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Invariant subvarieties of the 3-tensor space $C^2 \otimes C^2 \otimes C^2$

YOSHIO AGAOKA

Faculty of Integrated Arts and Sciences,
Hiroshima University,
Higashi-Hiroshima 724, Japan

Abstract : We classify G-invariant subvarieties of the 3-tensor space $C^2 \otimes C^2 \otimes C^2$ that are defined by polynomials with degree ≤ 6, where $G = GL(2,C) \times GL(2,C) \times GL(2,C)$. We also calculate the character of $S^p(C^2 \otimes C^2 \otimes C^2)$, determine the generators of each irreducible component of $S^p(C^2 \otimes C^2 \otimes C^2)$, and obtain some curious identities between them that play a fundamental role in classifying invariant subvarieties.

Key words : 3-tensor space, variety, invariant, representation, character, Schur function

Introduction

Let G be a Lie group acting on a vector space V. In many geometric situations, it is important to classify all G-invariant subvarieties of V and to obtain their defining equations. In fact, we often meet geometric problems where G-invariant subvarieties naturally appear. (See the examples below.)

In this note, as one simplest example of such problems, we classify G-invariant (irreducible) subvarieties of the 3-tensor space $C^2 \otimes C^2 \otimes C^2$ that are defined by the polynomials with degree ≤ 6, where $G = GL(2,C) \times GL(2,C) \times GL(2,C)$. In addition, we determine the explicit defining equations of these varieties. These defining equations constitute the G-invariant subspaces of the polynomial ring of $C^2 \otimes C^2 \otimes C^2$, and we first decompose the homogeneous polynomial ring $\sum_p S^p(C^2 \otimes C^2 \otimes C^2)$ into irreducible components, and express it as a sum of Schur functions. Then, by applying the method stated in [1], we can determine the generator of each irreducible component. As a result, up to degree 6, there appear six typical polynomials whose products generate all other invariant irreducible components of $S^p(C^2 \otimes C^2 \otimes C^2)$ $(p \leq 6)$.

In terms of these polynomials, we show that there are seven invariant irreducible varieties including $\{0\}$ and the whole space itself, and interesting to say, three of them that are defined by quadratic polynomials are mutually related in an essential way. For example, two of these varieties possess common defining equations to each other, and the algebraic set defined by one irreducible component of $S^2(C^2 \otimes C^2 \otimes C^2)$ decomposes into the union of the above two varieties. (For details, see § 3.) To prove these phenomena, curious identities between the generators play a fundamental

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role (cf. Lemma 10). In addition, in this note, we clarify the geometric meaning of each G-invariant subvariety, determine its dimension, and the inclusion relation of invariant subvarieties (Theorem 4).

We conjecture that all generators of irreducible components of $S^p\ (C^2\otimes C^2\otimes C^2)$ can be expressed as polynomials of the above six typical generators without the assumption $p \leq 6$. But unfortunately, we cannot prove this conjecture at present. (See § 4.)

There is a strong resemblance between the 3-tensor space $C^2\otimes C^2\otimes C^2$ and the classical results on the binary cubic forms $S^3\ (C^2)$. We summarize this resemblance in the final section of this note (§ 5). In a sense, the space $S^3\ (C^2)$ may be considered as a degeneration of the 3-tensor space $C^2\otimes C^2\otimes C^2$, and the classical theory of binary cubic forms has its root in $C^2\otimes C^2\otimes C^2$.

We usually meet general 3-tensor spaces $C^p\otimes C^q\otimes C^r$ in different geometric situations, such as partial Gauss equations defined in [3], r-tuple of polynomial valued 1-forms treated in [2], etc. Hence, it is desirable to study these general 3-tensor spaces in the same way as in the case of $C^2\otimes C^2\otimes C^2$. In the case of 2-tensor space $C^p\otimes C^q$, they are equivalent to the set of $p\times q$ matrices, and hence $GL\ (p,\ C)\ \times GL\ (q,\ C)$-invariant subvarieties of $C^p\otimes C^q$ are all classified in terms of the concept "rank" of matrices. But for general 3-tensor spaces, we do not yet obtain such unified concept to classify invariant subvarieties. (cf. [2; p.503].) In a near future, we want to treat more general class of 3-tensor spaces by introducing a unified systematic standpoint.

Finally, we explain the usefulness of the classification of G-invariant subvarieties stated at the top of this introduction, by giving some examples:

(A) In [2], we defined a complex which is defined by some polynomial valued 1-forms, and proved that the cohomology spaces vanish for generic polynomial valued 1-forms. But for singular 1-forms, they do not vanish in general, and those singular forms constitute several G-invariant algebraic sets, according as their degree of singularity. If we know all the invariant subvarieties of the space of polynomial valued 1-forms, we can determine all possible dimensions of the cohomology spaces of this complex for each singular case. (For details, see [2].)

(B) By using the Gauss equation that appears in the theory of submanifolds, we defined in [1] a $GL\ (V)$-equivariant quadratic map $\gamma : E \rightarrow K$. (E is the set of second fundamental forms, and K is the space of all curvature like tensors on V.) The defining equations of Im γ serve as the obstructions to the existence of local isometric imbeddings of Riemannian manifolds, and hence their explicit expressions are useful for geometric applications. Since the closure of Im γ is a G-invariant subvariety of K, it is desirable to classify all G-invariant subvarieties of K and to determine their defining equations. (See also [21].)

(C) Determination of the rank of multi-tensors is one of the important problems in linear algebra. It is connected with other fields of mathematics such as the theory of invariants, computational complexity, error-correcting codes, etc. (cf. [7], [16], [20].) Clearly, the closure of the set of all multi-tensors with rank $\leq k$ forms an invariant algebraic set, and the defining equations of this set are useful in actual determination of the rank of each multi-tensor. (For concrete examples, see Lemma 6 and § 5 of this note.)

Of course, if we know the normal form of each element under the action of G, we can give the answer to the above problems immediately. But in general, determination of normal forms (i.e., the classification of G-orbits) is a hard and often hopeless problem except some special cases. On the other hand, classification of G-invariant subvarieties gives us more natural and essential standpoint, because the answers to the above problems (A) $\sim$ (C) are all determined by only knowing invariant algebraic sets in which the element we are considering is contained. We want to treat these classification problems in forthcoming papers.

1. Character of the space $S^p(C^2 \otimes C^2 \otimes C^2)$

In this section, we determine the character of the space of homogeneous polynomials on the 3-tensor space $C^2 \otimes C^2 \otimes C^2$, under the natural action of the group $G = GL(2,C) \times GL(2,C) \times GL(2,C)$. This action is a tensor product of three representations of $GL(2,C)$ on C^2, and we denote by S_λ, T_μ, U_γ the Schur functions of $GL(2,C)$ acting on each component of $C^2 \otimes C^2 \otimes C^2$, where λ, μ, γ are some partitions. (For the definition of the Schur function, see [11], [14]). For example, the character of G acting on the linear polynomial of $C^2 \otimes C^2 \otimes C^2$ is equal to $S_1 T_1 U_1$, and each irreducible component of $S^p(C^2 \otimes C^2 \otimes C^2)$ is expressed as $S_\lambda T_\mu U_\gamma$ for some partitions λ, μ, γ of p. In the following, we often express the invariant irreducible subspace corresponding to the partition λ simply as S_λ etc.

Under these notations, we calculate the character of the space $S^p(C^2 \otimes C^2 \otimes C^2)$, which is the space of homogeneous polynomials on $C^2 \otimes C^2 \otimes C^2$ with degree p, by using several classical formulas. First, we denote by χ_α the character of the symmetric group $\mathfrak{S}_p$ corresponding to the partition α of p, and define the number $K_{\alpha\beta\lambda}$ by

$$\chi_\alpha \chi_\beta = \sum_\lambda K_{\alpha\beta\lambda} \chi_\lambda.$$

Then, we have

LEMMA 1. *The character of the p-th symmetric tensor space $S^p(C^2 \otimes C^2 \otimes C^2)$ is given by*

$$\sum K_{\alpha\beta\lambda} S_\lambda T_\alpha U_\beta,$$

where λ, α, β run all over the partitions of p with depth ≤ 2.

PROOF. In general, the character of the p-th symmetric tensor product of the 2-tensor space $V \otimes W$ is given by the formula

$$\sum_\lambda S_\lambda(V) S_\lambda(W),$$

where λ runs all over the partitions of p with depth $\leq \min\{\dim V, \dim W\}$, and $S_\lambda(V)$ (resp. $S_\lambda(W)$) denotes the character of the representation on V (resp. W) corresponding to the partition λ (cf. [13; p.103], [19; p.176]). Hence, the character of $S^p(C^2 \otimes C^2 \otimes C^2)$ is equal to

$$\sum S_\lambda(C^2)\, S_\lambda(C^2 \otimes C^2),$$

and using the formula

$$S_\lambda(V \otimes W) = \sum K_{\alpha\beta\lambda} S_\alpha(V)\, S_\beta(W)$$

(cf. [12; p.331] or [13; Appendix]), we have

$$S^p(C^2\otimes C^2\otimes C^2) = \Sigma K_{\alpha\beta\lambda} S_\lambda(C^2) S_\alpha(C^2) S_\beta(C^2)$$
$$= \Sigma K_{\alpha\beta\lambda} S_\lambda T_\alpha U_\beta. \qquad \text{q. e. d.}$$

The character table of the symmetric group $\mathfrak{S}_p$ is given in [13], and using this table, we can calculate the number $K_{\alpha\beta\lambda}$ for small p. We summarize the results in the following proposition.

PROPOSITION 2. *The character table of the polynomial ring of the* 3-*tensor space* $C^2\otimes C^2\otimes C^2$ *up to degree* 6 *is given by the following:*

$p=1$: $S_1T_1U_1$,

$p=2$: $S_2T_2U_2 + S_2T_{11}U_{11} + S_{11}T_2U_{11} + S_{11}T_{11}U_2$,

$p=3$: $S_3T_3U_3 + S_3T_{21}U_{21} + S_{21}T_3U_{21} + S_{21}T_{21}U_3 + S_{21}T_{21}U_{21}$,

$p=4$: $S_4T_4U_4 + S_4T_{31}U_{31} + S_{31}T_4U_{31} + S_{31}T_{31}U_4 + S_4T_{22}U_{22} + S_{22}T_4U_{22} + S_{22}T_{22}U_4 + S_{31}T_{31}U_{31}$
$+ S_{31}T_{31}U_{22} + S_{31}T_{22}U_{31} + S_{22}T_{31}U_{31} + S_{22}T_{22}U_{22}$,

$p=5$: $S_5T_5U_5 + S_5T_{41}U_{41} + S_{41}T_5U_{41} + S_{41}T_{41}U_5 + S_5T_{32}U_{32} + S_{32}T_5U_{32} + S_{32}T_{32}U_5 + S_{41}T_{41}U_{41}$
$+ S_{41}T_{41}U_{32} + S_{41}T_{32}U_{41} + S_{32}T_{41}U_{41} + S_{41}T_{32}U_{32} + S_{32}T_{41}U_{32} + S_{32}T_{32}U_{41} + S_{32}T_{32}U_{32}$,

$p=6$: $S_6T_6U_6 + S_6T_{51}U_{51} + S_{51}T_6U_{51} + S_{51}T_{51}U_6 + S_6T_{42}U_{42} + S_{42}T_6U_{42} + S_{42}T_{42}U_6 + S_{51}T_{51}U_{42}$
$+ S_{51}T_{42}U_{51} + S_{42}T_{51}U_{51} + S_{51}T_{51}U_{51} + S_6T_{33}U_{33} + S_{33}T_6U_{33} + S_{33}T_{33}U_6 + S_{51}T_{42}U_{33} + S_{51}T_{33}U_{42}$
$+ S_{42}T_{51}U_{33} + S_{42}T_{33}U_{51} + S_{33}T_{51}U_{42} + S_{33}T_{42}U_{51} + 2S_{42}T_{42}U_{42} + S_{51}T_{42}U_{42} + S_{42}T_{51}U_{42}$
$+ S_{42}T_{42}U_{51} + S_{42}T_{33}U_{33} + S_{33}T_{42}U_{33} + S_{33}T_{33}U_{42}$.

In actual calculations, we may omit the character S_λ in the case where the depth of λ exceeds 2 because we are considering only 2-dimensional spaces. We remark that in our situation the dimension of $S_{pq}T_{rs}U_{tu}$ is equal to $(p-q+1)(r-s+1)(t-u+1)$.

2. Generators of the irreducible components of $S^p(C^2\otimes C^2\otimes C^2)$

Each irreducible component $S_\lambda T_\mu U_\gamma$ of $S^p(C^2\otimes C^2\otimes C^2)$ possesses one generator, which generates the invariant subspace $S_\lambda T_\mu U_\gamma$ under the action of G. And using the results in § 1, and the method stated in [1; p.115] or [3; p.42], we can explicitly obtain the generators of the irreducible components of the space $S^p(C^2\otimes C^2\otimes C^2)$.

We express the element of $C^2\otimes C^2\otimes C^2$ as

$$a = (a_{ijk})_{1\le i,j,k\le 2},$$

i.e., $a = \Sigma a_{ijk}\, e_i\otimes e_j\otimes e_k$, where $\{e_1, e_2\}$ is a basis of C^2. Then, as one example, the generator of the space $S_2T_{11}U_{11}\subset S^2(C^2\otimes C^2\otimes C^2)$ is given by

$$\sum_{\sigma, \tau \in \mathfrak{S}_2} (-1)^{\sigma} (-1)^{\tau} a_{1\sigma(1)\tau(1)} a_{1\sigma(2)\tau(2)} = 2(a_{111}a_{122} - a_{112}a_{121}).$$

To state our results, we put

$$\begin{aligned}
p_1 &= a_{111}a_{122} - a_{112}a_{121}, \\
q_1 &= a_{111}a_{212} - a_{112}a_{211}, \\
r_1 &= a_{111}a_{221} - a_{121}a_{211}, \\
s_1 &= a_{111}^2a_{222} + 2a_{112}a_{121}a_{211} - a_{111}a_{112}a_{221} - a_{111}a_{121}a_{212} - a_{111}a_{122}a_{211}, \\
t &= a_{111}^2a_{222}^2 + a_{112}^2a_{221}^2 + a_{121}^2a_{212}^2 + a_{122}^2a_{211}^2 - 2a_{111}a_{112}a_{221}a_{222} - 2a_{111}a_{121}a_{212}a_{222} \\
&\quad - 2a_{111}a_{122}a_{211}a_{222} - 2a_{112}a_{121}a_{212}a_{221} - 2a_{112}a_{122}a_{211}a_{221} - 2a_{121}a_{122}a_{211}a_{212} \\
&\quad + 4a_{111}a_{122}a_{212}a_{221} + 4a_{112}a_{121}a_{211}a_{222}.
\end{aligned}$$

Then, we have

PROPOSITION 3. *The generators of irreducible components of S^p $(C^2 \otimes C^2 \otimes C^2)$ $(p \le 6)$ are given by the following:*

$p = 1$:	$S_1T_1U_1$	$: a_{111}$,		
$p = 2$:	$S_2T_2U_2$	$: a_{111}^2$,	$S_2T_{11}U_{11}$	$: p_1$,
	$S_{11}T_2U_{11}$	$: q_1$,	$S_{11}T_{11}U_2$	$: r_1$,
$p = 3$:	$S_3T_3U_3$	$: a_{111}^3$,	$S_3T_{21}U_{21}$	$: a_{111}p_1$,
	$S_{21}T_3U_{21}$	$: a_{111}q_1$,	$S_{21}T_{21}U_3$	$: a_{111}r_1$,
	$S_{21}T_{21}U_{21}$	$: s_1$,		
$p = 4$:	$S_4T_4U_4$	$: a_{111}^4$,	$S_4T_{31}U_{31}$	$: a_{111}^2p_1$,
	$S_{31}T_4U_{31}$	$: a_{111}^2q_1$,	$S_{31}T_{31}U_4$	$: a_{111}^2r_1$,
	$S_4T_{22}U_{22}$	$: p_1^2$,	$S_{22}T_4U_{22}$	$: q_1^2$,
	$S_{22}T_{22}U_4$	$: r_1^2$,	$S_{31}T_{31}U_{31}$	$: a_{111}s_1$,
	$S_{31}T_{31}U_{22}$	$: p_1q_1$,	$S_{31}T_{22}U_{31}$	$: p_1r_1$,
	$S_{22}T_{31}U_{31}$	$: q_1r_1$,	$S_{22}T_{22}U_{22}$	$: t$,
$p = 5$:	$S_5T_5U_5$	$: a_{111}^5$,	$S_5T_{41}U_{41}$	$: a_{111}^3p_1$,
	$S_{41}T_5U_{41}$	$: a_{111}^3q_1$,	$S_{41}T_{41}U_5$	$: a_{111}^3r_1$,
	$S_5T_{32}U_{32}$	$: a_{111}p_1^2$,	$S_{32}T_5U_{32}$	$: a_{111}q_1^2$,
	$S_{32}T_{32}U_5$	$: a_{111}r_1^2$,	$S_{41}T_{41}U_{41}$	$: a_{111}^2s_1$,
	$S_{41}T_{41}U_{32}$	$: a_{111}p_1q_1$,	$S_{41}T_{32}U_{41}$	$: a_{111}p_1r_1$,
	$S_{32}T_{41}U_{41}$	$: a_{111}q_1r_1$,	$S_{41}T_{32}U_{32}$	$: p_1s_1$,
	$S_{32}T_{41}U_{32}$	$: q_1s_1$,	$S_{32}T_{32}U_{41}$	$: r_1s_1$,
	$S_{32}T_{32}U_{32}$	$: a_{111}t$,		
$p = 6$:	$S_6T_6U_6$	$: a_{111}^6$,	$S_6T_{51}U_{51}$	$: a_{111}^4p_1$,
	$S_{51}T_6U_{51}$	$: a_{111}^4q_1$,	$S_{51}T_{51}U_6$	$: a_{111}^4r_1$,
	$S_6T_{42}U_{42}$	$: a_{111}^2p_1^2$,	$S_{42}T_6U_{42}$	$: a_{111}^2q_1^2$,

$S_{42}T_{42}U_{6}$	: $a_{111}{}^2r_1{}^2$,	$S_{51}T_{51}U_{42}$	: $a_{111}{}^2p_1q_1$,
$S_{51}T_{42}U_{51}$	: $a_{111}{}^2p_1r_1$,	$S_{42}T_{51}U_{51}$	: $a_{111}{}^2q_1r_1$,
$S_{51}T_{51}U_{51}$	: $a_{111}{}^3s_1$,	$S_{6}T_{33}U_{33}$	: p_1^3,
$S_{33}T_{6}U_{33}$	: q_1^3,	$S_{33}T_{33}U_{6}$	: r_1^3,
$S_{51}T_{42}U_{33}$	: $p_1^2q_1$,	$S_{51}T_{33}U_{42}$	: $p_1^2r_1$,
$S_{42}T_{51}U_{33}$	: $p_1q_1^2$,	$S_{42}T_{33}U_{51}$	: $p_1r_1^2$,
$S_{33}T_{51}U_{42}$	: $q_1^2r_1$,	$S_{33}T_{42}U_{51}$	: $q_1r_1^2$,
$2S_{42}T_{42}U_{42}$	: $a_{111}{}^2t$, $p_1q_1r_1$, s_1^2,		
$S_{51}T_{42}U_{42}$	: $a_{111}p_1s_1$,	$S_{42}T_{51}U_{42}$	: $a_{111}q_1s_1$,
$S_{42}T_{42}U_{51}$	: $a_{111}r_1s_1$,	$S_{42}T_{33}U_{33}$	: p_1t,
$S_{33}T_{42}U_{33}$	: q_1t,	$S_{33}T_{33}U_{42}$	: r_1t.

Note that the above generators are all expressed by the following six polynomials only:

$S_1T_1U_1$	: a_{111},	$S_2T_{11}U_{11}$	: p_1,
$S_{11}T_2U_{11}$	: q_1,	$S_{11}T_{11}U_2$	: r_1,
$S_{21}T_{21}U_{21}$	: s_1,	$S_{22}T_{22}U_{22}$	: t.

These polynomials play the fundamental role in the following arguments. Especially, the polynomial t is the invariant of the 3-tensor space $C^2\otimes C^2\otimes C^2$ with degree 4, which is explained in [8], [10] as the name of "hyperdeterminant". We also remark that the irreducible decomposition of the space $2S_{42}T_{42}U_{42} \subset S^6 (C^2\otimes C^2\otimes C^2)$ is not uniquely determined because the multiplicity is 2. The above three generators of $2S_{42}T_{42}U_{42}$ stated in Proposition 3 are not linearly independent, and satisfy the following curious identity

$$4p_1q_1r_1 + s_1^2 = a_{111}{}^2t$$

(cf. Lemma 10).

3. Invariant subvarieties of $C^2\otimes C^2\otimes C^2$

Let Σ be a G-invariant subvariety of $C^2\otimes C^2\otimes C^2$, i.e., Σ is an irreducible algebraic set of $C^2\otimes C^2\otimes C^2$ which is invariant under the action of $G = GL(2,C) \times GL(2,C) \times GL(2,C)$. We denote by $I(\Sigma)$ the defining ideal of Σ. Then, it is easy to see that $I(\Sigma)$ is invariant under the action of G, and hence $I(\Sigma)$ is expressed as a sum of homogeneous irreducible components $\Sigma S_\lambda T_\mu U_\gamma$. In addition, on account of Hilbert's basis theorem, $I(\Sigma)$ is generated by a finite number of invariant irreducible subspaces $S_\lambda T_\mu U_\gamma$. Our main results in this note are summarized in the following theorem.

THEOREM 4. *Let Σ be a G-invariant proper (irreducible) subvariety of the 3-tensor space $C^2\otimes C^2\otimes C^2$ which is defined by the polynomials with degree less than or equal to 6. Then, Σ is one of the varieties defined below:*

$$\begin{aligned}
\Sigma_0 &= \{\ 0\ \},\\
\Sigma_1 &= \{\ \langle p_1\rangle = \langle q_1\rangle = \langle r_1\rangle = 0\ \},\\
\Sigma_2 &= \{\ \langle p_1\rangle = \langle q_1\rangle = 0\ \},\\
\Sigma_3 &= \{\ \langle p_1\rangle = \langle r_1\rangle = 0\ \},\\
\Sigma_4 &= \{\ \langle q_1\rangle = \langle r_1\rangle = 0\ \},\\
\Sigma_5 &= \{\ t = 0\ \}.
\end{aligned}$$

(The symbol $\langle p_1\rangle$ implies the G-invariant subspace generated by the polynomial p_1, and the equality $\Sigma_2 = \{\langle p_1\rangle = \langle q_1\rangle = 0\}$ implies that Σ_2 is defined by the polynomials that are generated by p_1 and q_1.) The dimension of each variety is

$$\dim \Sigma_0 = 0,\quad \dim \Sigma_1 = 4,\quad \dim \Sigma_2 = \dim \Sigma_3 = \dim \Sigma_4 = 5,\quad \dim \Sigma_5 = 7.$$

In addition, these varieties satisfy the following inclusion relation:

$$\{\ 0\ \} \;\text{———}\; \Sigma_1 \begin{matrix} \nearrow \Sigma_2 \searrow \\ \text{—} \Sigma_3 \text{—}\!\!\to \\ \searrow \Sigma_4 \nearrow \end{matrix} \Sigma_5 \;\text{———}\; C^2 \otimes C^2 \otimes C^2.$$

Note that from this theorem, we have clearly $\Sigma_2 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_4 = \Sigma_3 \cap \Sigma_4 = \Sigma_1$. (We remark that two algebraic sets $\{\ \langle f\rangle = 0\ \}$ and $\{\ f = 0\ \}$ do not in general coincide. The latter set is defined by a single polynomial, though the former set may be defined by several polynomials generated by f.)

The algebraic set defined by one irreducible component of $S^P(C^2 \otimes C^2 \otimes C^2)$ is not in general an irreducible algebraic set. In our situation, we have the following proposition.

PROPOSITION 5. *Invariant algebraic sets defined by six typical generators a_{111}, p_1, q_1, r_1, s_1 and t of $S^P(C^2 \otimes C^2 \otimes C^2)$ (cf. Proposition 3) are given by*

$$\begin{aligned}
\{\ \langle a_{111}\rangle = 0\ \} &= \Sigma_0,\\
\{\ \langle p_1\rangle = 0\ \} &= \Sigma_2 \cup \Sigma_3,\\
\{\ \langle q_1\rangle = 0\ \} &= \Sigma_2 \cup \Sigma_4,\\
\{\ \langle r_1\rangle = 0\ \} &= \Sigma_3 \cup \Sigma_4,\\
\{\ \langle s_1\rangle = 0\ \} &= \Sigma_2 \cup \Sigma_3 \cup \Sigma_4,\\
\{\ t = 0\ \} &= \Sigma_5.
\end{aligned}$$

As stated in the above theorem and proposition, the invariant irreducible subspace $S_2T_{11}U_{11} = \langle p_1\rangle$ defines the reducible algebraic set $\Sigma_2 \cup \Sigma_3$, and irreducible subvariety Σ_2 is defined by two irreducible components $S_2T_{11}U_{11}$ and $S_{11}T_2U_{11}$, which implies that irreducible invariant subspaces of $S^P(C^2 \otimes C^2 \otimes C^2)$ do not in general correspond to irreducible subvarieties of $C^2 \otimes C^2 \otimes C^2$. In addition, three irreducible subspaces $S_2T_{11}U_{11}$, $S_{11}T_2U_{11}$ and $S_{11}T_{11}U_2$ are mutually related. For example, the condition $\langle p_1\rangle = 0$ necessarily induces one of the equality $\langle q_1\rangle = 0$ or $\langle r_1\rangle = 0$, and two of three varieties Σ_2, Σ_3, Σ_4 possess the common defining equations, from which the relation $\Sigma_2 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_4 = \Sigma_3 \cap \Sigma_4\ (= \Sigma_1)$ follows. Thus, these three invariant subspaces (or three invariant subvarieties) are not independent in a sense. Such curious phenomena also occur in another situation where the 3-tensor space is concerned. (For example, see [4].)

We also remark that among six typical generators of $S^p(C^2\otimes C^2\otimes C^2)$, the invariant subspace $S_{21}T_{21}U_{21}\subset S^3(C^2\otimes C^2\otimes C^2)$ does not define an essentially new subvariety, as the above proposition shows. This fact may be explained "symbolically" from the identity

$$4p_1q_1r_1+s_1^2=a_{111}^2t,$$

which implies that the square of the generator s_1 is expressed as a polynomial of a_{111}, p_1, q_1, r_1 and t, though s_1 itself is not. (See the proof of Proposition 5.)

Before proving Theorem 4 and Proposition 5, we must introduce new polynomials generated by p_1, q_1, etc. We put

$$\begin{aligned}
p_1 &= a_{111}a_{122}-a_{112}a_{121},\\
p_2 &= a_{111}a_{222}-a_{112}a_{221}-a_{121}a_{212}+a_{122}a_{211},\\
p_3 &= a_{211}a_{222}-a_{212}a_{221},\\
q_1 &= a_{111}a_{212}-a_{112}a_{211},\\
q_2 &= a_{111}a_{222}-a_{112}a_{221}+a_{121}a_{212}-a_{122}a_{211},\\
q_3 &= a_{121}a_{222}-a_{122}a_{221},\\
r_1 &= a_{111}a_{221}-a_{121}a_{211},\\
r_2 &= a_{111}a_{222}+a_{112}a_{221}-a_{121}a_{212}-a_{122}a_{211},\\
r_3 &= a_{112}a_{222}-a_{122}a_{212},\\
s_1 &= a_{111}^2a_{222}+2a_{112}a_{121}a_{211}-a_{111}a_{112}a_{221}-a_{111}a_{121}a_{212}-a_{111}a_{122}a_{211},\\
s_2 &= a_{122}a_{211}^2+2a_{111}a_{212}a_{221}-a_{111}a_{211}a_{222}-a_{112}a_{221}a_{211}-a_{121}a_{211}a_{212},\\
s_3 &= a_{121}^2a_{212}+2a_{111}a_{122}a_{221}-a_{111}a_{121}a_{222}-a_{112}a_{121}a_{221}-a_{121}a_{122}a_{211},\\
s_4 &= a_{112}a_{221}^2+2a_{121}a_{211}a_{222}-a_{111}a_{221}a_{222}-a_{121}a_{212}a_{221}-a_{122}a_{211}a_{221},\\
s_5 &= a_{112}^2a_{221}+2a_{111}a_{122}a_{212}-a_{111}a_{112}a_{222}-a_{112}a_{121}a_{212}-a_{112}a_{122}a_{211},\\
s_6 &= \mathrm{a}_{121}a_{212}^2+2a_{112}a_{211}a_{222}-a_{111}a_{212}a_{222}-a_{112}a_{212}a_{221}-a_{122}a_{211}a_{212},\\
s_7 &= a_{122}^2a_{211}+2a_{112}a_{121}a_{222}-a_{111}a_{122}a_{222}-a_{112}a_{122}a_{221}-a_{121}a_{122}a_{212},\\
s_8 &= a_{111}a_{222}^2+2a_{122}a_{212}a_{221}-a_{112}a_{221}a_{222}-a_{121}a_{212}a_{222}-a_{122}a_{211}a_{222},\\
t &= a_{111}^2a_{222}^2+a_{112}^2a_{221}^2+a_{121}^2a_{212}^2+a_{122}^2a_{211}^2-2a_{111}a_{112}a_{221}a_{222}-2a_{111}a_{121}a_{212}a_{222}\\
&\quad -2a_{111}a_{122}a_{211}a_{222}-2a_{112}a_{121}a_{212}a_{221}-2a_{112}a_{122}a_{211}a_{221}-2a_{121}a_{122}a_{211}a_{212}\\
&\quad +4a_{111}a_{122}a_{212}a_{221}+4a_{112}a_{121}a_{211}a_{222}.
\end{aligned}$$

(Some polynomials are already defined before.) It is easy to see that these polynomials span the following invariant irreducible subspaces of $S^p(C^2\otimes C^2\otimes C^2)$.

$$\begin{aligned}
S_1T_1U_1 &= \{\, a_{ijk} \,\}_{1\le i,j,k\le 2},\\
S_2T_{11}U_{11} &= \{\, p_1, p_2, p_3 \,\},\\
S_{11}T_2U_{11} &= \{\, q_1, q_2, q_3 \,\},\\
S_{11}T_{11}U_2 &= \{\, r_1, r_2, r_3 \,\},\\
S_{21}T_{21}U_{21} &= \{\, s_1, \cdots, s_8 \,\},
\end{aligned}$$

$$S_{22}T_{22}U_{22} = \{ t \}.$$

Now, using these polynomials, we first prove that the algebraic sets Σ_i defined in Theorem 4 are all irreducible. We first consider the case Σ_2. With respect to the basis $\{e_1, e_2\}$ of C^2, we define a linear map $f: C^2 \to C^2 \otimes C^2$ by the following matrix

$$\begin{pmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \\ a_{211} & a_{212} \\ a_{221} & a_{222} \end{pmatrix},$$

i.e., $f(e_k) = \sum_{ij} a_{ijk} e_i \otimes e_j$. Then, 2×2 minor determinants of this matrix span the 6-dimensional subspace of $S^2(C^2 \otimes C^2 \otimes C^2)$, which just coincides with the space $\{p_i, q_i\}_{1 \leq i \leq 3}$. (Note that this space corresponds to $\Lambda^2(C^2 \otimes C^2) \otimes \Lambda^2 C^2 = S_2T_{11}U_{11} + S_{11}T_2U_{11}$.) Hence, the algebraic set Σ_2 defined by the polynomials $S_2T_{11}U_{11}$ and $S_{11}T_2U_{11}$ coincides with the set of matrices with rank ≤ 1, and hence it is irreducible with dimension $2 \times 4 - (2-1)(4-1) = 5$. We can prove the same results for Σ_3 and Σ_4 by using similar linear maps defined by

$$\begin{pmatrix} a_{111} & a_{121} \\ a_{112} & a_{122} \\ a_{211} & a_{221} \\ a_{212} & a_{222} \end{pmatrix} \text{ and } \begin{pmatrix} a_{111} & a_{211} \\ a_{112} & a_{212} \\ a_{121} & a_{221} \\ a_{122} & a_{222} \end{pmatrix}.$$

Next, we consider the algebraic set Σ_5. To prove the irreducibility of Σ_5, we have only to show that the defining polynomial t is irreducible, and this fact can be easily checked by an elementary argument. Since Σ_5 is a hypersurface, we have clearly $\dim \Sigma_5 = 7$.

Next, we show that Σ_1 is irreducible. For this purpose, we give another characterization of Σ_1. We define a cubic map $\gamma: C^2 + C^2 + C^2 \to C^2 \otimes C^2 \otimes C^2$ by

$$\gamma(x, y, z) = x \otimes y \otimes z, \text{ for } x, y, z \in C^2.$$

Clearly, an element of $C^2 \otimes C^2 \otimes C^2$ is decomposable (or equivalently rank ≤ 1) if and only if it is contained in $\operatorname{Im} \gamma$. Then, we have

Lemma 6. $\operatorname{Im} \gamma = \Sigma_1$, *i.e., the set* Σ_1 *consists of all decomposable elements of the* 3*-tensor space* $C^2 \otimes C^2 \otimes C^2$.

Proof. We can easily show $\operatorname{Im} \gamma \subset \Sigma_1$ because elements of $\operatorname{Im} \gamma$ are expressed as $a_{ijk} = x_i y_j z_k$ for some $x_i, y_j, z_k \in C$, where $a = \sum a_{ijk} e_i \otimes e_j \otimes e_k$. And in this situation, we have immediately $p_i = q_i = r_i = 0$ for $1 \leq i \leq 3$. Conversely, we show the inclusion relation $\Sigma_1 \subset \operatorname{Im} \gamma$ by dividing into several cases. First, in the case $a_{111} \neq 0$, by using the equalities $p_i = q_i = r_i = 0$, we have

$$a_{122} = a_{112}a_{121}/a_{111}, \quad a_{212} = a_{112}a_{211}/a_{111},$$
$$a_{221} = a_{121}a_{211}/a_{111}, \quad a_{222} = a_{112}a_{121}a_{211}/a_{111}^2,$$

for $a \in \Sigma_1$. Then, the element a is expressed as

$$1/a_{111}{}^2 \cdot (a_{111}e_1 + a_{211}e_2) \otimes (a_{111}e_1 + a_{121}e_2) \otimes (a_{111}e_1 + a_{112}e_2),$$

and hence it belongs to Im γ. For other situations, we can show the decomposability of a by case by case check, and we omit the details. q.e.d.

By this lemma, the algebraic set Σ_1 is the image of $C^2 + C^2 + C^2$ under the polynomial map γ, and hence Σ_1 is irreducible. In addition, as stated in the above proof, generic elements of Σ_1 are expressed uniquely in terms of four parameters a_{111}, a_{112}, a_{121}, a_{211}, and hence we have dim $\Sigma_1 = 4$.

Next, to complete the proof of Theorem 4 and Proposition 5, we prepare general facts on the components of G-invariant algebraic sets of $C^2 \otimes C^2 \otimes C^2$.

PROPOSITION 7. *Let $S_\lambda T_\mu U_\gamma$ be an irreducible subspace of $S^p(C^2 \otimes C^2 \otimes C^2)$ generated by the polynomial $f_1^{c_1} \cdots f_k^{c_k}$ ($c_i \geq 1$), where f_i is one of a_{111}, p_1, q_1, r_1, s_1, t ($f_i \neq f_j$ for $i \neq j$.) Then, the invariant algebraic set defined by $S_\lambda T_\mu U_\gamma = 0$ decomposes into the union*

$$\{ <f_1> = 0 \} \cup \cdots\cdots \cup \{ <f_k> = 0 \},$$

where $<f_i>$ implies the irreducible component of $S^p(C^2 \otimes C^2 \otimes C^2)$ generated by f_i.

Note that up to degree 6, all irreducible components of $S^p(C^2 \otimes C^2 \otimes C^2)$ are of the above form, except the case $2S_{42}T_{42}U_{42}$. (cf. Proposition 3.) To prove this proposition, we prepare the following lemma for general invariant algebraic sets.

LEMMA 8. *Assume that a connected Lie group H acts on C^n, and let Σ be an H-invariant algebraic set of C^n, whose irreducible decomposition is given by $\Sigma = \Sigma_1 \cup \cdots\cdots \cup \Sigma_k$. Then, each Σ_i is also H-invariant.*

PROOF. Let h be an element of H. Then we have

$$\Sigma = h \cdot \Sigma = h \cdot \Sigma_1 \cup \cdots\cdots \cup h \cdot \Sigma_k = \Sigma_1 \cup \cdots\cdots \cup \Sigma_k.$$

From the uniqueness of irreducible decomposition of Σ, we have $h \cdot \Sigma_1 = \Sigma_i$ for some i, which may depend on the choice of h. Now, we take an element x of Σ_1 such that $x \notin \Sigma_2 \sim \Sigma_k$. Then for any $X \in$, where $\mathfrak{h}$ is the Lie algebra of H, the curve exp $(tX) \cdot x$ is contained in Σ_1, but not in $\Sigma_2 \sim \Sigma_k$ for small t. Hence, there exists an open subset U of H containing the unit element such that $h \cdot x \in \Sigma_1$ and $h \cdot x \notin \Sigma_2 \sim \Sigma_k$ for $h \in U$, and therefore, we have $h \cdot \Sigma_1 = \Sigma_1$. Since H is connected, every element of H is expressed as a product $h_1 \cdots h_\ell$ ($h_i \in U \cap U^{-1}$), and hence, we have $h \cdot \Sigma_1 = \Sigma_1$ for any $h \in H$. In the same way, we can show $h \cdot \Sigma_i = \Sigma_i$ for $i = 2 \sim k$.
q.e.d.

REMARK. In general, this lemma does not hold if H is not connected. For example, consider the following example: $H = \{ 1, \sigma \}$ ($\simeq Z_2$), and H acts on C^2 by $\sigma(x, y) = (y, x)$. Then, the algebraic set $\{ xy = 0 \} = \{ x = 0 \} \cup \{ y = 0 \}$ is H-invariant, but each irreducible component is not H-invariant.

As an immediate consequence of this lemma, we have

LEMMA 9. *Let H be a connected Lie group and let Σ be an H-invariant algebraic set of C^n. If Σ decomposes into the union of several algebraic sets Σ_i that are not necessary irreducible, then each Σ_i is also H-invariant.*

Note that in our situation the group G is a product of three connected Lie groups $GL(2,C)$, and hence the above lemmas are applicable.

Now, under these preliminaries, we prove Proposition 7. We put $\Sigma = \{ S_\lambda T_\mu U_\gamma = 0 \}$. Then, since the generator $f_1^{c_1} \cdots f_k^{c_k}$ is one of the defining equations of Σ, the algebraic set Σ decomposes into

$$\Sigma = \Sigma_1 \cup \cdots\cdots \cup \Sigma_k,$$

where $\Sigma_i = \{ x \in \Sigma \mid f_i(x) = 0 \}$. Then, by Lemma 9, each Σ_i is G-invariant, and hence, the polynomials contained in $<f_i>$ all vanish on Σ_i. Hence, we have $\Sigma_i \subset \{ <f_i> = 0 \}$. On the other hand, we have clearly $\{ <f_i> = 0 \} \subset \{ S_\lambda T_\mu U_\gamma = 0 \} \cap \{ f_i = 0 \} = \Sigma_i$, and hence we have $\Sigma_i = \{ <f_i> = 0 \}$, which completes the proof of the proposition. q.e.d.

To prove Theorem 4 and Proposition 5, we need the following fundamental (and curious) identities between the polynomials p_i, q_i, etc, defined before.

LEMMA 10. *The polynomials $a_{111} \sim t$ are related by the following identities:*

$$
\begin{aligned}
&4p_1q_1r_1 + s_1^2 = a_{111}^2 t,\\
&t = p_2^2 - 4p_1p_3 = q_2^2 - 4q_1q_3 = r_2^2 - 4r_1r_3,\\
&q_1r_1 = -(a_{211}^2 p_1 - a_{111}a_{211}p_2 + a_{111}^2 p_3),\\
&p_1r_1 = -(a_{121}^2 q_1 - a_{111}a_{121}q_2 + a_{111}^2 q_3),\\
&p_1q_1 = -(a_{112}^2 r_1 - a_{111}a_{112}r_2 + a_{111}^2 r_3),\\
&s_1 = a_{111}p_2 - 2a_{211}p_1 = a_{111}q_2 - 2a_{121}q_1 = a_{111}r_2 - 2a_{112}r_1,\\
&2q_1r_1 = a_{211}s_1 + a_{111}s_2,\\
&2p_1r_1 = a_{121}s_1 + a_{111}s_3,\\
&2p_1q_1 = a_{112}s_1 + a_{111}s_5.
\end{aligned}
$$

We can prove these identities by direct calculations. For example, the first identity can be checked in the following way by using remaining ones:

$$
\begin{aligned}
4p_1q_1r_1 + s_1^2 &= -4\,(a_{211}^2 p_1 - a_{111}a_{211}p_2 + a_{111}^2 p_3)\,p_1 + (a_{111}p_2 - 2a_{211}p_1)^2\\
&= a_{111}^2\,(p_2^2 - 4p_1p_3)
\end{aligned}
$$

$$= a_{111}{}^2 t.$$

Using these identities, we can show the inclusion relations stated in Theorem 4. For example, from the identity

$$t = p_2{}^2 - 4p_1p_3 = q_2{}^2 - 4q_1q_3 = r_2{}^2 - 4r_1r_3,$$

the inclusion relation $\Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \subset \Sigma_5$ follows immediately, and the remaining inclusion relations are trivial consequence of the definition.

Now, using these identities, we prove Proposition 5. First, the equality $\{ < a_{111} > = 0 \} = \Sigma_0$ clearly holds. Next, we show that the algebraic set Σ defined by $S_2T_{11}U_{11}$ $(= < p_1 >) = 0$ is equal to $\Sigma_2 \cup \Sigma_3$. On account of the identity

$$q_1r_1 = -(a_{211}{}^2p_1 - a_{111}a_{211}p_2 + a_{111}{}^2p_3)$$

in Lemma 10, the polynomial q_1r_1 which is the generator of $S_{22}T_{31}U_{31}$ vanishes on Σ. Since Σ is G-invariant, the polynomials in $< q_1r_1 >$ also vanish on Σ. Hence, by Proposition 7, we have

$$\Sigma \subset \{ < q_1r_1 > = 0 \} = \{ < q_1 > = 0 \} \cup \{ < r_1 > = 0 \},$$

which implies

$$\Sigma \subset \{ < p_1 > = < q_1 > = 0 \} \cup \} < p_1 > = < r_1 > = 0 \} = \Sigma_2 \cup \Sigma_3.$$

The converse inclusion relation clearly holds, and hence we have $\Sigma = \Sigma_2 \cup \Sigma_3$. Using the other identities in Lemma 10, we can prove the equality $\{ < q_1 > = 0 \} = \Sigma_2 \cup \Sigma_4$, and $\{ < r_1 > = 0 \} = \Sigma_3 \cup \Sigma_4$ completely in the same way.

Next, we consider the G-invariant algebraic set defined by $< s_1 > = 0$. In this case, from the identity

$$s_1 = a_{111}p_2 - 2a_{211}p_1 = a_{111}q_2 - 2a_{121}q_1 = a_{111}r_2 - 2a_{112}r_1,$$

we have

$$\{ < p_1 > = 0 \} \cup \{ < q_1 > = 0 \} \cup \{ < r_1 > = 0 \} \subset \{ < s_1 > = 0 \},$$

which implies $\Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \subset \{ < s_1 > = 0 \}$. On the other hand, from the identity

$$2q_1r_1 = a_{211}s_1 + a_{111}s_2,$$

we have

$$\begin{aligned} \{ < s_1 > = 0 \} &\subset \{ < q_1r_1 > = 0 \} \\ &= \{ < q_1 > = 0 \} \cup \{ < r_1 > = 0 \} \\ &= \Sigma_2 \cup \Sigma_3 \cup \Sigma_4, \end{aligned}$$

and hence we obtain the equality $\{ < s_1 > = 0 \} = \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, which completes the proof of Proposition 5. q.e.d.

Finally, we complete the proof of Theorem 4. For this purpose, we prepare one more lemma.

LEMMA 11. *The G-orbit of a point $a \in C^2 \otimes C^2 \otimes C^2$ contains an open neighborhood of a if and only if the invariant t does not vanish at a. In particular, all invariant proper subvarieties of $C^2 \otimes C^2 \otimes C^2$ are contained in the hypersurface Σ_5 defined by $t = 0$.*

REMARK. This lemma gives one of geometric characterization of the invariant variety Σ_5, i.e., the variety Σ_5 consists of all singular elements of $C^2 \otimes C^2 \otimes C^2$ with respect to the group action of G. (For another characterization of Σ_5, see [9], [10].) In addition, this lemma shows that the 3-tensor space $C^2 \otimes C^2 \otimes C^2$ is a prehomogeneous vector space in the sense of [17; p.35]. (See also, [25].)

PROOF. We have only to consider the Lie algebra action on $C^2 \otimes C^2 \otimes C^2$ to prove the lemma. For $a \in C^2 \otimes C^2 \otimes C^2$, we define a linear map $\varphi_a : \mathfrak{gl}(2,C) + \mathfrak{gl}(2,C) + \mathfrak{gl}(2,C) \to C^2 \otimes C^2 \otimes C^2$ by

$$\varphi_a(X_1, X_2, X_3) = X_1 \cdot a + X_2 \cdot a + X_3 \cdot a,$$

where $X_i \cdot a$ means the action of $X_i \in \mathfrak{gl}(2,C)$ on the i-th component of a $(1 \leq i \leq 3)$. Then, it is clear that the G-orbit of a contains an open neighborhood of a if and only if rank $\varphi_a = 8$ (= dim $C^2 \otimes C^2 \otimes C^2$), and with the aid of the computer, we can directly verify that the rank of φ_a is 8 if a_{111}, p_1, q_1, r_1, $t \neq 0$ at a. Hence, the set of singular elements which we denote by Σ are contained in the algebraic set defined by $a_{111} \cdot p_1 \cdot q_1 \cdot r_1 \cdot t = 0$. Since Σ is G-invariant, we have from Proposition 7 and Proposition 5

$$\begin{aligned} \Sigma &\subset \{ \langle a_{111} \cdot p_1 \cdot q_1 \cdot r_1 \cdot t \rangle = 0 \} \\ &= \{ \langle a_{111} \rangle = 0 \} \cup \{ \langle p_1 \rangle = 0 \} \cup \cdots\cdots \cup \{ t = 0 \} \\ &= \Sigma_0 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \\ &= \Sigma_5. \end{aligned}$$

Conversely, assume that $a \in C^2 \otimes C^2 \otimes C^2$ satisfies a_{111}, p_1, q_1, $r_1 \neq 0$ and $t = 0$. Then, as above, we can show that rank $\varphi_a = 7$, which implies that a is singular. In particular, there exists an open subset of Σ_5 consisting of singular elements. Since the singular elements constitute an algebraic set contained in Σ_5 and the hypersurface Σ_5 is irreducible, we obtain the desired equality $\Sigma = \Sigma_5$. The last statement in this lemma follows immediately from the first one because a is not contained in any invariant proper subvariety in the case $t \neq 0$. q.e.d.

Now, we prove Theorem 4. By Proposition 3, all generators of irreducible components of $S^p(C^2 \otimes C^2 \otimes C^2)$ $(p \leq 6)$ are expressed as a product of six polynomials a_{111}, p_1, q_1, r_1, s_1, t except $2S_{42}T_{42}U_{42}$. Hence, by Proposition 7 and Proposition 5, all invariant subvarieties defined by these components are the union of $\Sigma_0 \sim \Sigma_5$, with the above exception. Next, by Lemma 11, all invariant subvarieties satisfy the equality $t = 0$, and hence, one component of $2S_{42}T_{42}U_{42}$ which is generated by $a_{111}{}^2 t$ always vanishes on these varieties (cf. Proposition 3). If just one component of $2S_{42}T_{42}U_{42}$ vanishes, then the variety is contained in the hypersurface Σ_5, and if two components vanish on a variety Σ, the variety Σ is contained in $\Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ because the equalities $p_1q_1r_1 = s_1 = 0$ hold on Σ (cf. Proposition 3). Therefore, all irreducible proper subvarieties are exhausted by $\Sigma_0 \sim \Sigma_5$, and hence we complete the proof of Theorem 4. q.e.d.

4. A remark on the irreducible components of $S^p(C^2 \otimes C^2 \otimes C^2)$

If we can prove the following conjecture, we may drop the restriction on the degree of polynomials "$p \leq 6$" in Theorem 4.

CONJECTURE. *The coefficients of $x^{q+1}y^{r+1}z^{s+1}t^p$ in the following two formal polynomials coincide for $p/2 \leq q, r, s \leq p$:*

$$\frac{xyz(1+x^2y^2z^2t^3)}{(1-xyzt)(1-x^2yzt^2)(1-xy^2zt^2)(1-xyz^2t^2)(1-x^2y^2z^2t^4)},$$
$$\frac{(x-1)(y-1)(z-1)}{(1-t)(1-xt)(1-yt)(1-zt)(1-xyt)(1-yzt)(1-zxt)(1-xyzt)}.$$

The second polynomial is the generating function of the polynomial ring $\sum_p S^p(C^2 \otimes C^2 \otimes C^2)$, i.e., the coefficient of $x^{q+1}y^{r+1}z^{s+1}t^p$ of the second polynomial coincides with the multiplicity of $S_{q,p-q}T_{r,p-r}U_{s,p-s}$ in $S^p(C^2 \otimes C^2 \otimes C^2)$. We can prove this fact by using the formula stated in [6; p.14]. The first polynomial is the generating function of the subring of $\sum_p S^p(C^2 \otimes C^2 \otimes C^2)$ which is generated by six typical generators a_{111}, p_1, q_1, r_1, s_1 and t, i.e., generated by $S_1T_1U_1$, $S_2T_{11}U_{11}$, $S_{11}T_2U_{11}$, $S_{11}T_{11}U_2$, $S_{21}T_{21}U_{21}$, $S_{22}T_{22}U_{22}$. For example, the term $x^2y^2z^2t^3$ in the numerator of the first polynomial corresponds to $\langle s_1 \rangle = S_{21}T_{21}U_{21}$ ($p = 3$, $q = r = s = 2$), and the term $1/(1-x^2y^2z^2t^4)$ corresponds to the powers of $\langle t \rangle = S_{22}T_{22}U_{22}$ ($p = 4$, $q = r = s = 2$). Hence, if this conjecture actually holds, we can show that the generator of each irreducible component of $S^p(C^2 \otimes C^2 \otimes C^2)$ can be expressed in terms of only $a_{111} \sim t$ for general p, by applying the similar method developed in [15]. In addition, we can also show that the relations among these six generators are exhausted essentially by the identity

$$4p_1q_1r_1 + s_1^2 = a_{111}^2 t.$$

Unfortunately, we do not know the proof of the above conjecture at present. By Proposition 3, we know that this conjecture holds for $p \leq 6$, and with the aid of the computer we can check it for $p \leq 8$.

REMARK. It is already known that the 3-tensor space $C^2 \otimes C^2 \otimes C^2$ admits only seven G-orbits including $\{0\}$ and the space itself. (See [8; p.261].) It is easy to see that the closure of these orbits just coincide with the invariant subvarieties stated in Theorem 4, and by using this result, we can show that there exists only seven invariant proper subvarieties of $C^2 \otimes C^2 \otimes C^2$ without the assumption "degree ≤ 6". (In fact, for any invariant algebraic set Σ, we have $\Sigma = \bigcup_{x \in \Sigma} G \cdot x$, which implies that Σ is expressed as a finite union of G-orbits.) But, we want to show the finiteness of invariant varieties only in terms of defining polynomials and the identities between them, not using the normal forms.

We also remark that in general situations, G-invariant algebraic sets are not expressed as a closure of finite G-orbits. For example, consider the natural action of $GL(V)$ on the space of square matrices $V^* \otimes V$. In this case, the set of matrices with trace = 0 is the $GL(V)$-invariant algebraic set which consists of infinitely many $GL(V)$-invariant orbits. (Note that two elements of $V^* \otimes V$ belong to the same $GL(V)$-orbit if and only if their Jordan normal forms coincide.)

5. Invariant subvarieties of the space $S^3(C^2)$

In [15], the classical theory of invariants and covariants of the 3-symmetric tensor space $S^3(C^2)$ (binary cubic forms) is explained in detail. In this case, $GL(2,C)$-invariant irreducible subspaces of the polynomial ring $\sum_p S^p(S^3(C^2))$ are generated by

$$\begin{aligned}
p &= 1 \quad S_3 : b_{111},\\
p &= 2 \quad S_{42} : b_{111}b_{122} - b_{112}^2,\\
p &= 3 \quad S_{63} : b_{111}^2 b_{222} - 3b_{111}b_{112}b_{122} + 2b_{112}^3,\\
p &= 4 \quad S_{66} : b_{111}^2 b_{222}^2 - 6b_{111}b_{112}b_{122}b_{222} + 4b_{111}b_{122}^3 + 4b_{112}^3 b_{222} - 3b_{112}^2 b_{122}^2,
\end{aligned}$$

where we express $b = \sum_{1 \le i,j,k \le 2} b_{ijk} e_i \circ e_j \circ e_k \in S^3(C^2)$. ($\circ$ implies the symmetric tensor product.) In [15], the monomials b_{111}, b_{112}, b_{122}, b_{222} are expressed by the symbols $\xi^{(0)}$, $\xi^{(1)}$, $\xi^{(2)}$, $\xi^{(3)}$, respectively, and the above four generators are denoted by f, h, j and d. These generators satisfy the famous identity

$$4h^3 + j^2 = f^2 d,$$

which has a strong resemblance to

$$4p_1q_1r_1 + s_1^2 = a_{111}^2 t.$$

The quartic polynomial d is classically known as the name of "discriminant" . Using these results, we can show that there exist three $GL(2,C)$-invariant proper subvarieties of $S^3(C^2)$ defined by

$$\begin{aligned}
\bar{\Sigma}_0 &= \{ 0 \},\\
\bar{\Sigma}_2 &= \{ \langle h \rangle = 0 \},\\
\bar{\Sigma}_3 &= \{ d = 0 \},
\end{aligned}$$

which satisfy the following simple inclusion relation:

$$\{ 0 \} —— \bar{\Sigma}_2 —— \bar{\Sigma}_3 —— S^3(C^2).$$

(We can easily check the equalities $\dim \bar{\Sigma}_i = i$, and $\{ \langle j \rangle = 0 \} = \bar{\Sigma}_2$.) Clearly, there is a remarkable resemblance between the case $S^3(C^2)$ and our case $C^2 \otimes C^2 \otimes C^2$, concerning the irreducible decomposition of the polynomial ring, its generators (degree, expressions, identities), and their invariant subvarieties (geometric meaning, inclusion relations). In a sense, the space $S^3(C^2)$ may be considered as a degenerate space of the 3-tensor space $C^2 \otimes C^2 \otimes C^2$. For example, three subvarieties Σ_2, Σ_3, $\Sigma_4 \subset C^2 \otimes C^2 \otimes C^2$ degenerate to one variety $\bar{\Sigma}_2$, and three invariant subspaces $S_2T_{11}U_{11}$, $S_{11}T_2U_{11}$, $S_{11}T_{11}U_2$ of $S^2(C^2 \otimes C^2 \otimes C^2)$ degenerate to one irreducible space $S_{42} \subset S^2(S^3(C^2))$. In other words, the classical theory of binary cubic forms has its root in the 3-tensor space $C^2 \otimes C^2 \otimes C^2$.

We list up the character table of $S^p(S^3(C^2))$ ($p \leq 6$) in terms of the Schur functions in the following:

$$
\begin{aligned}
p &= 1 : S_3, \\
p &= 2 : S_6 + S_{42}, \\
p &= 3 : S_9 + S_{72} + S_{63}, \\
p &= 4 : S_{12} + S_{10,2} + S_{93} + S_{84} + S_{66}, \\
p &= 5 : S_{15} + S_{13,2} + S_{12,3} + S_{11,4} + S_{10,5} + S_{96}, \\
p &= 6 : S_{18} + S_{16,2} + S_{15,3} + S_{14,4} + S_{13,5} + 2S_{12,6} + S_{10,8}.
\end{aligned}
$$

We can determine the above characters by calculating the plethysm of the Schur functions. For example, see [13; Appendix], [1; p.111].

Finally, we give a geometrical characterization of the variety $\bar{\Sigma}_2$ as in Lemma 6. We define a cubic map $\bar{\gamma} : C^2 \to S^3(C^2)$ by

$$\bar{\gamma}(v) = v \circ v \circ v,$$

where $v \circ v \circ v$ is the symmetric 3-tensor product of the vector $v \in C^2$. Then, it is easy to see that the equality

$$\mathrm{Im}\ \bar{\gamma} = \bar{\Sigma}_2$$

holds, and hence, the subvariety $\bar{\Sigma}_2$ just coincides with the set of decomposable elements of $S^3(C^2)$. Consequently, a symmetric 3-tensor $b \in S^3(C^2)$ is decomposable if and only if the components of b satisfy the following equalities corresponding to the irreducible subspace $S_{42} \subset S^2(S^3(C^2))$:

$$
\begin{aligned}
b_{111}b_{122} - b_{112}^2 &= 0, \\
b_{111}b_{222} - b_{112}b_{122} &= 0, \\
b_{112}b_{222} - b_{122}^2 &= 0.
\end{aligned}
$$

Note that this condition is equivalent to

$$\mathrm{rank}\begin{pmatrix} b_{111} & b_{112} \\ b_{112} & b_{122} \\ b_{122} & b_{222} \end{pmatrix} \leq 1.$$

This result may be considered as a "symmetric" version of classical Plüker's quadratic relation for the Grassmann space, which characterizes the decomposability of an element of $\Lambda^p C^n$.

For another detailed explanation of the polynomial ring and invariant subvarieties of the space $S^3(C^2)$, see [18; p.1397 ~]. Concerning the rank of 3-tensor spaces, there are many interesting deep researches such as [7], [16], [20], etc. If the ground field is $\boldsymbol{R}$, almost same results as above hold, and normal form of $S^3(\boldsymbol{R}^2)$ is explained in [5; p.259], [24; p.276].

For 3-tensor spaces associated with higher dimensional vector space $\boldsymbol{C}^n$, we can theoretically study them by the same methods developed in this note. But, many calculations are needed for such study as the dimension of the base vector space increases, and for geometric applications, it is desirable to investigate more substantial method to study invariant subvarieties of general 3-tensor spaces. (The study of "normal forms" becomes hopeless in higher dimensional cases. See for example, [22], [23].)

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