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A table on the codimension of local isometric imbeddings of Riemannian symmetric spaces

Dedicated to Professor Tadashi Nagano on his 60th birthday

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Abstract. We summarize the conditions imposed on the curvature of Riemannian submanifolds of the Euclidean space $\boldsymbol{R}^N$, and as their applications, list up the best known estimates concerning the codimension of local isometric imbeddings of Riemannian symmetric spaces.

Key words. local isometric imbedding, Riemannian submanifold, Riemannian symmetric space, curvature

1. Introduction. For a given n-dimensional Riemannian manifold M, it is a natural and old question in differential geometry whether M can be realized as a Riemannian submanifold of a Euclidean space $\boldsymbol{R}^N$. Historically, Cartan [9] and Janet [15] first proved the existence of such local isometric imbeddings in $\boldsymbol{R}^{1/2\cdot n(n+1)}$ in the real analytic category. The global imbedding theorem was first proved by Nash [23] in the C^∞ category, and later the dimension of the ambient space $\boldsymbol{R}^N$ was improved little by little by Greene [11], Gromov-Rokhlin [13], Gromov [12] and Günther[14].

In the present note, apart from the above general theory, we restrict ourselves to the problem of local isometric imbeddings for a special class of manifolds "Riemannian symmetric spaces". Since symmetric spaces possess many elegant features, they can be realized globally in a comparatively low dimensional Euclidean spaces. (See Kobayashi [19].) But, for most spaces, it is not known yet whether these imbeddings give the least dimensional isometric imbeddings even in the local standpoint.

In order to determine the least dimension, we must as another approach to this problem, find a method to prove the non-existence of local isometric imbeddings into low dimensional Euclidean spaces. Concerning this problem, it is known that if M is realized as a Riemannian submanifold of $\boldsymbol{R}^N$, the curvature of M satisfies several pointwise algebraic conditions depending on the codimension of the imbedding, and these conditions serve as obstructions to the existence of local isometric imbeddings. In this note, we summarize these known conditions imposed on the curvature of Riemannian submanifolds of $\boldsymbol{R}^N$, and as their applications, list up the best known estimates concerning the codimension of local isometric imbeddings of symmetric spaces, which is summarized in Table II. But even combining all known results stated in this note, we do not yet obtain the best result for most symmetric spaces. And in order to complete the problem, we must find new additional conditions on the curvature of M which is useful in proving the non-existence of high codimensional imbeddings, or must find a new lower dimensional local realization of symmetric spaces. These two problems are both in general hard to solve, but we hope that in the near future we can make some progress for both directions and reach the final results.

Throughout this note, we assume the differentiability of class C^∞, unless otherwise stated. The author expresses his hearty thanks to professor Kaneda for giving valuable comments on the first version of this note.

2. We state several conditions imposed on the curvature of Riemannian submanifolds of the Euclidean space with small codimension, and their applications to the case of symmetric spaces. First, as a classical result, we consider the case of the spaces of constant curvature. The least dimensions for $\boldsymbol{R}^n$ (flat case) and S^n (positive case) are clear, and as for the remaining nontrivial case, we have

THEOREM 1. *Let H^n be the space of constant negative curvature of dimension n. Then H^n can be locally isometrically imbedded into $\boldsymbol{R}^{2n-1}$, but cannot be in $\boldsymbol{R}^{2n-2}$ even locally.*

The latter statement follows, for example, from the following theorem of Ôtsuki.

THEOREM 2 (cf. Ôtsuki[24]). *If M is of negative curvature, M cannot be isometrically immersed into $\boldsymbol{R}^{2n-2}$.*

Local realization of H^n in $\boldsymbol{R}^{2n-1}$ is well known, and can be found in several references such as Aminov [7], [8], Kaneda [16], Tachibana [26], etc.

For another symmetric spaces, Kobayashi [19] constructed many comparatively low dimensional global isometric imbeddings, using the fundamental property of symmetric R-spaces. For details, see [19].

3. Now, for general symmetric spaces, we first summarize the results in Agaoka-Kaneda [5]. For this purpose, we prepare several notations. Let $R(X,Y): T_xM \to T_xM$ (X, Y $\in T_xM$) be the curvature transformation of M at x, and we define a $\boldsymbol{Z}$-valued function $c(x)$ on M by

$$c(x) = 1/2 \cdot \max_{X,Y \in T_xM} \operatorname{rank} R(X,Y).$$

Note that rank $R(X,Y)$ is an even integer because $R(X,Y)$ is a skew symmetric endomorphism of T_xM. Then, we have

THEOREM 3 (Agaoka-Kaneda [5]). *If M is isometrically immersed into $\boldsymbol{R}^{n+r}$, then for each $x \in M$ the inequality $c(x) \leqq r$ holds. In particular, any open Riemannian submanifold of M containing x cannot be isometrically immersed into $\boldsymbol{R}^{n+c(x)-1}$.*

In the case M is a symmetric space, the function $c(x)$ takes a constant value, and we denote it by $c(M)$. Note that $c(M)$ is determined by the infinitesimal character of M. The integer $c(M)$ have the following fundamental properties.

PROPOSITION 4 (cf. [5; p.112]). (1) *Let* $M = M_1 \times \cdots \times M_k$ *be a product of Riemannian symmetric spaces. Then* $c(M) = \sum_{i=1}^{k} c(M_i)$.

(2) *Let* M *be a Riemannian symmetric space of compact type and let* M^* *be its non-compact dual space. Then* $c(M^*) = c(M)$.

Since every Riemannian symmetric space is locally a product of $\boldsymbol{R}^k$ and irreducible one, we can determine the value $c(M)$ for all symmetric spaces by combining Proposition 4 and the following theorem. (Note that $c(\boldsymbol{R}^k) = 0$.)

THEOREM 5 (cf. [5; p.112]). *Let* $M = G/K$ *be a simply connected irreducible Riemannian symmetric space of compact type. If* M *is not isomorphic to any real Grassmann manifold, then*

$$c(M) = 1/2 \cdot (\dim M - \operatorname{rank} G + \operatorname{rank} K).$$

For real Grassmann manifolds $SO(p+q)/SO(p) \times SO(q)$ $(p \geqq q \geqq 1)$,

$$c(M) = \begin{cases} [pq/2] & \text{if } q = \text{even or } 2q \geqq p \geqq q,\ q = \text{odd}, \\ p(q-1)/2 + q & \text{if } p > 2q \text{ and } q = \text{odd}, \end{cases}$$

where [ ] *is the Gauss symbol.*

In particular, by this theorem, it follows that most of the irreducible Riemannian symmetric spaces M cannot be isometrically immersed into the Euclidean space of dimension $\sim 3/2 \cdot \dim M$ even locally. We remark that many compact symmetric spaces can be globally isometrically imbedded into the Euclidean space of dimension $\sim 2 \cdot \dim M$ (Kobayashi [19]), and there is some gap between these two dimensions.

4. As a next condition, we state the results in Agaoka-Kaneda [6], by which we can in general obtain better estimates on the codimension. First, we fix a tangent vector $X \in T_xM$, and denote the complexification of T_xM by $(T_xM)^c$. We define two sets $\mathcal{N}(X)$ and $\mathcal{N}^c(X)$ consisting of linear subspaces of T_xM and $(T_xM)^c$ by

$$\mathcal{N}(X) = \{ W \subset T_xM \mid R(Y,Z)X = 0, \text{ for all } Y, Z \in W \},$$
$$\mathcal{N}^c(X) = \{ W \subset (T_xM)^c \mid R^c(Y,Z)X = 0, \text{ for all } Y, Z \in W \},$$

where $R^c : (T_xM)^c \times (T_xM)^c \times (T_xM)^c \to (T_xM)^c$ is the complexification of the curvature tensor R. Next, we put

$$d(X) = \max_{W \in \mathcal{N}(X)} \dim_{\boldsymbol{R}} W \quad \text{and} \quad d^c(X) = \max_{W \in \mathcal{N}^c(X)} \dim_{\boldsymbol{C}} W.$$

Under these notations, we define $\boldsymbol{Z}$-valued functions $p(x)$ and $p^c(x)$ on M by

$$p(x) = \min_{X \in T_xM} d(X) \quad \text{and} \quad p^c(x) = \min_{X \in T_xM} d^c(X).$$

Since there is a canonical inclusion $\mathcal{N}(X) \subset \mathcal{N}^c(X)$, the inequality $p(x) \leqq p^c(x)$ holds for $x \in M$. In this situation, we have the following theorem.

THEOREM 6 (Agaoka-Kaneda [6]). *If M is isometrically immersed into $\boldsymbol{R}^{n+r}$, then the inequality $p(x) \geqq n-r$ holds. In particular, any open Riemannian submanifold of M containing x cannot be isometrically immersed into $\boldsymbol{R}^{2n-p(x)-1}$.*

Since $p(x) \leqq p^c(x)$, the same statement holds if we replace $p(x)$ by $p^c(x)$ in this theorem. Of course, on account of this inequality, the estimates on the codimension obtained by $p(M)$ is better than the one obtained by $p^c(M)$. As in the previous case of $c(x)$, the functions $p(x)$ and $p^c(x)$ take constant values if M is a Riemannian symmetric space, and we denote them by $p(M)$ and $p^c(M)$, respectively. These constants satisfy completely the same properties as $c(M)$ stated in Proposition 4. But, we remark that for flat spaces, we have $p(\boldsymbol{R}^n) = p^c(\boldsymbol{R}^n) = n$ in this case.

The actual values for $p(M)$ and $p^c(M)$ are both in general hard to determine, and at the present time, we know only the partial results, which we state in the following. (It seems that these two values are quite different if the rank of M is sufficiently large.) First, for a special class of symmetric spaces, we have the following results.

THEOREM 7 (Agaoka-Kaneda [6]). *Let M be one of the following Riemannian symmetric space of compact type:*

AI	$SU(m)/SO(m)$,	**BI**	$SO(2m+1)/SO(m+1) \times SO(m)$,
CI	$Sp(m)/U(m)$,	**DI**	$SO(2m)/SO(m) \times SO(m)$,
EI	$E_6/Sp(4)$,	**EV**	$E_7/SU(8)$,
EVIII	$E_8/Spin(16)$,	**FI**	$F_4/Sp(3) \cdot SU(2)$,
G	$G_2/SO(4)$.		

Then, the equality $p(M) = \mathrm{rank}\, M$ holds.

Note that among the irreducible symmetric spaces $M = G/K$ of compact type, the above spaces can be characterized by the property "rank M = rank G". Or in terms of the Satake diagram, they correspond to irreducible diagrams without any black circles nor any arrows.

By this theorem, these spaces cannot be isometrically immersed into $\boldsymbol{R}^{2n-\mathrm{rank}\, M-1}$ even locally. In particular, among them, it follows that the canonical imbedding of **CI** $Sp(m)/U(m)$ into $\boldsymbol{R}^{m(2m+1)}$

constructed in Kobayashi [19] gives the least dimensional isometric imbedding even in the local standpoint.

For general symmetric spaces, we do not obtain such a simple formula at present. We summarize the remaining known results on the value $p(M)$ in the following table.

Table I

M	dim M	rank M	$p(M)$	
$SO(p+q)/SO(p)\times SO(q)$	pq	q	$\begin{cases} p \\ p-1 \end{cases}$	$(p=q)$ $(p\geqq q+1)$
$P^m(\boldsymbol{C})$	$2m$	1	$\begin{cases} 2 \\ m-1 \end{cases}$	$(m=2)$ $(m\geqq 3)$
$SU(4)/S(U(2)\times U(2))$	8	2	3	
$SU(2)\simeq SO(3)\simeq Sp(1)$	3	1	2	
$SU(3)$	8	2	3	
$SU(4)\simeq SO(6)$	15	3	5	
$SU(5)$	24	4	6	
$Sp(2)\simeq SO(5)$	10	2	4	
$Sp(3)$	21	3	6	
$SO(7)$	21	3	6	
$SO(8)$	28	4	8	
$SO(9)$	36	4	8	
G_2	14	2	4	

(The symbol "$\simeq$" in this table means a local isomorphism of Lie groups.)

In particular, since the space $Sp(m)$ is globally isometrically imbedded into $\boldsymbol{R}^{4m^2}$ (Kobayashi [19]), it follows from the above table that for the spaces $Sp(1)$, $Sp(2)$ and $Sp(3)$ these canonical imbeddings give the least dimensional local isometric imbeddings.

As for the integer $p^c(M)$, we determined its value completely for each compact simple Lie groups.

THEOREM 8 (Agaoka-Kaneda [6]). *The values* $p^c(G)$ *for compact simple Lie groups* G *are given in the following tables*:

G \ m		1	2	3	4		
A_{m-1}	$SU(m)$	0	2	3	5	$[m^2/4]$	$(m\geqq 5)$
B_m	$SO(2m+1)$	2	4	6	8	$1/2\cdot m(m-1)+1$	$(m\geqq 5)$
C_m	$Sp(m)$	2	4			$1/2\cdot m(m+1)$	$(m\geqq 3)$
D_m	$SO(2m)$	1	4	5	8	$1/2\cdot m(m-1)$	$(m\geqq 5)$

G	$p^c(G)$
E_6	16
E_7	27
E_8	36

G	$p^c(G)$
F_4	9
G_2	4

For each group G, the order of $p^c(G)$ is about $1/4\cdot\dim G$, and hence G cannot be locally isometrically immersed into the Euclidean space with dimension about $7/4\cdot\dim G$, which improves the previous results obtained by Theorem 5.

5. In the low codimensional case, there are detailed results obtained by Thomas [27], Vilms [28], Matsumoto [21], [22], Weinstein [29], Agaoka [2], etc. Among them, the conditions on the curvature in Vilms [28; p.198] (the case of hypersurfaces) and Matsumoto [22; p.185] are essentially equivalent to the one stated in Theorem 3, though they have completely different appearances. As another special condition, we state the results in Agaoka [2] in the following.

Let M be a 4-dimensional Riemannian manifold, and for $x \in M$, we fix an orientation of T_xM. Using the Riemannian metric, we may consider the curvature as the endomorphism $R: \wedge^2 T_xM \to \wedge^2 T_xM$ in a natural way. Then we have

THEOREM 9 (cf. [2; p.127]). *If a 4-dimensional Riemannian manifold M is isometrically immersed into $\boldsymbol{R}^6$, then for each $x \in M$, the equalities $\mathrm{Tr}\,(R\circ *)^3 = \mathrm{Tr}\,(R\circ *)^5 = 0$ hold, where $*: \wedge^2 T_xM \to \wedge^2 T_xM$ is the Hodge star operator.*

Note that if we change the orientation of T_xM, then the values $\mathrm{Tr}\,(R\circ *)^3$ and $\mathrm{Tr}\,(R\circ *)^5$ change only their signs. Using this theorem, we can easily show that the complex projective space $P^2(\boldsymbol{C})$ (with real dimension 4) cannot be locally isometrically immersed into $\boldsymbol{R}^6$. We can also prove this result by using Weinstein's condition in [29]. For details, see [2; p.130]. Concerning the space $P^2(\boldsymbol{C})$, it is known that $P^2(\boldsymbol{C})$ is globally isometrically imbedded into $\boldsymbol{R}^8$ ([19]), and has a solution of the Gauss equation in codimension 3 ([2; p.132]). But, at present, we do not know the least dimensional local imbeddings though there exist some attempts by Kaneda [17].

6. Finally, we summarize several fragmentary results on local isometric imbeddings of symmetric spaces. First, as for the general complex projective space $P^m(\boldsymbol{C})$ and the complex quadric $Q^m(\boldsymbol{C}) = SO(m+2)/SO(m)\times SO(2)$, we have

THEOREM 10 (cf. [3]). *If $P^m(\boldsymbol{C})$ (resp. $Q^m(\boldsymbol{C})$) is locally isometrically immersed into $\boldsymbol{R}^{2m+r}$, then the inequality $r \geqq 1/5\cdot(6m-4)$ (resp. $r \geqq 1/5\cdot(6m-2)$) holds.*

The proof for $Q^m(\boldsymbol{C})$ is not stated in [3], but can be proved in completely the same way as $P^m(\boldsymbol{C})$.

The above result for $P^m(\boldsymbol{C})$ is better than that of Theorem 3 in the case of $m \geqq 5$. For the space $P^3(\boldsymbol{C})$, applying the value $p(P^3(\boldsymbol{C}))$ in Table I to Theorem 6, it follows that $P^3(\boldsymbol{C})$ cannot be locally isometrically immersed into $\boldsymbol{R}^9$. (See also [4; p.19].)

It is proved in [1] that the symmetric space $SO(5,\boldsymbol{C})/SO(5)$ (the non-compact dual of the rotation group $SO(5)$) cannot be locally isometrically immersed into $\boldsymbol{R}^{16}$. In addition, in the same paper, it is proved that the least dimensional local isometric imbedding of $SO(5)$ into $\boldsymbol{R}^{16}$ is uniquely determined up to Euclidean transformations of $\boldsymbol{R}^{16}$.

As for the general references on the problem of isometric imbeddings, see Poznyak-Sokolov [25], where the brief historical survey is also summarized. For general existence theorem of local isometric imbeddings, see Cartan [9], Janet [15], Gasqui [10], and Kaneda-Tanaka [18] for analytic case, and see Greene [11] for C^∞ case, where the existence of local imbeddings into $\boldsymbol{R}^{1/2\cdot n(n+1)+n}$ is proved.

7. In the following, we list up the best known results on local isometric imbeddings of Riemannian symmetric spaces stated in this note.

Table II

	M	dim M	$M \subset \boldsymbol{R}^N$	$M \not\subset \boldsymbol{R}^N$	
$[A_{m-1}]$	$SU(m)$	m^2-1	$2m^2$	$2m^2-[m^2/4]-3$	$(m \geqq 5)$
$[B_m]$	$SO(2m+1)$	$m(2m+1)$	$(2m+1)^2$	$1/2\cdot(7m^2+5m-4)$	$(m \geqq 5)$
$[C_m]$	$Sp(m)$	$m(2m+1)$	$4m^2$	$1/2\cdot(7m^2+3m-2)$	$(m \geqq 3)$
$[D_m]$	$SO(2m)$	$m(2m-1)$	$4m^2$	$1/2\cdot(7m^2-3m-2)$	$(m \geqq 5)$
	$SU(3)$	8	18	12	
	$SU(4) \simeq SO(6)$	15	32	24	
	$SU(5)$	24	50	41	
	$Sp(2) \simeq SO(5)$ *)	10	16	15	
	$SO(5,\boldsymbol{C})/SO(5)$	10	?	16	
	$Sp(3)$ *)	21	36	35	
	$SO(7)$	21	49	35	
	$SO(8)$	28	64	47	
	$SO(9)$	36	81	63	
	E_6	78	?	139	
	E_7	133	?	238	
	E_8	248	?	459	
	F_4	52	?	94	
	G_2	14	?	23	
A I	$SU(m)/SO(m)$	$1/2\cdot(m-1)(m+2)$	$m(m+1)$	m^2-2	
A II	$SU(2m)/Sp(m)$	$(m-1)(2m+1)$	$2m(2m-1)$	$3m^2-2m-2$	
A III	$SU(p+q)/S(U(p)\times U(q))$	$2pq$	$(p+q)^2$	$3pq-1$	$(p \geqq q)$

Table II (continued)

	M	$\dim M$	$M \subset \boldsymbol{R}^N$	$M \not\subset \boldsymbol{R}^N$	
BD I	$SO(p+q)/SO(p)\times SO(q)$	pq	$1/2\cdot(p+q)(p+q+1)$	$2p^2-p-1$	$(p=q)$
				$2pq-p$	$(p\geqq q+1)$
B I	$SO(2m+1)/SO(m+1)\times SO(m)$	$m(m+1)$	$(m+1)(2m+1)$	$2m^2+m-1$	
C I	$Sp(m)/U(m)$ *)	$m(m+1)$	$m(2m+1)$	$2m^2+m-1$	
C II	$Sp(p+q)/Sp(p)\times Sp(q)$	$4pq$	$2(p+q)^2-(p+q)$	$6pq-1$	$(p\geqq q)$
D I	$SO(2m)/SO(m)\times SO(m)$	m^2	$m(2m+1)$	$2m^2-m-1$	
D III	$SO(2m)/U(m)$	$m(m-1)$	$m(2m-1)$	$3/2\cdot m(m-1)-1$	
BD II	S^n *)	n	$n+1$	n	
	H^n *)	n	$2n-1$	$2n-2$	
A III	$P^m(\boldsymbol{C})$	$2m$	$m(m+2)$	about $16m/5$	
	$P^2(\boldsymbol{C})$	4	8	6	
	$P^3(\boldsymbol{C})$	6	15	9	
	$Q^m(\boldsymbol{C})$	$2m$	$1/2\cdot(m+2)(m+3)$	about $16m/5$	
	$SU(4)/S(U(2)\times U(2))$	8	16	12	
E I	$E_6/Sp(4)$	42	?	77	
E II	$E_6/SU(2)\cdot SU(6)$	40	?	59	
E III	$E_6/Spin(10)\cdot SO(2)$	32	78	47	
E IV	E_6/F_4	26	54	37	
E V	$E_7/SU(8)$	70	?	132	
E VI	$E_7/Spin(12)\cdot SU(2)$	64	?	95	
E VII	$E_7/E_6\cdot SO(2)$	54	133	80	
E VIII	$E_8/Spin(16)$	128	?	247	
E IX	$E_8/E_7\cdot SU(2)$	112	?	167	
F I	$F_4/Sp(3)\cdot SU(2)$	28	?	51	
F II	$F_4/Spin(9)$	16	26	23	
G	$G_2/SO(4)$	8	?	13	

The symbol " $\simeq$ " indicates a local isomorphism of Lie groups. For the space with an asterisk *), the least dimension is completely determined. For example, from this table we know that $SO(5)$ can be locally isometrically imbedded into $\boldsymbol{R}^{16}$, but cannot be in $\boldsymbol{R}^{15}$, and hence the best result is determined for $SO(5)\simeq Sp(2)$.

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