma = 2\pi & (q \geq 2), \\
[C] & \alpha + (2p-1)\beta + \gamma = 2\pi & (p \geq 2), \\
[C'] & \alpha + \beta + (2q-1)\gamma = 2\pi & (q \geq 2), \\
[D] & 2p\beta + 2\gamma = 2\pi & (p \geq 2), \\
[D'] & 2\beta + 2q\gamma = 2\pi & (q \geq 2), \\
[E] & 2p\beta = 2\pi & (p \geq 3), \\
[E'] & 2q\gamma = 2\pi & (q \geq 3).
\end{array}
$$

REMARK. (1) Since each type contains only one parameter except [A], two kinds of vertices of the same type cannot exist simultaneously. For example, $2p\beta + 2\gamma = 2p'\beta + 2\gamma = 2\pi$ implies $p = p'$.

(2) Actually, odd-valent types [C] and [C'] do not appear in spherical tilings by right scalene triangles. We prove this fact during the classification in §6 (Proposition 6.7).

We prove this proposition by combining several lemmas. For this purpose, we first consider a situation where one vertex is surrounded by general (not necessary right) congruent scalene triangles with angles λ, μ, ν. (We do not assume that this triangle tiles the whole sphere.)

LEMMA 4.2. *The type of this vertex is*

$$2p\lambda + 2q\mu = 2\pi \quad (p, q \geq 0)$$

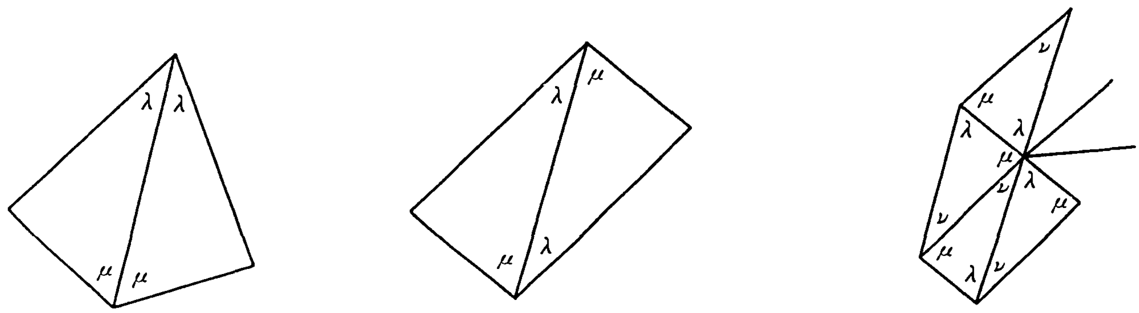
or

$$(2p-1)\lambda + (2q-1)\mu + \nu = 2\pi \quad (p, q \geq 1),$$

by changing the angles suitably.

PROOF. For a 3- or 4-valent vertex, as in Lemmas 3.2 and 3.3, we can easily prove that the type is $\lambda+\mu+\nu=2\pi$, $4\lambda=2\pi$ or $2\lambda+2\mu=2\pi$, by changing the angles λ, μ, ν suitably. Hence, the lemma holds in this case.

Now we consider a k-valent vertex with $k\geq 5$. Since three edges of a triangle have different length, any edge is of type (a) or (b) in the following figure. We assume that edges of type (a) do not appear around this vertex. Then, by drawing triangles around this point, we know that k is a multiple of 3. (See the figure below.)



(a) (b)

Figure 6

We put $k=3l$. Then, we have $l(\lambda+\mu+\nu)=2\pi$, and hence, $S=\lambda+\mu+\nu-\pi=(2-l)\pi/l>0$, which implies $l=1$. This contradicts the assumption $k\geq 5$, and therefore, there exists an edge of type (a) around this vertex. Then, we can drop two triangles containing this edge, and obtain a $k-2$-valent new combinatorial type of vertex, by changing the value of angles suitably.

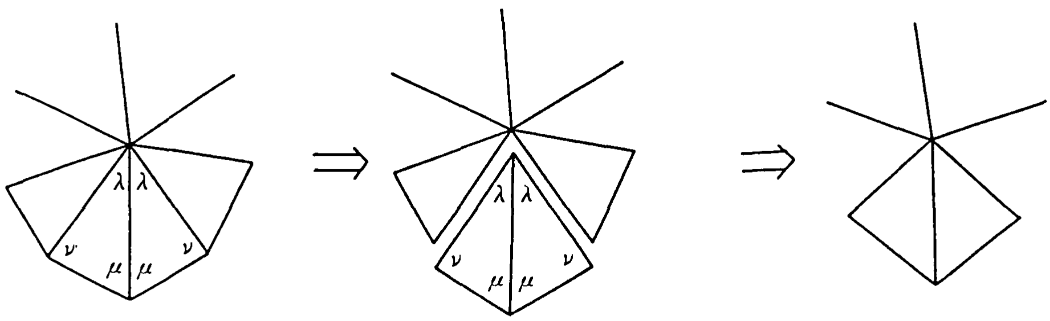


Figure 7

Repeating this procedure several times, we finally arrive at a 3- or 4- valent vertex. Hence, the type of the original vertex is obtained by adding 2λ, 2μ or 2ν to the type of 3- or 4-valent vertex. Therefore, in the case of odd-valent vertex, its type is expressed as

$$(2p-1)\lambda+(2q-1)\mu+(2r-1)\nu=2\pi.$$

We may assume $p \geq q \geq r \geq 1$. Then, we have

$$2\pi - 2(p-r)\lambda - 2(q-r)\mu = (2r-1)(\lambda + \mu + \nu) > (2r-1)\pi,$$

and hence, we have

$$(3-2r)\pi > 2(p-r)\lambda + 2(q-r)\mu \geq 0.$$

Therefore, we have $r=1$, and the type of odd-valent vertex must be $(2p-1)\lambda + (2q-1)\mu + \nu = 2\pi$. In the case of even-valent vertex, starting from the form $2p\lambda + 2q\mu + 2r\nu = 2\pi$ $(p \geq q \geq r \geq 0)$, we can prove $r=0$ in the same way. q. e. d.

Now, we return to the tiling by right scalene triangles with $F > 8$. As for odd-valent vertices, we have the following lemmas.

LEMMA 4.3. *Vertices of type* $3\alpha + (2p-1)\beta + \gamma = 2\pi$ *and* $3\alpha + \beta + (2q-1)\gamma = 2\pi$ *do not appear in the tiling.*

PROOF. Assume that a vertex of type $3\alpha + (2p-1)\beta + \gamma = 2\pi$ exists. Then, from this equality, we have $\alpha + \beta + \gamma - \pi = 2(1-p)\beta > 0$, which contradicts the assumption $p \geq 1$. The second type can be treated in the same way. q. e. d.

LEMMA 4.4. *Assume that there exists a vertex of type*

$$\alpha + (2p-1)\beta + (2q-1)\gamma = 2\pi \quad (p \geq q \geq 1).$$

Then, we have $p \geq 2$ *and* $q=1$.

PROOF. From the assumption, we have $(2p-1)\beta + (2q-1)\gamma = 3\pi/2$. On the other hand, from the formula $F = 4\pi/(\alpha + \beta + \gamma - \pi)$, we have $\beta + \gamma = (F+8)\pi/2F$. If $p=q$, then we have from these equalities $(p-2)(F+8) = -12$, which implies $p=1$ and $F=4$. But this contradicts the assumption $F > 8$. Hence, we have $p \neq q$. Then, from the above two equalities, we have

$$\beta = \frac{(2-q)F - 4(2q-1)}{2(p-q)F}\pi,$$

$$\gamma = \frac{4(2p-1) + (p-2)F}{2(p-q)F}\pi.$$

Substituting into $\alpha + \beta < \pi + \gamma$ and $\gamma + \alpha < \pi + \beta$ in Proposition 1.2, we have

$$(F+4)p+4q>2F+4,$$
$$4p+(F+4)q<2F+4.$$

From these inequalities, we have

$$\frac{2F+4-4q}{F+4}<p<\frac{2F+4-(F+4)q}{4},$$

which implies $q<(2F+4)/(F+8)<2$. Therefore, we have $q=1$ and $p\geq 2$. q. e. d.

Combining these three lemmas, it follows that the type of odd-valent vertex is $[C]\ \alpha+(2p-1)\beta+\gamma=2\pi\ (p\geq 2)$ or $[C']\ \alpha+\beta+(2q-1)\gamma=2\pi\ (q\geq 2)$ because the coefficient of α cannot exceed 4. This completes the proof of Proposition 4.1 for odd-valent case.

For even-valent vertices, we can prove the following lemma in the same way as Lemma 4.4.

LEMMA 4.5. *Assume that there exists a vertex of type*

$$2p\beta+2q\gamma=2\pi \quad (p\geq q\geq 1).$$

Then, we have $p\geq 2$ *and* $q=1$.

Then, Proposition 4.1 for even-valent case follows immediately from Lemmas 4.2 and 4.5.

5. Classification of spherical tilings by right scalene triangles with $F>8$. I

Now, using these results, we classify spherical tilings by right scalene triangles with $F>8$. We divide the proof into two cases:

I. The case where the right angle α appears only in vertices of type $[A]\ 4\alpha=2\pi$.
II. The case where there is a vertex containing α which is not of type $[A]$.

In this section, we treat the case I, and prove the following theorem.

THEOREM 5.1. *Assume* $F>8$ *and the right angle appears only in vertices of type* $[A]\ 4\alpha=2\pi$. *Then, the tiling by right scalene triangles is equal to* F_{48}, F_{120}, $TH_{16n-8}\ (n\geq 3)$ *or* $H_{8n}\ (n\geq 4)$.

Note that in the case I, any triangle is contained in one rhombus, and hence, the sphere is tiled by these $F/4$ congruent rhombuses with angles 2β, 2γ, 2β, 2γ.

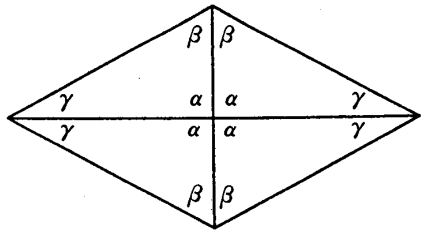


Figure 8

To prove the above theorem, we must prepare several lemma and propositions. First, note that, from Proposition 4.1, types of vertices are expressed as $4\alpha = 2\pi$ and $2p\beta + 2q\gamma = 2\pi$ ($p, q \geq 0$) because α appears only in type [A].

LEMMA 5.2. *Except the type* $4\alpha = 2\pi$, *there exist at least two types of vertices of the form* $2p\beta + 2q\gamma = 2\pi$.

PROOF. If the type of vertices of the form $2p\beta + 2q\gamma = 2\pi$ uniquely exists, we have $p = q$ because the number of β and γ appearing in the tiling must coincide. But, this type of vertices does not appear in Proposition 4.1. q. e. d.

Hence, the tiling must contain at least two types of the following vertices, except [A] $4\alpha = 2\pi$.

[D]	$2p\beta + 2\gamma = 2\pi$	($p \geq 2$),
[D']	$2\beta + 2q\gamma = 2\pi$	($q \geq 2$),
[E]	$2p\beta = 2\pi$	($p \geq 3$),
[E']	$2q\gamma = 2\pi$	($q \geq 3$).

PROPOSITION 5.3. *If the tiling contains vertices of type* [E] $2p\beta = 2\pi$ ($p \geq 3$) *and* [E'] $2q\gamma = 2\pi$ ($q \geq 3$) *simultaneously, then it is equal to* F_{48} *or* F_{120}.

PROOF. From the assumption, we have $\beta = \pi/p$ and $\gamma = \pi/q$. We may assume $p > q$ by the symmetry of β and γ. Then, we have $S = \alpha + \beta + \gamma - \pi = (2p + 2q - pq)\pi/2pq > 0$. Hence, we have $(p-2)(q-2) < 4$, which implies $(p, q) = (4, 3)$ or $(5, 3)$. If $(p, q) = (4, 3)$, then we have $\beta = \pi/4$, $\gamma = \pi/3$, $S = \pi/12$ and $F = 48$. Similarly, if $(p, q) = (5, 3)$, we have $\beta = \pi/5$, $\gamma = \pi/3$, $S = \pi/30$ and $F = 120$. For both cases, we can easily show that vertices of type [D] and [D'] cannot exist, and using this fact, we can uniquely construct the following development maps consisting of rhombuses with angles 2β, 2γ, 2β, 2γ. (We put $B = 2\beta$, $C = 2\gamma$ in the following figure.)

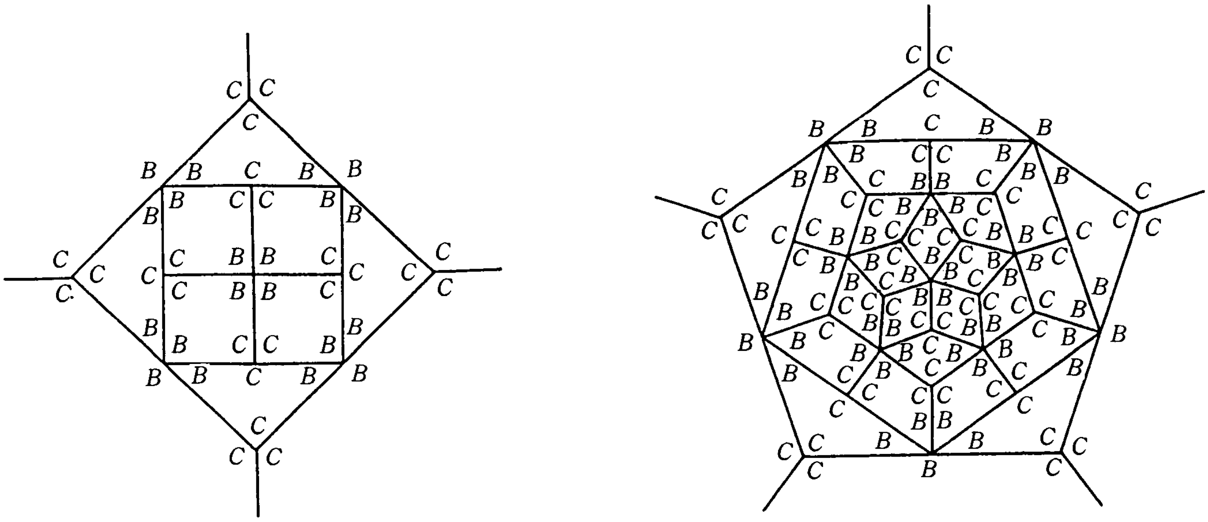


Figure 9

These tilings correspond to F_{48} and F_{120}, respectively. q. e. d.

PROPOSITION 5.4. *Assume that the tiling does not contain vertices of type* $[E]$ *nor* $[E']$. *Then it is equal to* $TH_{16n-8}\,(n \geq 3)$.

PROOF. From the assumption and Lemma 5.2, there exist vertices of type $[D]$ $2p\beta+2\gamma=2\pi$ and $[D']$ $2\beta+2q\gamma=2\pi$, and other types of vertices do not appear except $[A]$ $4\alpha=2\pi$. By the symmetry, we may assume $q \geq p \geq 2$. Then, from these equalities, we have $\beta=(q-1)\pi/(pq-1)$, $\gamma=(p-1)\pi/(pq-1)$, and $F=4\pi/(\alpha+\beta+\gamma-\pi)=8(pq-1)/(2p+2q-pq-3)>0$. Hence, we have $(p-2)(q-2)<1$, which implies $p=2$. If $q=2$, then we have $\beta=\gamma$, and this contradicts the assumption. Hence, we have $q \geq 3$. Then, from the above equality, we have $\beta=(q-1)\pi/(2q-1)$, $\gamma=\pi/(2q-1)$ and $F=16q-8$. Starting from the vertex of type $2\beta+q\cdot 2\gamma=2\pi$, we can draw the development map by rhombuses with angles 2β, 2γ, 2β, 2γ uniquely as follows:

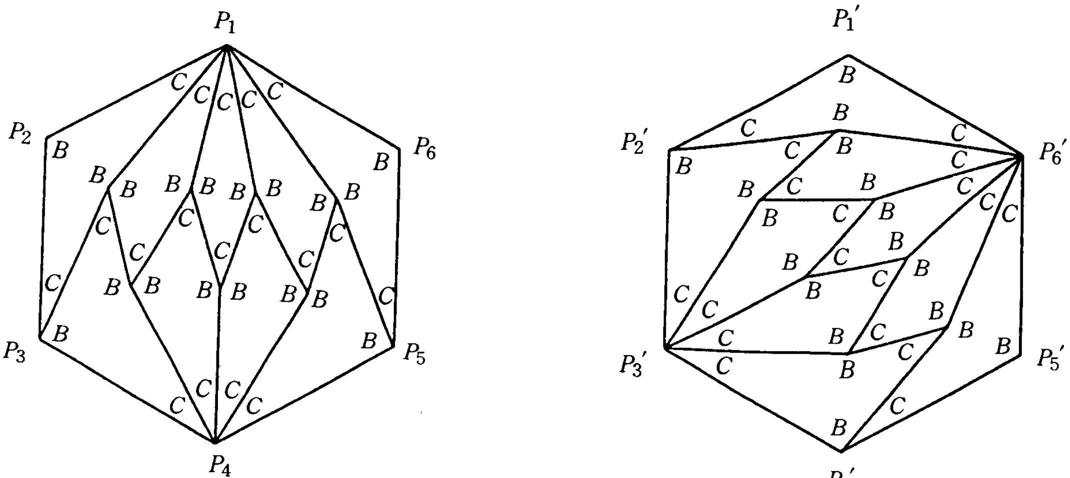


Figure 10

(Here, we identify P_i and P_i' and we put $B=2\beta$ and $C=2\gamma$, as above.) By construction, this tiling is TH_{16q-8}. q. e. d.

For remaining cases, the tiling must contain one of the vertices of type [E] and [E']. By the symmetry of β and γ, we may assume that a vertex of type [E'] exists, but [E] does not.

PROPOSITION 5.5. *Assume that the tiling contains vertices of type* [D] $2p\beta+2\gamma=2\pi$ $(p\geq 2)$ *and* [E'] $2q\gamma=2\pi\,(q\geq 3)$. *Then we have* $p=2$, $q\geq 4$ *and the tiling is equal to* H_{8q}.

PROOF. From the equality $F=4\pi/(\alpha+\beta+\gamma-\pi)$, we have $\beta+\gamma=(F+8)\pi/2F$. Combining with the equalities $2p\beta+2\gamma=2q\gamma=2\pi$, we have $F=8pq/(2p+2q-pq-2)>0$. Hence, we have $(p-2)(q-2)<2$, which implies $p=q=3$ or $p=2$. If $p=q=3$, then we have $\beta=2\pi/9$ and $\gamma=\pi/3$. Hence, the sphere can be tiled by the rhombus with angles $4\pi/9$, $2\pi/3$, $4\pi/9$, $2\pi/3$. It is easy to check that the possible types of vertices are $3\cdot 2\beta+2\gamma=2\pi$ and $3\cdot 2\gamma=2\pi$ in this situation. But, as the following figure shows, we cannot construct a tiling in this case.

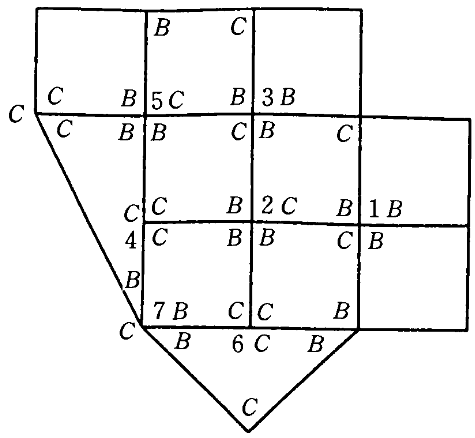


Figure 11

(The numbers of vertices indicate the order of drawing.) In the case $p=2$, we have $\beta=(q-1)\pi/2q$, $\gamma=\pi/q$ and $F=8q$. In this situation, if $q=3$, we have $\beta=\gamma=\pi/3$. Hence, we have $q\geq 4$. It is easy to see that the types of vertices except [A] are exhausted by [D] $4\beta+2\gamma=2\pi$, [D'] $2\beta+(q+1)\gamma=2\pi$ (this type exists only in the case $q=$odd) and [E'] $2q\gamma=2\pi$. From the assumption, a vertex of type $2q\gamma=2\pi$ actually exists, and we draw a development map consisting rhombuses by starting from the vertex $q\cdot 2\gamma=2\pi$. Since vertices containing four β's must be of type $2\cdot 2\beta+2\gamma=2\pi$, the development map necessary becomes the following form, which shows the uniqueness of the tiling. (Hence, as a result, vertices of type $2\beta+(q+1)\gamma=2\pi$ do not appear.)

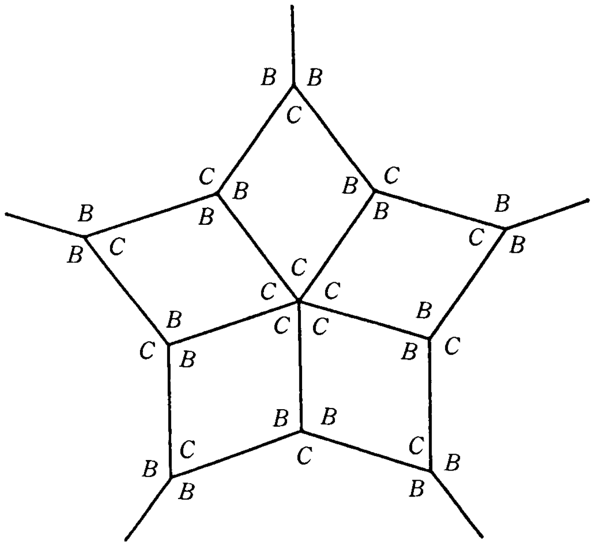


Figure 12

Clearly, this tiling corresponds to H_{8q}. q. e. d.

Now, to complete the proof of Theorem 5.1, we have only to consider the case where the tiling does not contain vertices of type [D] nor [E]. For this case, we have the following proposition.

PROPOSITION 5.6. *There is no tiling consisting of only three types of vertices* [A] $4\alpha = 2\pi$, [D'] $2\beta + 2p\gamma = 2\pi$ $(p \geq 2)$ *and* [E'] $2q\gamma = 2\pi$ $(q \geq 3)$.

PROOF. In this case, combined with the equality $\beta + \gamma = (F+8)\pi/2F$, we have $\beta = (q-p)\pi/q$, $\gamma = \pi/q$ and $F = 8q/(q-2p+2) > 0$. We denote by V_1, V_2, V_3 the number of vertices of type $4\alpha = 2\pi$, $2\beta + 2p\gamma = 2\pi$ and $2q\gamma = 2\pi$, respectively. Then, since the number of β and γ appearing in the tiling must be equal, we have $2V_2 = 2pV_2 + 2qV_3 = F = 8q/(q-2p+2)$. From these equalities, we have $V_3 = 4(1-p)/(q-2p+2) < 0$, which is a contradiction. Hence, this case does not occur. q. e. d.

6. Classification of spherical tilings by right scalene triangles with $F>8$. II

In this section, we treat the case II, and prove the following theorem.

THEOREM 6.1. *Assume* $F>8$ *and there is a vertex containing* α *which is not of type* [A] $4\alpha = 2\pi$. *Then, the tiling by right scalene triangles is equal to* TF_{48}.

First, we state the following combinatorial proposition. We often use this proposition in drawing development maps of tilings.

PROPOSITION 6.2. *In spherical tilings by congruent right scalene triangles, same angles appearing in a vertex are consecutively adjacent.*

By using Proposition 4.1, we can easily prove this proposition, and we left the proof to the reader.

Now, to prove Theorem 6.1, we must prepare several lemmas concerning the coexistence of types of vertices. We first remind that from the assumption in Theorem 6.1, there exists at least one type of vertices among the following:

[B] $2\alpha+2p\beta=2\pi \quad (p\geq 2)$,
[B′] $2\alpha+2q\gamma=2\pi \quad (q\geq 2)$,
[C] $\alpha+(2p-1)\beta+\gamma=2\pi \quad (p\geq 2)$,
[C′] $\alpha+\beta+(2q-1)\gamma=2\pi \quad (q\geq 2)$.

LEMMA 6.3. *Two types* [B] $2\alpha+2p\beta=2\pi$ $(p\geq 2)$ *and* [B′] $2\alpha+2q\gamma=2\pi$ $(q\geq 2)$ *cannot exist simultaneously.*

PROOF. From these equalities, we have $\beta=\pi/2p$, $\gamma=\pi/2q$. Hence, we have $S=\alpha+\beta+\gamma-\pi=(p+q-pq)\pi/2pq>0$, from which we have $(p-1)(q-1)<1$. This contradicts the assumption $p, q\geq 2$. q. e. d.

LEMMA 6.4. *Two types* [C] $\alpha+(2p-1)\beta+\gamma=2\pi$ $(p\geq 2)$ *and* [C′] $\alpha+\beta+(2q-1)\gamma=2\pi$ $(q\geq 2)$ *cannot exist simultaneously.*

PROOF. From these two equalities, we have $\beta=(3q-3)\pi/(4pq-2p-2q)$, $\gamma=(3p-3)\pi/(4pq-2p-2q)$. If $p=q$, then we have $\beta=\gamma=3\pi/4p$, which is a contradiction. Hence, we have $p\neq q$, and by the symmetry, we may assume $p>q$. Then, substituting to $\beta+\gamma=(F+8)\pi/2F$, we have $F=4(2pq-p-q)/(2p+2q-pq-3)>0$, from which we have $(p-2)(q-2)<1$. Therefore, we have $q=2$, $p\geq 3$, $\beta=3\pi/(6p-4)$ and $\gamma=(3p-3)\pi/(6p-4)$. In this situation, we can easily check that among the types [A] ~ [E′], only three types [A] $4\alpha=2\pi$, [C] $\alpha+(2p-1)\beta+\gamma=2\pi$ and [C′] $\alpha+\beta+3\gamma=2\pi$ can exist. Then, by drawing the development map from the vertex $\alpha+\beta+3\gamma=2\pi$, we can easily show that this case does not actually occur in the case $p\geq 3$, and hence two types [C] and [C′] do not exist simultaneously.

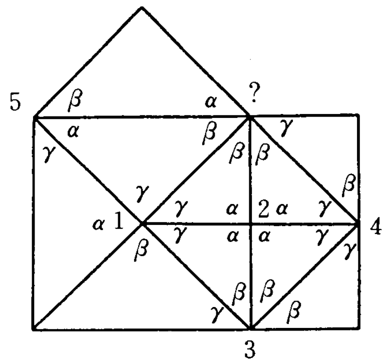


Figure 13

(The numbers of vertices indicate the order of drawing.) q. e. d.

LEMMA 6.5. *Two types* $[B']$ $2\alpha + 2p\gamma = 2\pi$ $(p \geq 2)$ *and* $[C]$ $\alpha + (2q-1)\beta + \gamma = 2\pi$ $(q \geq 2)$ *cannot exist simultaneously.*

PROOF. From these equalities, we have $\beta = (3p-1)\pi/2p(2q-1)$, $\gamma = \pi/2p$. Hence, we have $S = \alpha + \beta + \gamma - \pi = (2p+q-pq-1)\pi/p(2q-1) > 0$, from which we have $(p-1)(q-2) < 1$. Hence, we have $q=2$ and $\beta = (3p-1)\pi/6p$, $F=12p$. In this situation, we can easily show that possible types of vertices are $[A]$ $4\alpha = 2\pi$, $[B']$ $2\alpha + 2p\gamma = 2\pi$, $[C]$ $\alpha + 3\beta + \gamma = 2\pi$ and $[E']$ $4p\gamma = 2\pi$ $(p \geq 2)$, and other types cannot appear. We denote by V_1, V_2, V_3, V_4 the numbers of these vertices appearing in the tiling, respectively. Then, since numbers of α, β, γ in this tiling are equal to $F=12p$, we have

$$4V_1 + 2V_2 + V_3 = 3V_3 = 2pV_2 + V_3 + 4pV_4 = 12p,$$

and from these equalities, we can easily show that $V_1 > 0$. Hence, a vertex of type $[A]$ $4\alpha = 2\pi$ exists. Then, starting from this vertex, we draw a development map. But, we arrive at the contradiction as the following figure shows, and hence this case does not occur.

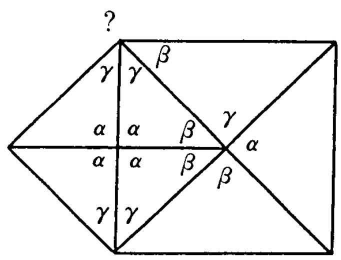


Figure 14

q. e. d.

By changing β and γ in this lemma, we have the following lemma.

LEMMA 6.6. *Two types* [B] $2\alpha+2p\beta=2\pi$ $(p\geq 2)$ *and* [C'] $\alpha+\beta+(2q-1)\gamma=2\pi$ $(q\geq 2)$ *cannot exist simultaneously.*

Using these lemmas, we prove the following proposition.

PROPOSITION 6.7. *There is no vertex of type* [C] *nor* [C'] *in the tiling of case* II.

PROOF. Assume there exists a vertex of type [C]. Then, by Lemmas 6.4, 6.5, there is no vertex of type [B'] nor [C']. Hence, vertices containing both α and γ must be of type [C]. Using this fact, we draw a development map by starting from the vertex of type [C] $\alpha+(2q-1)\beta+\gamma=2\pi$. But, as the following figure shows, this case does not occur.

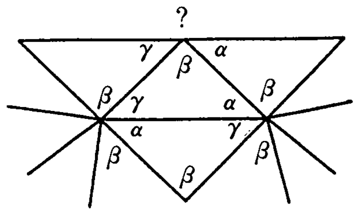


Figure 15

The case [C'] can be treated in the same way, by using Lemmas 6.4 and 6.6. q. e. d.

From this proposition, we know that tilings of case II consist of only even-valent vertices.

Under these preliminaries, we now prove Theorem 6.1. By Proposition 6.7, the type of vertices containing α is [A], [B] or [B'], and from Lemma 6.3, we may assume that there is a vertex of type [B] $2\alpha+2p\beta=2\pi$ $(p\geq 2)$, and the type [B'] does not exist. Then, from this equality, we have $\beta=\pi/2p$, and possible types of vertices are

$$\begin{array}{ll} [A] & 4\alpha=2\pi, \\ [B] & 2\alpha+2p\beta=2\pi, \\ [D] & 2q\beta+2\gamma=2\pi, \\ [D'] & 2\beta+2r\gamma=2\pi, \\ [E] & 4p\beta=2\pi, \\ [E'] & 2s\gamma=2\pi. \end{array} \qquad (p,\ q,\ r\geq 2,\ s\geq 3),$$

Now, we assume that there exists a vertex of type [E'] $2s\gamma=2\pi$. Then, we have $\gamma=\pi/s$, $S=(2p+s-ps)\pi/2ps>0$, from which we have $(p-1)(s-2)<2$. Therefore, we have $p=2$ and $s=3$, which implies $\beta=\pi/4$, $\gamma=\pi/3$, $S=\pi/12$ and $F=48$. In this situation, possible types of vertices are 4α, $2\alpha+4\beta$, 8β and 6γ. Then, starting from the vertex of type $[B]$ $2\alpha+4\beta=2\pi$, we can uniquely draw the development map of this tiling as follows:

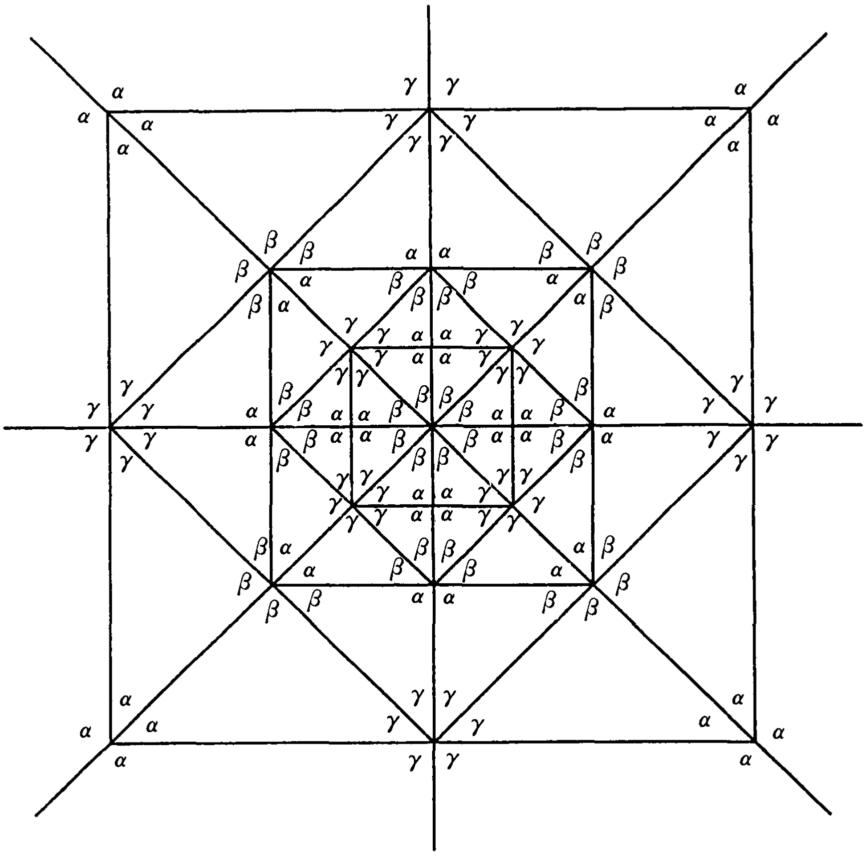


Figure 16

Clearly, this tiling is equal to TF_{48}.

Next, we consider the case where the type $[E']$ does not appear. In this case, if a vertex of type $[D']$ exists, then we have $r=2$. In fact, from the equality $2\beta+2r\gamma=2\pi$, we have $\gamma=(2p-1)\pi/2pr$. Hence, $S=(2p+r-pr-1)\pi/2pr>0$, which implies $(p-1)(r-2)<1$. Hence, we have $r=2$ and $\gamma=(2p-1)\pi/4p$. In addition, if a vertex of type $[D]$ exists, then from the equality $2q\beta+2\gamma=2\pi$, we have $2q-1=2p$, which is a contradiction. Hence, the type $[D]$ does not appear, and possible types of vertices are $[A]$ $4\alpha=2\pi$, $[B]$ $2\alpha+2p\beta=2\pi$, $[D']$ $2\beta+4\gamma=2\pi$ and $[E]$ $4p\beta=2\pi$. We draw a development map starting from the vertex $[B]$ $2\alpha+2p\beta=2\pi$. Then, as the following figure shows, this case does not occur.

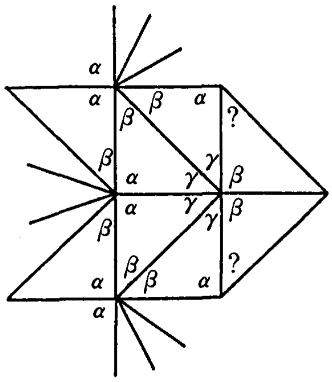


Figure 17

Therefore, vertices of type $[D']$ do not exist. Then, the remaining possible vertices are $[A]$, $[B]$, $[D]$ and $[E]$. In this situation, starting from the vertex $[B]$ $2\alpha + 2p\beta = 2\pi$, we draw a development map. But as the following figure shows, this case also does not occur.

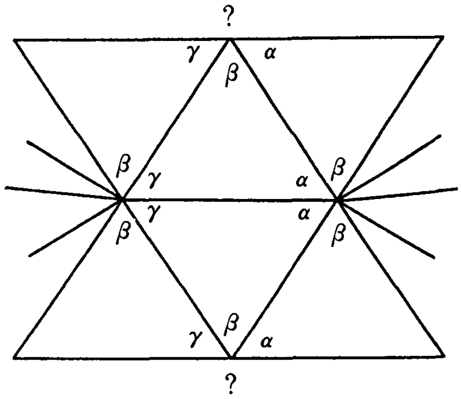


Figure 18

Hence, combining these cases, the tiling must be of type TF_{48}, and we complete the proof of Theorem 6.1. q. e. d.

Combining Theorems 2.1, 2.2, 3.1, 5.1 and 6.1, we complete the proof of Theorem 1.1.

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