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A note on growth estimates for positive solutions of nonlinear elliptic inequalities

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Abstract : We give some upper bounds for positive solutions of nonlinear elliptic inequalities specified below. Comparison principles are considerably used to analyze these problems.

Key words : elliptic inequality, positive solution

1. Introduction

Consider the simple sublinear elliptic equation

$$\Delta u = p(x)u^{\sigma}, \quad x \in \mathbb{R}^N, \tag{1. 1}$$

where $\sigma \in (0, 1)$ and p is positive and continuous. Then, it is easily seen from Jensen's inequality that the spherical mean $\bar{u}(r)$, $r \geq 0$, of a positive solution u of (1. 1) satisfies

$$[\bar{u}(r)]^{1-\sigma} \leq [u(0)]^{1-\sigma} + (1-\sigma)\int_0^r \int_0^s (t/s)^{N-1}\left(\max_{|x|=t} p(x)\right)dtds, \quad r \geq 0. \tag{1. 2}$$

In particular, $\min_{|x|=r} [u(x)]^{1-\sigma}$ is bounded by the right hand side of (1. 2). On the other hand, it can be shown that, occasionally, (1. 1) has a positive solution u such that $u^{1-\sigma}$ behaves like a positive constant multiple of the right hand side of (1. 2) as $|x| = r \to \infty$. For example, the function $(1+|x|^2)^{1/(1-\sigma)}$ solves (1. 1) with $p(x)$ given by

$$p(x) = \frac{2}{1-\sigma}\left[N + \frac{2\sigma|x|^2}{(1-\sigma)(1+|x|^2)}\right].$$

Accordingly we can conclude that (1. 2) gives an effective bound for positive solutions of (1. 1).

In the present paper we try to extend this fact for more general elliptic inequalities. In Section 2, first we treat the semilinear inequality

$$(L-c)u \equiv \sum_{i,j=1}^N a_{ij}(x)D_{ij}u + \sum_{i=1}^N b_i(x)D_iu - c(x)u \leq p(x)f(u), \quad x \in \Omega, \tag{1. 3}$$

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where Ω is an exterior domain, L is an elliptic operator, and $p \in C(\Omega)$ is positive. In Section 3 we consider the quasilinear inequality

$$Mu \equiv \operatorname{div}\left[\frac{Du}{(1+|Du|^2)^{\alpha}}\right] \le p(x)f(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $0 \le \alpha < 1/2$, and $p \in C(\mathbb{R}^N)$ is positive. When $0 < \alpha < 1/2$, M is often referred to as the generalized mean curvature operator. In these inequalities we always assume that $N \ge 2$, and $f \in C(0, \infty)$ is positive. Other detailed hypotheses are specified later.

Several works treated the problem of finding effective estimates for positive solutions of inequalities like (1. 3) or (1. 4). The cases of $\sigma > 1$, and $\sigma > 1$ and $p(x) \ge 0$ in (1. 1) were studied in [2] and [4], respectively. Other related results are also found in the papers [1, 3, 5].

2. Semilinear inequalities

First we investigate the semilinear inequality (1. 3). Throughout this section we assume the following:

(H_1) a_{ij}, b_i, $c \in C(\Omega)$ for all i, j, the symmetric matrix $(a_{ij}(x))$ is positive definite at each $x \in \Omega$, and $c(x) \ge 0$ in Ω.

(H_2) $f \in C^1(0, \infty), f'(u) \ge 0$ for $u > 0$, and

$$F(u) \equiv \int_0^u \frac{ds}{f(s)}, \quad u > 0,$$

exists.

Condition (H_2) means that f is a so-called sublinear function at $u = 0$.

We introduce notation by means of which our results are formulated. For $x = (x_i) \in \Omega$, $x \neq 0$, we put

$$A(x) = |x|^{-2} \sum_{i,j=1}^{N} a_{ij}(x) x_i x_j .$$

Let D_*, p^*, $\tilde{p}$, and c^* be continuous functions satisfying for large r

$$D_*(r) \le \min_{|x|=r} \frac{1}{A(x)} \sum_{i=1}^{N} \left(a_{ii}(x) + x_i b_i(x)\right)$$

and

$$p^*(r) \ge \max_{|x|=r} \frac{p(x)}{A(x)}, \quad \tilde{p}(r) \ge \max_{|x|=r} p(x), \quad c^*(r) \ge \max_{|x|=r} \frac{c(x)}{A(x)},$$

respectively. We also employ the function $K_*(r)$ defined by

$$K_*(r) = r \exp\left(-\int_{r_0}^{r} \frac{D_*(s)}{s} ds\right), \quad r \ge r_0,$$

where $r_0 > 0$ is an arbitrarily fixed constant such that $\{x : |x| \geq r_0\} \subset \Omega$. For example, if $L = \Delta$, the above functions can be taken to be those satisfying

$$A(x) \equiv 1, \quad D_*(r) \equiv N, \quad K_*(r) = \text{pos. const. } r^{1-N},$$

and

$$p^*(r) = \tilde{p}(r) \geq \max_{|x|=r} p(x), \quad c^*(r) \geq \max_{|x|=r} c(x).$$

THEOREM 2.1. *Suppose that there is a positive function $w(r)$ satisfying*

$$w'' + \frac{1}{r}(D_*(r)-1)w' - c^*(r)w \geq p^*(r), \quad r \geq r_0, \tag{2. 1}$$

$$w'(r_0) > 0; \quad w'(r) \geq 0, \quad r > r_0. \tag{2. 2}$$

Then, positive solutions u of (1. 3) *satisfy*

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} F(u(x))}{w(r)} < \infty. \tag{2. 3}$$

PROOF. Put $v(x) = F(u(x)), x \in \Omega$. Then we get

$$Lv(x) = \frac{Lu}{f(u)} - \frac{f'(u)}{[f(u)]^2} \sum_{i,j} a_{ij}(x) D_i u D_j u$$

$$\leq c(x)\frac{u}{f(u)} + p(x) \leq c(x)v(x) + p(x), \quad x \in \Omega.$$

Here the trivial inequality

$$v(x) \geq \int_0^{u(x)} \frac{ds}{f(u(x))}, \quad x \in \Omega,$$

is employed. On the other hand, $w(|x|)$ satisfies the inequality

$$Lw(r) - c(x)w(r) \geq p(x), \quad |x| = r \geq r_0.$$

Indeed, in view of $w'(r) \geq 0$, we have

$$\frac{1}{A(x)}\big(Lw(r) - c(x)w(r)\big)$$

$$= w''(r) + \frac{1}{r}\left[\frac{1}{A(x)} \sum_i \big(a_{ii}(x) + x_i b_i(x)\big) - 1\right] w'(r) - \frac{c(x)}{A(x)} w(r)$$

$$\geq w''(r) + \frac{1}{r}(D_*(r)-1)\, w'(r) - c^*(r)w(r) \geq p^*(r) \geq \frac{p(x)}{A(x)}$$

for $|x| = r \geq r_0$.

Now, to prove (2. 3) we suppose to the contrary that it never holds. Accordingly we have

$$\limsup_{|x|\to\infty}(v-mw)=\limsup_{|x|\to\infty} w\left(\frac{v}{w}-m\right)=\infty \tag{2. 4}$$

for all $m>0$. Choose $m>1$ large enough so that

$$D_n(v-mw)<0, \quad |x|=r_0$$

and

$$v-mw<0 \quad \text{at some } x, \quad |x|>r_0, \tag{2. 5}$$

where D_n denotes the directional derivative along to the outer normal vector n at x on the sphere $|x| = r_0$. This is possible by (2. 2). Combining this property with (2. 4) we see that the function $v - mw$ takes a local minimum at some point $\bar{x}$, $|\bar{x}| > r_0$. We may assume $v(\bar{x}) - mw(|\bar{x}|) < 0$ by (2. 5). However this leads us to a contradiction because

$$0 \le L(v-mw)(\bar{x}) \le c(\bar{x})(v(\bar{x})-mw(|\bar{x}|))+(1-m)p(\bar{x})<0.$$

This completes the proof.

Consider the case in which

$$\int^{\infty} K_*(r)\left(\int^{r}\frac{c^*(s)}{K_*(s)}ds\right)dr<\infty. \tag{2. 6}$$

Then, it is easily seen that the ordinary differential equation

$$w''+\frac{1}{r}(D_*(r)-1)w'-c^*(r)w=0$$

admits a positive solution $w_0 \in C^2[r_0, \infty)$ satisfying

$$w_0'(r)>0, \quad r>r_0, \quad \lim_{r\to\infty} w_0(r)=\ell=\text{const}\in(0,\infty), \tag{2. 7}$$

if r_0 is sufficiently large. In fact, we can find a positive solution w_0 of the integral equation

$$w_0(r)=\ell-\int_r^{\infty} K_*(s)\left(\int_{r_0}^{s}\frac{c^*(t)}{K_*(t)}w_0(t)dt\right)ds, \quad r\ge r_0,$$

satisfying (2. 7). Therefore inequality (2. 1) can be rewritten as

$$\frac{K_*(r)}{w_0(r)}\left[\frac{(w_0(r))^2}{K_*(r)}\left(\frac{w}{w_0(r)}\right)'\right]'\ge p^*(r), \quad r\ge r_0.$$

It is found from this observation that inequality (2. 1) has a positive solution $w \in C^2[r_0, \infty)$ explicitly given by

$$w(r)=w_0(r)\int_{r_0}^{r}\frac{K_*(s)}{(w_0(r))^2}\left(\int_{r_0}^{s}\frac{w_0(t)}{K_*(t)}p^*(t)dt\right)ds.$$

Combining this fact with Theorem 2.1 we get the following corollary.

COROLLARY 2.2. *Let* (2. 6) *hold. Then, positive solutions u of* (1. 3) *satisfy*

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} F(u(x))}{\int^r K_*(s)\left(\int^s \frac{p^*(t)}{K_*(t)}dt\right)ds} < \infty.$$

The following is a simple consequence of Corollary 2.2.

COROLLARY 2.3. *Let* (2. 6) *and the condition* $\lim_{u\to\infty} F(u) = \infty$ *hold. Then* (1. 3) *has no positive solutions tending to* ∞ *uniformly as* $|x| \to \infty$, *provided*

$$\int^\infty K_*(r)\left(\int^r \frac{p^*(s)}{K_*(s)}ds\right)dr < \infty. \tag{2. 8}$$

EXAMPLE 2.4. Let us consider the inequality

$$\left(\Delta + \sum_{i,j} \alpha_{ij}(x)D_{ij}\right)u + \sum_i b_i(x)D_i u \le p(x)u^\sigma, \quad x \in \Omega, \tag{2. 9}$$

where $0 < \sigma < 1$, $\alpha_{ij} = \alpha_{ji} \in C(\Omega)$. Suppose that there are continuous functions $\bar{\alpha}$ and $\bar{b}$ satisfying

$$|\alpha_{ij}(x)| \le \bar{\alpha}(|x|), \quad |\bar{b}_i(x)| \le \bar{b}(|x|), \quad x \in \Omega,$$

for all *i, j,* and

$$\lim_{r\to\infty} \bar{\alpha}(r) = 0, \quad \int^\infty \frac{\bar{\alpha}(r)}{r}dr < \infty, \quad \int^\infty \bar{b}(r)dr < \infty.$$

Then some computations show that we may take $D^*(r) = N - c_1(\bar{\alpha}(r) + r\bar{b}(r))$ for large r, with some $c_1 > 0$, from which we find that

$$c_2 r^{1-N} \le K_*(r) \le c_3 r^{1-N} \quad \text{for large } r$$

with some $c_2, c_3 > 0$. Therefore Corollary 2.2 asserts that positive solutions u of (2. 9) satisfy

$$\limsup_{r\to\infty} \frac{\min_{|x|=r}[u(x)]^{1-\sigma}}{\int^r \int^s (t/s)^{N-1} \tilde{p}(t)dtds} < \infty. \tag{2. 10}$$

Moreover, Corollary 2.3 asserts that, if $N \ge 3$ and $\int^\infty r\tilde{p}(r)\,dr < \infty$, then (2. 9) never has positive solutions tending to ∞ as $|x| \to \infty$.

Comparing (2. 10) with (1. 2), we find that our results above surely give an extension of the estimates for positive solutions of (1. 1) described in the Introduction.

The argument developed above is also applicable to the *linear* inequality

$$Lu \le p(x)u, \quad x \in \Omega. \tag{2. 11}$$

(Note that here we put $c(x) \equiv 0$ in (1. 3).) The following results are easily proved as in the proof of Theorem 2.1 by considering the function $v(x) = \log u(x)$, where u is a positive solution of (2. 11). Hence, the proofs are left to the reader.

THEOREM 2.5. *Suppose that there is a positive function* $w(r)$ *satisfying* (2. 1) (*with* $c^* \equiv 0$) and (2. 2). *Then, positive solutions* u *of* (2. 11) *satisfy*

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} \log u(x)}{w(r)} < \infty.$$

COROLLARY 2.6. *Positive solutions* u *of* (2. 11) *satisfy*

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} \log u(x)}{\int^r K_*(s)\left(\int^s \frac{p^*(t)}{K_*(t)}\,dt\right)ds} < \infty.$$

COROLLARY 2.7. *Let* (2. 8) *hold. Then* (2. 11) *has no positive solutions tending to* ∞ *uniformly as* $|x| \to \infty$.

3. Quasilinear inequalities

Next, we turn to the quasilinear inequality (1. 4). To explain our assumptions here, we introduce the function $\psi(s) = s/(1+s^2)^\alpha$, $s \in \mathbb{R}$, and denote its inverse function on $\mathbb{R}$ by ϕ. It is easily seen that the following inequalities hold:

$$\phi(s) \ge s^{1/(1-2\alpha)}, \quad s \ge 0,$$
$$\phi(s) \le Cs^{1/(1-2\alpha)}, \quad s \ge 1,$$

with some $C > 0$, and

$$\phi(s_1 s_2) \le \phi(s_1)\phi(s_2) \quad \text{for } s_1, s_2 \ge 0. \tag{3. 1}$$

For radially symmetric functions $h(|x|)$ of class C^2, we have

$$Mh(r) = r^{1-N}(r^{N-1}\psi(h'(r)))', \quad r = |x|.$$

This simple formula is often employed later.

Throughout this section we assume the following:

(H_3) f is strictly increasing, and satisfies

$$\int_1^\infty [f(s)]^{-1/(1-2\alpha)}\,ds = \infty, \tag{3. 2}$$

and

$$\int_0^1 \frac{ds}{f(s)} < \infty.$$

This assumption and the fact that $\psi'(0) = 1$ enable us to introduce the function

$$F_\phi(u) \equiv \int_0^u \frac{ds}{\phi(f(s))}, \quad u > 0.$$

A typical example of f satisfying (H_3) is the function $f(u) = u^\sigma$, $0 < \sigma < 1-2\alpha$. For this f, $F_\phi(u)$ behaves like a constant multiple of $u^{(1-2\alpha-\sigma)/(1-2\alpha)}$ as $u \to \infty$.

The main result of this section is as follows:

THEOREM 3.1. *Let $\tilde{p}$ be a continuous function such that $\tilde{p}(r) \geq \max_{|x|=r} p(x)$, $r \geq 0$. Then, for each positive solution u of* (1. 4) *we can find a positive constant m satisfying*

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} u(x)}{F_\phi^{-1}\left(m + \int_0^r \phi\left(\int_0^s (t/s)^{N-1} \tilde{p}(t)dt\right)ds\right)} \leq 1. \tag{3. 3}$$

To prove this theorem we need preliminary lemmas.

LEMMA 3.2. *Let u be a positive solution of* (1. 4), *and $v \in C^2(\mathbb{R}^N)$ be a positive solution of the inequality*

$$Mv \geq p(x)f(v), \quad x \in \mathbb{R}^N.$$

If, in addition,

$$\limsup_{r\to\infty} \min_{|x|=r} \frac{u(x)}{v(x)} > 1,$$

then, $u \geq v$ in $\mathbb{R}^N$.

PROOF. We remark first that for h of class C^2 we can express Mh in the form $\sum_{i,j} A_{ij}(Dh)\, D_{ij}h$, where the symmetric matrix $(A_{ij}(z))$, $z \in \mathbb{R}^N$, is positive definite.

Put $w = u - v$. To prove this lemma we suppose to the contrary that $w(x_0) < 0$ for some x_0. By the assumption we can find an $R > 0$ such that $w > 0$ on $|x| = R$ and $|x_0| < R$. Consider the point $\bar{x}$ at which w takes the minimum on the set $|x| \leq R$. Clearly $u(\bar{x}) < v(\bar{x})$ and $Du(\bar{x}) = Dv(\bar{x})$. Let $\bar{M}$ be the elliptic operator given by $\bar{M} = \sum_{i,j} A_{ij}(Du(\bar{x}))\, D_{ij}$. We then obtain a contradictory inequality:

$$\begin{aligned} 0 \leq \bar{M}w(\bar{x}) &= \sum_{i,j} A_{ij}(Du(\bar{x}))D_{ij}u(\bar{x}) - \sum_{i,j} A_{ij}(Dv(\bar{x}))D_{ij}v(\bar{x}) \\ &= Mu(\bar{x}) - Mv(\bar{x}) \leq p(\bar{x})[f(u(\bar{x})) - f(v(\bar{x}))] < 0. \end{aligned}$$

The proof is complete.

LEMMA 3.3. *Let $\lambda > 0$ and $v_\lambda(r)$ be a positive solution of the initial value problem for the ordinary differential equation*

$$r^{1-N}(r^{N-1}\psi(v'))' = \tilde{p}(r)f(v), \quad r > 0,$$
$$v(0) = \lambda, \quad v'(0) = 0.$$

Then v_λ exists on $[0, \infty)$ and satisfies

$$v_\lambda(r) \le F_\phi^{-1}\left(F_\phi(\lambda) + \int_0^r \phi\left(\int_0^s (t/s)^{N-1}\tilde{p}(t)dt\right)ds\right), \quad r \ge 0. \tag{3. 4}$$

PROOF. The local existence of v_λ is assured by the standard argument. Let $I = [0, R)$, $R \le \infty$, be the maximal interval of existence for v_λ.

First we shall show that $R = \infty$. To this end suppose the contrary that R is finite. Since

$$v_\lambda'(r) = \phi\left(\int_0^r (s/r)^{N-1}\tilde{p}(s)f(v_\lambda(s))ds\right), \quad r \in I, \tag{3. 5}$$

we have $v_\lambda'(r) \ge 0$, $r \in I$, and hence $\lim_{r \to R-0} v_\lambda(r) \equiv v_\lambda(R-0)$ exists in $(\lambda, \infty]$. Let us suppose $v_\lambda(R-0) < \infty$ for a moment. Then (3. 5) shows the existence of the finite limit $v_\lambda'(R-0)$. Hence we can extend v_λ as a solution of this IVP to the right of R by the standard argument. This contradicts the definition of R, which implies that $v_\lambda(R-0) = \infty$. From (3. 5), together with (3. 1), we obtain for $r \in I$

$$v_\lambda'(r) \le \phi(f(v_\lambda(r)))\phi\left(\int_0^r (s/r)^{N-1}\tilde{p}(s)ds\right),$$

that is,

$$F_\phi(v_\lambda(r)) - F_\phi(\lambda) \equiv \int_\lambda^{v_\lambda(r)} \frac{dz}{\phi(f(z))} \le \int_0^r \phi\left(\int_0^s (t/s)^{N-1}\tilde{p}(t)dt\right)ds.$$

Letting $r \to R-0$, we have a contradiction, since the left hand side of the above tends to ∞ by (3. 2). Therefore R must be ∞.

The validity of inequality (3. 4) immediately follows from the above inequality. The proof is complete.

We are now in a position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $v_\lambda(r)$, $\lambda > 0$, be the function introduced in Lemma 3.3. Clearly, we have

$$Mv_\lambda(|x|) \ge p(x)f(v_\lambda(|x|)), \quad x \in \mathbb{R}^N.$$

Now, we prove that

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} u(x)}{v_\mu(r)} \le 1 \quad \text{for some } \mu > 0. \tag{3. 6}$$

In fact, if this is not the case, then we have

$$\limsup_{r\to\infty} \frac{\min_{|x|=r} u(x)}{v_\lambda(r)} > 1 \quad \text{for all } \lambda > 0.$$

Hence Lemma 3.2 asserts that $u(x) \ge v_\lambda(|x|)$ in $\mathbb{R}^N$ for all $\lambda > 0$. In particular, we have $u(0) \ge v_\lambda(0) = \lambda$ for all $\lambda > 0$. Clearly, this is impossible, and so we have (3. 6). Combining (3. 4) with (3. 6), we can easily prove the validity of (3. 3) with $m = F_\phi(\mu)$. This completes the proof.

COROLLARY 3.4. *Let*

$$\int^\infty \phi\left(\int^r (s/r)^{N-1} \tilde{p}(s)ds\right)dr < \infty.$$

Then (1. 4) *has no positive solutions tending to* ∞ *uniformly as* $|x| \to \infty$.

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