COMPLICATED GENERALIZED TORSION ELEMENTS IN SEIFERT FIBERED SPACES WITH BOUNDARY

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Abstract. In a group, a non-trivial element is called a generalized torsion element if some non-empty finite product of its conjugates is equal to the identity. There are various examples of torsion-free groups which contain generalized torsion elements. We can define the order of a generalized torsion element as the minimum number of its conjugates required to generate the identity. In previous works, 3–manifold groups which contain a generalized torsion element of order two are determined. However, there are few previous studies that examine the order of a generalized torsion element bigger than two. In this paper, we focus on Seifert fibered spaces with boundary, including the torus knot exteriors, and construct concretely generalized torsion elements of order 3, 4, 6 and others in their fundamental groups.

1. Introduction

In a group G, a non-trivial element g is called a *generalized torsion element* if

$$g^{a_1}g^{a_2}\cdots g^{a_k}=1 \text{ for some } a_1,a_2,\dots,a_k\in G$$

where $g^{a_i}=a_i^{-1}ga_i$. For a generalized torsion element g, we define the *order* of g as the minimum number of its conjugates required to generate the identity, that is,

$$\min\{k \mid g^{a_1}g^{a_2}\cdots g^{a_k}=1 \text{ for some } a_1,a_2,\dots,a_k\in G\}.$$

Since $g\neq 1$, the order is at least two. If g is a generalized torsion element of order k, then so is any conjugate of g.

In this paper, we focus on the fundamental group of a compact orientable Seifert fibered space with boundary. For a compact orientable Seifert fibered space M, we denote the base orbifold of M as $B(\pm g,d;p_1,\dots,p_m)$, which has d (≥ 1) boundary components and m cone points of indices $p_1,p_2,\dots,p_m$, and $+g$ means that the underlying base surface is a genus g orientable surface, $-g$ means that one is a non-orientable surface with g cross-caps (see [7, 11], for examples). Note that if $d\geq 1$, then $\pi_1(M)$ is infinite and M is irreducible, and hence $\pi_1(M)$ is torsion-free ([1]).

There are many examples of torsion-free groups which have a generalized torsion element. Naylor and Rolfsen have given a generalized torsion element in torus knot groups [14]. Its construction is simple, and we now review it. The (p,q)–torus link exterior $E(T_{p,q})$ $(p,q\geq 2)$ is a Seifert fibered space over $B(+0,\gcd(p,q);p,q)$. When $\gcd(p,q)=1$, it is well known that $\pi_1(E(T_{p,q}))$ has a presentation $\langle a,b\mid a^p=b^q\rangle$

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and this group is torsion-free and non-abelian. Let $g = [a, b] := a^{-1}b^{-1}ab$. Then $[a, b^q] = 1$, since $b^q = a^p$. A straightforward calculation gives

$$[a, b^q] = [a, b][a, b]^b[a, b]^{b^2} \cdots [a, b]^{b^{q-1}}.$$

Since $g \neq 1$, g is a generalized torsion element. (Note that [14] does not mention the order of the generalized torsion element. This argument shows that the order of g is at most q. In fact, we can prove that it is equal to q if $p > q$. See Theorem 7.2.) Subsequently, Motegi and Teragaito extended this result to the fundamental groups of a Seifert fibered space using a similar construction method [12].

There are very few studies on the order of a generalized torsion element. In [10], the order of a generalized torsion element is introduced and they give an lower bound for the order in terms of stable commutator length (see Theorem 2.3). Generalized torsion elements of order two in 3–manifold groups are studied in [8, 9], where the determination of such 3–manifold groups is given. In particular, it implies that the fundamental group of a Seifert fibered space M over $B(\pm g, d; p_1, \dots, p_m)$, which is not the solid torus, contains a generalized torsion element of order two if and only if the underlying base surface is non-orientable or at least one of $p_1, \dots, p_m$ is even.

We now give an observation for abelian 3–manifold groups. Let M be a compact orientable 3–manifold whose $\pi_1(M)$ is abelian. Then M is one of the following (see [1]);

(1) $S^2 \times S^1$, $S^1 \times S^1 \times [0, 1]$, or $S^1 \times S^1 \times S^1$,
(2) the solid torus,
(3) the lens spaces $L(n, m)$ $(n \geq 2)$.

Note that M admits a Seifert fibration in any case. For (1) and (2), $\pi_1(M)$ contains no generalized torsion element, since it is free-abelian. For (3), it is well known that $L(n, m)$ admits infinitely many Seifert fibrations (see [11]). However, $\pi_1(L(n, m)) = \mathbb{Z}_n$ can contain only a (generalized) torsion element whose order is a divisor of n.

Next, we briefly summarize the situation without exceptional fiber. Let M be a Seifert fibered space which have no exceptional fibers and assume that $\partial M \neq \emptyset$. If the base surface B is orientable, then $\pi_1(M)$ cannot contain a generalized torsion element, because $\pi_1(M)$ is bi-orderable (see [2, 12]). Suppose that B is non-orientable. Let $g \in \pi_1(M)$ be a generalized torsion element satisfying $g^{a_1} \cdots g^{a_k} = 1$ and let μ be the projection $\pi_1(M) \to \pi_1(M)/\langle h \rangle \cong F$ where h is represented by a regular fiber and F is a free group with positive rank. Then, we have

$$\mu(g)^{\mu(a_1)} \cdots \mu(g)^{\mu(a_k)} = 1 \text{ in } F.$$

Since a free group admits no generalized torsion element, $\mu(g) = 1$. This implies that g is a power of h. Set $g = h^i$. By the presentation (N) of $\pi_1(M)$ in Section 2.1, we have $gg^{s_1} = h^i(h^i)^{s_1} = 1$, so g is a generalized torsion element of order two. Therefore, we focus on the case $m \geq 1$ in the following.

In this paper, we present infinitely many mutually non-conjugate generalized torsion elements in the fundamental group of a Seifert fibered space with boundary. These elements are more complicated than ones suggested in [12, 14]. Let M be a Seifert fibered space whose base orbifold is $B(\pm g, d; p_1, \dots, p_m)$.

Theorem 1.1. *Assume that $d \geq 1$ and $m \geq 1$. If at least one of $p_1, \dots, p_m$ is a multiple of three and M is not the solid torus, then $\pi_1(M)$ contains infinitely many mutually non-conjugate generalized torsion elements of order three.*

The solid torus admits a Seifert fibration over $B(+0,1;p_1)$, but its fundamental group admits no generalized torsion element as mentioned above.

Theorem 1.2. *Assume that $d \geq 1$ and $m \geq 2$. Let k be an odd integer, and let q be a multiple of prime number r satisfying $3 \leq r \leq 37$, or $r = 4$. If $\{2k, q\} \subset \{p_1, \ldots, p_m\}$, possibly as multi-sets, then $\pi_1(M)$ contains generalized torsion elements of order four and six.*

Remark 1.3. In Theorem 1.2, we believe that the conditions of k and q are redundant. However, we need to impose these at present, because we rely on computer program Scallop [16] for some part of the argument.

Table 1 is the list of the resulting generalized torsion elements in $\pi_1(M)$ and conditions of M. Moreover, we can verify that the order of E_1, I_1 (resp. H_1, J_1) is 4 (resp. 6) under certain conditions of indices using by the computer program Scallop.

Case	Conditions	Elements	Order
A	M is not the solid torus, $d \geq 1$ and at least one of $p_1, \ldots, p_m$ is a multiple of three	D_n	3
B	$d \geq 1$, $m \geq 2$ and $\{2k, q(\geq 4)\} \subset \{p_1, \ldots, p_m\}$, possibly as multisets	E_n	4^*
		H_n	6^*
C	$d \geq 1$, $m \geq 2$ and $\{2k, 3l\} \subset \{p_1, \ldots, p_m\}$, possibly as multisets	I_n	4^*
		J_n	6^*

TABLE 1. The list of the resulting generalized torsion elements and conditions of M. The mark $*$ indicates the expected value.

As a special case, we have the following corollary for torus knot or link groups.

Corollary 1.4.
1. *The fundamental group of the (p, q)–torus link exterior contains a generalized torsion element of order three, whenever p or q is a multiple of three.*
2. *Let k be odd, and let q be a multiple of prime number r satisfying $3 \leq r \leq 37$, or $r = 4$. The fundamental group of the $(2k, q)$–torus link exterior contains generalized torsion elements of order four and six.*

Proof. As mentioned above, the (p, q)–torus link exterior $E(T_{p,q})$ has a Seifert fibration over $B(+0, \gcd(p, q); p, q)$. Therefore, the conclusion immediately follows from Theorems 1.1 and 1.2. □

From the above observation, we come to mind the following conjecture.

Conjecture 1.5. *Let $k \geq 3$ and M be Seifert fibered space whose base orbifold is $B(\pm g, d; p_1, \ldots, p_m)$. Assume that $\pi_1(M)$ is non-abelian. Then $\pi_1(M)$ contains a generalized torsion element of order k if and only if at least one of $p_1 \ldots, p_m$ is not coprime to k.*

Throughout the paper, we use the notation $[g, h] = g^{-1}h^{-1}gh$ and $g^a = a^{-1}ga$ for a commutator and a conjugate in a group. The notation $\mathbb{Z}_0$ means $\mathbb{Z}$.

2. Preliminaries

2.1. **Fundamental group of Seifert fibered space.** Let M be a Seifert fibered space over the base orbifold $B(\pm g, d; p_1, \dots, p_m)$. Let B denote the underlying base surface. Then, $\pi_1(M)$ has a presentation as follows (see [11]).

(O) When B is an orientable surface,

$$\begin{aligned}\pi_1(M) = \langle s_1, t_1, \dots, s_g, t_g, u_1, \dots, u_m, v_1, \dots, v_d, h \mid \\ s_i h s_i^{-1} = h,\ t_i h t_i^{-1} = h,\ u_i h u_i^{-1} = h,\ v_i h v_i^{-1} = h, \\ u_j^{p_j} = h^{q_j},\ h^b = [s_1, t_1] \cdots [s_g, t_g] u_1 \cdots u_m v_1 \cdots v_d \rangle.\end{aligned}$$

(N) When B is a non-orientable surface,

$$\begin{aligned}\pi_1(M) = \langle s_1, \dots, s_g, u_1, \dots, u_m, v_1, \dots, v_d, h \mid \\ s_i h s_i^{-1} = h^{-1},\ u_i h u_i^{-1} = h,\ v_i h v_i^{-1} = h, \\ u_j^{p_j} = h^{q_j},\ h^b = s_1^2 \cdots s_g^2 u_1 \cdots u_m v_1 \cdots v_d \rangle.\end{aligned}$$

In either case, $b \in \mathbb{Z}$, and $q_j \in \mathbb{Z}$ satisfying $0 < q_j < p_j$ and $\gcd(p_j, q_j) = 1$.

Note that $h \in \pi_1(M)$ is a representative of a regular fiber of M, and the relations show that $u_j^{p_j} (= h^{q_j})$ commutes with u_i for any i. Moreover, $\langle h \rangle$ is a cyclic normal subgroup of $\pi_1(M)$, and we can obtain a presentation of $\pi_1(M)/\langle h \rangle$ for (O) and (N) as

$$\begin{aligned}\pi_1(M)/\langle h \rangle = \langle s_1, t_1, \dots, s_g, t_g, u_1, \dots, u_m, v_1, \dots, v_d \mid \\ u_j^{p_j} = 1,\ 1 = [s_1, t_1] \cdots [s_g, t_g] u_1 \cdots u_m v_1 \cdots v_d \rangle, \\ \pi_1(M)/\langle h \rangle = \langle s_1, \dots, s_g, u_1, \dots, u_m, v_1, \dots, v_d \mid \\ u_j^{p_j} = 1,\ 1 = s_1^2 \cdots s_g^2 u_1 \cdots u_m v_1 \cdots v_d \rangle,\end{aligned}$$

respectively.

If $d \geq 1$, then we can delete the generator v_d by the last relation of $\pi_1(M)/\langle h \rangle$ in either case. Then, we have

$$\pi_1(M)/\langle h \rangle \cong F_w * \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_m}$$

where F_w is the free group of rank w, which is $2g+d-1$ if B is orientable, $g+d-1$ otherwise. And, $\mathbb{Z}_{p_j}$ is generated by u_j.

Moreover, for given two cyclic factors $\mathbb{Z}_{p_i}$ and $\mathbb{Z}_{p_j}$, there is the natural projection $F_w * \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_m} \to \mathbb{Z}_{p_i} * \mathbb{Z}_{p_j}$. Therefore, there is the projection $\pi_1(M) \to \mathbb{Z}_{p_i} * \mathbb{Z}_{p_j}$. Similarly, for an infinite cyclic subgroup of F_w and a cyclic factor $\mathbb{Z}_{p_i}$, we can consider the projection $\pi_1(M) \to \mathbb{Z} * \mathbb{Z}_{p_i}$. These maps will be used in the remaining sections.

2.2. **Cyclically reduced form in a free product of groups.** Let A and B be groups. Recall that any element $g \in A * B$ has a unique presentation $g = g_1 g_2 \cdots g_r$ where $g_i \neq 1$, $g_i \in A$ or $g_i \in B$, and g_i, g_{i+1} are not in the same free factor. Such a presentation is called the *reduced form* of g. Additionally, when g_1, g_r are in different free factors, we call such a presentation a *cyclically reduced form*, and r is called the *syllable length* denoted by $\lambda(g)$ (see [13, Section 4.1]). Note that the cyclically reduced form of the identity is the empty word, and its syllable length is 0.

Using cyclically reduced forms, we can determine whether two elements in a free product of groups are conjugate to each other.

Theorem 2.1 (Theorem 4.2 of [13])**.** *If two cyclically reduced elements in a free product of groups are conjugate to each other, then they are cyclic permutations of each other, and hence, their syllable lengths coincide.*

From the above theorem, we can define the syllable length of $g \in A * B$, which is not presented as a cyclically reduced form, as $\lambda(g')$, where g' is presented as a cyclically reduced form and conjugate to g.

2.3. **Stable commutator length.** Let G be a group. We prepare basic facts on stable commutator length.

For $g \in [G, G]$, the *commutator length* on G, denoted $\mathrm{cl}_G(g)$, is the minimum number of commutators in G whose product yields g, and the *stable commutator length* on G, denoted $\mathrm{scl}_G(g)$, is defined to be

$$\mathrm{scl}_G(g) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}. \tag{2.1}$$

This limit exists, because the sequence $\{\mathrm{cl}_G(g^n)\}$ is non-negative and subadditive. Hence this definition is well defined. See [3] for details.

Furthermore we can extend this definition (2.1) for an element $g \notin [G, G]$ as

$$\mathrm{scl}_G(g) = \begin{cases} \frac{\mathrm{scl}_G(g^k)}{k} & \text{if } g^k \in [G,G] \text{ for some } k \geq 2, \\ \infty & \text{otherwise.} \end{cases} \tag{2.2}$$

This is independent of the choice of $k \geq 2$ such that $g^k \in [G, G]$ in (2.2); see [10].

The following proposition is obvious, but it is very useful.

Proposition 2.2 (scl's monotonicity [3])**.** *Let $\phi\colon G \to H$ be a homomorphism. Then $\mathrm{scl}_G(a) \geq \mathrm{scl}_H(\phi(a))$ holds for $a \in G$.*

For example, scl of the identity, or any torsion element, is 0. In general, computing $\mathrm{scl}_G(g)$ is very difficult. However, we can compute the scl on free product of two cyclic groups $\mathbb{Z}_p * \mathbb{Z}_q$ by using Walker's computer program *Scallop* [15].

For a generalized torsion element g, there is an inequality given in [10].

Theorem 2.3 ([10, Theorem 2.4])**.** *If $g \in G$ is a generalized torsion element of order k, then*

$$\mathrm{scl}_G(g) \leq \frac{1}{2} - \frac{1}{k}.$$

3. Case A: order 3 generalized torsion element D_n

Let M be a Seifert fibered space over $B(\pm g, d; p_1, \ldots, p_m)$, and assume that $d \geq 1$ and $m \geq 1$. If at least one of $p_1, \ldots, p_m$ is a multiple of three, without loss of generality, then we can write $p_m = 3l$.

Assume that M is not the solid torus. Then the case $(g, d, m) = (0, 1, 1)$ does not occur. In the presentations (O) and (N) of $\pi_1(M)$ in Section 2.1, set

$$a = \begin{cases} u_1 & \text{if } m \geq 2, \\ s_1^2 & \text{if } g \neq 0 \text{ and } m = 1, \\ v_1 & \text{if } g = 0,\ m = 1 \text{ and } d \geq 2, \end{cases}$$

and $b = u_m$. Note that the element a commutes with h in $\pi_1(M)$ in any case.

Proposition 3.1. *In $\pi_1(M)$, let*
$$D_n = [a, b^l][a, b^{2l}]^n[a, b^l]^{n+1}[a, b^{2l}]^{-1} \quad (n \in \mathbb{Z}).$$
Then there exist three conjugates of D_n whose product is the identity.

Proof. By a direct calculation, we have
$$\begin{aligned}
[a, b^l]^{b^l} &= b^{-l} \cdot (a^{-1}b^{-l}ab^l) \cdot b^l \\
&= b^{-l}a^{-1}b^{-l}ab^{2l} \\
&= b^{-l}a^{-1} \cdot b^l aa^{-1}b^{-l} \cdot b^{-l}ab^{2l} \\
&= b^{-l}a^{-1}b^l a \cdot a^{-1}b^{-2l}ab^{2l} \\
&= [a, b^l]^{-1}[a, b^{2l}].
\end{aligned}$$

Since the element $b^{3l}(= h^{q_m})$ commutes with a in $\pi_1(M)$, we have
$$\begin{aligned}
[a, b^l]^{b^{2l}} &= b^{-2l} \cdot a^{-1}b^{-l}ab^l \cdot b^{2l} \\
&= b^{-2l}a^{-1}b^{-l}(ab^{3l}) \\
&= b^{-2l}a^{-1}b^{-l}(b^{3l}a) \\
&= b^{-2l}a^{-1}b^{2l}a \\
&= [a, b^{2l}]^{-1}.
\end{aligned}$$

By similar calculations, we obtain $[a, b^{2l}]^{b^l} = [a, b^l]^{-1}$ and $[a, b^{2l}]^{b^{2l}} = [a, b^{2l}]^{-1}[a, b^l]$.

Let $\alpha,\ \beta$ be $[a, b^l],\ [a, b^{2l}]$ respectively. Then $D_n = \alpha\beta^n\alpha^{n+1}\beta^{-1}$ and
$$\alpha^{b^l} = \alpha^{-1}\beta,\ \alpha^{b^{2l}} = \beta^{-1},\ \beta^{b^l} = \alpha^{-1},\ \beta^{b^{2l}} = \beta^{-1}\alpha. \tag{3.1}$$
By (3.1), we have
$$\begin{aligned}
D_n^{b^{2l}\beta^{-1}} &= (\alpha\beta^n\alpha^{n+1}\beta^{-1})^{b^{2l}\beta^{-1}} \\
&= \beta \cdot \beta^{-1}(\beta^{-1}\alpha)^n\beta^{-n-1}(\beta^{-1}\alpha)^{-1} \cdot \beta^{-1} \\
&= (\beta^{-1}\alpha)^n\beta^{-n-1}\alpha^{-1}, \\
D_n^{b^l\alpha^{-1}} &= (\alpha\beta^n\alpha^{n+1}\beta^{-1})^{b^l\alpha^{-1}} \\
&= \alpha \cdot \alpha^{-1}\beta\alpha^{-n}(\alpha^{-1}\beta)^{n+1}\alpha \cdot \alpha^{-1} \\
&= \beta\alpha^{-n}(\alpha^{-1}\beta)^{n+1}.
\end{aligned}$$
Therefore, we obtain
$$\begin{aligned}
D_n^{b^{2l}\beta^{-1}} D_n D_n^{b^l\alpha^{-1}} &= (\beta^{-1}\alpha)^n\beta^{-n-1}\alpha^{-1} \cdot \alpha\beta^n\alpha^{n+1}\beta^{-1} \cdot \beta\alpha^{-n}(\alpha^{-1}\beta)^{n+1} \\
&= (\beta^{-1}\alpha)^n\beta^{-1}\alpha(\alpha^{-1}\beta)^{n+1} \\
&= (\beta^{-1}\alpha)^{n+1}(\alpha^{-1}\beta)^{n+1} \\
&= 1.
\end{aligned}$$
□

In the following, set $p = p_1$ if $a = u_1$, otherwise $p = 0$.

Lemma 3.2. *$D_n \in \pi_1(M)$ $(n \in \mathbb{Z})$ satisfies the following.*

(1) $D_n \neq 1$;

(2) Suppose $n, n' \notin \{-2, -1, 0, 1\}$. D_n and $D_{n'}$ are not conjugate if $n \neq n'$;

(3) Suppose $n \neq -1, 0$. D_n is not conjugate to D_n^{-1}.

Proof. Let $\mu\colon \pi_1(M) \to \mathbb{Z}_p * \mathbb{Z}_{3l} = \langle a, b \mid a^p = b^{3l} = 1\rangle$ be the projection (see Section 2.1). It suffices to show that $\mu(D_n)$ satisfies (1), (2) and (3).

The reduced form of $\mu(D_n)$ is

$$\begin{cases} a^{p-1}b^{2l}ab^{l}(a^{p-1}b^{l}ab^{2l})^{n}(a^{p-1}b^{2l}ab^{l})^{n}a^{p-1}b^{2l}ab^{2l}a^{p-1}b^{2l}a & (n \geq 0) \\ a^{p-1}b^{2l}ab^{2l}a^{p-1}b^{2l}a(b^{l}a^{p-1}b^{2l}a)^{-n-1}(b^{2l}a^{p-1}b^{l}a)^{-n-1}b^{l}a^{p-1}b^{2l}a & (n \leq -1). \end{cases}$$

Then, $\mu(D_n)$ is conjugate to a cyclically reduced form

$$X_n = \begin{cases} a^{p-1}b^{l}ab^{l}(a^{p-1}b^{l}ab^{2l})^{n}(a^{p-1}b^{2l}ab^{l})^{n}a^{p-1}b^{2l}ab^{2l} & (n \geq 0) \\ a^{p-1}b^{l}ab^{2l}a^{p-1}b^{2l}a(b^{l}a^{p-1}b^{2l}a)^{-n-1}(b^{2l}a^{p-1}b^{l}a)^{-n-1}b^{l} & (n \leq -1) \end{cases}$$

and its syllable length is

$$\lambda(\mu(D_n)) = \lambda(X_n) = \begin{cases} 8n+8 & (n \geq 0) \\ -8n & (n \leq -1). \end{cases}$$

Therefore, $\mu(D_n)$ is non-trivial by Theorem 2.1. Moreover, if $\mu(D_n)$ is conjugate to $\mu(D_{n'})$ $(n' \neq n)$, then $n' = -(n+1)$. When $n \geq 0$, $\mu(D_{-(n+1)})$ is conjugate to a cyclically reduced form

$$Y_n = a^{p-1}b^{l}ab^{2l}a^{p-1}b^{2l}a(b^{l}a^{p-1}b^{2l}a)^{n}(b^{2l}a^{p-1}b^{l}a)^{n}b^{l}.$$

When $n \geq 2$, we can find that X_n and Y_n are not cyclic permutations of each other as we focus on b. Thus, $\mu(D_{n'})$ $(n' \neq n)$ is not conjugate to $\mu(D_n)$ by Theorem 2.1.

Finally, the reduced form of $\mu(D_n^{-1})$ is

$$\begin{cases} a^{p-1}b^{l}ab^{l}a^{p-1}b^{l}a(b^{2l}a^{p-1}b^{l}a)^{n}(b^{l}a^{p-1}b^{2l}a)^{n}b^{2l}a^{p-1}b^{l}a & (n \geq 0) \\ a^{p-1}b^{l}ab^{2l}(a^{p-1}b^{2l}ab^{l})^{-n-1}(a^{p-1}b^{l}ab^{2l})^{-n-1}a^{p-1}b^{l}ab^{l}a^{p-1}b^{l}a & (n \leq -1) \end{cases}$$

and it is conjugate to a cyclically reduced form

$$Z_n = \begin{cases} b^{2l}ab^{l}a^{p-1}b^{l}a(b^{2l}a^{p-1}b^{l}a)^{n}(b^{l}a^{p-1}b^{2l}a)^{n}b^{2l}a^{p-1} & (n \geq 0) \\ b^{2l}ab^{2l}(a^{p-1}b^{2l}ab^{l})^{-n-1}(a^{p-1}b^{l}ab^{2l})^{-n-1}a^{p-1}b^{l}ab^{l}a^{p-1} & (n \leq -1). \end{cases}$$

When $n \neq 0, -1$, X_n and Z_n are not cyclic permutations of each other as we focus on b again. Therefore $\mu(D_n)$ is not conjugate to $\mu(D_n^{-1})$ by Theorem 2.1. □

Remark 3.3. For (2) and (3) of Lemma 3.2, since $\mu(D_{-2})$ is conjugate to $\mu(D_1)$, we do not know whether D_{-2} is conjugate to D_1. Moreover, when $p = 2$, we do not know whether D_{-1} is conjugate to D_0 as well, and D_n is conjugate to D_n^{-1} for $n = -1, 0$.

Proof of Theorem 1.1. Note that a non-trivial element g is a generalized torsion element of order two if and only if g is conjugate to g^{-1}.

Proposition 3.1 and Lemma 3.2 imply that the elements D_n $(n \neq -2, -1, 0, 1)$ are non-trivial and mutually non-conjugate generalized torsion elements of order three in $\pi_1(M)$. □

4. Case B

Again, let M be a Seifert fibered space over $B(\pm g, d; p_1, \ldots, p_m)$, and assume that $d \geq 1$ and $m \geq 2$. Assume that at least one of $p_1, \ldots, p_m$ is even, and then we may set $p_1 = 2k, p_2 = q$. Set $a = u_1$ and $b = u_2$ as before.

Proposition 4.1. *In $\pi_1(M)$, let*

$$E_n = [a^k, b]^n [a^k, b^2][a^k, b]^{n+1}[a^k, b^{q-1}] \quad (n \in \mathbb{Z}).$$

Then there exist four conjugates of E_n whose product is the identity.

Proof. Let $\alpha := [a^k, b],\ \beta := [a^k, b^2],\ \gamma := [a^k, b^3],\ \delta := [a^k, b^{q-1}]$. Then, $E_n = \alpha^n \beta \alpha^{n+1} \delta$. By a direct calculation, we have

$$\alpha^{a^k} = \alpha^{-1},\ \beta^{a^k} = \beta^{-1},\ \gamma^{a^k} = \gamma^{-1},\ \delta^{a^k} = \delta^{-1}, \tag{4.1}$$

$$\alpha^b = \alpha^{-1}\beta,\ \beta^b = \alpha^{-1}\gamma,\ \delta^b = \alpha^{-1}. \tag{4.2}$$

By (4.2),

$$\begin{aligned} E_n^b &= (\alpha^n \beta \alpha^{n+1} \delta)^b \\ &= (\alpha^{-1}\beta)^n \alpha^{-1} \gamma (\alpha^{-1}\beta)^{n+1} \alpha^{-1} \\ &= (\alpha^{-1}\beta)^n \alpha^{-1} \gamma \alpha^{-1} (\beta\alpha^{-1})^n \beta \alpha^{-1}. \end{aligned}$$

Since $E_n^{\alpha^n \beta} = \alpha^{n+1}\delta\alpha^n\beta$, we obtain

$$\begin{aligned} E_n^b E_n^{\alpha^n \beta} &= (\alpha^{-1}\beta)^n \alpha^{-1} \gamma \alpha^{-1} (\beta\alpha^{-1})^n \beta \alpha^{-1} \cdot \alpha^{n+1}\delta\alpha^n\beta \\ &= (\alpha^{-1}\beta)^n \alpha^{-1} \gamma \alpha^{-1} (\beta\alpha^{-1})^n \cdot \beta\alpha^n\delta\alpha^n\beta. \end{aligned}$$

Hence,

$$\begin{aligned} E_n^{b\beta^{-1}\alpha^{-n}\delta^{-1}\alpha^{-n}\beta^{-1}} E_n^{\delta^{-1}\alpha^{-n}\beta^{-1}} &= (E_n^b E_n^{\alpha^n\beta})^{\beta^{-1}\alpha^{-n}\delta^{-1}\alpha^{-n}\beta^{-1}} \\ &= \beta\alpha^n\delta\alpha^n\beta \cdot (\alpha^{-1}\beta)^n \alpha^{-1}\gamma\alpha^{-1}(\beta\alpha^{-1})^n. \end{aligned}$$

By (4.1),

$$E_n^{ba^k} E_n^{\alpha^n\beta a^k} = (\alpha\beta^{-1})^n \alpha\gamma^{-1}\alpha(\beta^{-1}\alpha)^n \cdot \beta^{-1}\alpha^{-n}\delta^{-1}\alpha^{-n}\beta^{-1}.$$

Therefore, we have

$$E_n^{ba^k} E_n^{\alpha^n\beta a^k} E_n^{b\beta^{-1}\alpha^{-n}\delta^{-1}\alpha^{-n}\beta^{-1}} E_n^{\delta^{-1}\alpha^{-n}\beta^{-1}} = 1.$$

□

Lemma 4.2. *When $q \geq 4$, $E_n \in \pi_1(M)$ $(n \in \mathbb{Z})$ satisfies the following.*

1. *(1) $E_n \neq 1$;*
2. *(2) E_n and $E_{n'}$ are not conjugate if $n \neq n'$.*

Proof. Let μ be the projection $\pi_1(M) \to \mathbb{Z}_{2k} * \mathbb{Z}_q$. Then $\mu(E_n)$ is conjugate to

$$\begin{cases} (a^k b^{q-1} a^k b)^n a^k b^{q-2} a^k b^2 (a^k b^{q-1} a^k b)^{n+1} a^k b a^k b^{q-1} & (n \geq 0) \\ a^k b^{q-1} a^k b^2 a^k b a^k b^{q-2} & (n = -1) \\ a^k b a^k b^{q-1} a^k b^{q-1} a^k b a^k b^2 a^k b^{q-2} & (n = -2) \\ (a^k b a^k b^{q-1})^{-n-1} a^k b^{q-1} a^k b a^k b a^k (b^{q-1} a^k b a^k)^{-n-3} b^{q-1} a^k b^2 a^k b^{q-2} & (n \leq -3) \end{cases}$$

and hence, its syllable length is

$$\lambda(\mu(E_n)) = \begin{cases} 8n+12 & (n \geq 0) \\ 8 & (n=-1) \\ 12 & (n=-2) \\ -8n-4 & (n \leq -3). \end{cases}$$

Claims can be shown in the same way as in the proof of Lemma 3.2. □

Lemmas 4.4, 5.2 and 5.4 below can be shown by the similar argument. Hence, we shall only give a cyclically reduced form and its syllable length in their proofs.

Lemma 4.3. *In $\pi_1(M)$, let*

$$H_n = [a^k, b]^n [a^k, b^2][a^k, b]^{n+1}[a^k, b^{q-1}]^2 \quad (n \in \mathbb{Z}).$$

Then there exist six conjugates of H_n whose product is the identity.

Proof. Let $\alpha = [a^k, b]$, $\beta = [a^k, b^2]$, $\gamma := [a^k, b^{q-1}]$. Then $H_n = \alpha^n\beta\alpha^{n+1}\gamma^2$. By a direct calculation, we have

$$\alpha^{a^k} = \alpha^{-1},\ \beta^{a^k} = \beta^{-1},\ \gamma^{a^k} = \gamma^{-1}, \tag{4.3}$$

$$\alpha^{b^{q-1}} = \gamma^{-1},\ \beta^{b^{q-1}} = \gamma^{-1}\alpha. \tag{4.4}$$

By (4.3),

$$\begin{aligned} H_n^{\gamma^{-2}a^k} &= (\gamma^2 \cdot \alpha^n\beta\alpha^{n+1}\gamma^2 \cdot \gamma^{-2})^{a^k} \\ &= (\gamma^2\alpha^n\beta\alpha^{n+1})^{a^k} \\ &= \gamma^{-2}\alpha^{-n}\beta^{-1}\alpha^{-n-1} \end{aligned}$$

Thus,

$$\begin{aligned} H_n H_n^{\gamma^{-2}a^k} &= \alpha^n\beta\alpha^{n+1}\gamma^2 \cdot \gamma^{-2}\alpha^{-n}\beta^{-1}\alpha^{-n-1} \\ &= \alpha^n\beta\alpha\beta^{-1}\alpha^{-n-1}. \end{aligned}$$

By (4.4),

$$\begin{aligned} (H_n H_n^{\gamma^{-2}a^k})^{\alpha^n b^{q-1}} &= (\beta\alpha\beta^{-1}\alpha^{-1})^{b^{q-1}} \\ &= \gamma^{-1}\alpha\gamma^{-1}(\gamma^{-1}\alpha)^{-1}\gamma \\ &= \gamma^{-1}\alpha\gamma^{-1}\alpha^{-1}\gamma^2. \end{aligned}$$

Since $H_n^{\alpha^{-1}} = \alpha^{n+1}\beta\alpha^{n+1}\gamma^2\alpha^{-1}$, we obtain

$$\begin{aligned} (H_n H_n^{\gamma^{-2}a^k})^{\alpha^n b^{q-1}} H_n^{\alpha^{-1}} &= \gamma^{-1}\alpha\gamma^{-1}\alpha^{-1}\gamma^2 \cdot \alpha^{n+1}\beta\alpha^{n+1}\gamma^2\alpha^{-1} \\ &= \gamma^{-1}\alpha\gamma^{-1} \cdot \alpha^{-1}\gamma^2\alpha^{n+1}\beta\alpha^{n+1}\gamma^2\alpha^{-1}. \end{aligned} \tag{4.5}$$

Moreover,

$$\begin{aligned} &((H_n H_n^{\gamma^{-2}a^k})^{\alpha^n b^{q-1}} H_n^{\alpha^{-1}})^{\gamma^{-1}\alpha\gamma^{-1}a^k} \\ &= \alpha\gamma^{-2}\alpha^{-n-1}\beta^{-1}\alpha^{-n-1}\gamma^{-2}\alpha \cdot \gamma\alpha^{-1}\gamma. \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we have

$$(H_n H_n^{\gamma^{-2}a^k})^{\alpha^n b^{q-1}} H_n^{\alpha^{-1}} ((H_n H_n^{\gamma^{-2}a^k})^{\alpha^n b^{q-1}} H_n^{\alpha^{-1}})^{\gamma^{-1}\alpha\gamma^{-1}a^k} = 1.$$

□

Lemma 4.4. *When $q \geq 4$, $H_n \in \pi_1(M)$ $(n \in \mathbb{Z})$ satisfies the following.*

(1) $H_n \neq 1$;

(2) H_n and $H_{n'}$ are not conjugate if $n \neq n'$.

Proof. Let μ be the projection $\pi_1(M) \to \mathbb{Z}_{2k} * \mathbb{Z}_q$. Then $\mu(H_n)$ is conjugate to

$$\begin{cases} (a^k b^{q-1} a^k b)^n a^k b^{q-2} a^k b^2 (a^k b^{q-1} a^k b)^{n+1} a^k b a^k b^{q-1} a^k b a^k b^{q-1} & (n \geq 0) \\ a^k b^{q-1} a^k b^2 a^k b a^k b^{q-1} a^k b a^k b^{q-2} & (n = -1) \\ a^k b a^k b^{q-1} a^k b^{q-1} a^k b a^k b^2 a^k b^{q-1} a^k b a^k b^{q-2} & (n = -2) \\ (a^k b a^k b^{q-1})^{-n-1} a^k b^{q-1} a^k b a^k b a^k (b^{q-1} a^k b a^k)^{-n-3} b^{q-1} a^k b^2 a^k b^{q-1} a^k b a^k b^{q-2} & (n \leq -3) \end{cases}$$

and hence, its syllable length is

$$\lambda(\mu(H_n)) = \begin{cases} 8n + 16 & (n \geq 0) \\ 12 & (n = -1) \\ 16 & (n = -2) \\ -8n & (n \leq -3). \end{cases}$$

□

5. Case C

Again, let M be a Seifert fibered space over $B(\pm g, d; p_1, \ldots, p_m)$, and assume that $d \geq 1$ and $m \geq 2$. If $\{2k, 3l\} \subset \{p_1, \ldots, p_m\}$, possibly as multi-sets, we may assume that $p_1 = 2k, p_2 = 3l$. Set $a = u_1$ and $b = u_2$ as before.

Lemma 5.1. *In $\pi_1(M)$, let*

$$I_n = [a^k, b^l]^n [a^k, b^{2l}]^{n+2} [a^k, b^l]^{n+1} [a^k, b^{2l}]^{n+1} \quad (n \in \mathbb{Z}).$$

Then there exist four conjugates of E_n whose product is identity.

Proof. Let $\alpha := [a^k, b^l]$, $\beta := [a^k, b^{2l}]$. Then $I_n = \alpha^n \beta^{n+2} \alpha^{n+1} \beta^{n+1}$. By a direct calculation, we have

$$\alpha^{a^k} = \alpha^{-1},\ \beta^{a^k} = \beta^{-1}, \tag{5.1}$$

$$\alpha^{b^l} = \alpha^{-1}\beta,\ \beta^{b^l} = \alpha^{-1}. \tag{5.2}$$

By (5.2),

$$I_n^{b^l} = (\alpha^{-1}\beta)^n \alpha^{-n-2} (\alpha^{-1}\beta)^{n+1} \alpha^{-n-1}.$$

Also,

$$I_n^{\alpha^n \beta^{n+2}} = \alpha^{n+1} \beta^{n+1} \alpha^n \beta^{n+2}.$$

Then we obtain

$$\begin{aligned} I_n^{b^l} E_n^{\alpha^n \beta^{n+2}} &= (\alpha^{-1}\beta)^n \alpha^{-n-2} (\alpha^{-1}\beta)^{n+1} \alpha^{-n-1} \cdot \alpha^{n+1} \beta^{n+1} \alpha^n \beta^{n+2} \\ &= (\alpha^{-1}\beta)^n \alpha^{-n-2} (\alpha^{-1}\beta)^{n+1} \beta^{n+1} \alpha^n \beta^{n+2} \\ &= (\alpha^{-1}\beta)^n \alpha^{-n-3} (\beta\alpha^{-1})^n \cdot \beta^{n+2} \alpha^n \beta^{n+2}. \end{aligned}$$

Moreover,

$$\begin{aligned} I_n^{b^l \beta^{-n-2} \alpha^{-n} \beta^{-n-2}} I_n^{\beta^{-n-2}} &= I_n^{b^l \beta^{-n-2} \alpha^{-n} \beta^{-n-2}} I_n^{\alpha^n \beta^{n+2} \cdot \beta^{-n-2} \alpha^{-n} \beta^{-n-2}} \\ &= (I_n^{b^l} I_n^{\alpha^n \beta^{n+2}})^{\beta^{-n-2} \alpha^{-n} \beta^{-n-2}} \\ &= \beta^{n+2} \alpha^n \beta^{n+2} \cdot (\alpha^{-1}\beta)^n \alpha^{-n-3} (\beta\alpha^{-1})^n. \end{aligned}$$

By (5.1),

$$I_n^{b^l\beta^{-n-2}\alpha^{-n}\beta^{-n-2}a^k} I_n^{\beta^{-n-2}a^k} = \beta^{-n-2}\alpha^{-n}\beta^{-n-2} \cdot (\alpha\beta^{-1})^n\alpha^{n+3}(\beta^{-1}\alpha)^n.$$

Therefore, we have

$$I_n^{b^l} I_n^{\alpha^n\beta^{n+2}} I_n^{b^l\beta^{-n-2}\alpha^{-n}\beta^{-n-2}a^k} I_n^{\beta^{-n-2}a^k} = 1.$$

□

Lemma 5.2. *$I_n \in \pi_1(M)$ $(n \in \mathbb{Z})$ satisfies the following.*

(1) $I_n \neq 1$;
(2) I_n and $I_{n'}$ are not conjugate if $n \neq n'$.

Proof. Let μ be the projection $\pi_1(M) \to \mathbb{Z}_{2k} * \mathbb{Z}_{3l}$. Then $\mu(I_n)$ is conjugate to

$$\begin{cases} (a^kb^{2l}a^kb^l)^n(a^kb^la^kb^{2l})^{n+2}(a^kb^{2l}a^kb^l)^{n+1}(a^kb^la^kb^{2l})^{n+1} & (n \geq 0) \\ a^kb^{2l}a^kb^l & (n = -1) \\ (b^{2l}a^kb^la^k)^{-n}(b^la^kb^{2l}a^k)^{-n-2}(b^{2l}a^kb^la^k)^{-n-1}(b^la^kb^{2l}a^k)^{-n-1} & (n \leq -2) \end{cases}$$

and hence, its syllable length is

$$\lambda(\mu(I_n)) = \begin{cases} 16n + 16 & (n \geq 0) \\ 4 & (n = -1) \\ -16n - 16 & (n \leq -2). \end{cases}$$

□

Lemma 5.3. *In $\pi_1(M)$, let*

$$J_n = [a^k, b^l][a^k, b^{2l}][a^k, b^l]^{-n}[a^k, b^{2l}]^{2n+3} \quad (n \in \mathbb{Z}).$$

Then there exist six conjugates of J_n whose product is identity.

Proof. Let $\alpha := [a^k, b^l]$, $\beta := [a^k, b^{2l}]$. Then $J_n = \alpha\beta\alpha^{-n}\beta^{2n+3}$. By a direct calculation, we have

$$\alpha^{a^k} = \alpha^{-1},\ \beta^{a^k} = \beta^{-1}, \tag{5.3}$$

$$\alpha^{b^{2l}} = \beta^{-1},\ \beta^{b^{2l}} = \beta^{-1}\alpha. \tag{5.4}$$

By (5.3) and (5.4),

$$J_n^{b^{2l}a^k} = \beta^2\alpha^{-1}\beta^{-n}(\beta\alpha^{-1})^{2n+3}.$$

Also,

$$\begin{aligned} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}} &= \beta^{-n+1}\alpha^{-1}\beta^2 \cdot \beta^{-2}\alpha\beta^n(\beta^{-1}\alpha)^{2n+3} \cdot \beta^{-2}\alpha\beta^{n-1} \\ &= \beta(\beta^{-1}\alpha)^{2n+3}\beta^{-2}\alpha\beta^{n-1} \\ &= (\alpha\beta^{-1})^{2n+3}\beta^{-1}\alpha\beta^{n-1}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}} &= \beta^2\alpha^{-1}\beta^{-n}(\beta\alpha^{-1})^{2n+3} \cdot (\alpha\beta^{-1})^{2n+3}\beta^{-1}\alpha\beta^{n-1} \\ &= \beta^2\alpha^{-1}\beta^{-n-1}\alpha\beta^{n-1}. \end{aligned}$$

Thus,

$$(J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}})^{\beta^2\alpha^{-1}} J_n = \beta^{-n-1}\alpha\beta^{n+1}\alpha^{-1}\cdot\alpha\beta\alpha^{-n}\beta^{2n+3} \tag{5.5}$$
$$= \beta^{-n-1}\alpha\beta^{n+2}\alpha^{-n}\beta^{2n+3}$$
$$= \beta^{-n-1}\alpha\beta^{-n-1}\cdot\beta^{2n+3}\alpha^{-n}\beta^{2n+3}.$$

Moreover,

$$((J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}})^{\beta^2\alpha^{-1}} J_n)^{\beta^{-n-1}\alpha\beta^{-n-1}} = \beta^{2n+3}\alpha^{-n}\beta^{2n+3}\cdot\beta^{-n-1}\alpha\beta^{-n-1}.$$

By (5.3),

$$((J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}})^{\beta^2\alpha^{-1}} J_n)^{\beta^{-n-1}\alpha\beta^{-n-1}a^k} \tag{5.6}$$
$$= \beta^{-2n-3}\alpha^n\beta^{-2n-3}\cdot\beta^{n+1}\alpha^{-1}\beta^{n+1}.$$

By (5.5) and (5.6), we have

$$(J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}})^{\beta^2\alpha^{-1}} J_n((J_n^{b^{2l}a^k} J_n^{b^{2l}\beta^{-2}\alpha\beta^{n-1}})^{\beta^2\alpha^{-1}} J_n)^{\beta^{-n-1}\alpha\beta^{-n-1}a^k} = 1.$$

□

Lemma 5.4. *$J_n \in \pi_1(M)$ $(n \in \mathbb{Z})$ satisfies the following.*

(1) $J_n \neq 1$;

(2) J_n and $J_{n'}$ are not conjugate if $n \neq n'$.

Proof. Let μ the projection $\pi_1(M) \to \mathbb{Z}_{2k} * \mathbb{Z}_{3l}$. Then $\mu(J_n)$ is conjugate to

$$\begin{cases} a^kb^{2l}a^kb^la^kb^la^kb^la^kb^la^k(b^{2l}a^kb^la^k)^{n-2}b^{2l}a^kb^{2l}a^kb^{2l}(a^kb^la^kb^{2l})^{2n+2} & (n \geq 2) \\ a^kb^{2l}a^kb^la^kb^la^kb^la^kb^{2l}a^kb^{2l}(a^kb^la^kb^{2l})^4 & (n = 1) \\ a^kb^{2l}a^kb^l(a^kb^la^kb^{2l})^4 & (n = 0) \\ a^kb^{2l}a^kb^la^kb^la^kb^{2l}a^kb^{2l}a^kb^la^kb^la^kb^{2l} & (n = -1) \\ a^kb^la^kb^la^kb^{2l}a^kb^{2l}a^kb^la^kb^{2l}a^kb^{2l}a^kb^l & (n = -2) \\ a^kb^la^kb^la^kb^{2l}(a^kb^{2l}a^kb^l)^{-n-1}a^kb^{2l}a^kb^{2l}a^kb^{2l}a^k(b^la^kb^{2l}a^k)^{-2n-5}b^la^kb^l & (n \leq -3) \end{cases}$$

and hence, its syllable length is

$$\lambda(\mu(J_n)) = \begin{cases} 12n+16 & (n \geq 2) \\ 28 & (n = 1) \\ 20 & (n = 0) \\ 16 & (n = -1) \\ 16 & (n = -2) \\ -12n-8 & (n \leq 3). \end{cases}$$

□

6. The order of E_1, H_1, I_1, J_1

In this section, we will determine the order of E_1, H_1, I_1, J_1 under some conditions. To see this, we use a stable commutator length; see Section 2.3.

Again, let M be a Seifert fibered space over $B(\pm g, d; p_1, \ldots, p_m)$, and assume that $d \geq 1$ and $m \geq 2$. Assume that at least one of $p_1, \ldots, p_m$ is even, and then we may set $p_1 = 2k, p_2 = q$.

Lemma 6.1. *Let r be a prime number satisfying $5 \le r \le 37$, or $r \in \{4, 6, 9\}$. If k is odd, and q is a multiple of r, then* $\mathrm{scl}_{\pi_1(M)}(E_1) \ge 1/4$.

Proof. Let $\mu\colon \pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_r = \langle a, b \mid a^2 = b^r = 1\rangle$ be the projection. Then, $\mu(E_1) = [a,b][a,b^2][a,b]^2[a,b^{-1}]$. The computer program Scallop verifies $\mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_r}(\mu(E_1)) = 1/4$ for $r = 4, 5, 6, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37$. Therefore, we obtain $\mathrm{scl}_{\pi_1(M)}(E_1) \ge \mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_r}(\mu(E_1)) = 1/4$ by scl's monotonicity. □

Lemma 6.2. *Let r be a prime number satisfying $5 \le r \le 37$, or $r \in \{4, 6, 9\}$. If k is odd, and q is a multiple of r, then* $\mathrm{scl}_{\pi_1(M)}(H_1) \ge 1/3$.

Proof. Let $\mu\colon \pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_r = \langle a, b \mid a^2 = b^r = 1\rangle$ be the projection. Then, $\mu(H_1) = [a,b][a,b^2][a,b]^2[a,b^{-1}]^2$. Again, the computer program Scallop verifies $\mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_r}(\mu(H_1)) = 1/3$ for $r = 4, 5, 7, 9, 11, 13, 17, 19, 23, 29, 31, 37$. Therefore, we obtain $\mathrm{scl}_{\pi_1(M)}(H_1) \ge \mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_r}(\mu(H_1)) = 1/3$ by scl's monotonicity. □

Suppose $q = 3l$.

Lemma 6.3. *If k is odd, and $l \not\equiv 0 \pmod 3$, then* $\mathrm{scl}_{\pi_1(M)}(I_1) \ge 1/4$.

Proof. Let μ be the natural projection $\pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1\rangle$. Then
$$\mu(I_1) = \begin{cases} [a,b][a,b^2]^3[a,b]^2[a,b^2]^2 & (l \equiv 1 \pmod 3) \\ [a,b^2][a,b]^3[a,b^2]^2[a,b]^2 & (l \equiv 2 \pmod 3). \end{cases}$$
Again, the computer program Scallop verifies that $\mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_3}(\mu(I_1)) = 1/4$, so we have $\mathrm{scl}_{\pi_1(M)}(I_1) \ge \mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_3}(\mu(I_1)) = 1/4$ by scl's monotonicity. □

Lemma 6.4. *If k is odd, and $l \not\equiv 0 \pmod 3$, then* $\mathrm{scl}_{\pi_1(M)}(J_1) \ge 1/3$.

Proof. Let μ be the projection $\pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1\rangle$. Then
$$\mu(J_1) = \begin{cases} [a,b][a,b^2][a,b]^{-1}[a,b^2]^5 & (l \equiv 1 \pmod 3) \\ [a,b^2][a,b][a,b^2]^{-1}[a,b]^5 & (l \equiv 2 \pmod 3). \end{cases}$$
Again, the computer program Scallop verifies that $\mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_3}(\mu(J_1)) = 1/3$, so we have $\mathrm{scl}_{\pi_1(M)}(J_1) \ge \mathrm{scl}_{\mathbb{Z}_2*\mathbb{Z}_3}(\mu(J_1)) = 1/3$ by scl's monotonicity. □

We remark that if a generalized torsion element $g \in G$ satisfies $\mathrm{scl}_G(g) \ge 1/4$, then the order of g is at least four by Theorem 2.3.

Proof of Theorem 1.2 for order four. First, set $r = 3$. Thus we can write $q = 3l$. By Proposition 5.1 and Lemma 6.3, I_1 gives a generalized torsion element of order four in $\pi_1(M)$ if $l \not\equiv 0 \pmod 3$. If $l \equiv 0 \pmod 3$, then q is a multiple of 9. Then Proposition 4.1 and Lemma 6.1 show that E_1 is a generalized torsion element of order four.

Second, assume that either r is a prime number with $5 \le r \le 37$, or $r = 4$. Then Proposition 4.1 and 6.1 again imply the conclusion. □

We remark that if a generalized torsion element $g \in G$ satisfies $\mathrm{scl}_G(g) \ge 1/3$, then the order of g is at least six by Theorem 2.3.

Proof of Theorem 1.2 for order six. First, set $r = 3$. Thus we can write $q = 3l$. By Proposition 4.3 and Lemma 6.2, H_1 gives a generalized torsion element of order six in $\pi_1(M)$ if $l \not\equiv 0 \pmod 3$. If $l \equiv 0 \pmod 3$, then q is a multiple of 9. Then

Proposition 5.3 and Lemma 6.4 show that J_1 is a generalized torsion element of order six.

Second, assume that either r is a prime number with $5 \leq r \leq 37$, or $r = 4$. Then Proposition 5.3 and Lemma 6.4 again imply the conclusion, and the proof of Theorem 1.2 has been completed. $\square$

Remark 6.5. When $q = 2$, E_n and H_n are generalized torsion elements of order two (although E_{-1} is the trivial element). Also, when $q = 3$, E_n is a generalized torsion element of order two again, and if k is odd, $\mathrm{scl}_{\mathbb{Z}_2 * \mathbb{Z}_3}(\mu(H_1)) = 1/4$ by Scallop. So, we don't know the order of H_n in this case.

On the other hand, it is easy to check that I_0, I_{-1}, I_{-2}, J_0 and J_{-1} are generalized torsion elements of order two. Moreover, by replacing a^k with a, J_{-2} becomes conjugate to D_1. By the same calculation as Proposition 3.1, it follows that three conjugates of J_{-2} is the identity hence J_{-2} is a generalized torsion element of order three.

7. Other results on the order of generalized torsion elements

Let M be a Seifert fibered space whose base orbifold is $B(\pm g, d; p_1, \ldots, p_m)$.

Theorem 7.1. *Let $k \geq 2$ and $l \geq 1$. Assume that M satisfies one of the following;*

(1) $d \geq 2$,
(2) $d = 1$ and $g \neq 0$.

For (1), set $a = v_1$ and $b = u_m$, otherwise, set $a = s_1^2$ and $b = u_m$.

If $p_m = kl$, then $[a^n, b^l] \in \pi_1(M)$ $(n \neq 0)$ is a generalized torsion element of order k.

Proof. Since b^{kl} $(= h^{q_m})$ commutes with a, $[a, b^{kl}] = 1$. As mentioned in Section 1, we have
$$[a, b^{kl}] = [a, b^l][a, b^l]^{b^l}[a, b^l]^{b^{2l}} \cdots [a, b^l]^{b^{(k-1)l}}.$$
Therefore, the product of k conjugates of $[a, b]$ yields the identity.

Let $\mu\colon \pi_1(M) \to \mathbb{Z} * \mathbb{Z}_{kl} = \langle a, b \mid b^{kl} = 1\rangle$ be the projection. By Proposition 5.6 of [5], $\mathrm{scl}_{\mathbb{Z}_p * \mathbb{Z}_q}(\mu([a^n, b^l])) = 1/2 - 1/k$. Thus the order of $[a^n, b^l]$ is equal to k by scl's monotonicity and Theorem 2.3. $\square$

Theorem 7.2. *Let $p \geq k \geq 2$ and $l \geq 1$. Assume that $d = 1$ and $g = 0$ (hence, B is the disk). Moreover, assume that $p_i = p$ and $p_j = kl$ $(i \neq j)$. Set $a = u_i$ and $b = u_j$. Then $[a, b^l] \in \pi_1(M)$ is a generalized torsion element of order k.*

Proof. The proof is similar to the proof of Theorem 7.1. $\square$

Lemma 7.3. *Let $h \in \pi_1(M)$ be represented by the regular fiber of M. If B is orientable and $d \geq 1$, then $[h] \in H_1(M) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$ is not a torsion element.*

Proof. By Section 2.1, $H_1(M)$ has a presentation
$$\begin{aligned} H_1(M) = \langle &[s_1], [t_1], \ldots, [s_g], [t_g], [u_1], \ldots, [u_m], [v_1], \ldots, [v_d], [h] \mid \\ &p_j[u_j] = q_j[h],\ b[h] = [u_1] + \cdots + [u_m] + [v_1] + \cdots + [v_d]\rangle \\ =&\langle [s_1], [t_1], \ldots, [s_g], [t_g], [v_1], \ldots, [v_{d-1}]\rangle \\ &\oplus \langle [u_1], \ldots, [u_m], [h] \mid p_j[u_j] = q_j[h]\rangle. \end{aligned}$$
Let $U = \langle [u_1],\ \ldots,\ [u_m],\ [h] \mid p_j[u_j] = q_j[h]\rangle$.

Assume that $[h] \in U$ has finite order, then U is finite, since the relations of U show that $[u_j]$ has finite order for any j.

On the other hand, the relations of U can be expressed as

$$\begin{pmatrix} p_1 & & & & -q_1 \\ & p_2 & & & -q_2 \\ & & \ddots & & \\ & & & p_m & -q_m \end{pmatrix} \begin{pmatrix} [u_1] \\ \vdots \\ [u_m] \\ [h] \end{pmatrix} = \mathbf{0},$$

where each empty entry is zero. Since the rank of the above $m \times (m+1)$ matrix is at most m, the degree of freedom is at least one. Therefore, U is infinite, a contradiction. □

Recall that if $d \geq 1$, then $\pi_1(M)$ is torsion-free.

Theorem 7.4. *Assume that $m \geq 1$, and let N (≥ 2) be the minimum of positive divisors of p_1, …, p_m. If B is orientable and $d \geq 1$, then the order of a generalized torsion element in $\pi_1(M)$ is at least N.*

Proof. Let $g \in \pi_1(M)$ be a generalized torsion element of order k, and let $\mu\colon \pi_1(M) \to \pi_1(M)/\langle h \rangle \cong F_{2g+d-1} * \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_m}$ be the projection. Let

$$g^{x_1} \cdots g^{x_k} = 1 \quad (x_i \in \pi_1(M)). \tag{7.1}$$

Then we have

$$\mu(g)^{\mu(x_1)} \cdots \mu(g)^{\mu(x_k)} = 1. \tag{7.2}$$

Thus, by the abelianization on $\pi_1(M)/\langle h \rangle$, $k[\mu(g)] = 0$ in $\mathbb{Z}^{2g+d-1} \oplus \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_m}$. If $\mu(g) \notin [\pi_1(M)/\langle h \rangle, \pi_1(M)/\langle h \rangle]$, k is a multiple of at least one of $p_1, \ldots, p_m$, in particular, $k \geq N$. Thus, it suffices to consider the case $\mu(g) \in [\pi_1(M)/\langle h \rangle, \pi_1(M)/\langle h \rangle]$.

Claim 7.5. *$\mu(g)$ is not conjugate into F_{2g+d-1}.*

Proof. Assume that $\mu(g)$ is conjugate into F_{2g+d-1}. Let $\mu(g) = y^{-1}fy = f^y$ for $f \in F_{2g+d-1},\ y \in \pi_1(M)/\langle h \rangle$. By (7.2),

$$f^{y\mu(x_1)} \cdots f^{y\mu(x_k)} = 1.$$

Let $\phi\colon \pi_1(M)/\langle h \rangle \to F_{2g+d-1}$ be the natural projection. Then, we obtain

$$f^{\phi(y\mu(x_1))} \cdots f^{\phi(y\mu(x_k))} = 1 \ \text{ in } F_{2g+d-1}.$$

Since the free group cannot contain a generalized torsion element (see [6]), $f = 1$. This implies $g \in \langle h \rangle$, and hence, $g^k = 1$ by (7.1). Since $\pi_1(M)$ is torsion-free, $g = 1$, a contradiction. □

Claim 7.6. *$\mu(g)$ is not conjugate into $\mathbb{Z}_{p_j}$ for any j.*

Proof. If $\mu(g)$ were conjugate into $\mathbb{Z}_{p_j}$, then $\mu(g)^{p_j} = \mu(g^{p_j}) = 1$ so $g^{p_j} \in \langle h \rangle$. By (7.1), $[g] \in H_1(M)$ is a torsion element of $H_1(M)$ (possibly trivial), hence $[h] \in H_1(M)$ is a torsion element (possibly trivial). This contradicts Lemma 7.3. □

Thus $\mu(g)$ is not conjugate into one factor of $F_{2g+d-1} * \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_m}$. After a conjugation and replacing subscripts, if necessary, we can express $\mu(g) = a_1 b_1 \cdots a_L b_L$ with $a_i \in A = F_{2g+d-1} * \mathbb{Z}_{p_1} * \cdots * \mathbb{Z}_{p_{m-1}}$, $b_i \in \mathbb{Z}_{p_m}$, $a_i \neq 1$, $b_i \neq 1$ and $L \geq 1$. Recall that $\mu(g) \in [\pi_1(M)/\langle h \rangle, \pi_1(M)/\langle h \rangle]$. Thus Theorem 3.1 of [4] shows

$$\mathrm{scl}_{A * \mathbb{Z}_{p_m}}(\mu(g)) \geq \frac{1}{2} - \frac{1}{N}.$$

By scl's monotonicity and Theorem 2.3, we have $1/2 - 1/N \leq 1/2 - 1/k$, so $k \geq N$. □

Remark 7.7. Theorem 7.4 does not hold when B is non-orientable, because then $\pi_1(M)$ contains a generalized torsion element of order two, regardless of the indices of exceptional fibers.

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