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On the Curvature of the Homogeneous Space $U(n+1)/U(n)$

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Abstract : We determine all torsion free invariant affine connections on the homogeneous space $S^{2n+1} = U(n+1)/U(n)$, and characterize their curvatures in terms of the polynomials of their components in the space of curvature-like tensors. The essential difference between the case $n = 1$ and $n \geq 2$ is explained in detail from the standpoint of flat affine geometry.

Key words : affine connection, curvature, homogeneous space, invariant

Introduction

Characterization of the curvature tensors, i.e., which curvature-like tensors are actually curvatures that are obtained from connections is one of the fundamental problem in differential geometry. Recently, several efforts have been achieved on this problem such as [2], [5], [6], [13], but the complete answer is not known yet. In this note, we consider a homogeneous affine version of this problem, and as a special example, we characterize the set of curvatures of torsion free invariant affine connections on the homogeneous odd dimensional sphere $S^{2n+1} = U(n+1)/U(n)$. It is well known that the sphere S^m considered as the homogeneous space $SO(m+1)/SO(m)$ admits a unique invariant connection (the canonical Riemannian connection). But in our case $U(n+1)/U(n)$, the degree of freedom of invariant connections is 4 (Theorem 1), and roughly speaking, their curvatures constitute a 4-dimensional subvariety in the space of all curvature-like tensors. Explicit determination of invariant connections on a homogeneous space is in general a hard problem. (See, for example, recent works [7], [10], [11].) In this note, we solve this problem with the aid of the character (the Schur function) of the general linear group $GL(n, \boldsymbol{C})$.

Concerning the characterization of the curvatures of S^{2n+1}, there exists an essential difference between the case $n = 1$ (i.e., S^3) and $n \geq 2$. In the case $n \geq 2$, the curvatures are characterized completely in terms of relatively simple quadratic polynomials of their components, while in the case $n = 1$, there exists no polynomial relations up to degree 4. In addition, in the case $n = 1$, S^3 admits invariant almost flat affine connections, i.e., the

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closure of the set of all curvatures contains the origin 0 corresponding to a flat affine connection, but it is not the case for $n \geq 2$ (cf. Theorem 5). It is classically well known that S^m does not admit a torsion free flat affine connection for $m \geq 2$ ([3; p.145]), and hence, the above difference is an interesting phenomenon in the standpoint of flat affine geometry.

1. Invariant connections on $U(n+1)/U(n)$.

In this section, we determine all invariant affine connections on the homogeneous space $S^{2n+1} = U(n+1)/U(n)$ $(n \geq 1)$, by using a representation theoretic method. To state the theorem, we first fix some notations.

Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of $U(n+1)$ (resp. $U(n)$), and $\mathfrak{m}$ be the canonical complementary subspace of $\mathfrak{k}$ in $\mathfrak{g}$, i.e.,

$$\mathfrak{g} = \mathfrak{u}(n+1),$$
$$\mathfrak{k} = \mathfrak{u}(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{u}(n) \right\},$$
$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & v \\ -{}^t\bar{v} & ki \end{pmatrix} \;\middle|\; v \in \mathbf{C}^n,\ k \in \mathbf{R} \right\}.$$

We express the above element of $\mathfrak{k}$ and $\mathfrak{m}$ simply as A and (v, k), respectively. The space $\mathfrak{m}$ is canonically identified with the tangent space at the origin of $U(n+1)/U(n)$. It is easy to see that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, and hence, the space $U(n+1)/U(n)$ is reductive. The linear isotropy representation $\rho : \mathfrak{k} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ of $U(n+1)/U(n)$ is given by

$$\begin{aligned}\rho(A)(v, k) &= \left[\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ -{}^t\bar{v} & ki \end{pmatrix} \right] \\ &= \begin{pmatrix} 0 & Av \\ {}^t\bar{v}A & 0 \end{pmatrix} \\ &= (Av,\ 0).\end{aligned}$$

We fix a basis $\{X_1, \cdots\cdots X_n, Y_1, \cdots\cdots, Y_n, W\}$ of $\mathfrak{m}$ by

$$X_k = \left(\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array} \right) \begin{array}{l} k\text{-th} \\ \\ \end{array}, \quad Y_k = \left(\begin{array}{c|c} 0 & i \\ \hline i & 0 \end{array} \right) \begin{array}{l} k\text{-th} \\ \\ \end{array},$$
$$\begin{array}{c} \ k\text{-th} \qquad\qquad\qquad\qquad\qquad k\text{-th} \end{array}$$

$$W = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & i \end{array} \right).$$

Then, the space $\mathfrak{m}$ has a canonical direct sum decomposition

$$\mathfrak{m} = C^n + \langle W \rangle,$$

where $C^n = \langle X_1, \cdots\cdots, X_n, Y_1, \cdots\cdots, Y_n \rangle$, and there exists a natural complex structure I on C^n defined by

$$IX_k = Y_k \quad \text{and} \quad IY_k = -X_k.$$

We introduce the inner product $(\ ,\ )$ on $\mathfrak{m}$ such that the vectors $X_1, \cdots\cdots, X_n, Y_1, \cdots\cdots, Y_n$, W form an orthonormal basis. Since the isotropy group $U(n)$ acts on $\mathfrak{m}$ as an isometry group, the metric $(\ ,\ )$ on $\mathfrak{m}$ is globally extendable to all points of $U(n+1)/U(n)$, which is nothing but the standard Riemannian metric on S^{2n+1} with constant curvature 1.

It is classically well known that there exists a one-to-one correspondence between the set of all torsion free invariant affine connections on a reductive homogeneous space with a connected isotropy group and the set of linear maps $f: \mathfrak{m} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ satisfying

$$(*)_1 \qquad f([A, X]) = [\rho(A), f(X)], \ A \in \mathfrak{k}, \ X \in \mathfrak{m},$$

$$(*)_2 \qquad f(X)Y - f(Y)X = [X, Y]_{\mathfrak{m}}, \quad X, Y \in \mathfrak{m},$$

where $[X, Y]_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of $[X, Y]$. In particular, the set of invariant connections may be considered as an affine subspace of $\mathfrak{m}^* \otimes \mathfrak{gl}(\mathfrak{m})$. (See [9; p.191 $\sim$ 192]. Note that the second condition $(*)_2$ indicates that the corresponding connection is torsion free.)

Under these notations, our first main theorem is stated as follows.

THEOREM 1. *The set of all torsion free invariant affine connections on* $U(n+1)/U(n)$ $(n \geq 1)$ *constitute a 4-dimensional affine subspace of* $\mathfrak{m}^* \otimes \mathfrak{gl}(\mathfrak{m})$. *In terms of real parameters* $a_1 \sim a_4$, *the corresponding linear map* $f: \mathfrak{m} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ *is expressed as*

$$f(X)Y = a_1(X, Y)W + (X, IY)W,$$

$$f(X)W = a_2 X + (a_3 + 1) IX,$$

$$f(W)X = a_2 X + a_3 IX,$$

$$f(W)W = (a_2 + a_4)W,$$

for $X, Y \in C^n \subset \mathfrak{m}$.

PROOF. To prove this theorem, we have only to show that the above linear map f

satisfies two conditions $(*)_1$, $(*)_2$, and that the degree of freedom of torsion free invariant affine connections is at most 4.

First, for $A \in \mathfrak{k}$ and $X, Y \in C^n \subset \mathfrak{m}$, we have

$$
\begin{aligned}
f([A, X])Y &= f(AX)Y = a_1(AX, Y)W + (AX, IY)W, \\
[\rho(A), f(X)]Y &= \rho(A)f(X)Y - f(X)\rho(A)Y \\
&= \rho(A)\{a_1(X, Y)W + (X, IY)W\} - f(X)AY \\
&= -f(X)AY \\
&= -a_1(X, AY)W - (X, IAY)W \\
&= -a_1(X, AY)W - (X, AIY)W \\
&= a_1(AX, Y)W + (AX, IY)W, \\
f([A, X])W &= f(AX)W = a_2AX + (a_3 + 1)IAX, \\
[\rho(A), f(X)]W &= \rho(A)f(X)W - f(X)\rho(A)W \\
&= \rho(A)\{a_2X + (a_3 + 1)IX\} \\
&= a_2AX + (a_3 + 1)AIX \\
&= a_2AX + (a_3 + 1)IAX,
\end{aligned}
$$

and hence, we have $f([A, X]) = [\rho(A), f(X)]$. Next, for $A \in \mathfrak{k}$ and $X \in C^n$, we have

$$
\begin{aligned}
&f([A, W]) = 0, \\
&[\rho(A), f(W)]X = \rho(A)f(W)X - f(W)\rho(A)X \\
&\qquad = \rho(A)(a_2X + a_3IX) - f(W)AX \\
&\qquad = (a_2AX + a_3AIX) - (a_2AX + a_3IAX) \\
&\qquad = 0, \\
&[\rho(A), f(W)]W = \rho(A)f(W)W - f(W)\rho(A)W \\
&\qquad = \rho(A)(a_2 + a_4)W \\
&\qquad = 0,
\end{aligned}
$$

and hence we have $f([A, W]) = [\rho(A), f(W)]$, which implies that f satisfies the condition $(*)_1$. Next, we show that f is torsion free. First, it is easy to show that

$$[X, Y]_{\mathfrak{m}} = 2(X, IY)W, \quad \text{for } X, Y \in C^n,$$

and $\quad [X, W]_{\mathfrak{m}} = IX, \quad$ for $X \in \mathbf{C}^n$.

Then, for $X, Y \in \mathbf{C}^n$, we have

$$\begin{aligned} f(X)Y - f(Y)X &= (X, IY)\,W - (Y, IX)\,W \\ &= 2\,(X, IY)\,W = [X, Y]_{\mathfrak{m}}, \\ f(X)W - f(W)X &= IX = [X, W]_{\mathfrak{m}}, \end{aligned}$$

and hence f satisfies the condition $(*)_2$.

Next, we show that the degree of freedom of torsion free invariant affine connections is at most 4. For this purpose, we fix one invariant torsion free affine connection f^0 once for all. Then, other connections f are expressed uniquely as

$$f(X)Y = f^0(X)\,Y + \alpha\,(X, Y),$$

where $\alpha : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ is a symmetric bi-linear map. Then, it is easy to see that f satisfies the condition $(*)_1$ if and only if

$$[A, \alpha\,(X, Y)] = \alpha\,([A, X], Y) + \alpha\,(X, [A, Y]),$$

for any $A \in \mathfrak{k}$, $X, Y \in \mathfrak{m}$, i.e., $\alpha \in S^2\,\mathfrak{m}^* \otimes \mathfrak{m}$ is invariant under the canonical action of $\mathfrak{k}$. (Note that the second condition $(*)_2$ is automatically satisfied in this situation because α is symmetric and f^0 is torsion free.) Clearly, the action of $\mathfrak{k} = \mathfrak{u}(n)$ on $\mathfrak{m}$ splits into two irreducible representations, namely, natural representation on $\mathbf{C}^n$ and the trivial representation on $<W>$. As a real representation, $A = B + iC \in \mathfrak{u}(n)$ acts on $\mathbf{C}^n = \mathbf{R}^{2n}$ by

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

We complexify this representation. Then, by putting

$$P = 1/2 \cdot \begin{pmatrix} E & E \\ -iE & iE \end{pmatrix},$$

we have

$$P^{-1} \begin{pmatrix} B & -C \\ C & B \end{pmatrix} P = \begin{pmatrix} B + iC & 0 \\ 0 & B - iC \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix},$$

and hence, the complexified representation of $\mathfrak{u}(n)^c$ $(\simeq \mathfrak{gl}(n, \mathbf{C}))$ on $(\mathbf{R}^{2n})^c = \mathbf{C}^{2n}$ is equivalent to the sum of the identity representation and its contragredient representation.

Hence, in terms of the Schur functions, the character of the complexified representation of $\mathfrak{k}^c$ on $\mathfrak{m}^c$ is given by $S_1 + S_{-1} + S_0$. (For the definition of the Schur function, see [8], [12].) Since the space $(\mathfrak{m}^c)^*$ is isomorphic to $\mathfrak{m}^c$ as a representation space, we can calculate the character of $S^2(\mathfrak{m}^c)^* \otimes \mathfrak{m}^c$ by using Littlewood-Richardson's rule and the formula $S^2(V+W) = S^2(V) + V \otimes W + S^2(W)$. The result is given by

$$S_3 + S_{21} + 2S_2 + S_{11} + 5S_1 + 2S_{2-1} + S_{11-1} + 4S_0 + 3S_{1-1} + S_{1-1-1} + 2S_{1-2} + 5S_{-1} + S_{-1-1} + 2S_{-2} + S_{-1-2} + S_{-3} \qquad (n \geq 3),$$

$$S_3 + S_{21} + 2S_2 + S_{11} + 5S_1 + 2S_{2-1} + 4S_0 + 3S_{1-1} + 2S_{1-2} + 5S_{-1} + S_{-1-1} + 2S_{-2} + S_{-1-2} + S_{-3} \qquad (n = 2),$$

$$S_3 + 2S_2 + 4S_1 + 4S_0 + 4S_{-1} + 2S_{-2} + S_{-3} \qquad (n = 1),$$

and for each case, the multiplicity of the invariant S_0 is 4. This implies that the degree of freedom of complex linear maps $f : \mathfrak{m}^c \longrightarrow \mathfrak{gl}(\mathfrak{m}^c)$ satisfying the complexified conditions corresponding to $(*)_1$ and $(*)_2$ is 4. Therefore, real linear maps satisfying $(*)_1$ and $(*)_2$ has degree of freedom at most 4, and hence, we complete the proof of Theorem 1. q.e.d.

REMARK. (1) Since the conditions $(*)_1$ and $(*)_2$ are linear on f, we can of course prove Theorem 1 by solving these linear equations on the components of f directly. But this method requires tremendous calculations.

(2) The canonical Riemannian connection on S^{2n+1} with standard constant curvature metric is given by the one corresponding to $a_i = 0$ $(i = 1 \sim 4)$. This fact can be easily checked by using Theorem 3.3 in [9; p.201].

(3) If we drop the assumption "torsion free" in Theorem 1, then the set of invariant affine connections constitutes a "linear" subspace of $\mathfrak{m}^* \otimes \mathfrak{gl}(\mathfrak{m})$. By a direct calculation, or by the character method, we can show that the dimension of this linear subspace is 7. It is explicitly given by

$$\begin{aligned} f(X)Y &= b_1(X, Y)W + b_2(X, IY)W, \\ f(X)W &= b_3X + b_4IX, \\ f(W)X &= b_5X + b_6IX, \\ f(W)W &= b_7W, \end{aligned}$$

where $X, Y \in \boldsymbol{C}^n$, and $b_1 \sim b_7 \in \boldsymbol{R}$. Needless to say, the above linear map f satisfies the condition $(*)_1$ only.

(4) If we consider the sphere S^{2n+1} as the homogeneous space $SU(n+1)/SU(n)$ $(n \geq 2)$, then the degree of freedom of torsion free invariant affine connections is equal to 4 (n

≥ 3) and 6 ($n = 2$). The two dimensional new freedom in the case $n = 2$ corresponds to the Schur functions S_{11} and S_{-1-1} in the above character table in the proof of Theorem 1. (Note that S_{11} and S_{-1-1} are the invariants of the special linear group $SL(n, \boldsymbol{C})$ only in the case $n = 2$.)

2. Characterization of the curvatures.

In this section, we calculate the curvature of the invariant connection in Theorem 1, and give the complete characterization of the set of curvatures in terms of their components in the case $n \geq 2$.

THEOREM 2. *The curvature of the invariant connection on $U(n+1)/U(n)$ ($n \geq 1$) stated in Theorem* 1 *is given by*

$$\begin{aligned}
R(X, Y)Z &= \{r_1(Y, Z) + r_2(Y, IZ)\}X - \{r_1(X, Z) + r_2(X, IZ)\}Y \\
&\quad + \{r_3(Y, Z) + r_4(Y, IZ)\}IX - \{r_3(X, Z) + r_4(X, IZ)\}IY \\
&\quad - 2r_2(X, IY)Z - 2r_4(X, IY)IZ, \\
R(X, W)Y &= \{-r_5(X, Y) + r_6(X, IY)\}W, \\
R(X, Y)W &= 2r_6(X, IY)W, \\
R(X, W)W &= r_7X - r_8IX,
\end{aligned}$$

where $X, Y, Z \in \boldsymbol{C}^n$, $\mathfrak{m} = \boldsymbol{C}^n + \langle W \rangle$, *and*

$$\begin{aligned}
r_1 &= a_1a_2 + 1, & r_5 &= a_1a_4 + a_3 + 1, \\
r_2 &= a_2, & r_6 &= a_1a_3 + a_1 - a_4, \\
r_3 &= a_1(a_3 + 1), & r_7 &= a_2a_4 + (a_3 + 1)^2, \\
r_4 &= a_3, & r_8 &= (a_3 + 1)(a_2 - a_4).
\end{aligned}$$

PROOF. It is known that the curvature of the invariant connection corresponding to f is given by

$$R(X, Y) = [f(X), f(Y)] - f([X, Y]_{\mathfrak{m}}) - \rho([X, Y]_{\mathfrak{k}}), \quad X, Y \in \mathfrak{m},$$

where $[X, Y]_{\mathfrak{k}}$ is the $\mathfrak{k}$-component of $[X, Y] \in \mathfrak{g}$ ([9; p.192]). Explicitly, the third term of the above equality is given by

$$\begin{aligned}
\rho([X, Y]_{\mathfrak{k}})Z &= (Y\,{}^t\overline{X} - X\,{}^t\overline{Y})Z \\
&= (X, Z)Y - (X, IZ)IY - (Y, Z)X + (Y, IZ)IX,
\end{aligned}$$

$$\rho([X, Y]_{\mathfrak{k}})W = 0,$$
$$\rho([X, W]_{\mathfrak{k}}) = 0,$$

for $X, Y, Z \in C^n$. We have only to calculate the curvature by using these formulas. First, for $X, Y, Z \in C^n \subset \mathfrak{m}$, we have

$$\begin{aligned}
R(X, Y)Z &= f(X)f(Y)Z - f(Y)f(X)Z - 2(X, IY)f(W)Z - \rho([X, Y]_{\mathfrak{k}})Z \\
&= f(X)\{a_1(Y, Z) + (Y, IZ)\}W - f(Y)\{a_1(X, Z) + (X, IZ)\}W \\
&\quad - 2(X, IY)(a_2 Z + a_3 IZ) - \rho([X, Y]_{\mathfrak{k}})Z \\
&= \{a_1(Y, Z) + (Y, IZ)\}\{a_2 X + (a_3 + 1)IX\} - \{a_1(X, Z) + (X, IZ)\}\{a_2 Y + (a_3 + 1)IY\} \\
&\quad -2(X, IY)(a_2 Z + a_3 IZ) - (X, Z)Y + (X, IZ)IY + (Y, Z)X - (Y, IZ)IX \\
&= \{r_1(Y, Z) + r_2(Y, IZ)\}X - \{r_1(X, Z) + r_2(X, IZ)\}Y \\
&\quad + \{r_3(Y, Z) + r_4(Y, IZ)\}IX - \{r_3(X, Z) + r_4(X, IZ)\}IY \\
&\quad - 2r_2(X, IY)Z - 2r_4(X, IY)IZ.
\end{aligned}$$

The remaining three equalities in this theorem can be verified completely in the same way as above, and we omit the detailed calculations. q.e.d.

From this theorem, the non-zero components of the curvature tensors are expressed in terms of 8 variables $r_1 \sim r_8$. In particular, the set of curvatures lie in the 8-dimensional linear subspace of curvature-like tensors on $\mathfrak{m}$. (By definition, the space of curvature-like tensors on an m-dimensional vector space V is the linear space

$$\{R \in \wedge^2 V^* \otimes V^* \otimes V \mid \underset{X, Y, Z}{\mathfrak{S}} R(X, Y)Z = 0\},$$

and it is known that the dimension of this space is equal to $1/3 \cdot m^2(m^2 - 1)$. It is a sum of three $GL(V)$-irreducible components with dimension $1/3 \cdot m^2(m^2 - 4)$, $1/2 \cdot m(m+1)$ and $1/2 \cdot m(m-1)$. For details, see for example [4; p.41]. In our case, since $m = 2n + 1$, its dimension is equal to $4/3 \cdot n(n+1)(2n+1)^2$.) Explicitly, the curvature may be expressed in the following form.

COROLLARY 3. *The non-zero components of the curvature of* $U(n+1)/U(n)$ *are exhausted by*

$$R(X_i, X_j)X_j = R(X_i, Y_j)Y_j = U_i,$$
$$R(Y_i, X_j)X_j = R(Y_i, Y_j)Y_j = IU_i,$$

$$
\begin{aligned}
R(X_j, X_i)Y_j &= R(X_i, Y_j)X_j = V_i,\\
R(X_j, Y_i)Y_j &= R(Y_i, Y_j)X_j = IV_i,\\
R(X_j, Y_j)X_i &= 2V_i,\\
R(X_j, Y_j)Y_i &= 2IV_i,\\
R(X_i, Y_i)X_i &= 3V_i - IU_i,\\
R(X_i, Y_i)Y_i &= U_i + 3IV_i,\\
R(X_i, W)X_i &= R(Y_i, W)Y_i = -r_5 W,\\
R(Y_i, W)X_i &= -R(X_i, W)Y_i = r_6 W,\\
R(X_i, Y_i)W &= -2r_6 W,\\
R(X_i, W)W &= r_7 X_i - r_8 Y_i,\\
R(Y_i, W)W &= r_8 X_i + r_7 Y_i,
\end{aligned}
$$

where $U_i = r_1 X_i + r_3 Y_i$ *and* $V_i = r_2 X_i + r_4 Y_i$. *(Throughout the above expressions, we assume* $i \neq j$.*)*

In particular, in the case $n = 1$, the curvatures lie in the 6-dimensional linear subspace of curvature-like tensors on $\mathfrak{m}$ spanned by the variables $r_1 - 3r_4$, $3r_2 + r_3$, r_5, r_6, r_7, r_8 because the terms such as $R(X_i, X_j)X_j$ $(i \neq j)$ do not appear in this case, and

$$U_i + 3IV_i = I(3V_i - IU_i) = (r_1 - 3r_4)X_i + (3r_2 + r_3)Y_i.$$

We can also show this fact by calculating the multiplicity of S_0 in the character of the space of curvature-like tensors on $\mathfrak{m}$ (for both cases $n = 1$ and $n \geq 2$), as in the case of determining the degree of freedom of invariant connections.

Now, we characterize the curvatures of the space $U(n+1)/U(n)$ by using the above expressions in the case $n \geq 2$. To express the result in a simple form, we change the variables of the curvatures as follows:

$$
\begin{aligned}
s_1 &= r_1 - 1, & s_5 &= r_5 - r_4 - 1,\\
s_2 &= r_2, & s_6 &= r_3 - r_6,\\
s_3 &= r_3, & s_7 &= r_7,\\
s_4 &= r_4 + 1, & s_8 &= r_8.
\end{aligned}
$$

Clearly, the components r_i are uniquely determined by s_i, and vice versa. In the following, we use $\{s_i\}$ as a coordinate of the 8-dimensional subspace of the curvature-like tensors on $\mathfrak{m}$, instead of $\{r_i\}$. To state the precise results, we introduce three real algebraic varieties Σ_0, Σ_3, Σ_4 in the 8-dimensional subspace by

$$\Sigma_0 = \{(0)\},$$

$$\Sigma_3 = \{(s_1, 0, s_3, 0, s_5, 0, 0, 0) \mid s_1, s_3, s_5 \in \boldsymbol{R}\},$$

$$\Sigma_4 = \left\{ (s_1, \cdots\cdots, s_8) \in \boldsymbol{R}^8 \,\middle|\, \begin{array}{l} s_1 s_4 = s_2 s_3,\ s_1 s_6 = s_2 s_5,\ s_3 s_6 = s_4 s_5, \\ s_7 = s_2 s_6 + s_4^2,\ s_8 = s_4 (s_2 - s_6) \end{array} \right\}.$$

Clearly, we have the inclusion relation:

$$\Sigma_0 \subset \Sigma_3 \subset \Sigma_4,$$

and it is easy to see that the dimension of the variety Σ_i is equal to i. Under these notations, we have

THEOREM 4. *In the case $n \geq 2$, the set of curvatures of torsion free invariant affine connections on $U(n+1)/U(n)$ is equal to the set $(\Sigma_4 \setminus \Sigma_3) \cup \Sigma_0$. In particular, the closure of this set constitutes the 4-dimensional variety Σ_4.*

PROOF. First, it is easy to see that the actual curvature (s_i) belongs to the variety Σ_4 because its components are expressed as

$$(**) \qquad \begin{array}{ll} s_1 = a_1 a_2, & s_5 = a_1 a_4, \\ s_2 = a_2, & s_6 = a_4, \\ s_3 = a_1 (a_3 + 1), & s_7 = a_2 a_4 + (a_3 + 1)^2, \\ s_4 = a_3 + 1, & s_8 = (a_3 + 1)(a_2 - a_4). \end{array}$$

Now, assume that the actual curvature (s_i) is not contained in Σ_0, i.e., $(s_i) \neq (0)$. If $s_2 = s_4 = s_6 = s_7 = s_8 = 0$, then we have $a_2 = a_4 = 0$, $a_3 = -1$ from the above expressions $(**)$. Hence, we have $s_1 = s_3 = s_5 = 0$, and this contradicts the assumption $(s_i) \neq (0)$. Therefore, one of s_2, s_4, s_6, s_7, s_8 is not zero, and we have $(s_i) \notin \Sigma_3$. Combining these facts, it follows that the actual curvature lies in the set $(\Sigma_4 \setminus \Sigma_3) \cup \Sigma_0$.

Conversely, let (s_i) be an element of $(\Sigma_4 \setminus \Sigma_3) \cup \Sigma_0$, and we show that it is actually a curvature. The case $(s_i) \in \Sigma_0$, (i.e., $(s_i) = (0)$) can be checked immediately. If $(s_i) \in \Sigma_4 \setminus \Sigma_3$, then, one of s_2, s_4, s_6, s_7, s_8 is not zero. In the case $s_2 \neq 0$, we consider the connection defined by

$$a_1 = s_1/s_2, \quad a_2 = s_2, \quad a_3 = s_4 - 1, \quad a_4 = s_6.$$

Then, from the expressions ($**$) and the quadratic relations which define the variety Σ_4, it is easy to see that the curvature corresponding to this connection is (s_i). In the case $s_4 \neq 0$, we put

$$a_1 = s_3/s_4, \quad a_2 = s_2, \quad a_3 = s_4 - 1, \quad a_4 = s_6.$$

Then, similarly, we obtain the same result. If $s_2 = s_4 = 0$, then we have $s_6 \neq 0$. In fact, if $s_2 = s_4 = s_6 = 0$, then from the above quadratic relations on s_i, we have $s_7 = s_8 = 0$ and this contradicts the assumption $(s_i) \notin \Sigma_3$. Hence, we have $s_6 \neq 0$, and in this case, from the quadratic relations, we have $s_i = 0$ except $i = 5, 6$. Then by putting

$$a_1 = s_5/s_6, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = s_6,$$

the same result holds as above. Combining these results, it follows that every element of $(\Sigma_4 \setminus \Sigma_3) \cup \Sigma_0$ is actually a curvature. q.e.d.

REMARK. (1) It is easy to see that the connection (a_i) is uniquely determined from the curvature (s_i) if and only if (s_i) is contained in the set $\Sigma_4 \setminus \Sigma_3$.

(2) By direct calculations, we can show that the quadratic polynomial relations of the curvature tensors (s_i) are exhausted by the five equations that are appeared in the definition of Σ_4. In addition, there exists just two cubic polynomial relations:

$$s_1s_3s_7 - s_1s_5s_8 - s_3^2s_8 - s_3s_5s_7 = 0,$$
$$s_1s_4s_7 - s_2s_5s_8 - s_3s_4s_8 - s_4s_5s_7 = 0,$$

that are not contained in the ideal generated by the above five quadratic polynomials. (To verify these facts, we used the algebraic programming system REDUCE 3.3.) But, we do not know at present whether the defining ideal of Σ_4 is generated by these seven polynomials. (From the general theory in algebra, this ideal is finitely generated.)

(3) If we rewrite the quadratic polynomial relations of $\{s_i\}$ in terms of $\{r_i\}$, then the non-zero constant terms appear in these expressions. For example, the relation $s_1s_4 - s_2s_3 = 0$ becomes $r_1r_4 - r_2r_3 + r_1 - r_4 - 1 = 0$. (This relation implies that for each i, the vectors $R(X_i, X_j)X_j - X_i$, $R(X_i, X_j)Y_j - Y_i$ $(j \neq i)$ are all parallel for any invariant connections.) These non-zero constant terms imply that $U(n+1)/U(n)$ does not admit torsion free flat invariant affine connections in the case $n \geq 2$ because the point $(r_i) = (0)$ which corresponds to the zero curvature does not satisfy the above quadratic relation. Of course, this result follows immediately from the fact that the fundamental group of a compact flat affine manifold is infinite. (See [3; p.145].) But, our result is stronger in the sense that even the closure of the set of curvatures does not contain the point $(r_i) = (0)$. (See also Theorem 5 and its remark in § 3.)

(4) In the case $n = 1$, the closure of the set of curvatures also constitutes a 4-dimensional variety because the rank of the differential of the map

$$(a_1, \cdots\cdots, a_4) \longrightarrow (r_1 - 3r_4,\ \ 3r_2 + r_3,\ \ r_5,\ \ r_6,\ \ r_7,\ \ r_8)$$

is 4 at generic points. But, unfortunately, we do not have the corresponding characterization of the curvatures as in Theorem 4. In this case, by using the system REDUCE 3.3, we can verify that there exist no polynomial relations on the curvature $\{r_1 - 3r_4,\ 3r_2 + r_3,\ r_5,\ r_6,\ r_7,\ r_8\}$ up to degree 4, and that there exist eight quintic polynomial relations, which are too long to write down all of them explicitly here. We exhibit one of (perhaps) the simplest relation among them as follows:

$$\begin{aligned}&(r_5^2 + r_6^2)\{u\,(u - r_6)^2 + 4\,(t-4)\,r_8 + 4\,r_6 r_7\}\\ &+ (r_5 r_8 - r_6 r_7)\{3\,(t-4)^2 + 5\,u^2 - 2\,ur_6 + r_5^2 + r_6^2 - 24\,r_7\}\\ &- 2\,(r_5 r_7 + r_6 r_8)\{5\,(t-4)\,u + 3\,ur_5 - 4\,r_8\} - 24\,u\,(r_7^2 + r_8^2)\,,\end{aligned}$$

where $t = r_1 - 3r_4$, $u = 3r_2 + r_3$. We do not know whether these eight polynomial relations are sufficient to characterize the curvature tensors in the case of $n = 1$.

3. Norm of the curvature.

In this final section, we calculate the norm of the curvature of invariant connections on $U(n+1)/U(n)$ and its infimum in the set of all invariant connections, where the norm $\| \ \|$ is determined by the standard constant curvature metric. Explicitly, $\| R \|^2$ is equal to

$$1/2 \cdot \sum_{ijkl} (R^l_{kij})^2,$$

where $R(e_i, e_j)\, e_k = \sum_l R^l_{kij}\, e_l$, and $\{e_i\}$ is an orthonormal basis of the tangent space. Clearly, the value $\inf \| R \|$ serves as an obstruction to the existence of torsion free "flat" invariant affine connections, and hence, it is an important and interesting problem to determine the value $\inf \| R \|$ explicitly for many homogeneous Riemannian manifolds. Precise statements in our case are summarized in the following theorem.

THEOREM 5. (1) *The norm of the curvature of the torsion free invariant affine connection on* $U(n+1)/U(n)$ *stated in Theorem* 1 *is given by*

$$\begin{aligned}\| R \|^2 = {} & 4n\,(n-1)\,(r_1^2 + r_3^2 + 3r_2^2 + 3r_4^2)\\ & + 2n\,\{r_5^2 + 3r_6^2 + r_7^2 + r_8^2 + (r_1 - 3r_4)^2 + (3r_2 + r_3)^2\}\,,\end{aligned}$$

where $r_1 \sim r_8$ *are the functions of* $a_1 \sim a_4$ *defined in Theorem* 2.

(2) *In the case* $n = 1$, *the infimum of* $\| R \|$ *is zero, while in the case* $n \geq 2$, *the value*

inf $\| R \|$ *is strictly positive.*

PROOF. The statement (1) is directly checked by using the expressions in Corollary 3 and the fact that $\{X_i, Y_i, W\}$ is orthonormal. We now give the proof of (2). In the case $n = 1$, we consider the connection corresponding to

$$a_1 = t, \quad a_2 = -4/t, \quad a_3 = -1, \quad a_4 = 0,$$

where t is a non-zero real parameter. Then, we have

$$r_1 = -3, \quad r_2 = -4/t, \quad r_3 = 0, \quad r_4 = -1, \quad r_5 = r_6 = r_7 = r_8 = 0,$$

and $\| R \|^2 = 288/t^2$. Hence, by putting $t \longrightarrow \infty$, we have inf $\| R \| = 0$. Next, consider the case $n \geq 2$. In terms of the variables $\{s_i\}$, the value $\| R \|^2$ is expressed as

$$\begin{aligned} &4n(n-1)\{(ks_2+1)^2 + k^2 s_4^2 + 3s_2^2 + 3(s_4-1)^2\} \\ &+ 2n\{(ks_6+s_4)^2 + 3(ks_4-s_6)^2 + (ks_2-3s_4+4)^2 \\ &+ (3s_2+ks_4)^2 + (s_2^2+s_4^2)(s_4^2+s_6^2)\}, \end{aligned}$$

where we put $k = a_1$. Hence, we have the inequality

$$\begin{aligned} \| R \|^2 &\geq 4n(n-1)\{(ks_2+1)^2 + k^2 s_4^2 + 3s_2^2 + 3(s_4-1)^2\} \\ &\quad + 2n\{(ks_6+s_4)^2 + 3(ks_4-s_6)^2 + (ks_2-3s_4+4)^2 + (3s_2+ks_4)^2\} \\ &= 2n\{(2n-1)k^2 + 3(2n+1)\}(s_2-\alpha)^2 \\ &\quad + 4n\{(n-1)(k^2+3) + 2k^2 + 3 + \frac{6}{k^2+3}\}(s_4-\beta)^2 \\ &\quad + 2n(k^2+3)(s_6-\gamma)^2 + \delta \\ &\geq \delta, \end{aligned}$$

where

$$\alpha = \frac{-2(n+1)k}{(2n-1)k^2 + 3(2n+1)}, \quad \beta = \frac{3(n+1)(k^2+3)}{(n+1)k^4 + 3(2n+1)k^2 + 3(3n+2)},$$

$$\gamma = \frac{2ks_4}{k^2+3},$$

$$\delta = \frac{12n(n+1)(k^2+1)\{2(n+1)(n-1)k^4 + 3(4n^2+n+1)k^2 + 3(3n-1)(2n+1)\}}{\{(2n-1)k^2 + 3(2n+1)\}\{(n+1)k^4 + 3(2n+1)k^2 + 3(3n+2)\}}.$$

By easy calculations, in the case $n \geq 2$, the function δ takes a minimum value $4n(n+1)$

$(3n-1)/(3n+2) = 4n^2 - 4n/(3n+2)$ at $k=0$, and hence, we have

$$\| R \|^2 \geq 4n^2 - \frac{4n}{3n+2}$$

for any invariant connections, and therefore, we have $\inf \| R \| > 0$.

q.e.d.

REMARK. (1) The canonical Riemannian connection corresponds to $a_1 = a_2 = a_3 = a_4 = 0$, as stated in § 1, and in this case, we have $\| R \|^2 = 2n(2n+1)$. Therefore, we have an estimate on the infimum

$$\sqrt{4n^2 - \frac{4n}{3n+2}} \leq \inf \| R \| \leq \sqrt{2n(2n+1)}$$

in the case $n \geq 2$. But, the actual value $\inf \| R \|$ is unknown at present. By calculating the Hessian of $\| R \|^2$ at $(a_i) = (0)$, we can easily show that the canonical Riemannian connection gives a local minimum of $\| R \|$ in the set of all torsion free invariant affine connections. We conjecture that this is actually the minimum, and the equality $\inf \| R \| = \sqrt{2n(2n+1)}$ holds in the case $n \geq 2$.

(2) For both cases $n=1$ and $n \geq 2$, we already know that the spheres S^{2n+1} do not admit flat affine structures. But, the above statement (2) in Theorem 5 implies the essential difference concerning the existence of flat homogeneous affine structures. Namely, in the case $n=1$, the above theorem implies that the 3-dimensional sphere S^3 is almost affinely flat in the sense of [1]. (This fact is already proved in [1].) But, it is not the case for S^{2n+1} $(n \geq 2)$ at least in the homogeneous category. In the general situation where we do not assume the invariance of connections, we do not know whether S^m $(m=2$ and $m \geq 4)$ admit such structures or not.

(3) As we stated at the end of § 1, in the special case $n=2$, the 5-dimensional sphere S^5 considered as the homogeneous space $SU(3)/SU(2)$ admits 2-dimensional additional freedom of torsion free invariant affine connections. Hence, the value $\inf \| R \|$ in this situation may be smaller than that of $U(3)/U(2)$. But, we can show that $\inf \| R \|$ is also strictly positive in this case, and we cannot decide whether S^5 is almost affinely flat or not.

In the case $n=1$, if we restrict ourselves to the situation where invariant connections are determined by some invariant Riemannian metrics (i.e., in the case of Riemannian connections), then the infimum of $\| R \|$ is strictly positive, in contrast to the above "affine" case. (We assume that the norm $\| \ \|$ is determined by the fixed standard metric throughout.) In fact, for general n, we have

PROPOSITION 6. *The set of invariant Riemannian connections on* $U(n+1)/U(n)$ $(n \geq 1)$ *determined by some invariant metrics is* 1*-dimensional, and it is expressed as*

$$\begin{aligned} f(X)Y &= (X, IY)\, W, \\ f(X)W &= kIX, \\ f(W)X &= (k-1)\, IX, \\ f(W)W &= 0, \end{aligned}$$

where k *is a positive parameter. The norm of the curvature of this connection takes a minimum value* $\sqrt{2n\,(2n+1)}$ *at* $k = 1$, *which corresponds to the canonical Riemannian connection.*

PROOF. We use the result in [9; p.201]. First, by an easy calculation, it follows that the invariant metric $B : \mathfrak{m} \times \mathfrak{m} \to R$ on $U(n+1)/U(n)$ is expressed as

$$\begin{aligned} B(X, Y) &= a\,(X, Y), \\ B(X, W) &= 0, \\ B(W, W) &= b, \end{aligned}$$

where $X, Y \in C^n$ and a, b are some positive constants (cf. [14; p.352]). Next, using the formulas

$$\begin{aligned} [X, Y]_{\mathfrak{m}} &= 2\,(X, IY)\, W, \\ [X, W]_{\mathfrak{m}} &= IX, \qquad\qquad X, Y \in C^n, \end{aligned}$$

it follows that the symmetric bi-linear map $U : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ satisfying the condition

$$2B\,(U(X, Y), Z) = B\,(X, [Z, Y]_{\mathfrak{m}}) + B\,([Z, X]_{\mathfrak{m}}, Y)$$

for any $X, Y, Z \in \mathfrak{m}$ is given by

$$\begin{aligned} U(X, Y) &= 0, \\ U(X, W) &= (k - 1/2)\, IX, \\ U(W, W) &= 0, \end{aligned}$$

where $X, Y \in C^n$ and $k = b/a > 0$. Then, by the formula stated in [9; p.201], its Riemannian connection is given by

$$f(X)\,Y = 1/2 \cdot [X, Y]_{\mathfrak{m}} + U(X, Y), \quad X, Y \in \mathfrak{m},$$

which shows the desired result. This connection corresponds to the case $a_1 = a_2 = a_4 =$

$0, a_3 = k-1$ in Theorem 1, and hence, we have

$$r_1 = 1, \quad r_4 = k-1, \quad r_5 = k, \quad r_7 = k^2,$$
$$r_2 = r_3 = r_6 = r_8 = 0.$$

In particular, by Theorem 5 (1), we have

$$\| R \|^2 = 4n(n-1)\{3(k-1)^2 + 1\} + 2n\{k^4 + k^2 + (3k-4)^2\},$$

and we can easily show that this function takes a minimum value $2n(2n+1)$ at $k=1$. In this case, we have $a = b$, which corresponds to the standard Riemannian metric on the sphere up to a positive constant. q.e.d.

The above proposition and Theorem 5 (2) in the case $n = 1$ indicate one of the essential difference between "affine" category and "Riemannian" category in characterizing the set of curvature tensors. (In contrast to the affine case, the curvatures in the above situation lie in the 3-dimensional linear subspace of the space of curvature-like tensors, defined by $r_2 = r_3 = r_6 = r_8 = 0$ and $r_1 + r_4 = r_5$ for any $n \geq 1$.)

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