A Study on Opportunity-Based Age Replacement Models and Their Applications

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degree of Ph.D. of Advanced Science and Engineering

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Abstract

Abstract

This thesis studies several opportunity-based age replacement models and their applications. In reliability theory, replacement opportunities often refer to delivering of spare parts with low cost, special service durations, or specific replacement time points. Overall, preventive maintenance can be performed flexibly and conveniently when the replacement opportunities arise.

In Chapter 2, we consider two classical age-based replacement models within a discrete time framework: a standard age replacement (AR) model and an opportunistic age replacement (DD) model. More specifically, we introduce the concept of replacement priority in situations where failure replacement and preventive replacement occur at a given age or opportunity. We explore two priority cases in each replacement model. First, we formulate the optimal preventive replacement policies minimizing the associated expected cost rates by the familiar renewal reward argument. Next, we extend the modellings taking account of net present value (NPV) method. We develop the expected total discounted costs over an infinite time horizon and obtain the optimal preventive replacement policies by minimizing these total expected costs. Also, we introduce unified stochastic models incorporating the probabilistic priority of replacement options. Besides, we propose a general framework for optimizing replacement policies in discrete time. The discrete time AR and DD models with/without discounting are reformulated under this framework. To provide practical insights, we present numerical illustrations using real failure data for pole air switches, comparing the performance of these optimal preventive policies.

In Chapter 3, we focus on discrete time opportunity-based age replacement models with replacement first (RF) and replacement last (RL) disciplines, where

the expected cost model under each discipline can be further classified into six cases by taking account of the priority of multiple replacement options. We characterize several optimal opportunity-based age replacement policies minimizing the relevant expected costs. We also apply the NPV method to formulate the expected total costs under RF and RL disciplines. In addition, we unify six discrete time opportunity-based age replacement models with deterministic priorities for each model. In numerical illustrations, we obtain and compare all the optimal scheduled preventive replacement times with RF and RL disciplines.

In Chapter 4, we concern about two opportunity-based age replacement problems in continuous/discrete time. Firstly, we formulate the opportunity-based age replacement models with RF and RL disciplines in continuous time. We also consider a restricted duration for the opportunity arrivals which obey a homogeneous Poisson process. Next, we reconsider these opportunity-based age replacement models in discrete time, where the inter-arrival times of replacement opportunities obey an independent and identical geometric distribution. The optimal two-phase opportunity-based age replacement policies are characterized by minimizing the long-run average costs. The numerical examples are presented to compare two replacement policies with RF and RL disciplines. The results indicates that RL policies could be better than RF policies in a few limited cases where the impact of failure replacement is relatively small.

In Chapter 5, we generalize the opportunity-based age replacement policies by introducing the NPV of expected total costs, where two cases are considered. First, we reformulate two basic opportunity-based age replacement models with RF and RL disciplines, in which the failure time and the arrival time of a replacement opportunity are statistically independent. Next, we take place the NPV analysis for the failure-correlated opportunity-based age replacement models with RF and RL disciplines. Since the NPV approach is useful to estimate more accurate replacement costs over a long-time planning in an unstable economic environment, we obtain the expected total discounted costs over an infinite time horizon, and derive the optimal preventive replacement policies by minimizing them in both cases. Numerical examples with the Farlie-Gumbel-Morgenstern bivariate copula are presented to investigate the dependence of correlation between the failure time and the opportunistic replacement time on the age opportunity-based replacement policies.

Finally, Chapter 6 concludes the thesis and give some remarks on the future studies.

The organization of this dissertation is as follows. Chapter 1 is introduction. We mainly introduce the background from three streams: opportunity in preventive maintenance, the discrete time replacement models and NPV approach in Section 1.1. In Section 1.2, we discuss the literature review from opportunity-based replacement models, discrete time replacement models and NPV method in replacement models. Chapter 2 studies the discrete time AR and DD models. Section 2.1 introduces some notations and assumptions about this thesis. Section 2.2 formulates the AR and DD models by minimizing the expected costs in steady state. Section 2.3 considers AR and DD models with discounting. In Section 2.4, AR and DD models are unified with probabilistic priority. Section 2.5 proposes a general optimizing framework for discrete time models. AR and DD models with/ without discounting are reformulated under this framework. Chapter 3 studies RF and RL models in discrete time. We formulate RF and RL models with renewal reward approach and propose the optimal preventive replacement polices in Section 3.1. Next, we discuss RF and RL models with NPV method in Section 3.2. Besides, we study the unified models with probabilistic priority under RF and RL disciplines in Section 3.3. A study on pole air switches is presented to obtain the optimal preventive replacement times in Section 3.4. Chapter 4 studies two-phase RF and RL models in continuous time and discrete time. Section 4.1 describes the continuous time RL model and gives the optimal preventive replacement policies. Section 4.2 and Section 4.3 consider two-phase RF and RL models in discrete time and obtain the existence of optimal preventive replacement policies. We also study the unified models with two-phase RF and RL disciplines in Section 4.4. Chapter 5 analyzes the failure-correlated- opportunity RF and RL models. Section 5.1 formulates the continuous time failure-correlated-opportunity RF and RL models with renewal reward approach. We further study the failure-correlated-opportunity RF and RL models with NPV method in Section 5.2. In Section 5.3, we analyze the correlation between lifetime and the arrival of opportunity in RF and RL models by numerical examples. Chapter 6 gives the conclusions and the future work.

The general framework of this study is shown in Figure 1.

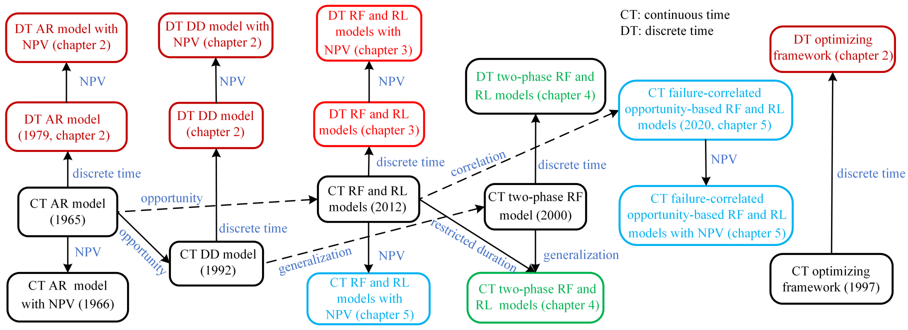

Figure 1: General framework of this study.

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Contents

Chapter 1

Introduction

1.1 Background

Opportunity-based age replacement models have been paid much attention in reliability theory. Preventive maintenance activities can be carried out flexibly and conveniently at an opportunity. A lot of opportunity-based age replacement policies have studied by related works. In reliability theory, the opportunity usually presents the space parts with low cost, the special service duration or the replacement time point [1]. The most opportunity-based age replacement models were discussed when the process of opportunity arrival obeys a continuous stochastic distribution.

As a significant expansion, Zhao and Nakagawa [2, 3], along with Zhao et al [4], proposed slightly different opportunity-based age replacement models: the replacement first (RF) and replacement last (RL) disciplines. Essentially, these models amalgamate the standard age replacement and random age replacement. In context, the RF discipline is described as a preventive replacement which is performed at an opportunity arrival time or a prescheduled replacement time, whichever occurs first. Conversely, the RL discipline is that a preventive replacement is performed at an arrival time of opportunity or a prescheduled replacement time, whichever occurs last. In other words, it can be seen that these replacement policies at random timing are essentially regarded as opportunistic replacement policies.

In fact, the RF and RL disciplines were applied to the minimal repair models [5], generalized models [6] and cumulative damage models [7] and among others. Especially, Iskandar and Sandoh [8] extended the seminal opportunity-based age replacement (DD) model in Dekker and Dijkstra [9] by introducing the opportunities in restricted duration, and dealt with the replacement first policies. In Chapter 4, we formulate the replacement last policies for Iskandar and Sandoh model [8] in the sense of Zhao and Nakagawa [2]. Chapter 4 also considers RF and RL models with the opportunities in restricted duration in discrete time. Another important extension in recent years, Dohi and Okamura [10] firstly found correlation between lifetime of the system and the occurrence of replacement opportunity. They studied RF and RL models in continuous time, where lifetime of the system and occurrence of replacement opportunity are correlated.

From the perspective of maintenance strategies, most opportunity-based age replacement policies in continuous time have been derived in the literature. However, less attention has been given to discrete time models. In certain real industrial scenarios, when the system or unit's lifetime is represented in cycles, discrete time models become effective [11]. For example, the lifetimes of jet tires are measured in terms of the number of flights [12]. Besides, the most early models [13–16] assumed that failure replacement should be selected with priority. However, this assumption may not hold true, as the cost of failure replacement is higher than that of preventive replacement. To tackle this problem, Chapter 2 primarily introduces the concept of replacement priority and formulates RF and RL models in discrete time. Except the extension in modelling, Chapter 2 explores new optimizing method to formulate the discrete time models. Many classic preventive replacement models, such as AR and DD models are reformulated under this optimizing method in discrete time.

Most studies on opportunity-based age replacement models implicitly assumed that the global economic environment remains stable during the maintenance plan, i.e., money does not have a time component and its value does not decrease over time. In many replacement models, the optimal preventive replacement policies were derived by minimizing the long-run average cost in the steady state. Indeed, in today's rapidly changing economic environment among countries, the net present value (NPV) method is more accurate in formulating preventive maintenance models.

Some new findings on opportunity-based models are studied in this thesis. In Chapter 2, the discrete time standard age replacement (AR) model and DD models are computed by priority of multiple replacement options. NPV method is applied to formulate these models. Besides, a general optimizing framework is proposed in discrete time. The discrete time AR and DD models with/without discounting are reformulated under this framework. In Chapter 3, two important opportunity-based models, RF and RL models, are considered under renewal reward theory and NPV method. The optimal opportunity-based age replacement policies minimizing the relevant expected costs are obtained. The unified model with deterministic priorities under renewal reward theory and NPV method is studied. In Chapter 4, two opportunity-based age replacement problems with RF and RF disciplines are studied in continuous/discrete time. In continuous time setting, two-phase RF and RF models are calculated, where the opportunity arrivals obey a homogeneous Poisson process. In discrete time models, the optimal preventive replacement policies are derived, where inter-arrival times of replacement opportunities obey an independent and identical geometric distribution. Chapter 5, we concern about the correlation between failure time and the arrival time of a replacement opportunity in RF and RL models. The optimal preventive replacement policies are formulated by NPV method. We summary the main contributions of this thesis as follows:

(1) The opportunity-based models are formulated in discrete time;

(2) NPV method is applied to the replacement first and last models;

(3) A general framework is proposed for the discrete time models.

(4) Two-phase opportunity-based age replacement models are discussed with RF and RL disciplines;

(5) Correlation between failure and the arrival of opportunity are analyzed in RF and RL models;

(6) The performance of RF and RL models is compared comprehensively.

1.2 Literature Review

We discuss three closely related streams of research: opportunity-based replacement models, discrete time replacement models and NPV method in replacement models. While reviewing the previous works, we also point out their main distinctions within this paper.

1.2.1 Opportunity-Based Replacement Models

Opportunity-based replacement models have been studied for over five decades (see a methodical survey in [17]). Most of the research on opportunity-based replacement models suppose that the arrival of opportunity obeys stochastic process, such as Poisson process. Radner and Jorgenson [18] firstly studied the opportunity arrival in age replacement policies for one-unit system. Berg [19] considered the opportunity arrival in two-units system. Dekker and Dijkstra [9] studied the opportunity-based age replacement model and proposed the well-known control limit policy. Jhang and Sheu [20] extended the model in [9] and formulated the opportunity-based age replacement policy with minimal repair. Dekker and Smeitink [21, 22] also studied the restricted duration in arrival of opportunity in preventive replacement model and the opportunity-based block replacement model. In recent years, Wang et al. [23] proposed a novel imperfect opportunistic maintenance model for a two-unit series system considering random repair time and two types of failures, where unit 1 and unit 2 are respectively subject to soft failure and hard failure. Si et al. [24] studied an agile framework which can quickly respond to organizational scheduling requirements while controlling service costs and not compromising.

For RF and RL models, Chen et al. [6] studied some modified age and block models with RF and RL disciplines, Dohi and Okamura [10] considered failure-correlated opportunity-based age replacement models with RF and RL disciplines using the bivariate copula of failure time and opportunity-arrival time distributions. Zheng et al. [1] generalized opportunity-based age replacement policies with RF and RL disciplines by introducing Markovian opportunity-arrival process. Mizutani et al. [25] took account of two failure modes in general replacement models under RF and RL disciplines. In addition, the mission duration was widely discussed in replacement models with RF and RL disciplines [5, 26, 27]. Chapter 4, we formulate the replacement last policies for Iskandar and Sandoh model [8] in the sense of Zhao and Nakagawa [2].

1.2.2 Discrete Time Replacement Models

Since the seminal work by Nakagawa and Osaki [13], various discrete time replacement models have been considered in the mid-1980s [13–16]. For example, Nakagawa [16] considered combined continuous and discrete replacement with minimal repair at failure, in which a unit is replaced at time T or at number N of uses. Recently, the discrete-time replacement models were studied from somewhat different viewpoint. Cha and Limnios [28] reformulated minimal repair models in discrete time under random environments. Eryilmaz [12] studied discrete-time age replacement policy when the lifetime of the system is modeled by a discrete phase-type distribution. Eryilmaz [29] investigated age-based preventive replacement policy for an arbitrary coherent system that consists of components which are independent and have common discrete lifetime distribution. Wei et al. [30] proposed an optimal opportunistic maintenance planning integrating discrete-and continuous-state information. These works mainly studied age-based replacement models. A methodical book was finished by Nair et al. [31]. This thesis generalizes more complex opportunity-based models in discrete time. More concretely, Chapter 2 introduces the replacement priority to deal with the case the simultaneous events of two distinct replacement activities come at same time point. The discrete time AR and DD models are computed with renewal reward and NPV methods. Chapter 3 formulates RF and RL models in discrete time. Chapter 4 cares about two-phase RF and RL models [8] in discrete time. What is more, we develop a general framework for discrete time models and purpose the optimal criterion for optimal preventive replacement times in Chapter 2.

1.2.3 NPV Method in Replacement Models

Early studies on NPV method focused on formulating and comparing classical maintenance models [13, 32–34]. Fox [32] firstly formulated the age replacement model with discounting and proposed the optimal preventive replacement policies. Nakagawa and Osaki [13] reformulated age replacement model with NPV method in discrete time. Nakagawa [33] also formulated several block type replacement models with discounting. Recently, from the similar motivation, Boomen et al. [35] developed a new life cycle costing approach for discounting in two classes of maintenance optimization models, the age replacement model, and the interval replacement model. Zhang et al. [36] considered mission-oriented systems with discounting. In this thesis, we also analyze the NPV in the RF and RL models [2] and failure-correlated opportunity RF and RL models [10] in Chapter 4.

Chapter 2

Discrete Time AR and DD Models

2.1 Preliminaries

Consider a single-unit system with a non-repairable item in discrete time setting. We suppose that the interval-arrival times between consecutive opportunities for replacements, X, are independent and identically distributed (i.i.d.) integer-valued random variables, having the probability mass function (p.m.f.) $\Pr\{X = x\} = g_X(x)$ $(x = 1, 2, \cdots)$. The failure times (lifetimes) of the item, Y, follow i.i.d. integer-valued random variables with the common p.m.f. $\Pr\{Y = y\} = f_Y(y)$ $(y = 1, 2, \cdots)$. It is general that we assume that $g_X(0) = f_Y(0) = 0$, where in general $\bar{G}_X(\cdot) = 1 - G_X(\cdot)$ and $\bar{F}_Y(\cdot) = 1 - F_Y(\cdot)$.

The cost components in this study are given as follows:

c_F: Corrective (failure) replacement cost for per failure;

c_T: Preventive replacement cost at prescheduled replacement time;

c_Y: Preventive replacement cost at a random opportunity.

Based on above notations, we make the following assumption:

Assumption 1: $c_F > c_T \geq c_Y$.

It would be reasonable to assume that the cost of failure replacement is the highest, while the cost of opportunistic replacement is lower than that of preventive replacement. This is because opportunistic replacement involves acquiring a spare part for an item at a cheaper cost, albeit at an unscheduled time. It is important to mention that the discrete time models should be considered carefully, because it has probability that one of three replacement options may arrive simultaneously: corrective (failure) replacement (C_a), scheduled preventive replacement (S_c), and opportunistic replacement (O_p). To clearly order the priority of three options, the following priority relationship is introduced:

Definition 1: The replacement option P has a priority to the replacement option Q, if $P \succ Q$.

2.2 Renewal Reward Approach

2.2.1 AR Model

First, we revisit the discrete time AR model [13]. Nakagawa and Osaki [13] implicitly assumed that failure replacement has priority to preventive replacement. As pointed out in Section 1.1, this opinion may not be right. Here we suppose that there are two options for replacement. If a system breaks down at time, then the unit is replaced by new one immediately, otherwise, the system is replaced preventively at a prescheduled preventive replacement time. The discrete-time AR model is illustrated in Figure 2.1. According to definition 1, one has possibility that two different models should be formulated in discrete time AR model:

(1) Model 1: $S_c \succ C_a$,

(2) Model 2: $C_a \succ S_c$.

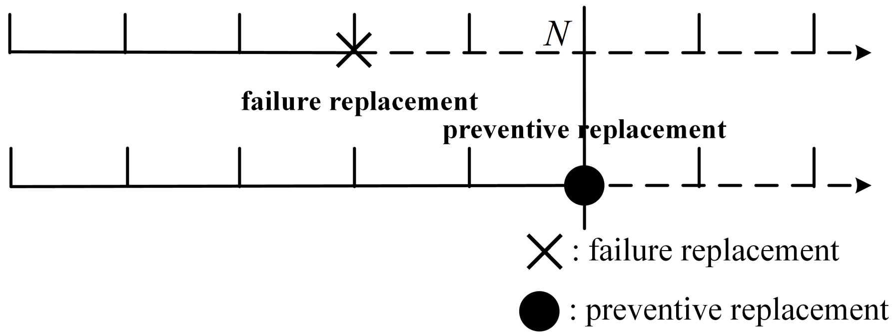

Figure 2.1: AR model.

For Model 1 and Model 2, we can calculate the probability that a system is replaced at time n $(= 0, 1, \cdots)$:

$$h_{aj}(n) = \begin{cases} f_Y(n) & 0 \le n \le N-1 \\ \bar{F}_Y(n-1) & n = N \\ 0 & n \ge N+1, \end{cases} \tag{2.1}$$

where $\sum_{n=0}^{\infty} h_{aj}(n) = 1$ for Model j $(= 1, 2)$.

From Eq. (2.1), we compute the expected lengths of one cycle, $A_a(N)$, which is the same in both models. We have

$$\begin{aligned} A_a(N) &= \sum_{n=1}^{N-1} n f_Y(n) + N\bar{F}_Y(N-1) \\ &= \sum_{n=1}^{N} \bar{F}_Y(n-1). \end{aligned} \tag{2.2}$$

We compute the expected costs of one cycle:

$$B_{a1}(N) = c_F \sum_{n=1}^{N-1} f_Y(n) + c_T \bar{F}_Y(N-1), \tag{2.3}$$

$$B_{a2}(N) = c_F \sum_{n=1}^{N} f_Y(n) + c_T \bar{F}_Y(N). \tag{2.4}$$

Then, the long-run costs per unit time in the steady state are denoted as $EC_{aj}(N)$ for Model j $(= 1, 2)$, from the renewal reward theory [37],

$$EC_{aj}(N) = \frac{B_{aj}(N)}{A_a(N)}, \tag{2.5}$$

and our interest is to find the optimal N^* minimizing $EC_{aj}(N)$.

We give the following the non-linear functions:

$$w_{a1}(N) = R_Y(N) \sum_{n=1}^{N} \bar{F}_Y(n-1) - F_Y(N-1), \tag{2.6}$$

$$w_{a2}(N) = r_Y(N+1) \sum_{n=1}^{N} \bar{F}_Y(n-1) - F_Y(N), \tag{2.7}$$

where $R_Y(n) = f_Y(n)/\bar{F}_Y(n)$ and $r_Y(n) = f_Y(n)/\bar{F}_Y(n-1)$ are failure rate and shifted failure rate functions, respectively.

For more detailed relationship between $R_Y(n)$ and $r_Y(n)$, see Lemma 7.1 in Appendix.

Theorem 2.1. *(I) Suppose that the lifetime Y is strictly increasing failure rate (IFR) under Assumption 1.*

(i) If $w_{aj}(\infty) > c_T/(c_F - c_T)$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{aj}(N^* - 1) < c_T/(c_F - c_T)$ and $w_{aj}(N^*) \geq c_T/(c_F - c_T)$.

(ii) If $w_{aj}(\infty) \leq c_T/(c_F - c_T)$, then the optimal scheduled preventive replacement time is $N^* \to \infty$, and it is optimal to carry out only the failure replacement.

(II) Suppose that the lifetime Y is strictly decreasing failure rate (DFR) under Assumption 1. Then the optimal scheduled preventive replacement time is given by $N^* \to \infty$ or $N^* = 1$.

For the proof, consult Appendix 7.4.1.

We can obtain the optimal expected costs per unit time in steady state from Theorem 2.1 straightforwardly.

Theorem 2.2. *For Model j $(= 1, 2)$, suppose that the lifetime Y is strictly IFR, and $w_{aj}(\infty) > c_T/(c_F - c_T)$, under Assumption 1. Then the minimum expected costs per unit time in the steady state have the lower and upper bounds:*

$$V_{aj}(N^* - 1) < EC_{aj}(N^*) \leq V_{aj}(N^*), \tag{2.8}$$

where

$$V_{a1}(N) = (c_F - c_T)R_Y(N), \tag{2.9}$$

$$V_{a2}(N) = (c_F - c_T)r_Y(N + 1). \tag{2.10}$$

2.2.2 DD Model

Next, we concern about the discrete time DD model. Dekker and Dijkstra [9] discussed the continues time DD model where the arrival of opportunities obeys a Poisson process. In addition, they also proposed a control-limit policy where the preventive replacement is made at first opportunity after a prescheduled replacement time. The discrete-time DD model is depicted in Figure 2.2. Here, it is assumed that the interval-arrival times between two consecutive opportunities for replacements, X, obey the i.i.d. geometric distribution $\Pr\{X =$

$x\} = f_X(x) = p(1-p)^{x-1}$ $(x = 1, 2, \cdots)$ with survivor function $\Pr\{X \geq x\} = f_X(x) = (1-p)^{x-1} = \bar{G}_X(n-1)$. According to the definition 1, we may consider two different models as follows:

(1) Model 1: $O_p \succ C_a$,

(2) Model 2: $C_a \succ O_p$.

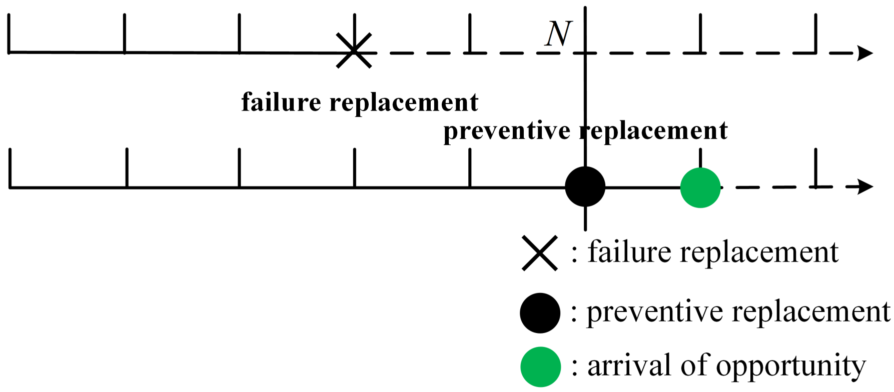

Figure 2.2: DD model.

For Model 1 and Model 2, we can calculate the probability that the system is replaced at time n $(= 0, 1, \cdots)$ is given by

$$h_{oj}(n) = \begin{cases} f_Y(n) & 0 \leq n \leq N \\ f_Y(n)(1-p)^{n-N} + p(1-p)^{n-N-1}\bar{F}_Y(n-1) & n \geq N+1, \end{cases} \tag{2.11}$$

where $\sum_{n=0}^{\infty} h_{oj}(n) = 1$ for Model j $(= 1, 2)$.

According to Eq. (2.11), we can obtain the expected time lengths of one cycle, $A_o(N)$, in two cases are exactly same:

$$A_o(N) = \sum_{n=1}^{N} \bar{F}_Y(n-1) + \sum_{n=N+1}^{\infty} \bar{F}_Y(n-1)(1-p)^{n-N-1}. \tag{2.12}$$

We also can compute the expected costs of one cycle, $B_{oj}(N)$, for Model j $(= 1, 2)$:

$$B_{o1}(N) = c_F \sum_{n=1}^{N} f_Y(n) + c_F \sum_{n=N+1}^{\infty} f_Y(n)(1-p)^{n-N} + c_Y \sum_{n=N+1}^{\infty} \bar{F}_Y(n-1)p(1-p)^{n-N-1}, \tag{2.13}$$

$$B_{o2}(N) = c_F \sum_{n=1}^{N} f_Y(n) + c_F \sum_{n=N+1}^{\infty} f_Y(n)(1-p)^{n-N-1} \\ + c_Y \sum_{n=N+1}^{\infty} \bar{F}_Y(n)p(1-p)^{n-N-1}. \tag{2.14}$$

Then, the expected costs per unit time in the steady state are formulated as $EC_{oj}(N)$ for Model j $(= 1, 2)$:

$$EC_{oj}(N) = \frac{B_{oj}(N)}{A_o(N)}, \tag{2.15}$$

and our objective is to determine the optimal N^* minimizing $EC_{oj}(N)$.

Similar to AR model in discrete time, we define the following non-linear functions:

$$w_{o1}(N) = H_Y(N)A_o(N) - \left[F_Y(N) + \sum_{n=N+1}^{\infty} f_Y(n)(1-p)^{n-N}\right], \tag{2.16}$$

$$w_{o2}(N) = h_Y(N+1)A_o(N) - \left[F_Y(N) + \sum_{n=N+1}^{\infty} f_Y(n)(1-p)^{n-N-1}\right], \tag{2.17}$$

where

$$H_Y(N) = \frac{\sum_{n=N+1}^{\infty} f_Y(n)(1-p)^{n-N}}{\sum_{n=N+1}^{\infty} \bar{F}_Y(n)(1-p)^{n-N}}, \tag{2.18}$$

$$h_Y(N+1) = \frac{\sum_{n=N+1}^{\infty} f_Y(n+1)(1-p)^{n-N}}{\sum_{n=N+1}^{\infty} \bar{F}_Y(n)(1-p)^{n-N}}. \tag{2.19}$$

For more detailed relationship between $R_Y(n)$ ($r_Y(n)$) and $H_Y(n)$ ($h_Y(n)$), see Lemma 7.2 and 7.3 in Appendix. We characterize the optimal replacement time limit N^* that minimizes $EC_{oj}(N)$. The proof can be found in Appendix 7.4.2.

Theorem 2.3. *(I) Suppose that the lifetime Y is strictly IFR under Assumption 1.*

(i) If $w_{oj}(\infty) > c_Y/(c_F - c_Y)$, then there exists at least one (at most two) optimal prevetive replacement time limit N^* satisfying $w_{oj}(N^* - 1) < c_Y/(c_F - c_Y)$ and $w_{oj}(N^*) \geq c_Y/(c_F - c_Y)$.

(ii) If $w_{oj}(\infty) \leq c_Y/(c_F - c_Y)$, then the optimal DD time limits are $N^* \to \infty$, and it is optimal to carry out only the failure replacement.

(II) Suppose that the lifetime Y is strictly DFR under Assumption 1. Then the optimal DD time limits are given by $N^* \to \infty$ or $N^* = 0$.

We also directly get the following theory from Theorem 2.3.

Theorem 2.4. *For Model j $(=1,2)$, suppose that the lifetime Y is strictly IFR, and $w_{oj}(\infty) > c_Y/(c_F - c_Y)$, under Assumption 1. Then the minimum expected costs per unit time in the steady state have the lower and upper bounds:*

$$V_{oj}(N^*-1) < EC_{oj}(N^*) \leq V_{oj}(N^*), \tag{2.20}$$

in which

$$V_{o1}(N) = (c_F - c_Y)H_Y(N), \tag{2.21}$$

$$V_{o2}(N) = (c_F - c_Y)h_Y(N+1). \tag{2.22}$$

2.3 NPV Method

2.3.1 AR Model

We denote the discounted factor β $(0 < \beta < 1)$ to represent the expected NPV of the unit cost. We first derive the preventive replacement policies in the AR model. In the NPV formulation, the expected total discounted costs over an infinite time horizon, $TC_{aj}(N, \beta)$, for Model j $(=1,2)$ are given by

$$\begin{aligned} TC_{aj}(N,\beta) = [c_F + TC_{aj}(N,\beta)] \sum_{n=1}^{N-1} \beta^n f_Y(n) \\ + [c_T + TC_{aj}(N,\beta)]\, \beta^N \bar{F}_Y(N-1). \end{aligned} \tag{2.23}$$

From a few algebraic manipulations, we can obtain,

$$TC_{aj}(N,\beta) = \frac{B_{aj}(N,\beta)}{1 - A_a(N,\beta)}. \tag{2.24}$$

In above function, $A_a(N, \beta)$ is the NPV of one unit cost during the renewal cycle:

$$A_a(N,\beta) = 1 - \frac{\beta}{1-\beta} \sum_{n=1}^{N} \beta^n \bar{F}_Y(n-1). \tag{2.25}$$

$B_{aj}(N, \beta)$ for Model j $(=1,2)$ are the expected total discounted costs during the renewal cycle :

$$B_{a1}(N,\beta) = c_F \sum_{n=1}^{N-1} \beta^n f_Y(n) + c_T \beta^N \bar{F}_Y(N-1), \tag{2.26}$$

$$B_{a2}(N,\beta) = c_F \sum_{n=1}^{N} \beta^n f_Y(n) + c_T \beta^N \bar{F}_Y(N). \tag{2.27}$$

It is evident from the well-known L'Hopital's theorem that

$$\lim_{\beta \to 1} (1-\beta) TC_{aj}(N,\beta) = EC_{aj}(N). \tag{2.28}$$

Next, we define the non-linear functions for a fixed β:

$$w_{a1}(N \mid \beta) = \left[\frac{(c_F - c_T)}{1-\beta} R_Y(N) - c_T\right] [1 - A_a(N,\beta)] - B_{a1}(N,\beta), \tag{2.29}$$

$$w_{a2}(N \mid \beta) = \left[\frac{\beta(c_F - c_T)}{1-\beta} r_Y(N+1) - c_T\right] [1 - A_a(N,\beta)] - B_{a2}(N,\beta). \tag{2.30}$$

In the NPV formulation, the optimal AR polices can be obtained (refer to the proof of Theorem 2.1 in Appendix).

Theorem 2.5. *(I) Suppose that the lifetime Y is strictly IFR under Assumption 1.*

(i) If $w_{aj}(\infty \mid \beta) > 0$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{aj}(N^* - 1 \mid \beta) < 0$ and $w_{aj}(N^* \mid \beta) \geq 0$.

(ii) If $w_{aj}(\infty \mid \beta) \leq 0$, then the optimal scheduled preventive replacement time is $N^* \to \infty$, and it is optimal to carry out only the failure replacement.

(II) Suppose that the lifetime Y is strictly DFR under Assumption 1. Then the optimal scheduled preventive replacement times are given by $N^* \to \infty$ or $N^* = 1$.

We can obtain the optimal expected costs per unit time in steady state from Theorem 2.5 straightforwardly.

Theorem 2.6. *For Model j $(= 1, 2)$, suppose that the lifetime Y is strictly IFR, and $w_{aj}(\infty \mid \beta) > 0$, under Assumption 1. Then the minimum $TC(N \mid \beta)$ have the lower and upper bounds for a fixed β:*

$$V_{aj}(N^* - 1 \mid \beta) < TC_{aj}(N^* \mid \beta) \leq V_{aj}(N^* \mid \beta), \tag{2.31}$$

where

$$V_{a1}(N \mid \beta) = \frac{(c_F - c_T)}{1-\beta} R_Y(N) - c_T, \tag{2.32}$$

$$V_{a2}(N \mid \beta) = \frac{\beta(c_F - c_T)}{1-\beta} r_Y(N+1) - c_T. \tag{2.33}$$

2.3.2 DD Model

We calculate the DD model with NPV approach. The expected NPV value of one unit cost during the renewal cycle, $A_o(N,\beta)$, for Model j $(=1,2)$ are obtained as

$$A_o(N,\beta) = \sum_{n=0}^{N} \beta^n f_Y(n) + \sum_{n=N+1}^{\infty} \beta^n \Big[f_Y(n)(1-p)^{n-N} + p(1-p)^{n-N-1} \bar{F}_Y(n-1) \Big]. \tag{2.34}$$

$B_{oj}(N,\beta)$ for Model j $(=1,2)$ are the expected total discounted costs during the renewal cycle :

$$B_{o1}(N,\beta) = c_F \sum_{n=0}^{N} \beta^n f_Y(n) + c_F \sum_{n=N+1}^{\infty} \beta^n f_Y(n)(1-p)^{n-N} + c_Y \sum_{n=N+1}^{\infty} \beta^n p(1-p)^{n-N-1} \bar{F}_Y(n-1), \tag{2.35}$$

$$B_{o2}(N,\beta) = c_F \sum_{n=0}^{N} \beta^n f_Y(n) + c_F \sum_{n=N+1}^{\infty} \beta^n f_Y(n)(1-p)^{n-N-1} + c_Y \sum_{n=N+1}^{\infty} \beta^n p(1-p)^{n-N-1} \bar{F}_Y(n). \tag{2.36}$$

Then, we obtain the NPV value of expected total costs, $TC_{oj}(N,\beta)$, for Model j $(=1,2)$:

$$TC_{oj}(N,\beta) = \frac{B_{oj}(N,\beta)}{1 - A_o(N,\beta)}. \tag{2.37}$$

It is evident from the well-known L'Hopital's theorem that

$$\lim_{\beta \to 1} (1-\beta)\, TC_{oj}(N,\beta) = EC_{oj}(N). \tag{2.38}$$

Next, we define the non-linear functions for a fixed β:

$$w_{o1}(N \mid \beta) = \left[\frac{(c_F - c_Y)}{1-\beta} H_Y(N,\beta) - c_Y \right] [1 - A_o(N,\beta)] - B_{o1}(N,\beta), \tag{2.39}$$

$$w_{o2}(N \mid \beta) = \left[\frac{\beta(c_F - c_Y)}{1-\beta} h_Y(N+1,\beta) - c_Y \right] [1 - A_o(N,\beta)] - B_{o2}(N,\beta), \tag{2.40}$$

where

$$H_Y(N,\beta) = \frac{\sum_{n=N+1}^{\infty} f_Y(n)\beta^n (1-p)^{n-N}}{\sum_{n=N+1}^{\infty} \bar{F}_Y(n)\beta^n (1-p)^{n-N}}, \tag{2.41}$$

$$h_Y(N+1,\beta) = \frac{\sum_{n=N+1}^{\infty} f_Y(n+1)\beta^n(1-p)^{n-N}}{\sum_{n=N+1}^{\infty} \bar{F}_Y(n)\beta^n(1-p)^{n-N}}. \tag{2.42}$$

For additional information regarding the monotonic relationship $R_Y(n)$ ($r_Y(n)$) and $H_Y(n,\beta)$ ($h_Y(n,\beta)$), refer to Lemma 7.4 and 7.5. In the NPV formulation, the optimal DD policies can be described as follows (refer to the proof of Theorem 2.2 in Appendix).

Theorem 2.7. *(I) Suppose that the lifetime Y is strictly IFR under Assumption 1.*

(i) If $w_{oj}(\infty \mid \beta) > 0$, then there exists at least one (at most two) optimal prevetive replacement time limit N^* which satisfies $w_{oj}(N^*-1 \mid \beta) < 0$ and $w_{oj}(N^* \mid \beta) \geq 0$.

(ii) If $w_{oj}(\infty \mid \beta) \leq 0$, then the optimal DD time limit is $N^* \to \infty$, and it is optimal to carry out only the failure replacement.

(II) Suppose that the lifetime Y is strictly DFR under Assumption 1. Then the optimal DD time limit are given by $N^* \to \infty$ or $N^* = 0$.

We can obtain the optimal expected costs per unit time in steady state from Theorem 2.7 straightforwardly.

Theorem 2.8. *For Model j $(=1,2)$, suppose that the lifetime Y is strictly IFR, and $w_{oj}(\infty \mid \beta) > 0$, under Assumption 1. Then the minimum $TC(N \mid \beta)$ have the lower and upper bounds for a fixed β:*

$$V_{oj}(N^*-1 \mid \beta) < TC_{oj}(N^* \mid \beta) \leq V_{oj}(N^* \mid \beta), \tag{2.43}$$

where

$$V_{o1}(N \mid \beta) = \frac{(c_F - c_Y)}{1-\beta} H_Y(N,\beta) - c_Y, \tag{2.44}$$

$$V_{o2}(N \mid \beta) = \frac{\beta(c_F - c_Y)}{1-\beta} r_Y(N+1,\beta) - c_Y. \tag{2.45}$$

2.4 Unification with Probabilistic Priority

2.4.1 Renewal Reward Method

We have discussed AR and DD polices in discrete time. However, it is worth noting that the priority of replacement is not always deterministic. That is to say, the occurrence of priority is probabilistic and may change at each decision point of replacement. Suppose that each priority corresponding to Model j ($=1,2$) occurs with probability p_j ($0 \leq p_j \leq 1$), where $\sum_{j=1}^{2} p_j = 1$.

First, let's consider a discrete time AR model. Since the mean time lengths of one cycle in Model j ($=1,2$) are all exactly the same, the associated mean time length in our unified model is given by $A_{a3}(N) = A_a(N)$ in Eq. (2.2). Instead, the expected total cost during one cycle, $B_{a3}(N)$, with probabilistic priority is given by $B_{a3}(N) = p_1 B_{a1}(N) + p_2 B_{a2}(N)$ with Eqs. (2.3) and (2.4). The underlying problem is simply formulated as $\min_N EC_{a3}(N)$, where $EC_{a3}(N) = B_{a3}(N)/A_a(N)$. Define $w_{a3}(N) = \sum_{j=1}^{2} p_j V_{aj}(N) A_a(N) - B_{a3}(N)$ with Eqs. (2.9) and (2.10). Then it can be seen that $w_{a3}(N+1) - w_{a3}(N) = \sum_{j=1}^{2} p_j \{w_{aj}(N+1) - w_{aj}(N)\} A_a(N+1)$. Hence, for $p_j > 0$, necessary conditions of strictly increasing $w_{aj}(N)$ are to hold all conditions in Theorem 2.1.

Theorem 2.9. *Suppose that the lifetime Y is strictly increasing, and $w_{a3}(\infty) > 0$ under Assumption 1. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds:*

$$V_{a3}(N^*-1) < EC_{a3}(N^*) \leq V_{a3}(N^*), \tag{2.46}$$

where

$$V_{a3}(N) = \sum_{j=1}^{2} p_j V_{aj}(N). \tag{2.47}$$

Next, let's consider the discrete-time DD model. Since the mean time length of one cycle and the expected total cost during one cycle are given by $A_{o3}(N) = A_o(N)$ in Eq. (2.12) and $B_{o3}(N) = p_1 B_{o1}(N) + p_2 B_{o2}(N)$ with Eqs. (2.13) and (2.14). The underlying problem is simply formulated as $\min_N EC_{o3}(N)$, where $EC_{o3}(N) = B_{o3}(N)/A_o(N)$. Define $w_{o3}(N) = \sum_{j=1}^{2} p_j V_{oj}(N) A_o(N) - B_{o3}(N)$ with Eqs. (2.21) and (2.22). Then, one has $w_{o3}(N+1) - w_{o3}(N) = \sum_{j=1}^{2} p_j \{w_{oj}(N+1) - w_{oj}(N)\} A_o(N+1)$, and finds that necessary conditions of strictly increasing $w_{o3}(N)$ are to hold all conditions in Theorem 2.3.

Theorem 2.10. *Suppose that $w_{o3}(n)$ is strictly increasing, $w_{o3}(\infty) > 0$, under Assumption 1. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds:*

$$V_{o3}(N^*-1) < EC_{o3}(N^*) \leq V_{o3}(N^*), \tag{2.48}$$

where

$$V_{o3}(N) = \sum_{j=1}^{2} p_j V_{oj}(N). \tag{2.49}$$

2.4.2 NPV Method

Here we calculate the unified AR and DD with discounting. First, let's consider a discrete time AR model. Since the mean time lengths of one cycle in Model j $(= 1, 2)$ are all exactly the same, the associated mean time length in our unified model is given by $A_{a3}(N, \beta) = A_a(N, \beta)$ in Eq. (2.25). Instead, the expected total cost during one cycle, $B_{a3}(N, \beta)$, with probabilistic priority is given by $B_{a3}(N, \beta) = p_1 B_{a1}(N, \beta) + p_2 B_{a2}(N, \beta)$ with Eqs. (2.26) and (2.27). The underlying problem is simply formulated as $\min_N TC_{a3}(N, \beta)$, where $TC_{a3}(N, \beta) = B_{a3}(N, \beta)/[1 - A_a(N, \beta)]$. Define $w_{a3}(N \mid \beta) = \sum_{j=1}^{2} p_j V_{aj}(N, \beta) A_a(N, \beta) - B_{a3}(N, \beta)$ with Eqs. (2.32) and (2.33). Then it can be seen that $w_{a3}(N + 1 \mid \beta) - w_{a3}(N \mid \beta) = \sum_{j=1}^{2} p_j \{w_{aj}(N + 1 \mid \beta) - w_{aj}(N \mid \beta)\} A_a(N + 1, \beta)$. Hence, for $p_j > 0$, necessary conditions of strictly increasing $w_{aj}(N \mid \beta)$ are to hold all conditions in Theorem 2.5.

Theorem 2.11. *Suppose that the lifetime Y is strictly increasing, and $w_{a3}(\infty \mid \beta) > 0$ under Assumption 1. Then the minimum $TC_{a3}(N \mid \beta)$ has the lower and upper bounds:*

$$V_{a3}(N^* - 1 \mid \beta) < TC_{a3}(N^* \mid \beta) \le V_{a3}(N^* \mid \beta), \tag{2.50}$$

where

$$V_{a3}(N \mid \beta) = \sum_{j=1}^{2} p_j V_{aj}(N \mid \beta). \tag{2.51}$$

Next, let's consider the discrete time DD model. Since the mean time length of one cycle and the expected total cost during one cycle are given by $A_{o3}(N, \beta) = A_o(N, \beta)$ in Eq. (2.34) and $B_{o3}(N, \beta) = p_1 B_{o1}(N, \beta) + p_2 B_{o2}(N, \beta)$ with Eqs. (3.26) and (2.36). The underlying problem is simply formulated as $\min_N TC_{o3}(N, \beta)$, where $TC_{o3}(N, \beta) = B_{o3}(N, \beta)/[1 - A_o(N, \beta)]$. Define $w_{o3}(N \mid \beta) = \sum_{j=1}^{2} p_j V_{oj}(N, \beta) A_o(N \mid \beta) - B_{o3}(N \mid \beta)$ with Eqs. (2.44) and (2.45). Then, one has $w_{o3}(N + 1, \beta) - w_{o3}(N, \beta) = \sum_{j=1}^{2} p_j \{w_{oj}(N + 1 \mid \beta) - w_{oj}(N \mid \beta)\} A_o(N + 1, \beta)$, and finds that necessary conditions of strictly increasing $w_{o3}(N \mid \beta)$ are to hold all conditions in Theorem 2.7.

Theorem 2.12. *Suppose that $w_{o3}(n \mid \beta)$ is strictly increasing, $w_{o3}(\infty \mid \beta) > 0$, under Assumption 1. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds:*

$$V_{o3}(N^* - 1 \mid \beta) < TC_{o3}(N^* \mid \beta) \leq V_{o3}(N^* \mid \beta), \tag{2.52}$$

where

$$V_{o3}(N \mid \beta) = \sum_{j=1}^{2} p_j V_{oj}(N \mid \beta). \tag{2.53}$$

2.5 A General Optimizing Framework

2.5.1 Introduction

Consider a system and a decision variable N (> 0), which affects the timing of maintenance action(s) renewing the system. In the simplest case of the framework, N represents the time at which the system is replaced preventively. Alternatively, N may also be a critical time after which the first suitable moment (opportunity) is awaited to renew the system. The problem is to determine the value of N that optimizes a given objective function. We consider both average and discounted costs. As the execution of the action(s) implies a renewal of the system, we can apply renewal theory and obtain for the long-term average costs for Model j

$$EC_j(N) = \frac{B_j(N)}{A(N)} \tag{2.54}$$

where $B_j(N)$ and $A(N)$ denote the expected cycle costs and length, respectively. We make the following definitions:

Definition 2: Both $B_j(N)$ and $A(N)$ are absolute discrete time functions of N, i.e., $B_j(N) = B_j(0) + \sum_{n=1}^{N} b_j(n)$ and $A(N) = A(0) + \sum_{n=1}^{N} a(n)$ for some functions $b_j(n)$ and $a(n)$.

Definition 3: $A(0) \geq 0$, $a(n) \geq 0$ for all $n > 0$ and $b_j(n) = 0$ if $a(n) = 0$.

From above assumptions, it follows that there exists a function $m_j(n)$ such that $b_j(n) = a(n)m_j(n)$, $n > 0$. To simplify notation, we rewrite b_j for $B_j(0)$ and a for $A(0)$. Accordingly, we can rewrite $EC_j(N)$ as

$$EC_j(N) = \frac{b_j + \sum_{n=1}^{N} a(n)m_j(n)}{a + \sum_{n=1}^{N} a(n)}. \tag{2.55}$$

The above equation is put central in this paper. Notice that for $N = 0$, the expected cycle cost is b_j, and typically this represents the cost of a preventive replacement c_T. The quantity $m_j(n)$ can be interpreted as the expected deterioration cost rate and $a(n)$ common denotes a survival function. The examples and the link to the marginal cost analysis given below. The framework covers quite some models.

The analysis of the framework follows similar the continuous time model. First, taking the difference of $EC_j(N)$ with respect to N, we have

$$\begin{aligned} EC_j(N+1) - EC_j(N) &= \frac{[m_j(N+1) - EC_j(N)]\, a(N+1)}{A(N+1)} \\ &= \frac{[\Psi_j(N) - b_j]\, a(N+1)}{A(N+1)A(N)}, \end{aligned} \tag{2.56}$$

where $\Psi_j(N) = m_j(N+1)A(N) - \sum_{n=1}^{N} m_j(n)a(n)$. Hence, if $m_j(N+1) - EC_j(N) \geq 0$, then $EC_j(N+1) - EC_j(N) \geq 0$. Alternatively, if $\Psi_j(N) - b_j \geq 0$, then $EC_j(N+1) - EC_j(N) \geq 0$. We give the following theorem.

Theorem 2.13. *Suppose $a(N+1) > 0$ for all $N > 0$,*
(i) If $m_j(N+1)$ is non-increasing on $[\underline{N}, \overline{N}]$ and $\Psi_j(\underline{N}) < b_j$, then $EC_j(N)$ is decreasing on $[\underline{N}, \overline{N}]$.
(ii) If $m_j(N+1)$ increases strictly for $N > \underline{N}$, where $\Psi_j(\underline{N}) < b_j$, and $\Psi_j(\overline{N}) > b_j$, then $EC_j(N)$ has at least one (at most two) minimum, say $EC_j(N^*)$ in N^*. Moreover, the optimal preventive replacement time N^* satifies
$\Psi_j(N^* - 1) < b_j$ and $\Psi_j(N^*) \geq b_j$. Further, we can obtain the following inequalities:

$$m_j(N^*) - EC_j(N^* - 1) < 0 \;\&\; m_j(N^* + 1) - EC_j(N^*) \geq 0. \tag{2.57}$$

$$\begin{aligned} &m_j(N) - EC_j(N-1) < 0, \;\; for\; \underline{N} < N \leq N^* - 1 \\ &\& \\ &m_j(N+1) - EC_j(N) \geq 0, \;\; for\; N^* < N \leq \overline{N}. \end{aligned} \tag{2.58}$$

$$\begin{aligned} &m_j(N) - EC_j(N^* - 1) < 0, for\; \underline{N} < N \leq N^* - 1 \\ &\& \\ &m_j(N+1) - EC_j(N^*) \geq 0, for\; N^* < N \leq \overline{N}. \end{aligned} \tag{2.59}$$

(iii) If $\Psi_j(N) < b_j$ *for all* $N > \underline{N}$*, then* $EC_j(N)$ *is decreasing for* $N > \underline{N}$.

(iv) Suppose that $m_j(N)$ *is increasing for* $N > \underline{N}$ *and that* $\Psi_j(\underline{N}) < b_j$ *and* $\Psi_j(\overline{N}) > b_j$*, if one of the following conditions hold*

1. $\lim_{N\to\infty} m_j(N) = \infty$.

2. $\lim_{N\to\infty} m_j(N) > \lim_{N\to\infty} EC_j(N)$.

3. $\lim_{N\to\infty} B_j(N) = \infty$, $\lim_{N\to\infty} m_j(N) = m$, *for* $m > 0$,
   and $\lim_{N\to\infty} \sum_{n=0}^{N} [m - m_j(n)]\, b_j(n) > b_j - am$.

For the proof, see the Appendix 7.4.3. Theorem 1 implies that for optimization one only needs to consider those regions where $m_j(N)$ is increasing. Furthermore, it says that a basic criterion in which at every moment we consider whether to delay the replacement or not, is average optimal. That is, the expected cost of delay the replacement to level $N+1$, being $m_j(N+1)a(N+1)$, should be compared to the minimal average costs over an interval of the same length, being $EC_j(N^*)a(N+1)$. Hence, if $m_j(N+1)$ is larger than or equals $EC_j(N^*)$, the deferment costs are larger and we should replace. This result gives a structuring of the optimal policy and it gives an explanation of why a policy is optimal. Next, we take the discrete time AR and DD models as example.

2.5.2 Examples

Example 2.1 (AR Model)**.** *According to Theorem 2.13, the parameters* b_j *are both preventive replacement* c_T *in Model 1 and 2. The functions* $m_j(n)$ *is* $(c_F - c_T)\, R_Y(n)$ *in Model 1 and* $(c_F - c_T)\, r_Y(n+1)$ *in Model 2. The function* $a(n)$ *is the survival function* $\bar{F}_Y(n-1)$ *with* $a = 0$.

Here, we take Model 1 as an example. If $R_Y(n)$ is strictly IFR, $m_1(\infty) = (c_F - c_T)\, R_Y(\infty)$. In addition, $EC_1(\infty) = c_T / \sum_{n=1}^{\infty} \bar{F}_Y(n-1)$. Hence, we can see that the condition $m_1(\infty) > EC_1(\infty)$ equals

$$R_Y(\infty) \sum_{n=1}^{\infty} \bar{F}_Y(n-1) > c_F / (c_F - c_T)\,. \tag{2.60}$$

It is clear that the analysis process above is exactly the same as Theorem 2.1.

Example 2.2 (AR Model with NPV). *According to Theorem 2.13, the parameters* b_j *are* 0 *in Model 1 and preventive replacement* c_T *in Model 2. The functions* $m_j(n)$ *is* $[(c_F - c_T)\, R_Y(n)/(1-\beta)] - c_T$ *in Model 1 and* $[\beta\, (c_F - c_T)\, r_Y(n+1)/(1-\beta)] - c_T$ *in Model 2. The function* $a(n)$ *is* $(1-\beta)\beta^n \bar{F}_Y(n-1)$ *with* $a = 0$.

Example 2.3 (DD Model). *In general framework, the parameters* b_j *are* $c_Y + (c_F - c_Y)\sum_{n=1}^{\infty} f_Y(n)(1-p)^n$ *in Model 1 and* $c_Y + (c_F - c_Y)\sum_{n=1}^{\infty} f_Y(n)(1-p)^{n-1}$ *in Model 2. The functions* $b_j(n)$ *are* $(c_F - c_Y)\sum_{n=1}^{\infty} \bar{F}_Y(n)(1-p)^{n-N-1}$ *in Model 1 and* $p(c_F - c_Y)\sum_{n=1}^{\infty} \bar{F}_Y(n+1)(1-p)^{n-N-1}$ *in Model 2. The functions* $m_j(n)$ *is* $(c_F - c_Y)\, H_Y(n)$ *in Model 1 and* $(c_F - c_Y)\, h_Y(n+1)$ *in Model 2. The function* $a(n)$ *is* $\sum_{n=N+1}^{\infty} \bar{F}_Y(n-1)(1-p)^{n-N-1}$ *with* $a = \sum_{n=1}^{\infty} \bar{F}_Y(n-1)(1-p)^{n-1}$.

Example 2.4 (DD Model with NPV). *In general framework, the parameters* b_j *are* $c_F \sum_{n=1}^{\infty} \beta^n f_Y(n)(1-p)^n + c_Y \sum_{n=1}^{\infty} \beta^n \bar{F}_Y(n-1)p(1-p)^{n-1}$ *in Model 1 and* $c_F \sum_{n=1}^{\infty} \beta^n f_Y(n)(1-p)^{n-1} + c_Y \sum_{n=1}^{\infty} \beta^n \bar{F}_Y(n)p(1-p)^{n-1}$ *in Model 2. The functions* $m_j(n)$ *is* $[(c_F - c_Y)\, H_Y(n,\beta)/(1-\beta)] - c_Y$ *in Model 1 and* $[\beta\, (c_F - c_Y)\, h_Y(n+1,\beta)/(1-\beta)] - c_Y$ *in Model 2. The function* $a(n)$ *is* $\sum_{n=N+1}^{\infty} \beta^n \bar{F}_Y(n-1)(1-p)^{n-N-1}$ *with* $a = \sum_{n=1}^{\infty} \beta^n \bar{F}_Y(n-1)(1-p)^{n-1}$.

2.6 Numerical Experiments

In this section we present a case study for the preventive replacement of long-life products. A pole air switch is a kind of section switches to distribute the power to several regions and is equipped on a pole (see Figure 2.3). The continuous-time replacement model for the similar electrical devices was reported in Holland and McLean [39]. In our example, the features of the product are summarized as follows.

(A) The pole air switch is a non-repairable item and is highly reliable with relatively long lifetime.

(B) If it fails, the power is down in the covered regions until the failed item is replaced by a new one.

(C) The preventive replacement is planned with the time unit of year and can be described as a discrete-time model.

(D) In addition to the scheduled preventive replacement, a certain amount of spare switches are ready in on-hand inventory, and the non-failed switches in the area are randomly selected for opportunistic replacement.

Figure 2.3: Pole air switch.

The 112 failure data of pole air switches are recorded during twenty-five years in Hiroshima City, Japan. Figure 2.4 illustrates the relative frequency of the failure data. Suppose that the (discrete) failure time obeys the following discrete Weibull distribution:

$$f_Y(n) = (1-r)^{(n-1)^\alpha} - (1-r)^{n^\alpha}, \tag{2.61}$$

where $0 < r < 1$, $\alpha > 0$, and $n = 1, 2\cdots$. From the definition above, the reliability function and its failure rate are given by

$$\overline{F}_Y(n-1) = (1-r)^{(n-1)^\alpha} \tag{2.62}$$

and

$$r_Y(n) = 1 - (1-r)^{n^\alpha - (n-1)^\alpha}, \tag{2.63}$$

respectively. When $\beta = 1$, then it can be reduced to the geometric distribution with the failure rate $r_Y(n) = r$. The discrete Weibull distribution in Eq. (2.61) was introduced first by Nakagawa and Osaki [40]. Later, Stein and Datter [41] defined a different discrete Weibull distribution which is not our case in this paper. AliKhan et al [42] developed a simple parameter estimation method as well as the moment method and the maximum likelihood method for the original

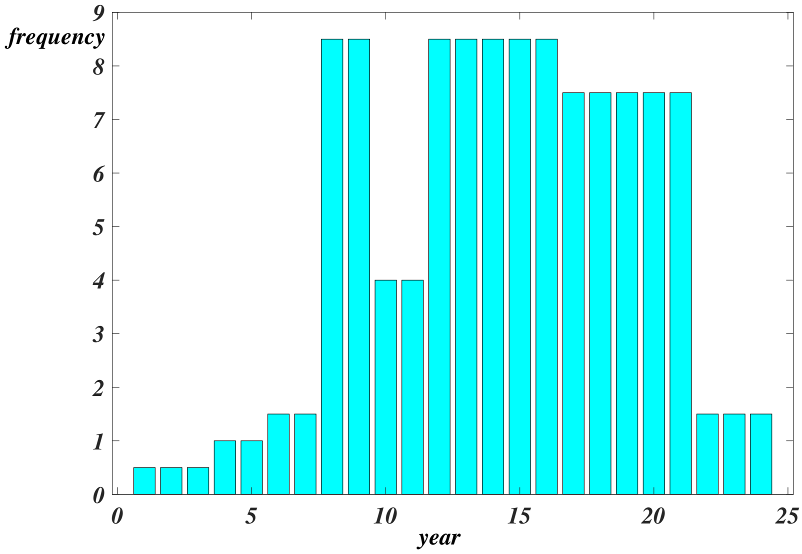

Figure 2.4: Failure time data of pole air switches.

discrete Weibull distribution. In practice, it is common to apply the classical moment method [42] to understand the mean and variance of the failure time in the power industry. We also applied the moment method to estimate r and α. For the failure data in Figure 2.4, we get $\mathrm{E}[Y] = 13.4$ (year) and $\mathrm{Var}[Y] = 24.36$ (year^2), so that

$$1 - \hat{r} = 0.9995, \quad \hat{\alpha} = 2.8547.$$

In this section, we compare two replacement models: RF policies and RL policies, under the assumption that the arrival time of opportunity for replacement obeys the geometric distribution with $h_X(x) = h = 0.95$ ($CaseA$). Throughout the example, we fix $c_T = 1.0$ (K dollar), and change the other cost parameters $c_F \in [1.5, 10.0]$ (K dollar), $c_Y = 0.8, 1.0$ (K dollar). In unified models, we set $p_1 = p_2 = 0.5$. The discounted factor is $\beta = 0.9$.

Tables 2.1–2.3 present the optimal AR time and DD time limit N^* and their associated $EC(N^*)$. We also provide the results for the unified models. From these results, we derive the following lessons learned from the numerical illustrations.

(1) When c_F increases, both the optimal AR time and the optimal DD time

Table 2.1: Optimal N^* and $EC(N^*)$ with Model 1 for AR and DD models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$EC_{a1}(N^*)$	N^*	$EC_{o1}(N^*)$	N^*	$EC_{a1}(N^*)$	N^*	$EC_{o1}(N^*)$
1.5	15	0.1083	8	0.1089	15	0.1083	12	0.1117
2.0	12	0.1296	6	0.1394	12	0.1296	8	0.1439
3.0	10	0.1575	4	0.1974	10	0.1575	5	0.2036
4.0	8	0.1769	3	0.2538	8	0.1769	3	0.2610
5.0	8	0.1926	3	0.3094	8	0.1926	3	0.3172
6.0	7	0.2049	2	0.3648	7	0.2049	2	0.3729
7.0	7	0.2166	1	0.4201	7	0.2166	2	0.4284
8.0	6	0.2264	1	0.4750	6	0.2264	2	0.4838
9.0	6	0.2345	1	0.5298	6	0.2345	1	0.5388
10.0	6	0.2437	1	0.5847	6	0.2437	1	0.5937

Table 2.2: Optimal N^* and $EC(N^*)$ with Model 2 for AR and DD models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$EC_{a2}(N^*)$	N^*	$EC_{o2}(N^*)$	N^*	$EC_{a2}(N^*)$	N^*	$EC_{o2}(N^*)$
1.5	16	0.1111	10	0.1106	16	0.1111	14	0.1125
2.0	12	0.1367	7	0.1427	12	0.1376	9	0.1465
3.0	9	0.1716	5	0.2037	9	0.1716	6	0.2093
4.0	8	0.1968	4	0.2631	8	0.1968	5	0.2697
5.0	7	0.2175	3	0.3216	7	0.2175	4	0.3288
6.0	7	0.2352	3	0.3800	7	0.2352	3	0.3874
7.0	6	0.2503	2	0.4380	6	0.2503	3	0.4458
8.0	6	0.2638	2	0.4957	6	0.2638	3	0.5041
9.0	6	0.2773	2	0.5535	6	0.2773	2	0.5619
10.0	5	0.2893	2	0.6113	5	0.2893	2	0.6197

Table 2.3: Optimal N^* and $EC(N^*)$ with unified Model for AR and DD models in discrete time, when $p_1 = 0.5 = p_2 = 0.5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$EC_{a3}(N^*)$	N^*	$EC_{o3}(N^*)$	N^*	$EC_{a3}(N^*)$	N^*	$EC_{o3}(N^*)$
1.5	16	0.1095	9	0.1098	16	0.1095	12	0.1121
2.0	12	0.1323	6	0.1418	12	0.1323	8	0.1452
3.0	9	0.1623	4	0.2006	9	0.1623	5	0.2065
4.0	8	0.1849	3	0.2584	8	0.1849	3	0.2654
5.0	7	0.2029	2	0.3155	7	0.2029	3	0.3230
6.0	7	0.2170	2	0.3724	7	0.2170	2	0.3802
7.0	6	0.2310	1	0.4290	6	0.2310	2	0.4371
8.0	6	0.2413	1	0.4853	6	0.2413	2	0.4939
9.0	6	0.2517	1	0.5417	6	0.2517	1	0.5504
10.0	6	0.2620	1	0.5980	6	0.2620	1	0.6067

Table 2.4: Optimal N^* and $TC(N^*)$ with Model 1 for AR and DD models in discrete time, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$TC_{a1}(N^*)$	N^*	$TC_{o1}(N^*)$	N^*	$TC_{a1}(N^*)$	N^*	$TC_{o1}(N^*)$
1.5	18	0.5800	11	0.5782	18	0.5800	15	0.5828
2.0	14	0.7410	8	0.7541	14	0.7410	10	0.7679
3.0	11	0.9802	5	1.0823	10	0.9802	7	1.1100
4.0	9	1.1548	4	1.3931	9	1.1548	5	1.4312
5.0	8	1.2968	3	1.6968	8	1.2968	4	1.7413
6.0	8	1.4195	3	1.9945	8	1.4195	3	2.0464
7.0	7	1.5190	2	2.2907	7	1.5190	3	2.3341
8.0	7	1.6131	2	2.5822	7	1.6131	3	2.6418
9.0	6	1.7028	2	2.8737	6	1.7028	2	2.9361
10.0	6	1.7706	2	3.1651	6	1.7706	2	3.2276

Table 2.5: Optimal N^* and $TC(N^*)$ with Model 2 for AR and DD models in discrete time, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$TC_{a2}(N^*)$	N^*	$TC_{o2}(N^*)$	N^*	$TC_{a2}(N^*)$	N^*	$TC_{o2}(N^*)$
1.5	22	0.5834	13	0.5822	22	0.5834	20	0.5835
2.0	15	0.7560	9	0.7665	15	0.7560	12	0.7748
3.0	11	1.0523	6	1.1116	11	1.0523	7	1.1346
4.0	9	1.2736	4	1.4399	9	1.2736	5	1.4732
5.0	8	1.4559	3	1.7596	8	1.4559	4	1.7998
6.0	7	1.6182	3	2.0722	7	1.6182	4	2.1196
7.0	7	1.7511	2	2.3843	7	1.7511	3	2.4339
8.0	7	1.8839	2	2.6908	7	1.8839	3	2.7465
9.0	6	1.9933	2	2.9974	6	1.9933	2	3.0568
10.0	6	2.0973	2	3.3040	6	2.0973	2	3.3634

Table 2.6: Optimal N^* and $TC(N^*)$ with unified model for AR and DD models in discrete time, when $\beta = 0.9$, $p_1 = p_2 = 0.5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	AR		DD		AR		DD	
c_F	N^*	$TC_{a3}(N^*)$	N^*	$TC_{o3}(N^*)$	N^*	$TC_{a3}(N^*)$	N^*	$TC_{o3}(N^*)$
1.5	19	0.5818	12	0.5804	19	0.5818	17	0.5833
2.0	14	0.7512	9	0.7606	14	0.7512	11	0.7716
3.0	11	1.0090	6	1.0970	11	1.0090	7	1.1224
4.0	9	1.2023	4	1.4165	9	1.2023	5	1.4523
5.0	8	1.3604	3	1.7282	8	1.3604	4	1.7706
6.0	8	1.4991	3	2.0334	8	1.4991	4	2.0831
7.0	7	1.6118	2	2.3375	7	1.6118	3	2.3891
8.0	7	1.7214	2	2.6365	7	1.7214	3	2.6943
9.0	6	1.8190	2	2.9355	6	1.8190	2	2.9966
10.0	6	1.9013	2	3.2346	6	1.9013	2	3.3296

limit N^* become smaller. This is because the preventive replacement tends to be set earlier if the corrective replacement cost c_F is higher.

(2) When $c_T = c_Y$, the AR policy is better than the DD policy. Additionally, the AR time is larger than the DD time.

(3) When $c_T < c_Y$, the DD policy is better than the AR policy in some cases where c_F is relatively smaller. For example, when $c_F = 1.5$, it is easy to confirm that the DD policy is better than the AR policy. In our actual application, under the assumption of $c_T = 2c_Y$, if $c_T < c_F < 1.5c_T$, the decision-maker should consider the opportunity in the preventive replacement. Otherwise, if $c_F > 1.5c_T$, the decision-maker should consider only the AR policy instead of the DD policy.

(4) In most cases, the optimal preventive replacement times for each priority model tend to converge to the same values. This phenomenon arises from the discretization of replacement times into integer values and the relatively subtle differences in replacement priorities.

(5) Comparing Tables 2.1, 2.2, and Table 2.3, notable discrepancies in the preventive replacement times are not evident. Moreover, the associated expected costs tend to converge towards similar values.

Tables 2.4–2.6 present the optimal AR time and DD time limit N^*, and their expected total discounted costs $TC(N^* \mid \beta)$ for Model 1 and Model 2, given a discount factor of $\beta = 0.90$. By observing the results carefully, we derive the following findings:

(6) The lessons (1)–(5) always hold in models with discounting.

(7) In terms of the optimal time in AR and DD policies, the optimal replacement time with discounting is longer than that without discounting. This indicates that when the economic environment is unstable, decision-makers will shorten the replacement times for their equipment.

(8) When the discount factor is relatively small, such as $\beta = 0.90$, the optimal preventive replacement times for each priority model often converge to similar values in most cases. In this scenario, the discount factor has a minimal impact on the optimal replacement times.

Chapter 3

Discrete Time RF and RL Models

3.1 Renewal Reward Approach

3.1.1 RF Model

Based on the notations and assumpations in section 2.1, we reformulat the RF and RL models in discrete time. Zhao and Nakagawa [2] initially proposed these models in continuous time. In discrete time RF model, if the system breaks down before an arrival of opportunity, X, and a scheduled preventive replacement time N, the failed item is replaced by a new one (see Figure 3.1 (i)). Otherwise, the system is replaced preventively at the time, the opportunity or the prescheduled preventive replacement time, whichever comes first. In contrast, in RL model, the system is replaced preventively at the time, the opportunity or the prescheduled preventive replacement time N, whichever comes last (see Figure 3.1 (ii)).

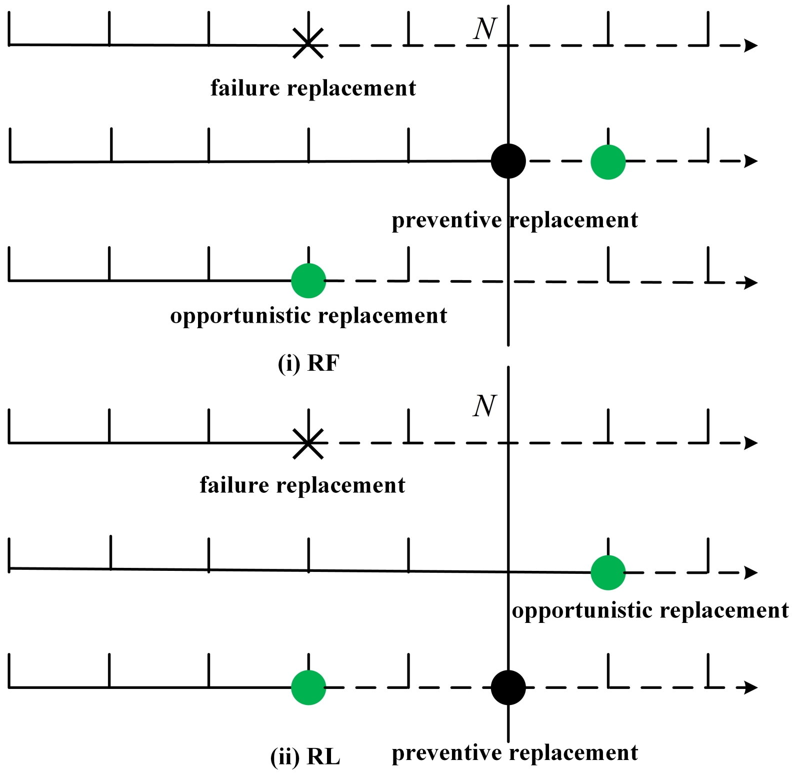

Figure 3.1: RF and RL disciplines.

According to definition 1, we may possibly consider six priority models for RF and RL disciplines:

(1) Model 1: $S_c \succ C_a \succ O_p$,

(2) Model 2: $C_a \succ S_c \succ O_p$,

(3) Model 3: $S_c \succ O_p \succ C_a$,

(4) Model 4: $O_p \succ S_c \succ C_a$,

(5) Model 5: $C_a \succ O_p \succ S_c$,

(6) Model 6: $O_p \succ C_a \succ S_c$.

First of all, the probability that the system is replaced at time n $(= 0, 1, 2, \ldots)$ is given by

$$h_{fj}(n) = \begin{cases} f_Y(n)\bar{G}_X(n-1) + g_X(n)\bar{F}_Y(n) & 0 \leq n \leq N-1 \\ \bar{G}_X(n-1)\bar{F}_Y(n-1) & n = N \\ 0 & n \geq N+1, \end{cases} \tag{3.1}$$

where $\sum_{n=0}^{\infty} h_{fj}(n) = 1$.

Let $A_f(N)$ denote the expected time length of one cycle for Model j. Then, we have

$$
\begin{aligned}
A_f(N) &= \sum_{n=1}^{N-1} n\left\{f_Y(n)\bar{G}_X(n-1) + g_X(n)\bar{F}_Y(n)\right\} + N\bar{G}_X(N-1)\bar{F}_Y(N-1) \\
&= \sum_{n=1}^{N} \bar{F}_Y(n-1)\bar{G}_X(n-1). \qquad (3.2)
\end{aligned}
$$

We calculate the expected total costs during one cycle, $B_{fj}(N)$, as follows:

$$
\begin{aligned}
B_{f1}(N) = c_F \sum_{n=1}^{N-1} f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=1}^{N-1} g_X(n)\bar{F}_Y(n) \\
+ c_T\bar{G}_X(N-1)\bar{F}_Y(N-1), \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
B_{f2}(N) = c_F \sum_{n=1}^{N} f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=1}^{N-1} g_X(n)\bar{F}_Y(n) \\
+ c_T\bar{G}_X(N-1)\bar{F}_Y(N), \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
B_{f3}(N) = c_F \sum_{n=1}^{N-1} f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=1}^{N-1} g_X(n)\bar{F}_Y(n-1) \\
+ c_T\bar{G}_X(N-1)\bar{F}_Y(N-1), \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
B_{f4}(N) = c_F \sum_{n=1}^{N-1} f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=1}^{N} g_X(n)\bar{F}_Y(n-1) \\
+ c_T\bar{G}_X(N)\bar{F}_Y(N-1), \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
B_{f5}(N) = c_F \sum_{n=1}^{N} f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=1}^{N} g_X(n)\bar{F}_Y(n) \\
+ c_T\bar{G}_X(N)\bar{F}_Y(N), \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
B_{f6}(N) = c_F \sum_{n=1}^{N} f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=1}^{N} g_X(n)\bar{F}_Y(n-1) \\
+ c_T\bar{G}_X(N)\bar{F}_Y(N). \qquad (3.8)
\end{aligned}
$$

From renewal reward theory [37], the expected costs per unit time in the steady state are given by

$$
EC_{fj}(N) = \frac{B_{fj}(N)}{A_f(N)}, \qquad (3.9)
$$

for Model j $(= 1, 2, \dots, 6)$. Then, our purpose is to find the optimal scheduled preventive replacement times N^* by which minimizes $EC_{fj}(N)$.

Taking the difference of $EC_{fj}(N)$ $(j = 1, 2, \dots, 6)$ with respect to N yields

$$EC_{fj}(N+1) - EC_{fj}(N) = \frac{\bar{F}_Y(N)\bar{G}_X(N)}{A_f(N+1)A_f(N)} w_{fj}(N), \tag{3.10}$$

where

$$w_{f1}(N) = \Big\{(c_F - c_T)R_Y(N)\left[1 + H_X(N)\right] - (c_F - c_T)H_X(N)\Big\}A_f(N) \\ - B_{f1}(N), \tag{3.11}$$

$$w_{f2}(N) = \Big\{(c_F - c_T)r_Y(N+1) - (c_F - c_T)H_X(N)\Big\}A_f(N) - B_{f2}(N), \tag{3.12}$$

$$w_{f3}(N) = \Big\{(c_F - c_T)R_Y(N) - (c_T - c_Y)\left[1 + R_Y(N)\right]H_X(N)\Big\}A_f(N) \\ - B_{f3}(N), \tag{3.13}$$

$$w_{f4}(N) = \Big\{(c_F - c_T)R_Y(N) - (c_T - c_Y)h_X(N+1)\Big\}A_f(N) - B_{f4}(N), \tag{3.14}$$

$$w_{f5}(N) = \Big\{(c_F - c_T)r_Y(N+1) - (c_T - c_Y)\left[1 - r_Y(N+1)\right]h_X(N+1)\Big\}A_f(N) \\ - B_{f5}(N), \tag{3.15}$$

$$w_{f6}(N) = \Big\{(c_1 - c_2)r_Y(N+1)\left[1 - h_X(N+1)\right] - (c_2 - c_3)h_X(N+1)\Big\}A_f(N) \\ - B_{f6}(N). \tag{3.16}$$

Next, we describe the optimal preventive replacement polices for RF model in discrete time and discuss the necessary conditions for the existence of a unique optimal policy.

Theorem 3.1. *(I) Suppose that $w_{fj}(n)$ $(j = 1, 2, \dots, 6)$ are strictly increasing functions in n under Assumption 1.*

- *(i) If $w_{fj}(\infty) > 0$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{fj}(N^* - 1) < 0$ and $w_{fj}(N^*) \geq 0$.*

(ii) If $w_{fj}(\infty) \leq 0$, then the optimal scheduled preventive replacement time is $N^* \to \infty$.

(II) Suppose $w_{fj}(n)$ are decreasing functions under Assumption 1. Then the optimal scheduled preventive replacement times are given by $N^* \to \infty$ or $N^* = 1$.

See Appendix 7.4.4 for proof of Theorem 3.1. We care about the necessary conditions the functions $w_{fj}(n)$ are strictly increasing. If the failure time Y is strictly IFR and the arrival time of opportunity X is DHR, Model 2 and Model 4 are strictly increasing. However, for other four models, additional monotonic properties are needed for the product of two hazard rate functions. Here we take the Model 1 as an example. If $w_{fj}(n)$ are strictly increasing, the additional condition is $R_Y(n+1)\,[1+H_X(n+1)] > R_Y(n)\,[1+H_X(n)]$. More detailed discussion can also be found in Lemma 7.6.

The following result can be derived from Theorem 3.1 straightforwardly.

Theorem 3.2. *For Model j $(=1,2,\ldots,6)$, suppose that $w_{fj}(n)$ is strictly increasing in n, and $w_{fj}(\infty) > 0$, under Assumption 1. Then the minimum expected costs per unit time in the steady state have the lower and upper bounds:*

$$V_{fj}(N^*-1) < EC_{fj}(N^*) \leq V_{fj}(N^*), \tag{3.17}$$

where

$$V_{f1}(N) = (c_F - c_T)R_Y(N)\,[1+H_X(N)] - (c_T - c_Y)H_X(N), \tag{3.18}$$

$$V_{f2}(N) = (c_F - c_T)r_Y(N+1) - (c_T - c_Y)H_X(N), \tag{3.19}$$

$$V_{f3}(N) = (c_F - c_T)R_Y(N) - (c_T - c_Y)\,[1+R_Y(N)]\,H_X(N), \tag{3.20}$$

$$V_{f4}(N) = (c_F - c_T)R_Y(N) - (c_T - c_Y)h_X(N), \tag{3.21}$$

$$V_{f5}(N) = (c_F - c_T)r_Y(N+1) - (c_T - c_Y)\,[1-r_Y(N+1)]\,h_X(N+1), \tag{3.22}$$

$$V_{f6}(N) = (c_F - c_T)r_Y(N+1)\,[1-h_X(N+1)] - (c_T - c_Y)h_X(N+1). \tag{3.23}$$

3.1.2 RL Model

Next, we concentrate on RL model in discrete time. Similar to RF discipline, we have the probability that the system is replaced at time n $(=0,1,2,\ldots)$ for

Model j $(= 1, 2, \ldots, 6)$:

$$h_{lj}(n) = \begin{cases} f_Y(n) & 0 \leq n \leq N-1 \\ f_Y(n)\bar{G}_X(n-1) + \bar{F}_Y(n-1)G_X(n-1) & \\ +\bar{F}_Y(n)g_X(n) & n = N \\ f_Y(n)\bar{G}_X(n-1) + g_X(n)\bar{F}_Y(n) & n \geq N+1, \end{cases} \tag{3.24}$$

where $\sum_{n=0}^{\infty} h_{lj}(n) = 1$.

We confirm that the expected time lengths of one cycle for Model j $(= 1, 2, \ldots, 6)$ are given by

$$\begin{aligned} A_l(N) &= \sum_{n=1}^{N-1} n f_Y(n) + N\bar{F}_Y(N-1)G_X(N-1) \\ &+ \sum_{n=N}^{\infty} n\Big[f_Y(n)\bar{G}_X(n-1) + g_X(n)\bar{F}_Y(n)\Big] \\ &= \sum_{n=1}^{N} \bar{F}_Y(n-1) + \sum_{n=N+1}^{\infty} \bar{F}_Y(n-1)\bar{G}_X(n-1). \end{aligned} \tag{3.25}$$

We also confirm that the expected total costs during one cycle for each model become

$$\begin{aligned} B_{l1}(N) = c_F &\sum_{n=1}^{N} f_Y(n) + c_T G_X(N-1)\bar{F}_Y(N-1) \\ &+ c_F \sum_{n=N}^{\infty} f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=N}^{\infty} g_X(n)\bar{F}_Y(n), \end{aligned} \tag{3.26}$$

$$\begin{aligned} B_{l2}(N) = B_{l5}(N) = c_F &\sum_{n=1}^{N} f_Y(n) + c_T G_X(N-1)\bar{F}_Y(N) \\ &+ c_F \sum_{n=N+1}^{\infty} f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=N}^{\infty} g_X(n)\bar{F}_Y(n), \end{aligned} \tag{3.27}$$

$$\begin{aligned} B_{l3}(N) = B_{l4}(N) = c_F &\sum_{n=1}^{N-1} f_Y(n) + c_T \bar{F}_Y(N-1)G_X(N-1) \\ &+ c_F \sum_{n=N}^{\infty} f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=N}^{\infty} g_X(n)\bar{F}_Y(n-1), \end{aligned} \tag{3.28}$$

$$\begin{aligned} B_{l6}(N) = c_F &\sum_{n=1}^{N-1} f_Y(n) + c_F f_Y(N)[1 - g_X(N)] + c_T \bar{F}_Y(N)G_X(N-1) \\ &+ c_T \sum_{n=N+1}^{\infty} f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=N}^{\infty} g_X(n)\bar{F}_Y(n-1). \end{aligned} \tag{3.29}$$

Then, the expected costs per unit time in the steady state, $EC_{lj}(N) = B_{lj}(N)/A_l(N)$, for Model j $(= 1, 2, \ldots, 6)$.

Next, we derive the optimal scheduled preventive replacement times N^* which minimize $EC_{lj}(N)$. Taking the difference of $EC_{lj}(N)$ $(j = 1, 2, \ldots, 6)$ with respect to N, one obtains

$$EC_{lj}(N+1) - EC_{lj}(N) = \frac{\bar{F}_Y(N)G_X(N)}{A_l(N+1)A_l(N)} w_{lj}(N), \tag{3.30}$$

where

$$w_{l1}(N) = \Big\{(c_F - c_T)R_Y(N)\Big[1 - \hat{H}_X(N)\Big] + (c_T - c_Y)\hat{H}_X(N)\Big\}A_l(N) - B_{l1}(N), \tag{3.31}$$

$$w_{l2}(N) = w_{l5}(N) = \Big\{(c_F - c_T)r_Y(N+1) + (c_T - c_Y)\hat{H}_X(N)\Big\} A_l(N) - B_{l2}(N), \tag{3.32}$$

$$w_{l3}(N) = w_{l4}(N) = \Big\{(c_F - c_T)R_Y(N) + (c_T - c_Y)\left[1 + R_Y(N)\right]\hat{H}_X(N)\Big\}A_l(N) - B_{l3}(N), \tag{3.33}$$

$$w_{l6}(N) = \Big\{(c_F - c_T)\Big[r_Y(N+1) + R_Y(N)\hat{H}_X(N)\Big] + (c_T - c_Y)\left[1 + R_Y(N)\right]\hat{H}_X(N)\Big\}A_l(N) - B_{l6}(N). \tag{3.34}$$

We give the optimal scheduled preventive replacement times N^* for RL model.

Theorem 3.3. *(I) Suppose that $w_{lj}(n)$ $(j = 1, 2, \ldots, 6)$ are strictly increasing functions under Assumption 1.*

- *(i) If $w_{lj}(\infty) > 0$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{lj}(N^* - 1) < 0$ and $w_{lj}(N^*) \geq 0$.*
- *(ii) If $w_{lj}(\infty) \leq 0$, then the optimal scheduled preventive replacement time is $N^* \to \infty$.*

(II) Suppose that $w_{lj}(n)$ $(j = 1, 2, \ldots, 6)$ are strictly increasing functions in n. Then the optimal scheduled preventive replacement times are given by $N^* \to \infty$ or $N^* = 0$.

For the proof, refer to Theorem 3.1. We can see that the necessary conditions of strictly increasing $w_{l1}(N)$, $w_{l2}(N)$, $w_{l5}(N)$ depend on the cost parameter under Assumption 1. At special case $c_2 = c_3$, the necessary conditions of strictly increasing $w_{lj}(N)$ $(j = 2, 3, 4, 5)$ is rather mild, except in Model 1 and Model 6. Note that $w_{lj}(N)$ depends on the reversed hazard rate $\hat{H}_X(N)$. It is well known that there does not exist the absolutely continuous distribution with constant reversed hazard rate on the positive real line [44]. Further detailed discussion is also available in Lemma 7.7.

We give the upper and lower bounds of the minimum expected cost per unit time in the steady state from Theorem 3.3.

Theorem 3.4. *For Model j $(= 1, 2, \ldots, 6)$, suppose that the functions $w_{lj}(n)$ $(j = 1, 2, \ldots, 6)$ are strictly increasing in n and $w_{lj}(\infty) > 0$ under Assumption 1. Then the minimum expected costs per unit time in the steady state have the lower and upper bounds:*

$$V_{lj}(N^* - 1) < EC_{lj}(N^*) \le V_{lj}(N^*), \tag{3.35}$$

where

$$V_{l1}(N) = (c_F - c_T)R_Y(N)\left[1 - \hat{H}_X(N)\right] + (c_T - c_Y)\hat{H}_X(N), \tag{3.36}$$

$$V_{l2}(N) = V_{l5}(N) = (c_F - c_T)r_Y(N+1) + (c_T - c_Y)\hat{H}_X(N), \tag{3.37}$$

$$V_{l3}(N) = V_{l4}(N) = (c_F - c_T)R_Y(N) + (c_T - c_Y)\left[1 + R_Y(N)\hat{H}_X(N)\right], \tag{3.38}$$

$$\begin{aligned} V_{l6}(N) = {} & (c_F - c_T)\left[r_Y(N+1) + R_Y(N)\hat{H}_X(N)\right] \\ & + (c_T - c_Y)\left[1 + R_Y(N)\right]\hat{H}_X(N). \end{aligned} \tag{3.39}$$

3.2 NPV Apporach

3.2.1 RF Model

First, we can calculate the NPV of one unit cost during the renewal cycle;

$$A_f(N, \beta) = \sum_{n=1}^{N-1} \beta^n \left[f_Y(n)\bar{G}(n-1) + g_X(n)\bar{F}_Y(n)\right] + \beta^N \bar{G}_X(N-1)\bar{F}_Y(N-1), \tag{3.40}$$

where $A_f(N, \beta)$ for Model j are all same.

We can compute the expected total discounted costs during the renewal cycle, $B_{fj}(N,\beta)$, for Model j $(=1,\cdots,6)$;

$$\begin{aligned} B_{f1}(N,\beta) &= c_F \sum_{n=1}^{N-1} \beta^n f_Y(n)\bar{G}(n-1) + c_Y \sum_{n=1}^{N-1} \beta^n g_X(n)\bar{F}_Y(n) \\ &+ c_T \beta^N \bar{G}_X(N-1)\bar{F}_Y(N-1), \end{aligned} \tag{3.41}$$

$$\begin{aligned} B_{f2}(N,\beta) &= c_F \sum_{n=1}^{N} \beta^n f_Y(n)\bar{G}(n-1) + c_Y \sum_{n=1}^{N-1} \beta^n g_X(n)\bar{F}_Y(n) \\ &+ c_T \beta^N \bar{G}_X(N-1)\bar{F}_Y(N), \end{aligned} \tag{3.42}$$

$$\begin{aligned} B_{f3}(N,\beta) &= c_F \sum_{n=1}^{N-1} \beta^n f_Y(n)\bar{G}(n) + c_Y \sum_{n=1}^{N-1} \beta^n g_X(n)\bar{F}_Y(n-1) \\ &+ c_T \beta^N \bar{G}_X(N-1)\bar{F}_Y(N-1), \end{aligned} \tag{3.43}$$

$$\begin{aligned} B_{f4}(N,\beta) &= c_F \sum_{n=1}^{N-1} \beta^n f_Y(n)\bar{G}(n) + c_Y \sum_{n=1}^{N} \beta^n g_X(n)\bar{F}_Y(n-1) \\ &+ c_T \beta^N \bar{G}_X(N)\bar{F}_Y(N-1), \end{aligned} \tag{3.44}$$

$$\begin{aligned} B_{f5}(N,\beta) &= c_F \sum_{n=1}^{N} \beta^n f_Y(n)\bar{G}(n-1) + c_Y \sum_{n=1}^{N} \beta^n g_X(n)\bar{F}_Y(n) \\ &+ c_T \beta^N \bar{G}_X(N)\bar{F}_Y(N), \end{aligned} \tag{3.45}$$

$$\begin{aligned} B_{f6}(N,\beta) &= c_F \sum_{n=1}^{N} \beta^n f_Y(n)\bar{G}(n) + c_Y \sum_{n=1}^{N} \beta^n g_X(n)\bar{F}_Y(n-1) \\ &+ c_T \beta^N \bar{G}_X(N)\bar{F}_Y(N), \end{aligned} \tag{3.46}$$

where discount factor $\beta(>0)$ represents the NPV of a unit cost component. In RF discipline, $TC_{fj}(N,\beta)$ denotes the NPV of the expected total discounted cost over an infinite time horizon. Then, we have

$$TC_{fj}(N,\beta) = \frac{B_{fj}(N,\beta)}{1-A_f(N,\beta)}, \tag{3.47}$$

for Model j $(=1,\cdots,6)$.

Then, the problem is to determine the optimal prescheduled preventive replacement times N^* by minimizing $TC_{fj}(N,\beta)$.

We determine the optimal prescheduled preventive replacement times N^* which minimize $TC_{fj}(N,\beta)$ for Mode j $(=1,\cdots,6)$ under RF discipline. We compute the difference in $TC_{fj}(N,\beta)$ for a fixed β, we have

$$TC_{fj}(N+1,\beta) - TC_{fj}(N,\beta) = \frac{\bar{F}_Y(N)\bar{G}_X(N)}{[1-A_f(N+1,\beta)]\,[1-A_f(N,\beta)]} w_{fj}(N \mid \beta), \tag{3.48}$$

in which

$$\begin{aligned} w_{f1}(N \mid \beta) &= \left[\frac{(c_F - c_T)\,R_Y(N)\,[1+H_X(N)]}{1-\beta} - \frac{(c_T - c_Y)\,H_X(N)}{1-\beta} - c_T\right] \\ &\times\ [1-A_f(N,\beta)] - B_{f1}(N,\beta), \end{aligned} \tag{3.49}$$

$$\begin{aligned} w_{f2}(N \mid \beta) &= \left[\frac{\beta\,(c_F - c_T)\,r_Y(N+1)}{1-\beta} - \frac{(c_T - c_Y)\,H_X(N)}{1-\beta} - c_T\right] \\ &\times\ [1-A_f(N,\beta)] - B_{f2}(N,\beta), \end{aligned} \tag{3.50}$$

$$\begin{aligned} w_{f3}(N \mid \beta) &= \left[\frac{(c_F - c_T)\,R_Y(N)}{1-\beta} - \frac{(c_T - c_Y)\,H_X(N)\,[R_Y(N)+1]}{1-\beta} - c_T\right] \\ &\times\ [1-A_f(N,\beta)] - B_{f13}(N,\beta), \end{aligned} \tag{3.51}$$

$$\begin{aligned} w_{f4}(N \mid \beta) &= \left[\frac{(c_F - c_T)\,R_Y(N)}{1-\beta} - \frac{\beta\,(c_T - c_Y)\,h_X(N+1)}{1-\beta} - c_T\right] \\ &\times\ [1-A_f(N,\beta)] - B_{f4}(N,\beta), \end{aligned} \tag{3.52}$$

$$\begin{aligned} w_{f5}(N \mid \beta) = \Bigg[&\frac{\beta\,(c_F - c_T)\,r_Y(N+1)}{1-\beta} - \frac{\beta\,(c_T - c_Y)\,h_X(N+1)\,[1-r_Y(N+1)]}{1-\beta} \\ &- c_T\Bigg]\,[1-A_f(N,\beta)] - B_{f5}(N,\beta), \end{aligned} \tag{3.53}$$

$$\begin{aligned} w_{f6}(N \mid \beta) = \Bigg[&\frac{\beta\,(c_F - c_T)\,r_Y(N+1)\,[1-h_X(N+1)]}{1-\beta} - \frac{\beta\,(c_T - c_Y)\,h_X(N+1)}{1-\beta} \\ &- c_T\Bigg]\,[1-A_f(N,\beta)] - B_{f6}(N,\beta). \end{aligned} \tag{3.54}$$

See Lemma 7.8 for the monotonicity of functions $w_{fj}(N \mid \beta)(j = 1, \cdots, 6)$.

We determine the optimal prescheduled preventive replacement times N^* that minimize the expected costs over an infinite time horizon, $TC_{fj}(N,\beta)$, for Model j $(= 1, \cdots, 6)$.

Theorem 3.5. *(I) Suppose that $w_{fj}(N \mid \beta)$ is a strictly increasing function of N for a fixed β.*

(i) If $w_{fj}(\infty \mid \beta) > 0$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{fj}(N^* - 1 \mid \beta) < 0$ and $w_{fj}(N^* \mid \beta) \geq 0$.

(ii) If $w_{fj}(\infty \mid \beta) \leq 0$, then the optimal prescheduled preventive replacement time is given by $N^* \to \infty$, and so the failure replacement or opportunistic replacement is better than the preventive replacement.

(II) If $w_{fj}(N \mid \beta)$ is a decreasing function of N, the optimal prescheduled preventive replacement time is always $N^* \to \infty$ or $N^* = 1$.

For the proof of Theorem 3.5, see the Appendix 7.4.5.
We can derive the results directly from Theorem 1 as follows:

Theorem 3.6. *For Model j $(= 1, \cdots, 6)$, suppose that $w_{fj}(N \mid \beta)$ is a strictly increasing function and $w_{fj}(N \mid \beta) > 0$, under Assumption 1. Then the minimum expected total discounted costs over an infinite horizon have the lower and upper bounds:*

$$V_{fj}(N^* - 1 \mid \beta) < TC_{fj}(N^* \mid \beta) \leq V_{fj}(N^* \mid \beta), \tag{3.55}$$

where

$$V_{f1}(N \mid \beta) = \frac{(c_F - c_T)\, R_Y(N)\, [1 + H_X(N)]}{1 - \beta} - \frac{(c_T - c_Y)\, H_X(N)}{1 - \beta} - c_T, \tag{3.56}$$

$$V_{f2}(N \mid \beta) = \frac{\beta\, (c_F - c_T)\, r_X(N + 1)}{1 - \beta} - \frac{(c_T - c_Y)\, H_X(N)}{1 - \beta} - c_T, \tag{3.57}$$

$$V_{f3}(N \mid \beta) = \frac{(c_F - c_T)\, R_Y(N)}{1 - \beta} - \frac{(c_T - c_Y)\, H_X(N)\, [1 + R_Y(N)]}{1 - \beta} - c_T, \tag{3.58}$$

$$V_{f4}(N \mid \beta) = \frac{(c_F - c_T)\, R_Y(N)}{1 - \beta} - \frac{\beta\, (c_T - c_Y)\, h_X(N + 1)}{1 - \beta} - c_T, \tag{3.59}$$

$$V_{f5}(N \mid \beta) = \frac{\beta\, (c_F - c_T)\, r_X(N + 1)}{1 - \beta} - \frac{\beta\, (c_T - c_Y)\, h_X(N + 1)\, [1 - r_X(N + 1)]}{1 - \beta} - c_T, \tag{3.60}$$

$$V_{f6}(N \mid \beta) = \frac{\beta\, (c_F - c_T)\, r_X(N + 1)\, [1 - h_X(N + 1)]}{1 - \beta} - \frac{\beta\, (c_T - c_Y)\, h_X(N + 1)}{1 - \beta} - c_T. \tag{3.61}$$

3.2.2 RL Model

Like RF discipline, $A_l(N,\beta)$ are all same for Model j $(=1,\cdots,6)$ in RL discipline, so we can obtain

$$A_l(N,\beta) = \sum_{n=1}^{N-1} \beta^n f_Y(n) + \beta^N \bar{F}_Y(N-1) G_X(N-1) + \sum_{n=N}^{\infty} \beta^n \left[f_Y(n)\bar{G}(n-1) + \bar{F}_Y(n) g_X(n) \right]. \tag{3.62}$$

We can obtian the expected total discounted over an infinite horizon, $B_{lj}(N,\beta)$, for Model j $(=1,\cdots,6)$;

$$B_{l1}(N,\beta) = c_F \sum_{n=1}^{N-1} \beta^n f_Y(n) + c_T \beta^N G_X(N-1)\bar{F}_Y(N-1) + c_F \sum_{n=N}^{\infty} \beta^n f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=N}^{\infty} \beta^n g_X(n)\bar{F}_X(n), \tag{3.63}$$

$$B_{l2}(N,\beta) = B_{l5}(N,\beta) = c_F \sum_{n=1}^{N} \beta^n f_Y(n) + c_T \beta^N G_X(N-1)\bar{F}_Y(N) + c_F \sum_{n=N+1}^{\infty} \beta^n f_Y(n)\bar{G}_X(n-1) + c_Y \sum_{n=N}^{\infty} \beta^n g_X(n)\bar{F}_X(n), \tag{3.64}$$

$$B_{l3}(N,\beta) = B_{l4}(N,\beta) = c_F \sum_{n=1}^{N-1} \beta^n f_Y(n) + c_T \beta^N G_X(N-1)\bar{F}_Y(N-1) + c_F \sum_{n=N}^{\infty} \beta^n f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=N}^{\infty} \beta^n g_X(n)\bar{F}_X(n-1), \tag{3.65}$$

$$B_{l6}(N,\beta) = c_F \sum_{n=1}^{N-1} \beta^n f_Y(n) + c_F \beta^N f_Y(N)[1-g_X(N)] + c_T \beta^N G_X(N-1)\bar{F}_Y(N) + c_F \sum_{n=N+1}^{\infty} \beta^n f_Y(n)\bar{G}_X(n) + c_Y \sum_{n=N}^{\infty} \beta^n g_X(n)\bar{F}_X(n-1). \tag{3.66}$$

Then, the problem is to minimize the expected total costs over an infinite horizon,

$$TC_{lj}(N,\beta) = \frac{B_{lj}(N,\beta)}{1-A_l(N,\beta)}. \tag{3.67}$$

Our interest is to find the optimal prescheduled preventive replacement times N^* which minimizes $TC_{lj}(N,\beta)$.

We formulate the optimal preventive replacement times N^* which minimize $TC_{lj}(N,\beta)$ for Model j $(=1,\cdots,6)$ with RL discipline. Taking the difference of $TC_{lj}(N,\beta)$ with respect to N, we have

$$TC_{lj}(N,\beta)-TC_{lj}(N,\beta)=\frac{\bar{F}_Y(N)G_X(N)}{[1-A_l(N+1,\beta)]\,[1-A_l(N,\beta)]}w_{lj}(N\mid\beta),\quad(3.68)$$

where

$$w_{l1}(N\mid\beta)=\left[\frac{(c_F-c_T)R_Y(N)[1-\hat{H}_X(N)]}{1-\beta}+\frac{(c_T-c_Y)\hat{H}_X(N)}{1-\beta}-c_T\right]\times[1-A_l(N,\beta)]-B_{l1}(N,\beta),\quad(3.69)$$

$$w_{l2}(N\mid\beta)=w_{l5}(N\mid\beta)=\left[\frac{\beta(c_F-c_T)r_Y(N+1)}{1-\beta}+\frac{(c_T-c_Y)\hat{H}_X(N)}{1-\beta}-c_T\right]\times[1-A_l(N,\beta)]-B_{l2}(N,\beta),\quad(3.70)$$

$$w_{l3}(N\mid\beta)=w_{l4}(N\mid\beta)=\left[\frac{(c_F-c_T)R_Y(N)}{1-\beta}+\frac{(c_T-c_Y)\hat{H}_X(N)[1+R_Y(N)]}{1-\beta}-c_T\right][1-A_l(N,\beta)]-B_{l3}(N,\beta),\quad(3.71)$$

$$w_{l6}(N\mid\beta)=\left[\frac{(c_F-c_T)[R_Y(N)\hat{H}_X(N)+\beta r_Y(N+1)]}{1-\beta}+\frac{(c_T-c_Y)\hat{H}_X(N)[1+R_Y(N)]}{1-\beta}-c_T\right][1-A_l(N,\beta)]-B_{l6}(N,\beta).\quad(3.72)$$

Taking the subsequent difference of $w_{lj}(N\mid\beta)$ for $j=1,\cdots,6$ in Eqs. (3.69) – (3.72), we can get

$$
\begin{aligned}
w_{l1}(N+1 \mid \beta) - w_{l1}(N \mid \beta) = &\left\{ \frac{(c_F - c_T)\left[R_Y(N+1)\left(1 - \hat{H}_X(N+1)\right) - \right.}{1-\beta} \right. \\
&- \frac{\left. R_Y(N)\left(1 - \hat{H}_X(N)\right)\right]}{1-\beta} \\
&+ \left. \frac{(c_T - c_Y)\left[\hat{H}_X(N+1) - \hat{H}_X(N)\right]}{1-\beta} \right\} \\
&\times \left[1 - A_l(N+1, \beta)\right],
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
w_{l2}(N+1 \mid \beta) - w_{l2}(N \mid \beta) &= w_{l5}(N+1 \mid \beta) - w_{l5}(N \mid \beta) \\
&= \left\{ \frac{\beta\,(c_F - c_T)\left[r_Y(N+2) - R_Y(N+1)\right]}{1-\beta} \right. \\
&+ \left. \frac{(c_T - c_Y)\left[\hat{H}_X(N+1) - \hat{H}_X(N)\right]}{1-\beta} \right\} \\
&\times \left[1 - A_l(N+1, \beta)\right],
\end{aligned}
\tag{3.74}
$$

$$
\begin{aligned}
w_{l3}(N+1 \mid \beta) - w_{l3}(N \mid \beta) &= w_{l4}(N+1 \mid \beta) - w_{l4}(N \mid \beta) \\
&= \left\{ \frac{(c_F - c_T)\left[R_Y(N+1) - R_Y(N)\right]}{1-\beta} \right. \\
&+ \frac{(c_T - c_Y)\left[\hat{H}_X(N+1)\left(1 + R_Y(N+1)\right)\right.}{1-\beta} \\
&- \left. \frac{\left.\hat{H}_X(N)\left(1 + R_Y(N)\right)\right]}{1-\beta} \right\} \\
&\times \left[1 - A_l(N+1, \beta)\right],
\end{aligned}
\tag{3.75}
$$

$$
\begin{aligned}
w_{l6}(N+1 \mid \beta) - w_{l6}(N \mid \beta) = \Bigg\{ & \frac{(c_F - c_T)\left[R_Y(N+1)\hat{H}_X(N+1) - R_Y(N)\hat{H}_X(N) + \right]}{1-\beta} \\
& + \frac{\beta r_Y(N+2) - \beta r_Y(N+1)}{1-\beta} \\
& + \frac{(c_T - c_Y)\left[\hat{H}_X(N+1)\,(1 + R_Y(N+1))\right.}{1-\beta} \\
& - \frac{\left.\hat{H}_X(N)\,(1 + R_Y(N))\right]}{1-\beta} \Bigg\} [1 - A_l(N+1, \beta)] \,.
\end{aligned} \tag{3.76}
$$

From Eqs. (3.73)–(3.76), the monotone properties of $w_{lj}(N \mid \beta)$ $(j = 1, \cdots, 6)$ depend on the $R_Y(N)$ and the cost parameters in our modeling. It was proved in [43] that if the lifetime X is DFR, then the $H_X(N)$ is decreasing in N. The necessary conditions that $w_{lj}(N \mid \beta)$ $(j = 1, \cdots, 6)$ is a strictly increasing functions in N are given in Lemma 7.9.

Theorem 3.7. *(I) Suppose that $w_{lj}(N \mid \beta)$ is a strictly increasing function of N for a fixed β.*

(i) If $w_{lj}(\infty \mid \beta) > 0$, then there exists at least one (at most two) optimal prescheduled preventive replacement time N^*, which satisfies $w_{lj}(N^* - 1 \mid \beta) < 0$ and $w_{lj}(N^* \mid \beta) \geq 0$.

(ii) If $w_{lj}(\infty \mid \beta) > 0$, then there exists at least one (at most two) optimal prescheduled preventive replacement time N^*, which satisfies $w_{lj}(N^* - 1 \mid \beta) < 0$ and $w_{lj}(N^* \mid \beta) \geq 0$.

(II) If $w_{lj}(N \mid \beta)$ is a decreasing function of N , the optimal prescheduled preventive replacement time is always $N^* \to \infty$ or $N^* = 1$.

For the proof of Theorem 3.7, see the proof of Theorem 3.5 in Appdenix.

We can directly have the following result from Theorem 3.7 .

Theorem 3.8. *For Model $j(= 1, \cdots, 6)$, suppose that $w_{lj}(N \mid \beta)$ is a strictly increasing function and $w_{lj}(N \mid \beta) > 0$, unde Assumption 1. Then the minimum expected total discounted costs over an infinite horizon have the lower and upper bounds:*

$$V_{lj}(N^* - 1 \mid \beta) < TC_{lj}(N^* \mid \beta) \leq V_{lj}(N^* \mid \beta), \tag{3.77}$$

where

$$V_{l1}(N \mid \beta) = \frac{(c_F - c_T)\, R_Y(N) \left[1 - \hat{H}_X(N)\right]}{1-\beta} + \frac{(c_T - c_Y)\, \hat{H}_X(N)}{1-\beta} - c_T, \quad (3.78)$$

$$V_{l2}(N \mid \beta) = V_{l5}(N \mid \beta) = \frac{\beta\,(c_F - c_T)\, r_Y(N+1)}{1-\beta} + \frac{(c_T - c_Y)\, \hat{H}_X(N)}{1-\beta} - c_T, \quad (3.79)$$

$$V_{l3}(N \mid \beta) = V_{l4}(N \mid \beta) = \frac{(c_F - c_T)\, R_Y(N)}{1-\beta} + \frac{(c_T - c_Y)\, \hat{H}_X(N)\,[1 + R_Y(N)]}{1-\beta} - c_T, \quad (3.80)$$

$$V_{l6}(N \mid \beta) = \frac{(c_F - c_T)\left[R_Y(N)\hat{H}_X(N) + \beta r_Y(N+1)\right]}{1-\beta} + \frac{(c_T - c_Y)\, \hat{H}_X(N)\,[1 + R_X(N)]}{1-\beta} - c_T. \quad (3.81)$$

3.3 Unified Models with Probabilistic Priority

3.3.1 Renewal Reward Approach

In the previous argument on RF and RL policies, we have derived the optimal scheduled preventive replacement times in respective cases. Suppose that each priority corresponding to Model j $(= 1, 2, \ldots, 6)$ occurs with probability p_j $(0 \le p_j \le 1)$, where $\sum_{j=1}^{6} p_j = 1$.

First consider RF model. Since the mean time lengths of one cycle in Model j $(= 1, 2, \ldots, 6)$ are all exactly same, the associated mean time length in our unified model with probability p_j is given by $A_{f7}(N) = A_f(N)$ in Eq. (3.2). Instead, the expected total cost during one cycle, $B_{f7}(N)$, with the probabilistic priority is given by $B_{f7}(N) = \sum_{j=1}^{6} p_j B_{fj}(N)$ with Eqs. (3.3)–(3.8). The underlying problem is simply formulated as $\min_N EC_{f7}(N)$, where $EC_{f7}(N) = B_{f7}(N)/A_f(N)$.

Define $w_{f7}(N) = \sum_{j=1}^{6} p_j V_{fj}(N) A_f(N) - B_{f7}(N)$ with Eqs. (3.18)–(3.23). Then it can be seen that $w_{f7}(N+1) - w_{f7}(N) = \sum_{j=1}^{6} p_j \{w_{fj}(N+1) - w_{fj}(N)\} A_f(N+1)$. Hence, for $p_j \ne 0$ $(j = 1, 2, \ldots, 6)$, necessary conditions of strictly increasing $w_{f7}(N)$ are to hold all conditions in Lemma 7.6.

Theorem 3.9. *Suppose that $w_{f7}(n)$ is strictly increasing, and $w_{f7}(\infty) > 0$ under Assumption 1. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds:*

$$V_{f7}(N^* - 1) < EC_{f7}(N^*) \leq V_{f7}(N^*), \tag{3.82}$$

where

$$V_{f7}(N) = \sum_{j=1}^{6} p_j V_{fj}(N). \tag{3.83}$$

Next consider RL model. Since the mean time length of one cycle and the expected total cost during one cycle are given by $A_{l7}(N) = A_l(N)$ in Eq. (3.25) and $B_{l7}(N) = \sum_{j=1}^{6} p_j B_{lj}(N)$ with Eqs. (3.26)–(3.29), define $w_{l7}(N) = \sum_{j=1}^{6} p_j V_{lj}(N) A_l(N) - B_{l7}(N)$ with Eqs. (3.36)–(3.39). Then, one has $w_{l7}(N+1) - w_{l7}(N) = \sum_{j=1}^{6} p_j \{w_{lj}(N+1) - w_{lj}(N)\} A_l(N+1)$, and finds that necessary conditions of strictly increasing $w_{l7}(N)$ are to hold all conditions in Lemma 7.7 with Assumption 1, when the failure time Y is strictly IFR.

Theorem 3.10. *Suppose that $w_{l7}(n)$ is strictly increasing and $w_{l7}(\infty) > 0$, under Assumption 1. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds:*

$$V_{l7}(N^* - 1) < EC_{l7}(N^*) \leq V_{l7}(N^*), \tag{3.84}$$

where

$$V_{l7}(N) = \sum_{j=1}^{6} p_j V_{lj}(N). \tag{3.85}$$

3.3.2 NPV Method

First, we calculate the unified model under RF discipline. Since the $A_f(N)$ for Model j $(= 1, \cdots, 6)$ are all same, $A_f(N)$ in the unified model with probability p_j is given by $A_{f7}(N, \beta) = A_f(N, \beta)$ in Eq. (3.40). In addition, the total expected discounted costs during the renewal cycle can be calculated by $TC_{f7}(N, \beta) = \sum_{j=1}^{6} p_j B_{fj}(N, \beta)$ with Eqs. (3.41)–(3.46). The underlying problem can be thought as $min_N TC_{f7}(N, \beta)$, where $TC_{f7}(N) = B_{f7}(N, \beta) / [1 - A_f(N, \beta)]$.

We define $w_{f7}(N \mid \beta) = \sum_{j=1}^{6} p_j V_{fj}(N) A_f(N) - B_{fj}(N, \beta)$ with Eqs. (3.56)–(3.61). Then we can obtain that $w_{f7}(N + 1 \mid \beta) - w_{f7}(N \mid \beta) =$

$\sum_{j=1}^{6} p_j \{w_{f7}(N+1 \mid \beta) - w_{f7}(N \mid \beta)\} A_f(N,\beta) - B_{fj}(N,\beta)$. Therefore, for $p_j \neq 0$, necessary conditions of strictly increasing $w_{f7}(N \mid \beta)$ for a given are to hold all conditions in Lemma 7.8 under Assumption 1. In unified model, we in position can have the optimal preventive replacement policies under RF discipline as follows.

Theorem 3.11. *Suppose that $w_{f7}(0 \mid \beta) < 0$ and $w_{f7}(\infty \mid \beta) > 0$ for a given β, under Assumption 1. Then the expected total discounted cost over an infinite horizon has the lower and upper bounds:*

$$V_{f7}(N^* - 1 \mid \beta) < TC_{f7}(N^* \mid \beta) \leq V_{f7}(N^* \mid \beta), \tag{3.86}$$

where

$$V_{f7} = \sum_{j=1}^{6} p_j V_{fj}(N \mid \beta). \tag{3.87}$$

Next, we examine the unified RL model. Since the net present value (NPV) of one unit cost during the renewal cycle, denoted as $A_l(N,\beta)$ for Model j $(= 1, \cdots, 6)$, remains constant, the expression for $A_l(N,\beta)$ in the unified model, accounting for probability p_j, is equivalent to $A_{l7}(N,\beta) = A_l(N,\beta)$ as defined in Eq. (3.62). Consequently, the total expected discounted costs throughout the renewal cycle can be determined by $TC_{l7}(N,\beta) = \sum_{j=1}^{6} p_j B_{lj}(N,\beta)$ using Eqs. (3.63)–(3.66). The underlying problem can be thought as $min_N TC_{l7}(N,\beta)$, where $TC_{l7}(N,\beta) = B_{l7}(N,\beta)/\left[1 - A_l(N,\beta)\right]$. Define $w_{l7}(N \mid \beta) = \sum_{j=1}^{6} p_j V_{lj}(N \mid \beta) A_l(N,\beta) - B_{lj}(N,\beta)$ with Eqs. (3.78)–(3.81). Then we can obtain that $w_{l7}(N+1 \mid \beta) - w_{l7}(N \mid \beta) = \sum_{j=1}^{6} p_j \Big\{ w_{f7}(N+1 \mid \beta) - w_{f7}(N \mid \beta) \Big\} A_f(N,\beta) - B_{fj}(N,\beta)$. The necessary conditions of strictly increasing $w_{l7}(N \mid \beta)$ for a given are to hold all conditions in Lemma 7.9 under Assumption 1. In unified model, we in position can have the optimal preventive replacement policies under RL discipline as follows.

Theorem 3.12. *Suppose that $w_{l7}(0 \mid \beta) < 0$ and $w_{l7}(\infty \mid \beta) > 0$ for a given β, under Assumption 1. Then the expected total discounted cost over an infinite horizon has the lower and upper bounds:*

$$V_{l7}(N^* - 1 \mid \beta) < TC_{l7}(N^* \mid \beta) \leq V_{l7}(N^* \mid \beta), \tag{3.88}$$

where

$$V_{l7} = \sum_{j=1}^{6} p_j V_{lj}(N \mid \beta). \tag{3.89}$$

3.4 Numerical Experiments

The model parameters and cost parameter as set in section 2.6. In unified model, we suppose that $p_1 = p_2 = p_3 = p_4 = 0.2$ and $p_5 = p_6 = 0.1$.

3.4.1 Renewal Reward Method

Table 3.1: Optimal N^* and $EC(N^*)$ with Model 1 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f1}(N^*)$	N^*	$EC_{l1}(N^*)$	N^*	$EC_{f1}(N^*)$	N^*	$EC_{l1}(N^*)$
1.5	16	0.1221	15	0.1107	16	0.1315	16	0.1111
2.0	13	0.1418	12	0.1402	13	0.1511	13	0.1411
3.0	10	0.1688	11	0.1959	10	0.1779	11	0.1973
4.0	9	0.1884	10	0.2504	9	0.1974	10	0.2523
5.0	8	0.2035	9	0.3049	8	0.2124	10	0.3067
6.0	7	0.2162	9	0.3589	7	0.2249	10	0.3612
7.0	7	0.2273	9	0.4130	7	0.2361	9	0.4153
8.0	6	0.2379	9	0.4670	6	0.2464	9	0.4693
9.0	6	0.2458	9	0.5210	6	0.2543	9	0.5234
10.0	6	0.2615	9	0.5750	6	0.2621	9	0.5774

First, we compare two replacement disciplines; RF policies and RL policies, under the assumption that the arrival time of opportunity for replacement obeys the geometric distribution with $h_X(x) = h = 0.95$ ($CaseA$). Throughout the example, we fix $c_T = 1.0$ (K dollar), and change the other cost parameters $c_F \in [1.5, 10.0]$ (K dollar), $c_Y = 0.8, 1.0$ (K dollar). In Tables 3.1–3.6, we compare two disciplines, RF policies and RL policies in Model 1 $\sim$ Model 6.

Table 3.2: Optimal N^* and $EC(N^*)$ with Model 2 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f2}(N^*)$	N^*	$EC_{l2}(N^*)$	N^*	$EC_{f2}(N^*)$	N^*	$EC_{l2}(N^*)$
1.5	18	0.1234	16	0.1122	18	0.1328	16	0.1125
2.0	13	0.1468	12	0.1431	13	0.1561	13	0.1441
3.0	10	0.1802	10	0.2004	10	0.1893	10	0.2023
4.0	8	0.2052	9	0.2564	8	0.2141	9	0.2588
5.0	8	0.2254	9	0.3123	8	0.2342	9	0.3147
6.0	7	0.2420	8	0.3677	7	0.2507	9	0.3705
7.0	6	0.2581	8	0.4232	6	0.2666	8	0.4261
8.0	6	0.2707	8	0.4786	6	0.2792	8	0.4816
9.0	6	0.2832	8	0.5341	6	0.2917	8	0.5370
10.0	6	0.2958	8	0.5896	6	0.3043	8	0.5924

Table 3.3: Optimal N^* and $EC(N^*)$ with Model 3 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f3}(N^*)$	N^*	$EC_{l3}(N^*)$	N^*	$EC_{f3}(N^*)$	N^*	$EC_{l3}(N^*)$
1.5	18	0.1234	16	0.1122	17	0.1303	15	0.1108
2.0	13	0.1397	12	0.1389	13	0.1494	12	0.1402
3.0	10	0.1663	10	0.1925	10	0.1757	10	0.1947
4.0	9	0.1854	9	0.2451	9	0.1946	10	0.2476
5.0	8	0.2004	9	0.2974	8	0.2094	9	0.3001
6.0	7	0.2133	9	0.3496	7	0.2222	9	0.3524
7.0	7	0.2239	9	0.4028	7	0.2327	9	0.4046
8.0	7	0.2344	8	0.4541	7	0.2433	9	0.4568
9.0	7	0.2450	8	0.5060	6	0.2511	9	0.5090
10.0	7	0.2555	8	0.5580	6	0.2586	9	0.5613

Table 3.4: Optimal N^* and $EC(N^*)$ with Model 4 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f4}(N^*)$	N^*	$EC_{l4}(N^*)$	N^*	$EC_{f4}(N^*)$	N^*	$EC_{l4}(N^*)$
1.5	18	0.5800	16	0.1122	18	0.5800	15	0.1108
2.0	14	0.7410	12	0.1389	14	0.1494	12	0.1402
3.0	11	0.9802	10	0.1925	11	0.1757	10	0.1947
4.0	9	1.1548	9	0.2451	9	0.1946	10	0.2476
5.0	8	1.2968	9	0.2974	8	0.2094	9	0.3001
6.0	8	1.4195	9	0.3496	8	0.2222	9	0.3524
7.0	7	1.5190	9	0.4028	7	0.2327	9	0.4046
8.0	7	1.6131	8	0.4541	7	0.2433	9	0.4568
9.0	6	1.7028	8	0.5060	6	0.2511	9	0.5090
10.0	6	1.7706	8	0.5580	6	0.2586	9	0.5613

Table 3.5: Optimal N^* and $EC(N^*)$ with Model 5 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f5}(N^*)$	N^*	$EC_{l5}(N^*)$	N^*	$EC_{f5}(N^*)$	N^*	$EC_{l5}(N^*)$
1.5	18	0.1234	16	0.1122	18	0.1328	16	0.1125
2.0	13	0.1465	12	0.1431	12	0.1561	13	0.1441
3.0	10	0.1797	10	0.2004	10	0.1893	10	0.2023
4.0	8	0.2043	9	0.2564	8	0.2141	9	0.2588
5.0	8	0.2245	9	0.3123	8	0.2343	9	0.3147
6.0	7	0.2409	8	0.3677	7	0.2507	9	0.3705
7.0	6	0.2567	8	0.4232	6	0.2666	8	0.4261
8.0	6	0.2693	8	0.4786	6	0.2792	8	0.4816
9.0	6	0.2818	8	0.5341	6	0.2917	8	0.5370
10.0	6	0.2944	8	0.5896	6	0.3043	8	0.5924

Table 3.6: Optimal N^* and $EC(N^*)$ with Model 6 for RF and RL models in discrete time.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f6}(N^*)$	N^*	$EC_{l6}(N^*)$	N^*	$EC_{f6}(N^**)$	N^*	$EC_{l6}(N^*)$
1.5	18	0.1234	16	0.1122	18	0.1315	16	0.1123
2.0	13	0.1465	12	0.1431	12	0.1544	12	0.1430
3.0	10	0.1797	10	0.2004	10	0.1865	10	0.1991
4.0	8	0.2043	9	0.2564	8	0.2108	9	0.2543
5.0	8	0.2245	9	0.3123	8	0.2303	8	0.3073
6.0	8	0.2409	8	0.3677	7	0.2466	8	0.3607
7.0	6	0.2567	8	0.4232	7	0.2621	8	0.4141
8.0	6	0.2693	8	0.4786	6	0.2748	8	0.4675
9.0	6	0.2818	8	0.5341	6	0.2867	8	0.5210
10.0	6	0.2986	8	0.5896	6	0.2944	7	0.5744

Table 3.7: Optimal N^* and $EC(N^*)$ with unifid Model for RF and RL models in discrete time, when $p_1 = p_2 = p_3 = p_4 = 0.2$ and $p_5 = p_6 = 0.1$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$EC_{f7}(N^*)$	N^*	$EC_{l7}(N^*)$	N^*	$EC_{f7}(N^*)$	N^*	$EC_{l7}(N^*)$
1.5	17	0.1218	15	0.1112	17	0.1315	15	0.1114
2.0	13	0.1426	12	0.1407	13	0.1523	13	0.1419
3.0	10	0.1718	10	0.1961	10	0.1813	10	0.1980
4.0	9	0.1933	9	0.2503	9	0.2028	10	0.2526
5.0	8	0.2102	9	0.3042	7	0.2196	9	0.3067
6.0	7	0.2245	9	0.3580	7	0.2309	9	0.3606
7.0	7	0.2374	8	0.4118	6	0.2467	9	0.4144
8.0	6	0.2514	8	0.4653	6	0.2580	9	0.4683
9.0	6	0.2609	8	0.5189	6	0.2675	8	0.5221
10.0	6	0.2679	8	0.5725	6	0.2770	8	0.5757

(1) When c_F is relatively small ($c_F = 1.5, 2.0$), it can be shown in all priority models that RL policies are better than RF policies in both Assumption 1. On the other hand, when c_F is larger and the impact of system failure becomes more remarkable, we find that RF policies are better than RL policies. From the results above, it is confirmed that RL policies can be motivated even in the plausible case of $c_F > c_Y$. However, when the failure impact is remarkable with large c_F, as expected, RF policies always outperform RL policies. In the sensitivity of the cost parameter c_F, as c_F increases, the optimal scheduled preventive replacement time N_0^* and its associated minimum expected cost decreases and increases, respectively. As the cost parameter c_Y increases from 0.8 to 1.0, the optimal scheduled preventive replacement times N^* are not sensitive to the change of c_Y, but the resulting minimum expected costs increase in almost all cases. A few exceptions arise in RL policies for Model 3 ($c_F = 1.5$), Model 4 ($c_F = 1.5$) and Model 6 ($c_F = 2.0$).

(2) In comparison of six priority models, when $c_F = 1.5$ and $c_Y = 0.8$, Model 1 provides the minimum expected cost uniquely. When $c_F = 1.5$ and $c_Y = 0.8$, Model 3 and Model 4 give the same minimum expected cost values. In the other combinations of c_F and c_Y, it can be observed that Model 3 and Model 4 minimize the expected cost functions per unit time in the steady state and give the exactly same cost values. Thus, if the replacement priority is selective, Model 3 ($S_c \succ O_p \succ C_a$) and Model 4 ($O_p \succ S_c \succ C_a$) show the similar cost performance and are better than the other priority models in almost all cases. This is because the failure replacement has the lowest priority, so that Model 3 and Model 4 tend to select cheaper replacement options even though the cost parameter c_F increases.

Next, we consider the unified model with probabilistic priority, where $p_1 = p_2 = p_3 = p_4 = 0.2$ and $p_5 = p_6 = 0.1$ are assumed. Table 3.7 presents the optimal scheduled preventive replacement times and their associated minimum expected costs for probabilistic priority models.

(3) Similar to the deterministic priority models, it is found that RL policies

are better than RF policies when c_F is small ($c_F = 1.5, 2.0$). Also, we can see the similar monotone properties of the optimal scheduled preventive replacement times and their associated minimum expected costs, as c_F and c_Y increase. In both deterministic and probabilistic priority models, we could find that RL policies can outperform RF policies when the cost impact of failure replacement is relatively small. Conversely, as c_F becomes larger, the minimum expected costs with RF discipline are much smaller than those with RL discipline. For instance, it is seen that the minimum expected costs with RL policy are almost double of those with RF policy with $c_F = 9.0, 10.0$.

3.4.2 NPV Method

We calculate the optimal preventive replacement times N^* and their associated expected costs $TC(N^* \mid \beta)$ under RF and PL disciplines for each model, when $c_T = 1$, $c_Y = 0.4, 1$ and discounted factor $\beta = 0.9$. The results are shown in Tables 3.8 – 3.14. Comparing Tables 3.8–3.14 and Tables 3.1–3.7, we can find:

(4) The discounted factor cannot affect the structure of optimal preventive replacement policy. i.e., The above lessons (1) – (3) always hold with discounted factor $\beta = 0.9$.

(5) When the economic environment is unstable, such as $\beta = 0.9$, the optimal preventive replacement times N^* will be delayed. When the economic environment is more unstable, for example, during times of economic uncertainty, the optimal timing for performing preventive equipment replacements will be postponed. This means that in uncertain economic conditions, people may delay replacing equipment or machinery to reduce costs or mitigate risks.

(6) The associated expected costs $TC(N^* \mid \beta)$ under RF and PL disciplines for each model have experienced a significant expansion in their overall size. We take Model 1 as an example. When $c_F = 2.0$, $c_Y = 0.8$, the optimal preventive replacement times N^* and their associated expected costs $EC(N^*)$ are given by $N^* = 13$, $EC_{f1}(N^*) = 0.1418$ and $N^* = 12$, $EC_{l1}(N^*) = 0.1402$ in Table 3.1. When $c_F = 2.0$, $c_Y = 0.8$, $\beta = 0.9$, the

optimal preventive replacement times N^* and their associated expected costs $TC(N^* \mid \beta)$ are given by $N^* = 14$, $TC_{f1}(N^* \mid \beta) = 0.9319$ and $N^* = 14$, $TC_{l1}(N^* \mid \beta) = 0.7568$ in Table 3.8

Table 3.8: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 1, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f1}(N^*)$	N^*	$TC_{l1}(N^*)$	N^*	$TC_{f1}(N^*)$	N^*	$TC_{l1}(N^*)$
1.5	19	0.7962	18	0.5812	19	0.8832	18	0.5816
2.0	14	0.9319	14	0.7568	14	1.0184	14	0.7588
3.0	11	1.1463	11	1.0740	11	1.2320	11	1.0798
4.0	9	1.3127	10	1.3742	9	1.3971	10	1.3826
5.0	8	1.4488	9	1.6665	8	1.5321	9	1.6780
6.0	8	1.5620	8	1.9577	8	1.6453	9	1.9695
7.0	7	1.6625	8	2.2443	7	1.7445	8	2.2598
8.0	7	1.7506	8	2.5310	7	1.8328	8	2.5465
9.0	7	1.8392	8	2.8176	7	1.9211	8	2.8331
10.0	7	1.9082	8	3.1043	6	1.9882	8	3.1198

Table 3.9: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 2, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f2}(N^*)$	N^*	$TC_{l2}(N^*)$	N^*	$TC_{f2}(N^*)$	N^*	$TC_{l2}(N^*)$
1.5	25	0.7971	22	0.5834	25	0.8841	22	0.5835
2.0	16	0.9437	15	0.7702	16	1.0305	15	0.7702
3.0	12	1.1966	11	1.1073	12	1.2827	11	1.1073
4.0	10	1.4008	10	1.4217	10	1.4859	10	1.4217
5.0	9	1.5739	9	1.7255	9	1.6538	9	1.7255
6.0	8	1.7219	8	2.0243	8	1.8053	8	2.0243
7.0	7	1.8559	8	2.3212	7	1.9378	8	2.3212
8.0	7	1.9764	7	2.6171	7	2.0584	7	2.6171
9.0	6	2.0938	7	2.9095	6	2.1737	7	2.9095
10.0	6	2.1899	7	3.2018	6	2.2697	7	3.2018

Table 3.10: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 3, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f3}(N^*)$	N^*	$TC_{l3}(N^*)$	N^*	$TC_{f3}(N^*)$	N^*	$TC_{l3}(N^*)$
1.5	20	0.7860	18	0.5808	19	0.8759	18	0.5813
2.0	15	0.9165	13	0.7535	15	1.0058	14	0.7563
3.0	11	1.1258	10	1.0615	11	1.2134	11	1.0694
4.0	10	1.2889	9	1.3502	10	1.3757	10	1.3631
5.0	9	1.4220	8	1.6328	9	1.5078	9	1.6469
6.0	8	1.5325	8	1.9103	8	1.6170	8	1.9277
7.0	7	1.6351	8	2.1879	7	1.7180	8	2.2052
8.0	7	1.7190	7	2.4637	7	1.8019	8	2.4828
9.0	7	1.8030	7	2.7374	7	1.8858	7	2.7602
10.0	6	1.8785	7	3.0111	6	1.9591	7	3.0339

Table 3.11: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 4, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f4}(N^*)$	N^*	$TC_{l4}(N^*)$	N^*	$TC_{f4}(N^*)$	N^*	$TC_{l4}(N^*)$
1.5	19	0.7859	18	0.5808	19	0.8759	18	0.5813
2.0	15	0.9158	13	0.7535	15	1.0058	14	0.7563
3.0	11	1.1234	10	1.0615	11	1.2134	11	1.0694
4.0	10	1.2858	9	1.3502	10	1.3757	10	1.3631
5.0	9	1.4178	8	1.6328	9	1.5078	9	1.6469
6.0	8	1.5270	8	1.9103	8	1.6170	8	1.9277
7.0	7	1.6280	8	2.1879	7	1.7180	8	2.2052
8.0	7	1.7119	7	2.4637	7	1.8019	8	2.4828
9.0	7	1.7958	7	2.7374	7	1.8858	7	2.7602
10.0	6	1.8691	7	3.0111	6	1.9591	7	3.0339

Table 3.12: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 5, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f5}(N^*)$	N^*	$TC_{l5}(N^*)$	N^*	$TC_{f5}(N^*)$	N^*	$TC_{l5}(N^*)$
1.5	24	0.7971	22	0.5834	25	0.8841	22	0.5835
2.0	16	0.9433	15	0.7702	16	1.0305	15	0.7702
3.0	12	1.1950	11	1.1073	12	1.2827	11	1.1073
4.0	10	1.3979	10	1.4217	10	1.4859	10	1.4217
5.0	9	1.5700	9	1.7255	9	1.6538	9	1.7255
6.0	8	1.7167	8	2.0243	8	1.8053	8	2.0243
7.0	7	1.8490	8	2.3212	7	1.9378	8	2.3212
8.0	7	1.9696	7	2.6171	7	2.0584	7	2.6171
9.0	6	2.0847	7	2.9095	6	2.1737	7	2.9095
10.0	6	2.1807	7	3.2018	6	2.2697	7	3.2018

Table 3.13: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with Model 6, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f6}(N^*)$	N^*	$TC_{l6}(N^*)$	N^*	$TC_{f6}(N^*)$	N^*	$TC_{l6}(N^*)$
1.5	25	0.7866	20	0.5833	25	0.8766	21	0.5834
2.0	17	0.9263	14	0.7660	17	1.0163	15	0.7683
3.0	12	1.1697	10	1.0876	12	1.2597	11	1.0967
4.0	10	1.3672	9	1.3858	10	1.4572	9	1.3988
5.0	9	1.5345	8	1.6737	9	1.6245	8	1.6910
6.0	8	1.6790	7	1.9575	8	1.7690	8	1.9788
7.0	7	1.8117	7	2.2394	7	1.9017	7	2.2622
8.0	7	1.9262	7	2.5214	7	2.0161	7	2.5441
9.0	7	2.0407	6	2.8004	7	2.1307	7	2.8261
10.0	6	2.1365	6	3.0788	6	2.2265	7	3.1080

Table 3.14: Optimal prescheduled preventive times N^* and associated expected discounted costs $TC(N^* \mid \beta)$ with unified model, when $\beta = 0.9$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N^*	$TC_{f7}(N^*)$	N^*	$TC_{l7}(N^*)$	N^*	$TC_{f7}(N^*)$	N^*	$TC_{l7}(N^*)$
1.5	21	0.7916	19	0.5822	21	0.8801	19	0.5824
2.0	15	0.9290	14	0.7602	15	1.0173	14	0.7623
3.0	11	1.1556	10	1.0791	11	1.2431	11	1.0856
4.0	10	1.3346	9	1.3769	10	1.4218	10	1.3885
5.0	9	1.4834	8	1.6683	9	1.5702	9	1.6815
6.0	8	1.6083	8	1.9545	8	1.6984	8	1.9709
7.0	7	1.7224	8	2.2407	7	1.8076	8	2.2571
8.0	7	1.8212	7	2.5243	7	1.9064	8	2.5433
9.0	7	1.9201	7	2.8065	7	2.0053	7	2.8282
10.0	6	2.0008	7	3.0887	6	2.0898	7	3.1104

Chapter 4

Two-Phase Opportunity-Based Age Replacement Models

4.1 RL Model in Continuous Time

4.1.1 Models Description

Let us consider a single-unit system with a non-repairable item. It is assumed that the time interval between opportunity arrivals for replacement obeys a homogeneous Poisson process with rate $\lambda(> 0)$, so that the inter-arrival time of replacement opportunities follows the exponential distribution:

$$G(t) = 1 - e^{-\lambda t} \tag{4.1}$$

with the p.d.f. is given by $g(t) = dG(t)/d = \lambda e^{-\lambda t}$. Let $F(t)$, $f(t)$, and $\bar{F}(t)$ denote the c.d.f., p.d.f., and the survivor function of the failure time of a unit, respectively. It is convenient to introduce the failure rate, denoted as $r(t) = \frac{f(t)}{\bar{F}(t)}$. Besides, the cost parameters are same with section 2.1.

Next, we describe the Iskandar and Sandoh model [8] and reformulate the replacement policies. Similar to [8], let S denote the restricted duration of replacement opportunity arrivals, during which no preventive replacement is made even if opportunities arrive. That is, if the unit fails during the time interval $(0, S]$, a failure replacement is made. After the time S passes, it is assumed that the unfailed unit is replaced at a pre-specified time T $(\geq S)$ or upon the arrival of a replacement opportunity. More specifically, in the replacement first model in the Iskandar and Sandoh model [8] , the unit is preventively replaced at time T or at the first arrival of an opportunity after the time S, whichever occurs first. On the other hand, under the replacement last discipline [2], the unit is preventively replaced at time T or at the first arrival of an opportunity after the time S, whichever occurs last (see the Figure 4.1). Let $EC_l(S,T)$ denote the long-run average cost. Then we have

$$EC_l(S,T) = \frac{B_l(S,T)}{A_l(S,T)}, \tag{4.2}$$

where $A_l(S,T)$ and $B_l(S,T)$ are the expected cycle length and the expected cost per cycle, respectively. Here, each cycle corresponds to the time interval between consecutive replacement actions, including both preventive replacements and failure replacements. It is straightforward to derive $A_l(S,T)$ and $B_l(S,T)$ as follows:

$$\begin{aligned} A_l(S,T) &= \int_0^T t f(t)\,dt + T\bar{F}(T)\int_S^T g(t-S)\,dt \\ &\quad + \int_T^\infty t\bar{G}(t-S)f(t)\,dt + \int_T^\infty t\bar{F}(t)g(t-S)\,dt \\ &= \int_0^T \bar{F}(t)\,dt + \int_T^\infty \bar{F}(t)e^{\lambda(t-S)}\,dt, \end{aligned} \tag{4.3}$$

$$\begin{aligned} B_l(S,T) &= c_F\int_0^T f(t)\,dt + c_T\bar{F}(T)\int_S^T g(t-S)\,dt \\ &\quad + c_F\int_T^\infty \bar{G}(t-S)f(t)\,dt + c_Y\int_T^\infty \bar{F}(t)g(t-S)\,dt \\ &= c_F + (c_F - c_T)\left[F(T) + \int_T^\infty \bar{G}(t-S)f(t)\,dt\right] \\ &\quad - (c_T - c_Y)\int_T^\infty g(t-S)\bar{F}(t)\,dt. \end{aligned} \tag{4.4}$$

From the above equations, our purpose is to minimize the long-run average cost $EC_l(S,T)$ with respect to (S,T). This minimization problem can be solved numerically.

In this paper, we focus on the one-dimensional optimization problems with respect to S for fixed T and T for fixed S. In some realistic cases, we may seek an optimal S^* for a given T when the preventive replacement time T is scheduled, or an optimal T^* for a given S when the restricted duration S is

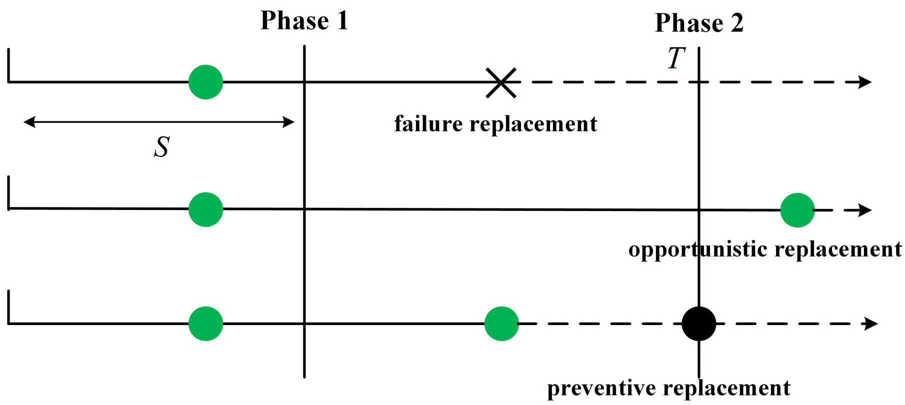

Figure 4.1: Configuration of two-phase RL discipline.

decided in accordance with supply chain management. Hence, our problems are to minimize $EC_l(S \mid T)$ and $EC_l(T \mid S)$, respectively, with respect to S and T.

4.1.2 Optimal Preventive Replacement Policies

First, we investigate the existence of an optimal S that minimizes $EC_l(S \mid T)$ for a fixed value of T. Two special cases with $S = 0$ and $S = T$ are given in the following:

$$EC_l(0 \mid T) = \frac{\begin{array}{c} c_F + (c_F - c_T)\left[F(T) + \int_T^\infty \bar{G}(t) f(t)\, dt\right] \\ -(c_T - c_Y)\int_T^\infty g(t)\bar{F}(t)\, dt \end{array}}{\int_0^T \bar{F}(t)\, dt + \int_T^\infty \bar{F}(t) e^{\lambda(t)}\, dt}, \tag{4.5}$$

$$EC_l(T \mid T) = \frac{\begin{array}{c} c_F + (c_F - c_T)\left[F(T) + \int_T^\infty \bar{G}(t-T) f(t)\, dt\right] \\ -(c_T - c_Y)\int_T^\infty g(t-T)\bar{F}(t)\, dt \end{array}}{\int_0^T \bar{F}(t)\, dt + \int_T^\infty \bar{F}(t) e^{\lambda(t-T)}\, dt}. \tag{4.6}$$

To obtain the optimal S^* minimizing $EC_l(S \mid T)$, taking the differentiation of $EC_l(S \mid T)$ with respect to S, we have

$$\begin{aligned} \frac{dEC_l(S \mid T)}{dS} &= (c_F - c_T)\left[\int_0^T \bar{F}(t)dt \int_T^\infty e^{-\lambda t} f(t)dt - F(T)\int_T^\infty e^{-\lambda t}\bar{F}(t)\right] \\ &\quad - [c_T - c_Y]\,\lambda \int_0^T \bar{F}(t)dt \int_T^\infty e^{-\lambda t}dt - c_T \int_T^\infty e^{-\lambda t}\bar{F}(t)dt. \end{aligned} \tag{4.7}$$

Let $w(S \mid T)$ denote the right-hand side of Eq. (4.7). Then it is evident to see that the function $w(S \mid T)$ does not contain S, so that $w(S \mid T)$ is regarded as a constant, and that the long-run average cost $EC_l(S \mid T)$ becomes a linear function.

Theorem 4.1. *(I) If $w(S \mid T) \geq 0$, then the optimal restricted duration time is $S^* = 0$, and it is optimal to trigger the preventive replacement at time T or at the first arrival of a replacement opportunity, whichever occurs last. The associated long-run average cost is given by Eq.* (4.5).
(II) If $w(S \mid T) < 0$, then the optimal restricted duration time is $S^* = T$, and it is optimal to trigger the preventive replacement at the first arrival of an opportunity. The minimized long-run average cost is given by Eq. (4.6).

The above result indicates that the optimal replacement last policies do not depend on the non-zero and finite S, and can be reduced to two trivial policies [2].

Next, we examine the existence of an optimal T that minimizes $EC(T|S)$ for a fixed S. Two special cases with $T = S$ and $T \to \infty$ are given by

$$EC_l(S \mid S) = \frac{\begin{array}{c} c_F + (c_F - c_T)\left[F(S) + \int_S^\infty \bar{G}(t-S) f(t)\, dt\right] \\ -(c_T - c_Y)\int_S^\infty g(t-S)\bar{F}(t)\, dt \end{array}}{\int_0^S \bar{F}(t)\, dt + \int_S^\infty \bar{F}(t) e^{\lambda(t-S)}\, dt}, \tag{4.8}$$

$$EC_l(\infty \mid S) = \frac{c_F}{\int_0^T \bar{F}(t) dt}. \tag{4.9}$$

Taking the differentiation of $EC(T|S)$ with respect to T yields

$$\frac{dEC_l(T \mid S)}{dT} = \frac{w(T \mid S)}{A_l(T \mid S)^2}, \tag{4.10}$$

where

$$w(T \mid S) = \Big[(c_F - c_T)\, r(T) + (c_T - c_Y)\, \hat{H}(T-S)\Big]\, A_l(S,T) - B_l(S,T). \tag{4.11}$$

Theorem 4.2. *(I) Suppose that $(c_F - c_T)\, r'(t) + (c_T - c_Y)\, \hat{H}'(t-S) > 0$ under Assumption 1.*

(i) If $w(\infty \mid S) > 0$, then there exists a finite and unique optimal time T^* which satisfies $w(T \mid S) = 0$ and the optimal expected cost in steady state is:

$$EC_l(T^* \mid S) = (c_F - c_T)\, r(T^*) + (c_T - c_Y)\, \hat{H}(T^* - S). \tag{4.12}$$

(ii) If $w(\infty \mid S) \leq 0$, then the optimal scheduled preventive replacement time is $T^* \to \infty$ and the optimal expected cost in steady state can be calculated by Eq. (4.9).

(II) Suppose that $(c_F - c_T)\, r'(t) + (c_T - c_Y)\, \hat{H}'(t - S) < 0$. Then the optimal scheduled preventive replacement times are given by $T^* \to \infty$ or $T^* = S$.

The proof of above theorem see the Appendix 7.4.6.

4.2 RF Model in Discrete Time

In our two-phase opportunity-based age replacement problems, if the failure occurs during the first phase ($0 < n \leq N_0$), then the failure replacement is made like a common age replacement, even though opportunity arrivals for replacement come before the failure. After the time N_0, an unfailed item is replaced at the time when an opportunistic arrival or a scheduled preventive replacement time N_1 ($\geq N_0$) comes, whichever occurs first. This discipline is called the *replacement first* (RF) discipline with preventive replacement time in the second phase ($N_0 \leq n \leq N_1$). On the other hand, Zhao and Nakagawa [2] proposed RL discipline, where an unfailed item is replaced preventively at the time when an opportunistic arrival or a scheduled preventive replacement time N_1 comes, whichever occurs last in the second phase ($N_0 \leq n \leq N_1$). The configurations of RF and RL disciplines are shown in Figure 4.2.

Here we concentrate on RF policies with six priority models and formulate the expected costs per unit time in the steady state. According to Figure 2 (i), we can derive the probability that the item is replaced at time n ($= 0, 1, 2, \ldots$) for all Model j ($= 1, \cdots, 6$) by

$$h_{fj}(n) = \begin{cases} f_Y(n) & 0 \leq n \leq N_0 \\ f_Y(n)\bar{G}_X(n-1-N_0) + g_X(n-N_0)\bar{F}_Y(n) & N_0+1 \leq n < N_1 - 1 \\ \bar{G}_X(N_1-1-N_0)\bar{F}_Y(N_1-1) & n = N_1 \\ 0 & n \geq N_1+1, \end{cases} \tag{4.13}$$

where $\sum_{i=0}^{\infty} h_{fj}(i) = 1$.

Then, it is evident to see that the mean time lengths of one cycle in Model j ($= 1, 2, \ldots, 6$) are all same. Let $A_f(N_0, N_1)$ be the mean time length of one

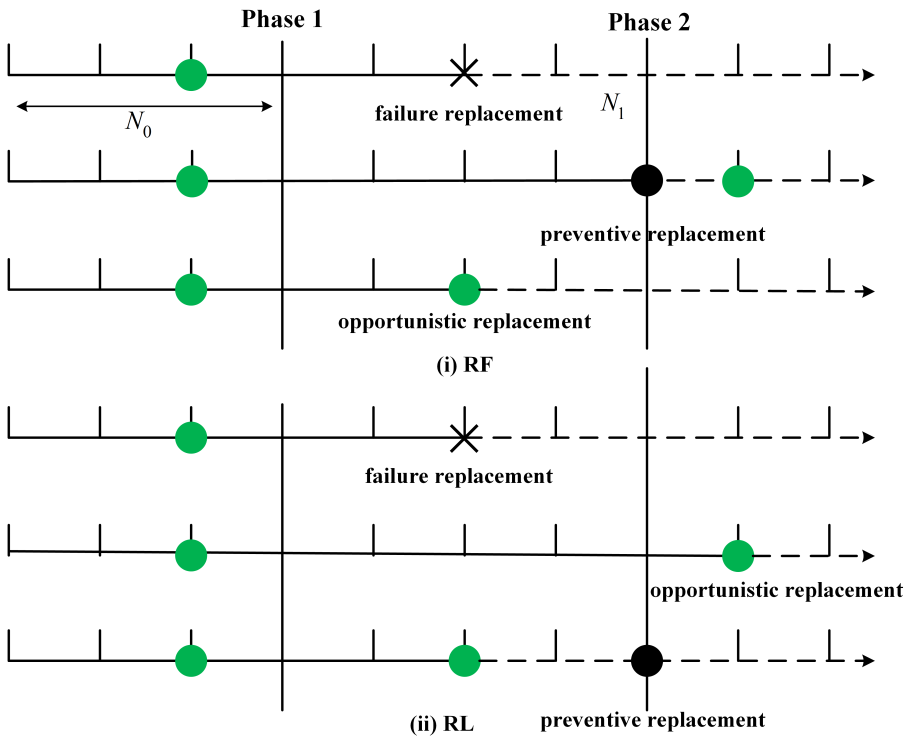

Figure 4.2: Configurations of two-phase RF and RL disciplines.

cycle in each Model j as a function of N_0 and N_1. From a few algebras, we have

$$
\begin{aligned}
A_f(N_0, N_1) &= \sum_{n=0}^{N_0} n f_Y(n) + \sum_{n=N_0+1}^{N_1-1} n\Big[f_Y(n)\bar{G}_X(n-1-N_0) + g_X(n-N_0)\bar{F}_Y(n)\Big] \\
&\quad + N_1\bar{G}_X(N_1-1-N_0)\bar{F}_Y(N_1-1) \\
&= \sum_{n=1}^{N_0} \bar{F}_Y(n-1) + \sum_{n=N_0+1}^{N_1} (1-q)^{n-1-N_0}\bar{F}_Y(n-1), \qquad (4.14)
\end{aligned}
$$

which is statistically independent of the kind of replacement priority.

The total expected costs during one cycle, $B_{fj}(N_0, N_1)$, for Model j $(= 1, 2, \ldots, 6)$ are given by

$$
\begin{aligned}
B_{f1}(N_0, N_1) &= c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{n=N_0+1}^{N_1-1} (1-q)^{n-1-N_0} f_Y(n)\right] \\
&\quad - (c_T - c_Y)\sum_{n=N_0+1}^{N_1-1} q(1-q)^{n-1-N_0}\bar{F}_Y(n), \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
B_{f2}(N_0, N_1) &= c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{N_0+1}^{N_1} (1-q)^{n-1-N_0} f_Y(n)\right] \\
&\quad - (c_T - c_Y)\sum_{n=N_0+1}^{N_1-1} q(1-q)^{n-1-N_0}\bar{F}_Y(n), \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
B_{f3}(N_0, N_1) &= c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{n=N_0+1}^{N_1-1} (1-q)^{n-N_0} f_Y(n)\right] \\
&\quad - (c_T - c_Y)\sum_{n=N_0+1}^{N_1-1} q(1-q)^{n-1-N_0}\bar{F}_Y(n-1), \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
B_{f4}(N_0, N_1) &= c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{n=N_0+1}^{N_1-1} (1-q)^{n-N_0} f_Y(n)\right] \\
&\quad - (c_T - c_Y)\sum_{n=N_0+1}^{N_1} q(1-q)^{n-1-N_0}\bar{F}_Y(n-1), \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
B_{f5}(N_0, N_1) &= c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{n=N_0+1}^{N_1} (1-q)^{n-N_0-1} f_Y(n)\right] \\
&\quad - (c_T - c_Y)\sum_{n=N_0+1}^{N_1} q(1-q)^{n-1-N_0}\bar{F}_Y(n), \qquad (4.19)
\end{aligned}
$$

$$B_{f6}(N_0, N_1) = c_T + (c_F - c_T)\left[\sum_{n=0}^{N_0} f_Y(n) + \sum_{n=N_0+1}^{N_1} (1-q)^{n-N_0} f_Y(n)\right] - (c_T - c_Y)\sum_{n=N_0+1}^{N_1} q(1-q)^{n-1-N_0}\bar{F}_Y(n-1), \tag{4.20}$$

for Model j $(= 1, 2, \ldots, 6)$, respectively.

The expected costs per unit time in the steady state are given by

$$EC_{fj}(N_0, N_1) = \frac{B_{fj}(N_0, N_1)}{A_f(N_0, N_1)}. \tag{4.21}$$

Then, the problems are to determine the optimal pair of the restricted duration and the preventive replacement time (N_0^*, N_1^*), which minimizes $EC_{fj}(N_0, N_1)$ with $1 \leq N_0 \leq N_1$.

We minimize the long-run average costs $EC_{fj}(N_0, N_1)$ $(j = 1, \ldots, 6)$ with respect to (N_0, N_1). The simultaneous minimization problem in discrete time can be solved numerically. In this section, we focus on two one-dimensional optimization problems with respect to N_0 for a fixed N_1 and N_1 for a fixed N_0. In some realistic cases, we may seek an optimal N_0^* for a given N_1 when the preventive replacement time N_1 is scheduled, or an optimal N_1^* for a given N_0, when the restricted duration N_0 is decided in accordance with the supply chain management.

First, we consider the case with N_0 for a fixed N_1, for Model j $(= 1, 2, \ldots, 6)$, and derive the respective optimal opportunistic age replacement policies which minimize the expected costs per unit time in the steady state, $EC_{fj}(N_0 \mid N_1)$. Unfortunately, here we discuss the only case $c_T = c_Y$. The other case $c_T > c_Y$ is analyzed by numerical experiment.

Taking the difference of $EC_{fj}(N_0 \mid N_1)$ with respect to N_0 yields

$$EC_{fj}(N_0 + 1 \mid N_1) - EC_{fj}(N_0 \mid N_1) = \frac{A_{f0}(N_0 \mid N_1) w_{fj}(N_0 \mid N_1)}{A_f(N_0, N_1) A_f(N_0 + 1, N_1)}, \tag{4.22}$$

where

$$A_{f0}(N_0 \mid N_1) = A_f(N_0 + 1, N_1) - A_f(N_0, N_1) = \sum_{n=N_0+2}^{N_1} q(1-q)^{n-N_0-2}\bar{F}_Y(n-1), \tag{4.23}$$

and

$$w_{f1}(N_0 \mid N_1) = (c_F - c_T)\left[h_Y(N_0, N_1) - \frac{q(1-q)^{N_1-N_0-2} f_Y(N_1)}{A_{f0}(N_0 \mid N_1)}\right] \times A_f(N_0, N_1) - B_{f1}(N_0, N_1), \tag{4.24}$$

$$w_{f2}(N_0 \mid N_1) = (c_F - c_T) h_Y(N_0, N_1) A_f(N_0, N_1) - B_{f2}(N_0, N_1), \tag{4.25}$$

$$w_{f3}(N_0 \mid N_1) = (c_F - c_T) H_Y(N_0 + 1, N_1 - 1) A_f(N_0, N_1) - B_{f3}(N_0, N_1), \tag{4.26}$$

$$w_{f4}(N_0 \mid N_1) = (c_F - c_T) H_Y(N_0 + 1, N_1 - 1) A_f(N_0, N_1) - B_{f4}(N_0, N_1), \tag{4.27}$$

$$w_{f5}(N_0 \mid N_1) = (c_F - c_T) h_Y(N_0, N_1) A_f(N_0, N_1) - B_{f5}(N_0, N_1), \tag{4.28}$$

$$w_{f6}(N_0 \mid N_1) = (c_F - c_T)\left[H_Y(N_0 + 1, N_1 - 1) - \frac{q(1-q)^{N_1-N_0-1} f_Y(N_1)}{A_{f0}(N_0 \mid N_1)}\right] \times A_f(N_0, N_1) - B_{f6}(N_0, N_1), \tag{4.29}$$

and

$$h_Y(N_0, N_1) = \frac{\sum_{n=N_0+2}^{N_1} (1-q)^{n-N_0} f_Y(n)}{\sum_{n=N_0+2}^{N_1} (1-q)^{n-N_0} \bar{F}_Y(n-1)}, \tag{4.30}$$

$$H_Y(N_0 + 1, N_1 - 1) = \frac{\sum_{n=N_0+1}^{N_1-1} (1-q)^{n-N_0} f_Y(n)}{\sum_{n=N_0+1}^{N_1-1} (1-q)^{n-N_0} \bar{F}_Y(n)}. \tag{4.31}$$

Next, we care about the necessary conditions the functions $w_{fj}(n)$ are strictly increasing. If the failure time Y is strictly IFR, both $h_Y(N_0, N_1)$ and $H_Y(N_0 + 1, N_1 - 1)$ are strictly increasing in N_0 (the proof can be seen in Lemma 7.10). Hence, if the failure time Y is strictly IFR, $w_{fj}(n)$ $(j = 2, 3, 4, 5)$ are strictly increasing. For Model 1 and More 6, the additional condition is required. More detailed discussion can also be found in Lemma 7.11.

Theorem 4.3. *(I) Suppose that the functions $w_{fj}(N_0 \mid N_1)$ $(j = 1, 2, \ldots, 6)$ are strictly increasing in N_0 at the special case $c_T = c_Y$.*

(i) If $w_{fj}(N_1 \mid N_1) > 0$, then there exists at least one (at most two) optimal restricted duration N_0^* which satisfies $w_{fj}(N_0^* - 1 \mid N_1) < 0$ and $w_{fj}(N_0^* \mid N_1) \geq 0$.

(ii) If $w_{fj}(N_1 \mid N_1) \leq 0$, *then the optimal restricted duration is given by* $N_0^* = N_1$ *and it is optimal to carry out only the age replacement.*

(II) Suppose that the functions $w_{fj}(N_0 \mid N_1)$ *are decreasing in* N_0. *Then the optimal restricted duration is given by* $N_0^* = 1$ *or* $N_0^* = N_1$.

For the proof, see the Appendix 7.4.5.

The following result can be derived from Theorem 4.3 straightforwardly.

Theorem 4.4. *For Model* j $(= 1, 2, \ldots, 6)$, *suppose that the functions* $w_{fj}(N_0 \mid N_1)$ *are strictly increasing in* N_0, *in addition to* $w_{fj}(N_0|N_1) < 0$ *and* $w_{fj}(N_1|N_1) > 0$. *Then the minimum expected costs per unit time in the steady state have the lower and upper bounds;*

$$V_{fj}(N_0^* - 1 \mid N_1) < EC_{fj}(N_0^* \mid N_1) \leq V_{fj}(N_0^* \mid N_1), \tag{4.32}$$

where

$$V_{f1}(N_0 \mid N_1) = (c_F - c_T)\left[h_Y(N_0, N_1) - \frac{q(1-q)^{N_1 - N_0 - 2} f_Y(N_1)}{A_{f0}(N_0 \mid N_1)}\right], \tag{4.33}$$

$$V_{f2}(N_0 \mid N_1) = (c_F - c_T)h_Y(N_0, N_1), \tag{4.34}$$

$$V_{f3}(N_0 \mid N_1) = (c_F - c_T)H_Y(N_0 + 1, N_1 - 1), \tag{4.35}$$

$$V_{f4}(N_0 \mid N_1) = (c_F - c_T)H_Y(N_0 + 1, N_1 - 1), \tag{4.36}$$

$$V_{f5}(N_0 \mid N_1) = (c_F - c_T)h_Y(N_0, N_1), \tag{4.37}$$

$$V_{f6}(N_0 \mid N_1) = (c_F - c_T)\left[H_Y(N_0 + 1, N_1 - 1) - \frac{q(1-q)^{N_1 - N_0 - 1} f_Y(N_1)}{A_{f0}(N_0 \mid N_1)}\right]. \tag{4.38}$$

Next, we consider the problem $\min_{N_1} EC_{fj}(N_0, N_1)$ for a fixed N_0. Define the functions:

$$w_{f1}(N_1 \mid N_0) = \frac{1}{1-q}\Big\{(c_F - c_T)R_Y(N_1) - q(c_T - c_Y)\Big\}A_f(N_0, N_1) - B_{f1}(N_0, N_1), \tag{4.39}$$

$$w_{f2}(N_1 \mid N_0) = \Big[(c_F - c_T)r_Y(N_1 + 1) - \frac{q}{1-q}(c_T - c_Y)\Big]A_f(N_0, N_1) - B_{f2}(N_0, N_1), \tag{4.40}$$

$$w_{f3}(N_1 \mid N_0) = \Big\{(c_F - c_T)R_Y(N_1) - \frac{q}{1-q}(c_T - c_Y)\left[1 + R_Y(N_1)\right]\Big\} \times A_f(N_0, N_1) - B_{f3}(N_0, N_1), \tag{4.41}$$

$$w_{f4}(N_1 \mid N_0) = \Big[(c_F - c_T)R_Y(N_1) - q(c_T - c_Y)\Big] A_f(N_0, N_1) - B_{f4}(N_0, N_1), \tag{4.42}$$

$$\begin{aligned} w_{f5}(N_1 \mid N_0) = & \Big\{(c_F - c_T)r_Y(N_1 + 1) - q(c_T - c_Y)\left[1 - r_Y(N_1 + 1)\right]\Big\} \\ & \times A_f(N_0, N_1) - B_{f5}(N_0, N_1), \end{aligned} \tag{4.43}$$

$$\begin{aligned} w_{f6}(N_1 \mid N_0) = & \Big\{(1 - q)(c_F - c_T)r_Y(N_1 + 1) - q(c_T - c_Y)\Big\} \\ & \times A_f(N_0, N_1) - B_{f6}(N_0, N_1), \end{aligned} \tag{4.44}$$

satisfying

$$EC_{fj}(N_1 + 1 \mid N_0) - EC_{fj}(N_1 \mid N_0) = \frac{A_{f1}(N_1 \mid N_0)q_{fj}(N_1 \mid N_0)}{A_f(N_0, N_1)A_f(N_0, N_1 + 1)}, \tag{4.45}$$

where

$$\begin{aligned} A_{f1}(N_1 \mid N_0) &= A_f(N_0, N_1 + 1) - A_f(N_0, N_1) \\ &= \bar{F}_Y(N_1)\bar{G}_X(N_1 - N_0). \end{aligned} \tag{4.46}$$

Theorem 4.5. *(I) For Model 1, Model 2, Model 4, Model 5 and Model 6, suppose the failure time distribution is strictly (IFR). For Model 3, suppose that the failure time distribution is strictly IFR and* $(c_F - c_T) \geq (c_T - c_Y)q/(1 - q)$.

(i) If $w_{fj}(\infty \mid N_0) > 0$*, then there exists at least one (at most two) optimal preventive replacement time* N_1^* *which satisfies* $w_{fj}(N_1^* - 1 \mid N_0) < 0$ *and* $w_{fj}(N_1^* \mid N_0) \geq 0$.

(ii) If $w_{fj}(\infty \mid N_0) \leq 0$*, then the optimal preventive replacement time is* $N_1^* \to \infty$*, and it is optimal to carry out either the failure replacement or the opportunistic one, whichever occurs first.*

(II) For Model 1, Model 2, Model 4, Model 5 and Model 6, suppose that the failure time distribution is decreasing failure rate (DFR). For Model 3, suppose that the failure time distribution is DFR and $(c_F - c_T) \leq (c_T - c_Y)q/(1 - q)$ *holds. Then the optimal preventive replacement time is given by* $N_1^* \to \infty$ *or* $N_1^* = N_0$.

see the Appendix 7.4.5.

The following result can be derived from Theorem 4.5 directly without the proof.

Theorem 4.6. *For Model j $(= 1, 2, \ldots, 6)$, suppose that $w_{fj}(N_1 \mid N_0)$ are strictly increasing in N_1, $w_{fj}(N_0|N_0) < 0$ and $w_{fj}(\infty \mid N_0) > 0$. Then the minimum expected costs per unit time in the steady state have the lower and upper bounds;*

$$V_{fj}(N_1^* - 1 \mid N_0 - 1) < EC_{fj}(N_1^* \mid N_0) \leq V_{fj}(N_1^* \mid N_0), \tag{4.47}$$

where

$$V_{f1}(N_1 \mid N_0) = \frac{1}{1-q}\left[(c_F - c_T)R_Y(N_1) - q(c_T - c_Y)\right], \tag{4.48}$$

$$V_{f2}(N_1 \mid N_0) = (c_F - c_T)r_Y(N_1 + 1) - \frac{q}{1-q}(c_T - c_Y), \tag{4.49}$$

$$V_{f3}(N_1 \mid N_0) = (c_F - c_T)R_Y(N_1) - \frac{q}{1-q}(c_T - c_Y)\left[1 + R_Y(N_1)\right], \tag{4.50}$$

$$V_{f4}(N_1 \mid N_0) = (c_F - c_T)R_Y(N_1) - q(c_T - c_Y), \tag{4.51}$$

$$V_{f5}(N_1 \mid N_0) = (c_F - c_T)r_Y(N_1 + 1) - q(c_T - c_Y)\left[1 - r_Y(N_1 + 1)\right], \tag{4.52}$$

$$V_{f6}(N_1 \mid N_0) = (1 - q)(c_F - c_T)r_Y(N_1 + 1) - q(c_T - c_Y). \tag{4.53}$$

4.3 RL Model in Discrete Time

Next, we concern about RL policies with 6 priority models (see Figure 2 (ii)). Similar to the RF discipline, the probabilities that the item is replaced at time n $(= 0, 1, 2, \ldots)$ for Model j $(= 1, 2, \ldots, 6)$ are given by

$$h_{lj}(n) = \begin{cases} f_Y(n) & 0 \leq n \leq N_1 - 1 \\ f_Y(N_1)\bar{G}_X(N_1 - 1 - N_0) + & \\ \bar{F}_Y(N_1 - 1)G_X(N_1 - 1 - N_0) & \\ +\bar{F}_Y(N_1)g_X(N_1 - N_0) & n = N_1 \\ f_Y(n)\bar{G}_X(n - 1 - N_0) + g_X(n - N_0)\bar{F}_Y(n) & n \geq N_1 + 1, \end{cases} \tag{4.54}$$

where $\sum_{i=0}^{\infty} h_{lj}(i) = 1$ $(j = 1, 2, \ldots, 6)$.

From Eq. (4.54), it is confirmed that the mean time lengths of one cycle for Model j $(= 1, 2, \ldots, 6)$ are all same and given by

$$\begin{aligned} A_l(N_0, N_1) &= \sum_{n=0}^{N_1-1} n f_Y(n) + N_1 \Big[f_Y(N_1)\bar{G}_X(N_1 - 1 - N_0) \\ &\quad + \bar{F}_Y(N_1 - 1)G_X(N_1 - 1 - N_0) + \bar{F}_Y(N_1)g_X(N_1 - N_0) \Big] \\ &\quad + \sum_{n=N_1+1}^{\infty} n \Big[f_Y(n)\bar{G}_X(n - 1 - N_0) \\ &\quad + g_X(n - N_0)\bar{F}_Y(n) \Big] \\ &= \sum_{n=1}^{N_1} \bar{F}_Y(n-1) + \sum_{n=N_1+1}^{\infty} (1-q)^{n-N_0-1}\bar{F}_Y(n-1). \end{aligned} \tag{4.55}$$

The expected total costs during one cycle, $B_{lj}(N_0, N_1)$, for Model j $(= 1, 2, \ldots, 6)$ are given by

$$\begin{aligned} B_{l1}(N_0, N_1) &= c_T + (c_F - c_T) \left[\sum_{n=0}^{N_1-1} f_Y(n) + \sum_{n=N_1}^{\infty} (1-q)^{n-N_0-1} f_Y(n) \right] \\ &\quad - (c_T - c_Y) \sum_{n=N_1}^{\infty} q(1-q)^{n-N_0-1}\bar{F}_Y(n), \end{aligned} \tag{4.56}$$

$$\begin{aligned} B_{l2}(N_0, N_1) &= B_{l5}(N_0, N_1) \\ &= c_T + (c_F - c_T) \left[\sum_{n=0}^{N_1} f_Y(n) + \sum_{n=N_1+1}^{\infty} (1-q)^{n-N_0-1} f_Y(n) \right] \\ &\quad - (c_T - c_Y) \sum_{n=N_1}^{\infty} q(1-q)^{n-N_0-1}\bar{F}_Y(n), \end{aligned} \tag{4.57}$$

$$\begin{aligned} B_{l3}(N_0, N_1) &= B_{l4}(N_0, N_1) \\ &= c_T + (c_F - c_T) \left[\sum_{n=0}^{N_1-1} f_Y(n) + \sum_{n=N_1}^{\infty} (1-q)^{n-N_0} f_Y(n) \right] \\ &\quad - (c_T - c_Y) \sum_{n=N_1}^{\infty} q(1-q)^{n-N_0-1}\bar{F}_Y(n-1), \end{aligned} \tag{4.58}$$

$$
\begin{aligned}
B_{l6}(N_0, N_1) = c_T + (c_F - c_T)\Bigg[& \sum_{n=0}^{N_1-1} f_Y(n) + f_Y(N_1)[1 - q(1-q)^{n-N_0-1}] \\
& + \sum_{n=N_1}^{\infty} (1-q)^{n-N_0} f_Y(n)\Bigg] \\
& - (c_T - c_Y) \sum_{n=N_1}^{\infty} q(1-q)^{n-N_0-1} \bar{F}_Y(n-1).
\end{aligned}
\tag{4.59}
$$

Then, the problems are to minimize the expected costs per unit time in the steady state under RL discipline, $EC_{lj}(N_0, N_1) = B_{lj}(N_0, N_1)/A_l(N_0, N_1)$, for Model j $(= 1, 2, \dots, 6)$.

In a fashion similar to the RF policies, we consider $\min_{N_0} EC_{lj}(N_0, N_1)$ for a fixed N_1 and $\min_{N_1} EC_{lj}(N_0, N_1)$ for a fixed N_0 under RL discipline.

First, let $w_{lj}(N_0 \mid N_1)$ satisfy

$$
EC_{lj}(N_0+1 \mid N_1) - EC_{lj}(N_0 \mid N_1) = \frac{A_{l0}(N_0 \mid N_1) q_{lj}(N_0 \mid N_1)}{A_l(N_0, N_1) A_l(N_0+1, N_1)}, \tag{4.60}
$$

where

$$
A_{l0}(N_0 \mid N_1) = A_l(N_0+1, N_1) - A_l(N_0, N_1) = \sum_{n=N_1}^{\infty} q(1-q)^{n-N_0-1} \bar{F}_Y(n). \tag{4.61}
$$

Then it is immediate to derive

$$
\begin{aligned}
w_{l1}(N_0 \mid N_1) = & \left\{ (c_F - c_T) \frac{1}{1-q} H_Y(N_1 - 1) - \frac{q}{1-q}(c_T - c_Y) \right\} \\
& \times \sum_{n=1}^{N_1-1} \bar{F}_Y(n-1) - c_T,
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
w_{l2}(N_0 \mid N_1) = w_{l5}(N_0 \mid N_1) = & \left\{ (c_F - c_T) H_Y(N_1 - 1) - \frac{q}{1-q}(c_T - c_Y) \right\} \\
& \times \sum_{n=1}^{N_1} \bar{F}_Y(n-1) - c_T,
\end{aligned}
\tag{4.63}
$$

$$
\begin{aligned}
w_{l3}(N_0 \mid N_1) = w_{l4}(N_0 \mid N_1) = & \Bigg\{ \left[(c_F - c_T) \frac{1}{1-q} - \frac{q}{1-q}(c_T - c_Y) \right] \\
& \times H(N_1 - 1) - \frac{q}{1-q}(c_T - c_Y) \Bigg\} \sum_{n=1}^{N_1-1} \bar{F}_Y(n-1) - c_T,
\end{aligned}
\tag{4.64}
$$

$$\begin{aligned}
w_{l6}(N_0 \mid N_1) = & \Bigg\{ \left[(c_F - c_T) \frac{1}{1-q} - \frac{q}{1-q} (c_T - c_Y) \right] H_Y(N_1 - 1) \\
& - \frac{q}{1-q} (c_F - c_T) \frac{f_Y(N_1)(1-q)^n}{\sum_{n=N_1}^{\infty} \bar{F}_Y(n)(1-q)^n} \\
& - \frac{q}{1-q} (c_T - c_Y) \Bigg\} \sum_{n=1}^{N_1 - 1} \bar{F}_Y(n-1) + (c_F - c_T) f_Y(N_1) \\
& - c_T, \qquad (4.65)
\end{aligned}$$

where

$$H_Y(N_1 - 1) = \frac{\sum_{n=N_1}^{\infty} f_Y(n)(1-q)^n}{\sum_{n=N_1}^{\infty} \bar{F}_Y(n)(1-q)^n}. \qquad (4.66)$$

As shown in the above results, the functions $w_{lj}(N_0|N_1)$ $(j = 1, 2, \ldots, 6)$ do not contain N_0, i.e., the functions $w_{lj}(N_0|N_1)$ are constant values for Model j $(= 1, 2, \ldots, 6)$.

Theorem 4.7. *(I) If $w_{lj}(N_0|N_1) < 0$ $(j = 1, 2, \ldots, 6)$, then the optimal restricted duration is given by $N_0^* = N_1$ and it is optimal to carry out only the age replacement.*
(II) If $w_{lj}(N_0|N_1) \geq 0$ $(j = 1, 2, \ldots, 6)$, then the optimal restricted duration is $N_0^* = 0$ and it is optimal to carry out either the age replacement or the opportunistic one, whichever occurs last.

The following result can be derived from Theorem 4.7 directly.

Theorem 4.8. *The minimum expected costs per unit time in the steady state* *are given by*

$$EC_{l1}(N_0^* \mid N_1) = (c_F - c_T)\,\frac{1}{1-q} H_Y(N_1 - 1) - \frac{q}{1-q}\,(c_T - c_Y)\,, \tag{4.67}$$

$$EC_{l2}(N_0^* \mid N_1) = EC_{l5}(N_0^* \mid N_1) = (c_F - c_T)\,H_Y(N_1 - 1) - \frac{q}{1-q}\,(c_T - c_Y)\,, \tag{4.68}$$

$$\begin{aligned} EC_{l3}(N_0^* \mid N_1) = EC_{l4}(N_0^* \mid N_1) &= \left[(c_F - c_T)\,\frac{1}{1-q} - \frac{q}{1-q}\,(c_T - c_Y)\right] \\ &\times H_Y(N_1 - 1) - \frac{q}{1-q}\,(c_T - c_Y)\,, \end{aligned} \tag{4.69}$$

$$\begin{aligned} EC_{l6}(N_0^* \mid N_1) = &\left[(c_F - c_T)\,\frac{1}{1-q} - \frac{q}{1-q}\,(c_T - c_Y)\right] H_Y(N_1 - 1) \\ &- \frac{q}{1-q}\,(c_F - c_T)\,\frac{f_Y(N_1)\,(1-q)^n}{\sum_{n=N_1}^{\infty} \bar{F}_Y(n)\,(1-q)^n} \\ &- \frac{q}{1-q}\,(c_T - c_Y)\,. \end{aligned} \tag{4.70}$$

Next, we examine the existence of an optimal preventive replacement time N_1^* that minimizes $EC_{lj}(N_0, N_1)$ for a fixed N_0. Taking the difference of $EC_{lj}(N_0, N_1)$ with respect to N_1 leads to

$$EC_{lj}(N_1 + 1 \mid N_0) - EC_{lj}(N_1 \mid N_0) = \frac{A_{l1}(N_1 \mid N_0) w_{lj}(N_1 \mid N_0)}{A_l(N_0, N_1) A_l(N_0, N_1 + 1)}, \tag{4.71}$$

where

$$\begin{aligned} A_{l1}(N_1 \mid N_0) &= A_l(N_0, N_1 + 1) - A_l(N_0, N_1) \\ &= \left[1 - (1-q)^{n-N_0}\right] \bar{F}_Y(N_1), \end{aligned} \tag{4.72}$$

and

$$\begin{aligned} w_{l1}(N_1 \mid N_0) = &\left\{(c_F - c_T) R_Y(N_1) \left[1 - \hat{H}(N_0, N_1)\right] + (c_T - c_Y)\hat{H}(N_0, N_1)\right\} \\ &\times A_{l1}(N_0, N_1) - B_{l1}(N_0, N_1), \end{aligned} \tag{4.73}$$

$$\begin{aligned} w_{l2}(N_1 \mid N_0) = w_{l5}(N_1 \mid N_0) = &\left[(c_F - c_T) r_Y(N_1 + 1) + (c_T - c_Y)\hat{H}(N_0, N_1)\right] \\ &\times A_l(N_0, N_1) - B_{l2}(N_0, N_1), \end{aligned} \tag{4.74}$$

$$
\begin{aligned}
w_{l3}(N_1 \mid N_0) = w_{l4}(N_1 \mid N_0) = \Big\{ &(c_F - c_T) R_Y(N_1) \\
&+ (c_T - c_Y)\,[1 + R_Y(N_1)]\, \hat{H}(N_0, N_1) \Big\} A_l(N_0, N_1) \\
&- B_{l3}(N_0, N_1),
\end{aligned} \tag{4.75}
$$

$$
\begin{aligned}
w_{l6}(N_1 \mid N_0) = \Big\{ &(c_F - c_T) \Big[r_Y(N_1 + 1) + R_Y(N_1) \hat{H}(N_0, N_1) \Big] \\
&+ (c_T - c_Y)\,[1 + R_Y(N_1)]\, \hat{H}(N_0, N_1) \Big\} A_l(N_0, N_1) \\
&- B_{l6}(N_0, N_1),
\end{aligned} \tag{4.76}
$$

$$
\hat{H}(N_0, N_1) = \frac{q(1-q)^{N_1 - 1 - N_0}}{1 - (1-q)^{N_1 - N_0}}. \tag{4.77}
$$

The monotone property of $\hat{H}(N_0, N_1)$ is shown in [43]. In addition, the monotonicity of the functions $w_{lj}(N_1 \mid N_0)$ $(j = 1, 2, \ldots, 6)$ depends on the function $\hat{H}(N_0, N_1)$. Then, additional necessary conditions for strictly increasing $w_{lj}(N_1 \mid N_0)$ $(j = 1, 2, \ldots, 6)$ are given in Lemma 7.12 in Appendix.

Theorem 4.9. *(I) Suppose that the functions $w_{lj}(N_1|N_0)$ $(j = 1, 2, \ldots, 6)$ are strictly increasing in N_1 for a fixed N_0.*

(i) If $w_{lj}(\infty|N_0) > 0$, then there exists at least one (at most two) optimal preventive replacement time N_1^* which satisfies $w_{lj}(N_1^* - 1|N_0) < 0$ and $w_{lj}(N_1^*|N_0) \geq 0$.

(ii) If $w_{lj}(\infty|N_0) \leq 0$, then the optimal preventive replacement time is given by $N_1^* \to \infty$, and it is optimal to carry out only the failure replacement or opportunistic one.

(II) Suppose that $w_{lj}(N_1|N_0)$ $(j = 1, 2, \ldots, 6)$ are strictly decreasing in N_1. The optimal preventive replacement time is given by $N_1^* \to \infty$ or $N_1^* = N_0$.

For the proof of Theorem 4.9, see the Appendix 7.4.5. We also obtain the following result without the proof.

Theorem 4.10. *If $w_{lj}(N_1|N_0)$ are strictly increasing in N_1, $w_{lj}(N_0|N_0) < 0$ and $w_{lj}(\infty|N_0) > 0$, then the lower and upper bounds of the minimum expected costs per unit time in the steady state are given by*

$$
V_{lj}(N_1^* - 1 \mid N_0) < EC_{lj}(N_1^* \mid N_0) \leq V_{lj}(N_1^* \mid N_0), \tag{4.78}
$$

where

$$
\begin{aligned}
V_{l1}(N_1 \mid N_0) =& (c_F - c_T)R_Y(N_1)\left[1 - \hat{H}(N_0, N_1)\right] \\
&+ (c_T - c_Y)\hat{H}(N_0, N_1), \qquad (4.79)\\
V_{l2}(N_1 \mid N_0) =& V_{l5}(N_1 \mid N_0) = (c_F - c_T)r_Y(N_1 + 1) \\
&+ (c_T - c_Y)\hat{H}(N_0, N_1), \qquad (4.80)\\
V_{l3}(N_1 \mid N_0) =& V_{l4}(N_1 \mid N_0) = (c_F - c_T)R_Y(N_1) \\
&+ (c_T - c_Y)\left[1 + R_Y(N_1)\right]\hat{H}(N_0, N_1), \qquad (4.81)\\
V_{l6}(N_1 \mid N_0) =& (c_F - c_T)\left[r_Y(N_1 + 1) + R_Y(N_1)\hat{H}(N_0, N_1)\right] \\
&+ (c_T - c_Y)\left[1 + R_Y(N_1)\right]\hat{H}(N_0, N_1). \qquad (4.82)
\end{aligned}
$$

4.4 Unification with Probabilistic Priority in Discrete Time

In the previous argument on the RF and RL policies, we classified 6 priority cases and derived the optimal two-phase opportunity-based age replacement times in respective cases. We also suppose that each replacement priority corresponding to Model j $(= 1, 2, \ldots, 6)$ occurs with probability p_j $(0 \le p_j \le 1)$, where $\sum_{j=1}^{6} p_j = 1$.

4.4.1 RF Model

First of all, we consider the RF policy. Since the mean time lengths of one cycle in Model j $(= 1, 2, \ldots, 6)$ are all exactly same, the associated mean time length in our unified model with probability p_j is given by $A_{f7}(N_0, N_1) = A_f(N_0, N_1)$ in Eq. (4.14) Instead, the expected total cost during one cycle, $B_{f7}(N_0, N_1)$, with the probabilistic priority is given by $B_{f7}(N_0, N_1) = \sum_{j=1}^{6} p_j B_{fj}(N_0, N_1)$ with Eqs. (4.15)–(4.20). The underlying problem is to determine the optimal (N_0^*, N_1^*), which minimizes $EC_{f7}(N_0, N_1)$, where $EC_{f7}(N_0, N_1) = B_{f7}(N_0, N_1)/A_f(N_0, N_1)$.

Define $w_{f7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j V_{fj}(N_0 \mid N_1)A_f(N_0, N_1) - B_{f7}(N_0, N_1)$ with Eqs. (4.33)–(4.38). Then it can be seen that $w_{f7}(N_0 + 1 \mid N_1) - w_{f7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j\{w_{fj}(N_0 + 1 \mid N_1) - w_{fj}(N_0 \mid N_1)\}A_f(N_0 + 1, N_1)$.

Theorem 4.11. *Suppose that* $w_{f7}(N_0 \mid N_1)$ *is strictly increasing in* N_0, $w_{f7}(0 \mid N_1) < 0$ *and* $w_{f7}(N_1 \mid N_1) > 0$. *Then the minimum expected cost per unit time in the steady state has the lower and upper bounds;*

$$V_{f7}(N_0^* - 1 \mid N_1) < EC_{f7}(N_0^* \mid N_1) \leq V_{f7}(N_0^* \mid N_1), \tag{4.83}$$

where

$$V_{f7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j V_{fj}(N_0 \mid N_1). \tag{4.84}$$

Next, we define $w_{f7}(N_1 \mid N_0) = \sum_{j=1}^{6} p_j V_{fj}(N_1 \mid N_0) A_f(N_0, N_1) - B_{f7}(N_0, N_1)$ with Eqs. (4.48)–(4.53). Then we have $w_{f7}(N_1 + 1 \mid N_0) - w_{f7}(N_1, N_0) = \sum_{j=1}^{6} p_j \{w_{fj}(N_1 + 1 \mid N_0) - w_{fj}(N_1 \mid N_0)\} A_f(N_0, N_1 + 1)$.

Theorem 4.12. *Suppose that* $w_{f7}(N_1 \mid N_0)$ *is strictly increasing in* N_1, $w_{f7}(N_0 \mid N_0) < 0$ *and* $w_{f7}(\infty \mid N_0) > 0$. *Then the minimum expected cost per unit time in the steady state has the lower and upper bounds;*

$$V_{f7}(N_1^* - 1 \mid N_0) < EC_{f7}(N_1 \mid N_0) \leq V_{f7}(N_1^* \mid N_0), \tag{4.85}$$

where

$$V_{f7}(N_1 \mid N_0) = \sum_{j=1}^{6} p_j V_{fj}(N_1 \mid N_0). \tag{4.86}$$

4.4.2 RL Model

Next, we consider the RL discipline. Since the mean time length of one cycle and the expected total cost during one cycle are given by $A_{l7}(N_0, N_1) = A_l(N_0, N_1)$ in Eq. (4.55) and $B_{l7}(N_0, N_1) = \sum_{j=1}^{6} p_j B_{lj}(N_0, N_1)$ with Eqs. (4.56)–(4.59), define $w_{l7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j EC_{lj}(N_0 \mid N_1) A_l(N_0, N_1) - B_{l7}(N_0, N_1)$ with Eqs. (4.67)–(4.70). Then, one has $w_{l7}(N_0+1 \mid N_1) - w_{l7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j \{w_{lj}(N_0 + 1 \mid N_1) - w_{lj}(N_0 \mid N_1)\} A_l(N_0 + 1, N_1)$.

Theorem 4.13. *The minimum expected cost per unit time in the steady state is given by*

$$EC_{l7}(N_0 \mid N_1) = \sum_{j=1}^{6} p_j EC_{lj}(N_0 \mid N_1). \tag{4.87}$$

Next, we define $w_{l7}(N_1 \mid N_0) = \sum_{j=1}^{6} p_j V_{lj}(N_1 \mid N_0) A_l(N_0, N_1) - B_{l7}(N_0, N_1)$ with Eqs. (4.79)–(4.82). Then, one obtains $w_{l7}(N_1 + 1 \mid N_0) - w_{l7}(N_1 \mid N_0) = \sum_{j=1}^{6} p_j \{w_{lj}(N_1 + 1 \mid N_0) - w_{lj}(N_1 \mid N_0)\} A_l(N_0, N_1 + 1)$.

Theorem 4.14. *Suppose that $w_{l7}(N_1 \mid N_0)$ is strictly increasing in N_1, $w_{l7}(N_0 \mid N_0) < 0$ and $w_{l7}(\infty \mid N_0) > 0$. Then the minimum expected cost per unit time in the steady state has the lower and upper bounds;*

$$V_{l7}(N_1^* - 1 \mid N_0) < EC_{l7}(N_1^* \mid N_0) \leq V_{l7}(N_1^* \mid N_0), \tag{4.88}$$

where

$$V_{l7}(N_1 \mid N_0) = \sum_{j=1}^{6} p_j V_{lj}(N_1 \mid N_0). \tag{4.89}$$

4.5 Numerical Experiment

4.5.1 Continuous Time Models

This section presents numerical examples to illustrate the theoretical underpinnings of the proposed preventive replacement policies for each case. We take the same assumptions with RF model [8]. The failure time Y of the unit obeys a Gamma distribution. We can obtain the related c.d.f. and p.d.f.:

$$F(t) = 1 - (1 + \theta t)e^{-\theta t}, \tag{4.90}$$

$$f(t) = \theta^2 t e^{-\theta t}. \tag{4.91}$$

The cost parameters are gvien: $c_F = 3.0, 5.0, 6.0, 7.0, 8.0, 9.0, 10.0, 11.0, 12.0$, $c_T = 1.0$, and $c_Y = 0.8, 1.0$. Here, we calculate the S^* with a fixed $T = 4$, T^* with a fixed $S = 1$, and associated $EC(S^* \mid T)$, $EC(T^* \mid S)$. In addition, we compare our model (RL) and the RF [8]. Here, we review the RF [8] at the case $c_Y \leq c_T$.

$$EC_f(S, T) = \frac{B_f(S, T)}{A_f(S, T)}, \tag{4.92}$$

where

$$A_f(S, T) = \int_0^S \bar{F}(t)dt + \int_S^T \bar{G}(t - S)\bar{F}(t)dt, \tag{4.93}$$

$$B_f(S,T) = c_T + (c_F - c_T)\left[\int_0^S f(t)dt + \int_S^T \bar{G}(t-S)f(t)dt\right]$$
$$- (c_T - c_Y)\int_S^T g(t-S)\bar{F}(t)dt. \tag{4.94}$$

Table 4.1: Optimal restricted duration S^* and associated expected costs $EC(S^* \mid T)$ in RF and RL models, when $\theta = 1$, $\lambda = 1$, $T = 4$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	S^*	$EC_f(S^*)$	S^*	$EC_l(S^*)$	S^*	$EC_f(S^*)$	S^*	$EC_l(S^*)$
3.0	1.4597	1.4569	4	1.4889	2.4798	1.4876	0	1.4904
4.0	0.8960	1.8626	0	1.9711	1.3037	1.9252	0	1.9713
5.0	0.6487	2.2426	0	2.4519	0.8979	2.3275	0	2.4521
6.0	0.5093	2.6087	0	2.9326	0.6873	2.7095	0	2.9327
7.0	0.4195	2.9661	0	3.4134	0.5575	3.0789	0	3.4135
8.0	0.3567	3.3178	0	3.8942	0.4693	3.4400	0	3.8943
9.0	0.3104	3.6655	0	4.3750	0.4045	3.7954	0	4.3751
10.0	0.2747	4.0104	0	4.8558	0.3568	4.1466	0	4.8559
11.0	0.2465	4.3533	0	5.3366	0.3188	4.4947	0	5.3367
12.0	0.2235	4.6946	0	5.8173	0.2880	4.8403	0	5.8174

Table 4.1 presents the optimal restricted duration S^* and the minimum long-run cost $EC(S^* \mid T)$ under both RF and RL disciplines, with the assumption that $c_T \geq c_Y$. It can be observed that RL outperforms RF regarding the optimal restricted duration S.

Table 4.2 shows the optimal preventive replacement time T^* and its associated long-run average cost $EC(T^* \mid S)$ under RF and RL disciplines. Except for the case where $c_F = 3$, RF is generally superior to RL in terms of the optimal preventive replacement time T^*.

4.5.2 Discrete Time Models

Here, we take the same models parameters and cost parameters with Section 2.6. We calculate the optimal prescheduled preventive replacement times N_1^* for a fixed N_0 and the optimal restricted durations N_0^* for a fixed N_1 under RF and

Table 4.2: Optimal prescheduled preventive replacement time T^* and associated expected costs $EC(T^* \mid S)$ in RF and RL models, when $\theta = 1$, $\lambda = 1$, $S = 1$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	T^*	$EC_f(T^*)$	T^*	$EC_l(T^*)$	T^*	$EC_f(T^*)$	T^*	$EC_l(T^*)$
3.0	4.9896	1.4661	1	1.4660	3.2880	1.5336	2.8963	1.4863
4.0	2.1866	1.8584	1	1.8644	1.7685	1.9164	1.7751	1.9190
5.0	1.5422	2.2266	1	2.2627	1.3082	2.2670	1.3910	2.3271
6.0	1.2392	2.5671	1	2.6609	1.0772	2.5871	1.2001	2.7280
7.0	1.0566	2.8825	1	3.0592	1	2.8844	1.0884	3.1269
8.0	1	3.1792	1	3.4575	1	3.1792	1.0146	3.5253
9.0	1	3.4739	1	3.8558	1	3.4740	1	3.9236
10.0	1	3.7688	1	4.2541	1	3.7688	1	4.3219
11.0	1	4.0636	1	4.6523	1	4.0635	1	4.7201
12.0	1	4.3583	1	5.0506	1	4.3583	1	5.1184

RL disciplines. Besides, we seek the pair of restricted duration and preventive replacement time (N_0^*, N_1^*).

In Tables 4.3–4.9, we present the comparison results of the optimal restricted durations N_0 for a fixed N_1, when $N_1 = 8$, $c_Y = 0.8, 1.0$. We can see that:

(1) In all tables, the optimal prescheduled restricted durations for respective priority models often converge to similar values in most cases. This is primarily since these optimal restricted durations are discretized as integer values, and the differences in replacement priorities are not particularly significant.

(2) When the cost of corrective replacement c_F is become big, the optimal restricted durations N_0^* become small. In addition, in RL policies, when c_F is relatively small, such as $c_F = 1.5, 2.0$, the N_0^* are constant values 8. At this case, two-phase RF model will degenerate into DD model in Section 2.2.2. When $c_F \geq 3$, the N_0^* is constant value 0. This is indicates that two-phase models in RL discipline can degenerate into RF model in Section 3.1.2.

(3) When the cost of the opportunistic replacement c_Y is become big, the optimal prescheduled optimal restricted durations N_0^* become small.

(4) RL policies are only better than RF policies in some limited cases where the failure replacement cost c_F is relatively small. In our example, when $c_F = 1.5$, RL policies are better than RF policies. Conversely, when c_F is large and the impact of system failure becomes more remarkable, we can find that RF policies are better than RL policies.

Tables 4.10–4.16 present the optimal preventive replacement times N_1^* and their associated long-run average costs $EC(N_1^* \mid N_0)$, when $N_0 = 5$, $c_Y = 0.8, 1.0$. We can find that:

(5) When the cost of the corrective replacement c_Y is become big, the optimal prescheduled preventive replacement times N_1^* become small. When the cost of repairing or replacing equipment becomes expensive, it means that it's no longer cost-effective to wait for the equipment to break down and then perform emergency repairs. In such situations, it's more economical to perform preventive maintenance, which involves regular maintenance or replacement of components before they fail.

(6) Even in these cases, RL policies could outperform the RF policies only when c_F is small enough. From the results above, it can be recognized that the efficiency of the RL policies is rather limited in the case where the failure replacement cost is relatively low compared to the preventive replacement and opportunistic replacement costs.

In Tables 4.17–4.24, we numerically derive the joint optimal policies (N_0^*, N_1^*) in the two-phase opportunity-based age replacement models under the RF and RL disciplines, where we assume $c_Y = 0.8, 1.0$. It can be seen that:

(7) In the realm of RF discipline, with $c_Y = 0.8$, the jointly optimal policy is $(4, 8)$. However, when both $c_T = c_Y = 1.0$, the optimal joint policy shifts to $(7, 7)$.

(8) Within the domain of RL discipline, with $c_Y = 0.8$, the optimal combination of restricted duration and preventive replacement time is $(0, 9)$. As $c_Y = 1.0$, the jointly optimal policy transitions to $(0, 9)$.

Table 4.3: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 1, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f1}(N_0^*)$	N_0^*	$EC_{l1}(N_0^*)$	N_0^*	$EC_{f1}(N_0^*)$	N_0^*	$EC_{l1}(N_0^*)$
1.5	6	0.1372	8	0.1101	7	0.1376	8	0.1140
2.0	6	0.1451	8	0.1428	7	0.1454	8	0.1465
3.0	6	0.1609	0	0.1969	7	0.1612	0	0.1999
4.0	6	0.1767	0	0.2504	7	0.1769	0	0.2534
5.0	5	0.1925	0	0.3039	7	0.1926	0	0.3069
6.0	5	0.2082	0	0.3574	7	0.2084	0	0.3603
7.0	4	0.2238	0	0.4108	7	0.2241	0	0.4138
8.0	4	0.2393	0	0.4643	7	0.2398	0	0.4673
9.0	3	0.2548	0	0.5178	7	0.2556	0	0.5208
10.0	3	0.2700	0	0.5713	7	0.2713	0	0.5742

Table 4.4: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 2, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f2}(N_0)$	N_0^*	$EC_{l2}(N_0^*)$	N_0^*	$EC_{f2}(N_0^*)$	N_0^*	$EC_{l2}(N_0^*)$
1.5	6	0.1403	8	0.1100	7	0.1409	8	0.1130
2.0	5	0.1514	8	0.1426	7	0.1521	8	0.1465
3.0	5	0.1732	0	0.1198	7	0.1745	0	0.2028
4.0	4	0.1949	0	0.2547	7	0.1968	0	0.2577
5.0	4	0.2163	0	0.3096	6	0.2191	0	0.3126
6.0	3	0.2376	0	0.3645	5	0.2411	0	0.3675
7.0	3	0.2586	0	0.4195	4	0.2628	0	0.4224
8.0	3	0.2796	0	0.4743	4	0.2843	0	0.4773
9.0	2	0.3005	0	0.5293	3	0.3056	0	0.5322
10.0	2	0.3211	0	0.5841	3	0.3266	0	0.5872

Table 4.5: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 3, when the prescheduled preventive replacement time duration $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f3}(N_0^*)$	N_0^*	$EC_{l3}(N_0^*)$	N_0^*	$EC_{f3}(N_0^*)$	N_0^*	$EC_{l3}(N_0^*)$
1.5	6	0.1370	8	0.1082	7	0.1376	8	0.1126
2.0	5	0.1447	8	0.1395	7	0.1454	8	0.1440
3.0	5	0.1599	0	0.1923	7	0.1612	0	0.1957
4.0	5	0.1752	0	0.2437	7	0.1769	0	0.2471
5.0	4	0.1902	0	0.2950	7	0.1926	0	0.2984
6.0	4	0.2051	0	0.3464	6	0.2082	0	0.3498
7.0	3	0.2199	0	0.3978	5	0.2236	0	0.4012
8.0	3	0.2345	0	0.4491	4	0.2388	0	0.4525
9.0	3	0.2491	0	0.5005	4	0.2537	0	0.5039
10.0	3	0.2637	0	0.5519	4	0.2687	0	0.5554

Table 4.6: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 4, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f4}(N_0^*)$	N_0^*	$EC_{l4}(N_0^*)$	N_0^*	$EC_{f4}(N_0^*)$	N_0^*	$EC_{l4}(N_0^*)$
1.5	6	0.1358	8	0.1082	7	0.1376	8	0.1126
2.0	5	0.1435	8	0.1395	7	0.1454	8	0.1440
3.0	5	0.1588	0	0.1923	7	0.1612	0	0.1957
4.0	5	0.1741	0	0.2437	7	0.1769	0	0.2471
5.0	4	0.1891	0	0.2950	7	0.1926	0	0.2984
6.0	4	0.2040	0	0.3464	6	0.2082	0	0.3498
7.0	3	0.2189	0	0.3978	5	0.2236	0	0.4012
8.0	3	0.2335	0	0.4491	4	0.2388	0	0.4525
9.0	3	0.2481	0	0.5005	4	0.2537	0	0.5039
10.0	3	0.2627	0	0.5519	4	0.2687	0	0.5553

Table 4.7: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 5, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f5}(N_0^*)$	N_0^*	$EC_{l5}(N_0^*)$	N_0^*	$EC_{f5}(N_0^*)$	N_0^*	$EC_{l5}(N_0^*)$
1.5	6	0.1393	8	0.1100	7	0.1409	8	0.1139
2.0	5	0.1503	8	0.1426	7	0.1521	8	0.1465
3.0	5	0.1721	0	0.1998	7	0.1745	0	0.2028
4.0	4	0.1939	0	0.2547	7	0.1968	0	0.2577
5.0	4	0.2153	0	0.3096	6	0.2191	0	0.3126
6.0	3	0.2366	0	0.3645	5	0.2411	0	0.3675
7.0	3	0.2576	0	0.4195	4	0.2628	0	0.4224
8.0	3	0.2786	0	0.4743	4	0.2843	0	0.4773
9.0	2	0.2995	0	0.5193	3	0.3056	0	0.5322
10.0	2	0.3202	0	0.5841	3	0.3266	0	0.5872

Table 4.8: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in Model 6, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f6}(N_0^*)$	N_0^*	$EC_{l6}(N_0^*)$	N_0^*	$EC_{f6}(N_0^*)$	N_0^*	$EC_{l6}(N_0^*)$
1.5	6	0.1388	8	0.1101	7	0.1407	8	0.1146
2.0	5	0.1494	8	0.1434	7	0.1518	8	0.1479
3.0	5	0.1704	0	0.2015	6	0.1736	0	0.2049
4.0	4	0.1910	0	0.2574	5	0.1951	0	0.2609
5.0	3	0.2115	0	0.3134	5	0.2162	0	0.3168
6.0	3	0.2315	0	0.3694	4	0.2369	0	0.3728
7.0	3	0.2515	0	0.4253	4	0.2575	0	0.4287
8.0	3	0.2706	0	0.4813	3	0.2778	0	0.4847
9.0	2	0.2913	0	0.5372	3	0.2978	0	0.5406
10.0	2	0.3109	0	0.5932	3	0.3179	0	0.5966

Table 4.9: Optimal restricted duration N_0^* and associated expected costs $EC(N_0^* \mid N_1)$ in unified Model, when the prescheduled preventive replacement time $N_1 = 8$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_0^*	$EC_{f7}(N_0^*)$	N_0^*	$EC_{l7}(N_0^*)$	N_0^*	$EC_{f7}(N_0^*)$	N_0^*	$EC_{l7}(N_0^*)$
1.5	6	0.1379	8	0.1093	7	0.1389	8	0.1135
2.0	5	0.1470	8	0.1415	7	0.1481	8	0.1457
3.0	5	0.1649	0	0.1964	7	0.1664	0	0.1996
4.0	5	0.1828	0	0.2497	7	0.1848	0	0.2529
5.0	4	0.2003	0	0.3030	7	0.2031	0	0.3062
6.0	4	0.2179	0	0.3563	6	0.2214	0	0.3595
7.0	3	0.2352	0	0.4096	5	0.2394	0	0.4128
8.0	3	0.2524	0	0.4629	4	0.2572	0	0.4661
9.0	3	0.2696	0	0.5163	4	0.2748	0	0.5195
10.0	2	0.2868	0	0.5696	4	0.2923	0	0.5727

Table 4.10: Optimal preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 1, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f1}(N_1^*)$	N_1^*	$EC_{l1}(N_1^*)$	N_1^*	$EC_{f1}(N_1^*)$	N_1^*	$EC_{l1}(N_1^*)$
1.5	16	0.1097	15	0.1105	15	0.1148	16	0.1110
2.0	12	0.1297	12	0.1418	12	0.1341	13	0.1429
3.0	10	0.1572	11	0.1969	9	0.1606	12	0.2045
4.0	8	0.1767	10	0.2028	8	0.1792	11	0.2654
5.0	8	0.1924	10	0.3240	7	0.1945	11	0.3262
6.0	7	0.2048	10	0.3846	7	0.2063	10	0.3869
7.0	7	0.2166	10	0.4542	7	0.2180	10	0.4475
8.0	6	0.2263	10	0.5057	6	0.2264	10	0.5080
9.0	6	0.2345	10	0.5663	6	0.2345	10	0.5686
10.0	6	0.2427	10	0.6269	6	0.2427	10	0.6292

Table 4.11: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 2, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f2}(N_1^*)$	N_1^*	$EC_{l2}(N_1^*)$	N_1^*	$EC_{f2}(N_1^*)$	N_1^*	$EC_{l2}(N_1^*)$
1.5	17	0.1114	5	0.1115	17	0.1166	16	0.1120
2.0	13	0.1353	5	0.1421	12	0.1399	13	0.1448
3.0	10	0.1696	5	0.2034	9	0.1731	10	0.2073
4.0	8	0.1950	5	0.2647	8	0.1975	10	0.2689
5.0	7	0.2162	5	0.3260	7	0.2177	9	0.3303
6.0	7	0.2338	5	0.3873	7	0.2352	9	0.3916
7.0	7	0.2503	5	0.4486	7	0.2503	9	0.4529
8.0	6	0.2638	5	0.5098	6	0.2638	9	0.5142
9.0	6	0.2773	5	0.5711	6	0.2773	8	0.5754
10.0	5	0.2893	5	0.6324	5	0.2893	8	0.6636

Table 4.12: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 3, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f3}(N_1^*)$	N_1^*	$EC_{l3}(N_1^*)$	N_1^*	$EC_{f3}(N_1^*)$	N_1^*	$EC_{l3}(N_1^*)$
1.5	16	0.1082	5	0.1095	16	0.1138	15	0.1106
2.0	12	0.1280	5	0.1387	12	0.1327	12	0.1416
3.0	10	0.1551	5	0.1971	10	0.1590	11	0.2010
4.0	9	0.1751	5	0.2556	8	0.1777	10	0.2596
5.0	8	0.1905	5	0.3140	8	0.1930	10	0.3181
6.0	7	0.2036	5	0.3724	7	0.2051	9	0.3764
7.0	7	0.2152	5	0.4308	7	0.2167	9	0.4347
8.0	6	0.2264	5	0.4893	6	0.2264	9	0.4930
9.0	6	0.2345	5	0.5477	6	0.2345	9	0.5513
10.0	6	0.2427	5	0.6061	6	0.2427	9	0.6096

Table 4.13: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 4, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f4}(N_1^*)$	N_1^*	$EC_{l4}(N_1^*)$	N_1^*	$EC_{f4}(N_1^*)$	N_1^*	$EC_{l4}(N_1^*)$
1.5	16	0.1080	5	0.1095	16	0.1138	15	0.1106
2.0	12	0.1275	5	0.1387	12	0.1327	12	0.1416
3.0	10	0.1544	5	0.1971	10	0.1590	11	0.2010
4.0	8	0.1741	5	0.2556	8	0.1777	10	0.2596
5.0	8	0.1874	5	0.3140	8	0.1930	10	0.3181
6.0	7	0.2023	5	0.3724	7	0.2051	9	0.3764
7.0	7	0.2138	5	0.4308	7	0.2167	9	0.4347
8.0	6	0.2247	5	0.4893	6	0.2264	9	0.4930
9.0	6	0.2328	5	0.5477	6	0.2345	9	0.5513
10.0	6	0.2410	5	0.6061	6	0.2427	9	0.6096

Table 4.14: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 5, when the restricted duration $N_1 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f5}(N_1^*)$	N_1^*	$EC_{l5}(N_1^*)$	N_1^*	$EC_{f5}(N_1^*)$	N_1^*	$EC_{l5}(N_1^*)$
1.5	17	0.1113	5	0.1115	17	0.1166	16	0.1120
2.0	13	0.1350	5	0.1421	12	0.1399	13	0.1448
3.0	10	0.1690	5	0.2034	9	0.1731	10	0.2073
4.0	8	0.1940	5	0.2647	8	0.1975	10	0.2689
5.0	7	0.2150	5	0.3260	7	0.2177	9	0.3303
6.0	7	0.2325	5	0.3873	7	0.2352	9	0.3916
7.0	7	0.2487	5	0.4486	7	0.2503	9	0.4529
8.0	6	0.2622	5	0.5098	6	0.2638	9	0.5142
9.0	6	0.2757	5	0.5711	6	0.2773	8	0.5754
10.0	6	0.2892	5	0.6324	5	0.2893	8	0.6636

Table 4.15: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in Model 6, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f6}(N_1^*)$	N_1^*	$EC_{l6}(N_1^*)$	N_1^*	$EC_{f6}(N_1^*)$	N_1^*	$EC_{l6}(N_1^*)$
1.5	18	0.1095	5	0.1104	17	0.1154	19	0.1129
2.0	13	0.1327	5	0.1405	13	0.1381	5	0.1471
3.0	10	0.1664	5	0.200 8	10	0.1709	5	0.2074
4.0	8	0.1915	5	0.2611	8	0.1951	5	0.2677
5.0	8	0.2122	5	0.3214	7	0.2156	5	0.3280
6.0	7	0.2298	5	0.3817	7	0.2326	5	0.3882
7.0	7	0.2469	5	0.4419	6	0.2487	5	0.4485
8.0	6	0.2603	5	0.5022	6	0.2619	5	0.5088
9.0	6	0.2735	5	0.5625	6	0.2752	5	0.5691
10.0	6	0.2868	5	0.6228	6	0.2884	5	0.6293

Table 4.16: Optimal prescheduled preventive replacement time N_1^* and associated expected costs $EC(N_1^* \mid N_0)$ in unified Model, when the restricted duration $N_0 = 5$.

	$c_Y = 0.8$				$c_Y = 1.0$			
	RF		RL		RF		RL	
c_F	N_1^*	$EC_{f7}(N_1^*)$	N_1^*	$EC_{l7}(N_1^*)$	N_1^*	$EC_{f7}(N_1^*)$	N_1^*	$EC_{l7}(N_1^*)$
1.5	16	0.1096	5	0.1106	16	0.1150	16	0.1114
2.0	12	0.1309	5	0.1406	12	0.1357	13	0.1434
3.0	10	0.1608	5	0.2007	9	0.1648	11	0.2048
4.0	8	0.1828	5	0.2608	8	0.1857	10	0.2653
5.0	8	0.2007	5	0.3208	7	0.2032	9	0.3256
6.0	7	0.2152	5	0.3809	7	0.2171	9	0.3857
7.0	7	0.2291	5	0.4409	6	0.2309	9	0.4459
8.0	6	0.2405	5	0.5010	6	0.2412	9	0.5060
9.0	6	0.2508	5	0.5611	6	0.2514	9	0.5662
10.0	6	0.2610	5	0.6211	6	0.2617	9	0.6263

Table 4.17: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RF discipline with $c_F = 5$ and $c_Y = 0.8$.

	Model 1		Model 2		Model 3	
N_0	N_1^*	$EC_{f1}(N_0^*, N_1^*)$	N_1^*	$EC_{f2}(N_0^*, N_1^*)$	N_1^*	$EC_{f3}(N_0^*, N_1^*)$
1	8	0.1985	7	0.2205	8	0.1954
2	8	0.1954	7	0.2175	8	0.1924
3	8	0.1936	7	0.2160	8	0.1907
4	8	0.1927	7	**0.2157**	8	**0.1902**
5	8	0.1927	7	0.2163	8	0.1905
6	8	**0.1925**	7	0.2174	8	0.1914
7	8	0.1926	7	0.2174	8	0.1927
8	9	0.1979	8	0.2192	9	0.1979
9	10	0.2075	9	0.2262	10	0.2075
10	11	0.2203	10	0.2368	11	0.2203

Table 4.18: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RF discipline with $c_F = 5$ and $c_Y = 0.8$.

	Model 4		Model 5		Model 6	
N_0	N_1^*	$EC_{f4}(N_0^*, N_1^*)$	N_1^*	$EC_{f5}(N_0^*, N_1^*)$	N_1^*	$EC_{f6}(N_0^*, N_1^*)$
1	8	0.1944	7	0.2193	8	0.2154
2	8	0.1914	7	0.2163	8	0.2127
3	8	0.1897	7	0.2147	8	0.2115
4	8	**0.1891**	7	**0.2144**	8	**0.2115**
5	8	0.1894	7	0.2150	8	0.2126
6	8	0.1903	7	0.2161	8	0.2144
7	8	0.1914	7	0.2174	8	0.2167
8	9	0.1969	8	0.2192	8	0.2192
9	10	0.2067	9	0.2262	9	0.2262
10	11	0.2195	10	0.2368	10	0.2368

Table 4.19: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RF discipline with $c_F = 5$ and $c_Y = 1.0$.

	Model 1		Model 2		Model 3	
N_0	N_1^*	$EC_{f1}(N_0^*, N_1^*)$	N_1^*	$EC_{f2}(N_0^*, N_1^*)$	N_1^*	$EC_{f3}(N_0^*, N_1^*)$
1	8	0.2063	7	0.2279	8	0.2034
2	8	0.2018	7	0.2233	8	0.1989
3	8	0.1986	7	0.2203	8	0.1959
4	7	0.1962	7	0.2185	8	0.1940
5	7	0.1945	7	0.2177	8	0.1930
6	7	0.1932	7	0.2174	8	0.1927
7	7	**0.1926**	7	**0.2174**	8	**0.1926**
8	9	0.1979	8	0.2192	9	0.1979
9	10	0.2075	9	0.2262	10	0.2075
10	10	0.2203	10	0.2368	11	0.2203

Table 4.20: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RF discipline with $c_F = 5$ and $c_Y = 1.0$.

	Model 4		Model 5		Model 6	
N_0	N_1^*	$EC_{f4}(N_0^*, N_1^*)$	N_1^*	$EC_{f5}(N_0^*, N_1^*)$	N_1^*	$EC_{f6}(N_0^*, N_1^*)$
1	8	0.2034	7	0.2279	8	0.2244
2	8	0.1989	7	0.2233	7	0.2201
3	8	0.1959	7	0.2203	7	0.2173
4	8	0.1940	7	0.2185	7	0.2158
5	8	0.1930	7	0.2177	7	**0.2156**
6	8	0.1927	7	0.2174	7	0.2162
7	8	**0.1926**	7	**0.2174**	7	0.2174
8	9	0.1979	8	0.2192	8	0.2192
9	10	0.2075	9	0.2262	9	0.2262
10	10	0.2203	10	0.2368	10	0.2368

Table 4.21: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RL discipline with $c_F = 5$ and $c_Y = 0.8$.

	Model 1		Model 2		Model 3	
N_0	N_1^*	$EC_{l1}(N_0^*, N_1^*)$	N_1^*	$EC_{l2}(N_0^*, N_1^*)$	N_1^*	$EC_{l3}(N_0^*, N_1^*)$
0	9	**0.3049**	9	**0.3123**	9	**0.2974**
1	10	0.3036	8	0.3128	9	0.2983
2	10	0.3105	8	0.3162	9	0.3022
3	10	0.3148	8	0.3197	9	0.3063
4	10	0.3193	4	0.3230	4	0.3105
5	10	0.3239	5	0.3258	5	0.3140
6	10	0.3286	6	0.3294	6	0.3178
7	11	0.3334	7	0.3336	7	0.3224
8	11	0.3382	8	0.3382	8	0.3275
9	12	0.3432	9	0.3429	9	0.3330
10	13	0.3479	10	0.3477	10	0.3385

Table 4.22: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RL discipline with $c_F = 5$ and $c_Y = 0.8$.

	Model 4		Model 5		Model 6	
N_0	N_1^*	$EC_{l4}(N_0^*, N_1^*)$	N_1^*	$EC_{l5}(N_0^*, N_1^*)$	N_1^*	$EC_{l6}(N_0^*, N_1^*)$
0	9	**0.2974**	9	**0.3123**	6	0.3120
1	9	0.2983	8	0.3128	1	**0.3096**
2	9	0.3022	8	0.3162	2	0.3097
3	9	0.3063	8	0.3197	3	0.3119
4	4	0.3105	4	0.3230	4	0.3159
5	5	0.3140	5	0.3258	5	0.3214
6	6	0.3178	6	0.3294	6	0.3280
7	7	0.3224	7	0.3336	7	0.3354
8	8	0.3275	8	0.3382	8	0.3433
9	9	0.3330	9	0.3429	9	0.3512
10	10	0.3385	10	0.3477	10	0.3589

Table 4.23: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RL discipline with $c_F = 5$ and $c_Y = 1.0$.

	Model 1		Model 2		Model 3	
N_0	N_1^*	$EC_{l1}(N_0^*, N_1^*)$	N_1^*	$EC_{l2}(N_0^*, N_1^*)$	N_1^*	$EC_{l3}(N_0^*, N_1^*)$
0	10	**0.3041**	9	**0.3119**	9	**0.2972**
1	10	0.3082	9	0.3154	9	0.3011
2	10	0.3125	9	0.3189	9	0.3052
3	10	0.3169	9	0.3226	9	0.3094
4	10	0.3214	9	0.3263	9	0.3137
5	11	0.3261	9	0.3301	10	0.3181
6	11	0.3306	9	0.3340	10	0.3226
7	11	0.3354	9	0.3380	10	0.3272
8	12	0.3401	9	0.3421	10	0.3320
9	12	0.3448	9	0.3463	9	0.3368
10	13	0.3494	10	0.3505	10	0.3418

Table 4.24: Simultaneous optimal restricted duration and prescheduled preventive replacement time (N_0^*, N_1^*) under RL discipline with $c_F = 5$ and $c_Y = 1.0$.

	Model 4		Model 5		Model 6	
N_0	N_1^*	$EC_{l4}(N_0^*, N_1^*)$	N_1^*	$EC_{l5}(N_0^*, N_1^*)$	N_1^*	$EC_{l6}(N_0^*, N_1^*)$
0	9	**0.2972**	9	**0.3119**	7	**0.3164**
1	9	0.3011	9	0.3154	1	0.3197
2	9	0.3052	9	0.3189	2	0.3188
3	9	0.3094	9	0.3226	3	0.3201
4	9	0.3137	9	0.3263	4	0.3233
5	10	0.3181	9	0.3301	5	0.3280
6	10	0.3226	9	0.3340	6	0.3338
7	10	0.3272	9	0.3380	7	0.3406
8	10	0.3320	9	0.3421	8	0.3478
9	9	0.3368	9	0.3463	9	0.3551
10	10	0.3418	10	0.3505	10	0.3622

Chapter 5

An NPV Analysis of Failure-Correlated Opportunity-Based Age Replacement Models

5.1 Models Description

5.1.1 Assumptions and Notations

We discuss a single-unit system consisting of a non- repairable item. Suppose that the failure times (lifetimes) of the item, Y, follow i.i.d.. When the system fails, we denote the c.d.f. by $F(t)$, the p.d.f. by $f(t)$ and the survivor function of the failure times by $\bar{F}(t)$. The failure rate of this system is $r(t) = f(t)/\bar{F}(t)$. In our assumption, the random time intervals, X, between two consecutive opportunities for replacement follow a common distribution with p.d.f. $g(t)$ and c.d.f. $G(t)$. Additionally, we define the hazard rate function of X as $h(t) = g(t)/\bar{G}(t)$ and the reversed hazard rate function as $\hat{H}(t) = g(t)/G(t)$.

5.1.2 Renewal Reward Approach

Zhao and Nakagawa [2] proposed some slightly different opportunistic preventive replacement models from Dekker and Smeitink [21, 22] and Dekker and Dijkstra [9]. More specifically, the authors in [2] proposed that opportunities for replacement obey a renewal process. Based on this assumption, they introduced RF discipline. For better understanding the underlying model, we formulate the expected cost per unit time in steady state with RF discipline. Let T (> 0) denote the pre-determined age measured from the new installation of the system or the replacement time of the unit. Also, we define the time length from the replacement point of failed/unfailed unit to the next one as one cycle. In the RF model, the decision for preventive replacement is determined by the age threshold T or the occurrence of a random opportunity X for replacement.

The expected time length of one cycle $A_f(T)$ with RF discipline is given by

$$\begin{aligned} A_f(T) &= \int_0^T t\bar{G}(f)f(t)dt + \int_0^T t\bar{F}(t)g(t)dt + T\bar{F}(T)\bar{G}(T) \\ &= \int_0^T \bar{G}(t)\bar{F}(t)dt. \end{aligned} \tag{5.1}$$

The expected cost for one cycle $B_f(T)$ is

$$\begin{aligned} B_f(T) &= c_F \int_0^T \bar{G}(f)f(t)dt + c_Y \int_0^T \bar{F}(t)g(t)dt + c_T\bar{G}(T)\bar{F}(T) \\ &= c_T + (c_F - c_T)\int_0^T \bar{G}(f)f(t)dt - (c_T - c_Y)\int_0^T \bar{F}(t)g(t)dt. \end{aligned} \tag{5.2}$$

Based on the renewal reward theorem, we can give the expected cost per unit time in the steady state (expected cost rate) in RF model by

$$EC_f(T) = \frac{B_f(T)}{A_f(T)}. \tag{5.3}$$

Our aim is to find the optimal preventive replacement time T which minimizes $EC_f(T)$. From a few algebraic manipulations, it is immediate to see that an optimal preventive replacement time T^* minimizing $EC_f(T)$ satisfies

$$\begin{aligned} &(c_F - c_T)\left[r(T)\int_0^T \bar{F}(t)\bar{G}(t)dt - \int_0^T \bar{G}(t)dF(t)\right] \\ &- (c_T - c_Y)\left[h(T)\int_0^T \bar{F}(t)\bar{G}(t)dt - \int_0^T \bar{F}(t)dG(t)\right] = c_T. \end{aligned} \tag{5.4}$$

Zhao and Nakagawa [2] gave the minimal expected cost rate function as

$$EC_f(T^*) = (c_F - c_T)r(T^*) - (c_T - c_Y)h(T^*). \tag{5.5}$$

Next, we consider the RL model. In RL discipline, the non-failed unit is taken place by new one at the pre-determined time T^* or the occurrence of the opportunity X, whichever comes last [2]. Similarly, the expected cost rate in RL discipline becomes

$$EC_l(T) = \frac{B_l(T)}{A_l(T)}, \tag{5.6}$$

where

$$A_l(T) = \int_0^T \bar{F}(t)dt + \int_T^\infty \bar{F}(t)\bar{G}(t)dt \tag{5.7}$$

and

$$B_l(T) = c_T + (c_F - c_T)\left[F(T) + \int_T^\infty \bar{G}(t)f(t)dt\right] - (c_T - c_Y)\int_T^\infty \bar{F}(t)dG(t). \tag{5.8}$$

From the similar manipulations to Eq. (5.5), our interest is to derive an optimal preventive replacement time T^* minimizing $E_l(t)$ which satisfies

$$\left[(c_F - c_T)r(T) + (c_T - c_Y)\hat{H}(T)\right] A_l(T) - B_l(T) = 0. \tag{5.9}$$

Zhao and Nakagawa [2] gave the minimum expected cost rate function, when $c_T = c_Y$. In our case, it is seen that

$$EC_l(T^*) = (c_F - c_T)r(T^*) + (c_T - c_Y)\hat{H}(T^*). \tag{5.10}$$

5.1.3 NPV Approach

In our models, we denote the discount factor β (> 0) to represent the NPV of the expected total cost over an operating horizon. In the RF model, $TC_{\beta f}(T, \beta)$ denotes the NPV of the expected total cost. Then, we have

$$TC_{\beta f}(T, \beta) = \frac{B_{\beta f}(T, \beta)}{1 - A_{\beta f}(T, \beta)}, \tag{5.11}$$

where $A_{\beta f}(T, \beta)$ and $B_{\beta f}(T, \beta)$ are the expected discounted value of one unit cost during one cycle and the expected discounted cost during one cycle, respectively. We straightforwardly derive $A_{\beta f}(T, \beta)$ and $B_{\beta f}(T, \beta)$ as

$$A_{\beta f}(T, \beta) = \int_0^T e^{-\beta t}\bar{G}(t)dF(t) + \int_0^T e^{-\beta t}\bar{F}(t)dG(t) + e^{-\beta T}\bar{G}(T)\bar{F}(T), \tag{5.12}$$

$$B_{\beta f}(T, \beta) = c_F\int_0^T e^{-\beta t}\bar{G}(t)dF(t) + c_Y\int_0^T e^{-\beta t}\bar{F}(t)dG(t) + c_T e^{-\beta T}\bar{G}(T)\bar{F}(T). \tag{5.13}$$

It is evident from the well-known l'Hopital's theorem that

$$EC_f(T) = \lim_{\beta \to 0} \beta TC_{\beta f}(T, \beta). \tag{5.14}$$

Taking the differentiation of $TC_{\beta f}(T, \beta)$ with respect to T and equaling it to zero, we can get

$$[(c_F - c_T)r(T) - (c_T - c_Y)h(T) - \beta c_T]\,[1 - A_{\beta f}(T, \beta)] - B_{\beta f}(T, \beta) = 0. \tag{5.15}$$

Let $w_{\beta f}(T, \beta)$ denote the left-hand side of Eq. (5.15). Further differentiating $w_{\beta f}(T, \beta)$ with respect to T, we have

$$w'_{\beta f}(T, \beta) = [(c_F - c_T)r'(T) - (c_T - c_Y)h'(T)]\,[1 - A_{\beta f}(T, \beta)]\,. \tag{5.16}$$

Then the optimal replacement policy in RF discipline with discounting can be described as follows.

Theorem 5.1. *(I) Suppose that* $(c_F - c_T)r'(T) - (c_T - c_Y)h'(T) > 0$.

(i) If $w_{\beta f}(\infty \mid \beta) > 0$*, then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.15) *and its resulting expected total discounted cost rate is*

$$TC_{\beta f}(T^* \mid \beta) = \frac{(c_F - c_T)r(T^*) - (c_T - c_Y)h(T^*)}{\beta} - c_T. \tag{5.17}$$

(ii) If $w_{\beta f}(\infty \mid \beta) \leq 0$*, then the optimal preventive replacement time is given by* $T^* \to \infty$*, so the decision-maker should take the failure replacement or opportunistic replacement.*

(II) Suppose that $(c_F - c_T)r'(T) - (c_T - c_Y)h'(T) \leq 0$*. Then, the decision-maker should perform the failure replacement or opportunistic replacement, whichever comes first.*

The proof is similar to that for Theorem 4.2 in Appendix 7.4.6.

Next, we formulate the RL model with NPV approach. In the RL discipline, $TC_{\beta l}(T, \beta)$ denotes the NPV of the total expected cost over an infinite time horizon. Then, we have

$$TC_{\beta l}(T, \beta) = \frac{B_{\beta l}(T, \beta)}{1 - A_{\beta l}(T, \beta)}, \tag{5.18}$$

where

$$A_{\beta l}(T,\beta) = \int_0^T e^{-\beta t} f(t)dt + \int_T^\infty e^{-\beta t}\bar{G}(t)dF(t) + \int_T^\infty e^{-\beta t}\bar{F}(t)dG(t) + e^{-\beta T}G(T)\bar{F}(T), \tag{5.19}$$

$$B_{\beta l}(T,\beta) = c_F \int_0^T e^{-\beta t} f(t)dt + c_F \int_T^\infty e^{-\beta t}\bar{G}(t)dF(t) + c_Y \int_T^\infty e^{-\beta t}\bar{F}(t)dG(t) + c_T e^{-\beta T}G(T)\bar{F}(T). \tag{5.20}$$

It is evident to confirm that

$$EC_l(T) = \lim_{\beta\to 0} \beta TC_{\beta l}(T,\beta). \tag{5.21}$$

Calculating $dTC_{\beta l}(T,\beta)/dT = 0$, we have

$$\Big[(c_F - c_T)r(T) + (c_T - c_Y)\hat{H}(T) - \beta c_T\Big]\left[1 - A_{\beta l}(T,\beta)\right] - B_{\beta l}(T,\beta) = 0. \tag{5.22}$$

Let $w_{\beta l}(T,\beta)$ denote the left-hand side of Eq. (5.22). Further differentiating $w_{\beta l}(T,\beta)$ with respect to T, we have

$$w'_{\beta l}(T,\beta) = \Big[(c_F - c_T)r'(T) + (c_T - c_Y)\hat{H}'(T)\Big]\left[1 - A_{\beta l}(T,\beta)\right]. \tag{5.23}$$

Then the optimal replacement policy in RL discipline with discounting can be described as follows.

Theorem 5.2. *(I) Suppose that* $(c_F - c_T)r'(T) + (c_T - c_Y)\hat{H}'(T) > 0$.

(i) If $w_{\beta l}(\infty \mid \beta) > 0$*, then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.22) *and its resulting expected total discounted cost rate is*

$$TC_{\beta l}(T^* \mid \beta) = \frac{(c_F - c_T)r(T^*) + (c_T - c_Y)\hat{H}(T^*)}{\beta} - c_T. \tag{5.24}$$

(ii) If $w_{\beta l}(\infty \mid \beta) \le 0$*, then the optimal preventive replacement time is given by* $T^* \to \infty$*, so the decision-maker should take the failure replacement or opportunistic replacement.*

(II) Suppose that $(c_F - c_T)r'(T) + (c_T - c_Y)\hat{H}'(T) \le 0$*. Then, the optimal preventive replacement time is given by* $T^* \to \infty$.

The proof of Theorem 5.2 is similar to Theorem 4.2 in Appendix 7.4.6.

5.2 Failure-Correlated Opportunity Models

In the conventional opportunistic replacement models [9, 21, 22], it is assumed that the lifetime of the system and the coming time of replacement opportunities are statistically independent from each other. However, this assumption is quite strong. Recognizing the importance to account for the correlation between the lifetime and the coming time of replacement opportunities, Dohi and Okamura [10] introduced a bivariate copula with the marginal distributions. The copula allows for a more accurate representation of the correlation in models when the age of the system and the occurrence of the opportunity are dependent.

5.2.1 Renewal Reward Approach

Following Dohi and Okamura [10], we first formulate the failure-correlated opportunity-based RF model. Suppose that the random variables Y and X are statistically dependent. The joint c.d.f. of variables Y and X is

$$\Pr\{Y \le v, X \le u\} = C(v,u) = \int_0^y \int_0^x c_{Y,X}(s,t)dsdt, \tag{5.25}$$

where $c_{y,x} = \partial^2 C(y,x)/\partial y \partial x$ is the bivariate p.d.f. of (X,Y), $\lim_{x\to\infty} = F(y)$ and $\lim_{y\to\infty} = G(x)$ are marginal c.d.f.'s. The bivariate survival function is given by:

$$\Pr\{Y > v, X > u\} = M(y,x) = 1 - F(y) - G(x) + C(y,x). \tag{5.26}$$

When the random variables Y and X are statistically dependent, the expected cost for one cycle in RF model is given by [10]

$$\begin{aligned} B_F(T) = {} & c_F \int_0^T \int_y^\infty c_{Y,X}(y,x)dxdy + c_Y \int_0^T \int_x^\infty c_{Y,X}(y,x)dydx \\ & + c_T \int_T^\infty \int_T^\infty c_{Y,X}(y,x)dydx \\ & = c_T + (c_F - c_T) \int_0^T \int_y^\infty c_{Y,X}(y,x)dxdy \\ & - (c_T - c_Y) \int_0^T \int_x^\infty c_{Y,X}(y,x)dydx. \end{aligned} \tag{5.27}$$

The expected time length of one cycle with RF discipline is

$$\begin{aligned}A_F(T) &= \int_0^T \int_y^\infty y c_{Y,X}(y,x)dxdy + \int_0^T \int_x^\infty x c_{Y,X}(y,x)dydx \\ &+ T \int_T^\infty \int_T^\infty c_{Y,X}(y,x)dydx \\ &= \int_0^T M(t,t)dt. \end{aligned} \tag{5.28}$$

The expected cost rate can get formulated as

$$EC_F(T) = \frac{B_F(T)}{A_F(T)}. \tag{5.29}$$

Calculating $dEC_F(T)/dT$, we can get

$$\frac{dEC_F(T)}{dT} = \frac{w_F(T)}{A_F^2(T)}, \tag{5.30}$$

where

$$w_F(T) = \left[(c_F - c_T)\Lambda_Y(T) - (c_T - c_Y)\Lambda_X(T)\right] A_F(T) - B_F(T). \tag{5.31}$$

In the bivariate model, the concept of the bivariate hazard rate function, as defined by Basu [45], is widely known. However, in our models, the bivariate hazard rate function does not adequately capture the optimality condition in the dependent case.

In Eq. (5.31), the expressions

$$\Lambda_Y(t) = \frac{\int_t^\infty c_{X,Y}(t,x)dx}{M(t,t)}, \tag{5.32}$$

$$\Lambda_X(t) = \frac{\int_t^\infty c_{X,Y}(t,y)dy}{M(t,t)}, \tag{5.33}$$

are called the initial hazard rate functions for the bivariate random variable Y and X [45]. We are in the position to characterize the optimal failure-correlated opportunity-based RF policy with the initial hazard rate functions.

Theorem 5.3. *(I) Suppose that* $(c_F - c_T)\Lambda'_Y(T) - (c_T - c_Y)\Lambda'_X(T) > 0$.

(i) If $w_F(\infty) > 0$*, then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.31) *and its resulting optimal expected total cost rate is given by*

$$EC_F(T^*) = (c_F - c_T)\Lambda_Y(T^*) - (c_T - c_Y)\Lambda_X(T^*). \tag{5.34}$$

(ii) If $w_F(\infty) \leq 0$, then the optimal preventive replacement time is given by $T^* \to \infty$, so the decision-maker should take the failure replacement or opportunistic replacement.

(II) Suppose that $(c_F - c_T)\Lambda'_Y(T) - (c_T - c_Y)\Lambda'_X(T) \leq 0$. Then, the optimal preventive replacement time is given by $T^* \to \infty$.

Next, we consider the failure-correlated opportunity-based RL model. The expected time length of one cycle becomes

$$\begin{aligned} B_L(T) = c_F \int_0^T tf(t)dt + c_F \int_T^\infty \int_y^\infty c_{Y,X}(y,x)dxdy \\ + c_Y \int_T^\infty \int_x^\infty c_{Y,X}(y,x)dydx + c_T \int_0^T \int_T^\infty c_{Y,X}(y,x)dydx \\ = c_T + (c_F - c_T)\left[F(T) + \int_T^\infty \int_y^\infty c_{Y,X}(y,x)dxdy\right] \\ - (c_T - c_Y)\int_T^\infty \int_x^\infty c_{Y,X}(y,x)dydx. \end{aligned} \tag{5.35}$$

The expected time length of one cycle with RL discipline is

$$\begin{aligned} A_L(T) = \int_0^T tf(t)dt + \int_T^\infty \int_y^\infty yc_{Y,X}(y,x)dxdy \\ + \int_T^\infty \int_x^\infty xc_{Y,X}(y,x)dydx + T\int_0^T \int_T^\infty c_{Y,X}(y,x)dydx \\ = \int_0^T \bar{F}(t)dt + \int_T^\infty M(t,t)dt. \end{aligned} \tag{5.36}$$

The expected cost rate can ge formulated as

$$EC_L(T) = \frac{B_L(T)}{A_L(T)}. \tag{5.37}$$

Calculating $dEC_L(T)/dT$, we can get

$$\frac{EC_L(T)}{dT} = \frac{w_L(T)}{A_L^2(T)}, \tag{5.38}$$

where

$$w_L(T) = [(c_F - c_T)\Phi_Y(T) + (c_T - c_Y)\Phi_X(T)]\, A_L(T) - B_L(T), \tag{5.39}$$

$$\Phi_Y(t) = \frac{f(t) - \int_t^\infty c_{X,Y}(t,x)dx}{\bar{F}(t) - M(t,t)}, \tag{5.40}$$

$$\Phi_X(t) = \frac{\int_t^\infty c_{X,Y}(x,t)dx}{\bar{F}(t) - M(t,t)}. \tag{5.41}$$

Then the optimal failure-correlated replacement policies with RL discipline can be described as follows.

Theorem 5.4. *(I) Suppose that* $(c_F - c_T)\Phi'_Y(T) + (c_T - c_Y)\Phi'_X(T) > 0$.

(i) If $w_L(\infty) > 0$, *then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.39) *and its resulting optimal expected total cost rate is given by*

$$EC_F(T^*) = (c_F - c_T)\Phi_Y(T^*) + (c_T - c_Y)\Phi_X(T^*). \tag{5.42}$$

(ii) If $w_L(\infty) \leq 0$, *then the optimal preventive replacement time is given by* $T^* \to \infty$.

(II) Suppose that $(c_F - c_T)\Phi'_Y(T) + (c_T - c_Y)\Phi'_X(T) \leq 0$. *Then, the optimal preventive replacement time is given by* $T^* \to \infty$.

5.2.2 NPV Approach

Similarly, we formulate the failure-correlated opportunity-based RF model with discounting. Let $TC_{\beta F}(T, \beta)$ denote the NPV of the expected total cost. We can get

$$TC_{\beta F}(T, \beta) = \frac{B_{\beta F}(T, \beta)}{1 - A_{\beta F}(T, \beta)}, \tag{5.43}$$

$$\begin{aligned} B_{\beta F}(T, \beta) = {} & c_F \int_0^T \int_y^\infty e^{-\beta y} c_{Y,X}(y, x) dx dy + c_Y \int_0^T \int_x^\infty e^{-\beta x} c_{Y,X}(y, x) dy dx \\ & + c_T e^{-\beta T} \int_T^\infty \int_T^\infty c_{Y,X}(y, x) dy dx, \end{aligned} \tag{5.44}$$

$$\begin{aligned} A_{\beta F}(T, \beta) = {} & \int_0^T \int_y^\infty e^{-\beta y} c_{Y,X}(y, x) dx dy + \int_0^T \int_x^\infty e^{-\beta x} c_{Y,X}(y, x) dy dx \\ & + e^{-\beta T} \int_T^\infty \int_T^\infty c_{Y,X}(y, x) dy dx. \end{aligned} \tag{5.45}$$

It is evident to confirm that

$$EC_F(T) = \lim_{\beta \to 0} \beta TC_{\beta F}(T, \beta). \tag{5.46}$$

Calculating $dTC_{\beta F}(T \mid \beta)/dT$, we can get

$$\frac{dTC_{\beta F}(T \mid \beta)}{dT} = \frac{w_{\beta F}(T \mid \beta)}{[1 - A_{\beta F}(T, \beta)]^2}, \tag{5.47}$$

where

$$w_{\beta F}(T \mid \beta) = \left[\frac{(c_F - c_T)\Lambda_Y(T) - (c_T - c_Y)\Lambda_X(T)}{\beta} - c_T\right] \times [1 - A_{\beta F}(T, \beta)] - B_{\beta F}(T, \beta). \tag{5.48}$$

Theorem 5.5. *(I) Suppose that* $(c_F - c_T)\Lambda'_Y(T) - (c_T - c_Y)\Lambda'_X(T) > 0$.

(i) If $w_{\beta F}(\infty \mid \beta) > 0$, *then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.48) *and its resulting optimal expected total cost rate is given by*

$$TC_{\beta F}(T^* \mid \beta) = \frac{(c_F - c_T)\Lambda_Y(T^*) - (c_T - c_Y)\Lambda_X(T^*)}{\beta} - c_T. \tag{5.49}$$

(ii) If $w_{\beta F}(\infty \mid \beta) \leq 0$, *then the optimal preventive replacement time is given by* $T^* \to \infty$, *so the decision-maker should take the failure replacement or opportunistic replacement.*

(II) Suppose that $(c_F - c_T)\Lambda'_Y(T) - (c_T - c_Y)\Lambda'_X(T) \leq 0$. *Then, the optimal preventive replacement time is given by* $T^* \to \infty$.

Next, we derive the optimal failure-correlated replacement policy in RL model with NPV approach. The NPV of the expected total cost over an infinite time horizon is

$$TC_{\beta L}(T, \beta) = \frac{B_{\beta L}(T, \beta)}{1 - A_{\beta L}(T, \beta)}, \tag{5.50}$$

where

$$B_{\beta L}(T, \beta) = c_F \int_0^T e^{-\beta t} f(t)dt + c_F \int_T^\infty \int_y^\infty e^{-\beta y} c_{Y,X}(y, x)dxdy + c_Y \int_T^\infty \int_x^\infty e^{-\beta x} c_{Y,X}(y, x)dydx + c_T e^{-\beta T} \int_0^T \int_T^\infty c_{Y,X}(y, x)dydx, \tag{5.51}$$

$$A_{\beta L}(T, \beta) = \int_0^T e^{-\beta t} f(t)dt + \int_T^\infty \int_y^\infty e^{-\beta y} c_{Y,X}(y, x)dxdy + \int_T^\infty \int_x^\infty e^{-\beta x} c_{Y,X}(y, x)dydx + e^{-\beta T} \int_0^T \int_T^\infty c_{Y,X}(y, x)dydx. \tag{5.52}$$

It is evident to confirm that

$$EC_L(T) = \lim_{\beta \to 0} \beta TC_{\beta L}(T, \beta). \tag{5.53}$$

Calculating $dTC_{\beta L}(T \mid \beta)/dT$, we can get

$$\frac{dTC_{\beta L}(T \mid \beta)}{dT} = \frac{w_{\beta L}(T \mid \beta)}{[1 - A_{\beta L}(T, \beta)]^2}, \tag{5.54}$$

where

$$w_{\beta L}(T \mid \beta) = \left[\frac{(c_F - c_T)\Phi_Y(T) + (c_T - c_Y)\Phi_X(T)}{\beta} - c_T \right] \times [1 - A_{\beta L}(T, \beta)] - B_{\beta L}(T, \beta). \tag{5.55}$$

Theorem 5.6. *(I) Suppose that* $(c_F - c_T)\Phi'_Y(T) + (c_T - c_Y)\Phi'_X(T) > 0$.

(i) If $w_{\beta L}(\infty \mid \beta) > 0$*, then there exists a finite and unique optimal preventive replacement time* T^* $(0 < T^* < \infty)$ *which satisfies Eq.* (5.55) *and its resulting optimal expected total cost rate is given by*

$$TC_{\beta F}(T^* \mid \beta) = \frac{(c_F - c_T)\Phi_Y(T^*) + (c_T - c_Y)\Phi_X(T^*)}{\beta} - c_T. \tag{5.56}$$

(ii) If $w_{\beta F}(\infty \mid \beta) \leq 0$*, then the optimal preventive replacement time is given by* $T^* \to \infty$*, so the decision-maker should take the failure replacement or opportunistic replacement.*

(II) Suppose that $(c_F - c_T)\Phi'_Y(T) + (c_T - c_Y)\Phi'_X(T) \leq 0$*. Then, the optimal preventive replacement time is given by* $T^* \to \infty$.

5.3 Numerical Examples

Based on the previous mathematical theorems, we analyze the correlation between failure time and opportunity arrival in opportunity-based age replacement models. In our examples, the FGM bivariate copula is utilized to capture the dependence between the system lifetime and the occurrence of an opportunity. For more details of bivariate copula, see Dohi and Okamura [10]. The FGM bivariate copula function is given by

$$C[F(y), F(x)] = F(y)G(x)[1 + \gamma(1 - F(y))(1 - G(x))], \tag{5.57}$$

where $-1 \leq \gamma \leq 1$ is a parameter for the strength of correlation. Spearman's rank correlation coefficient becomes $\rho_S = \gamma/3$.

We suppose the failure time Y of the unit obeys a Gamma distribution. We can obtain the related c.d.f. and p.d.f.:

$$F(t) = 1 - (1 + \theta t)e^{-\theta t}, \tag{5.58}$$

$$f(t) = \theta^2 t e^{-\theta t}. \tag{5.59}$$

When $G(t) = 1 - e^{-\theta t}$, we can have $g(t) = \theta e^{-\theta t}$. The cost parameters are gvien: $c_T = 15, 17, 19, 21, 23, 25, 27, 29, 31, 35$, $c_F = 150$, and $c_Y = 10, 15$. The discounted factor is set as $\beta = 0.6, 0.9$.

Table 5.1 shows the optimal preventive replacement times T^* and their associated expected cost rates $EC(T^*)$ under RF and RL disciplines, when $\theta = 1$, $\lambda = 1$ and $c_F = 150$. It can be seen that the optimal preventive replacement times T^* under under RF and RL disciplines increase, as the preventive replacement cost c_T increases significantly. When the opportunistic replacement cost is relatively small, such as $c_Y = 10$, RF policy always outperforms RL policy. When the opportunistic replacement cost is big enough, such as $c_Y = 15$, RF policy is better than RL policy in some specific cases where the preventive replacement cost is very big, such as $c_T = 31, 35$.

Tables 5.2 and 5.3 present the optimal preventive replacement times T^* and their expected total discounted costs $TC(T^*)$ over an infinite horizon under RF and RL disciplines, when the discount factor is given by $\beta = 0.9, 0.6$. From the tables, when the economic environment is not stable, the optimal preventive replacement times T^* are delayed. That is, when the economic environment is unstable, the decision-maker can reduce the frequency of preventive replacements to save money. Moreover, the tendency becomes more remarkable when the economic environment is more unstable.

Table 5.4 presents the optimal preventive replacement times T^* and their associated expected cost rates $EC(T^*)$ with FGM copula, when $\theta = 1$, $\lambda = 1$ and $c_F = 150$. We can find that when there is a positive correlation, the expected cost rate decreases in both disciplines. Conversely, a negative correlation indicates the opposite scenario. In such cases, if the unit remains operational for an extended period, the chance for replacement may arise sooner. In the discipline of RL, a positive correlation leads to a longer optimal replacement time, while a negative correlation results a smaller optimal replacement time. However, this trend is observed in the RF discipline only when the cost of preventive replacement c_T is sufficiently low.

Tables 5.5 and 5.6 present the optimal preventive replacement times T^* and their expected total discounted costs $TC(T^*)$ over an infinite horizon with FGM copula, when the discount factor $\beta = 0.6, 0.9$. It is seen that when the lifetime of system and the arrival of replacement opportunity are statistically correlated, the optimal preventive replacement times T^* are delayed in the unstable economic environment.

Table 5.1: Optimal RF and RL policies without discounting, when $\lambda = 1$, $\theta = 1$ and $c_F = 150$.

	$c_Y = 10$				$c_Y = 15$			
	RF		RL		RF		RL	
c_T	T^*	$EC(T^*)$	T^*	$EC(T^*)$	T^*	$EC(T^*)$	T^*	$EC(T^*)$
15	0.973	56.6	0	57.5	0.973	66.6	0.715	56.2
17	1.145	57.0	0	57.5	1.145	67.0	0.732	57.9
19	1.351	57.3	0	57.5	1.351	67.3	0.769	59.6
21	1.602	57.4	0	57.5	1.602	67.5	0.794	61.2
23	1.917	57.5	0	57.5	1.917	67.5	0.842	62.7
25	2.332	57.5	0	57.5	2.332	67.5	0.912	64.2
27	2.906	57.5	0	57.5	2.906	67.5	0.945	65.6
29	3.745	57.5	0.778	66.8	3.745	67.5	1.111	66.9
31	5.103	57.5	1.052	67.8	5.103	67.5	1.170	68.1
35	14.33	57.5	1.391	69.9	14.33	67.5	1.421	70.1

Table 5.2: Optimal RF and RL policies with discounting, when $\beta = 0.6$, $\lambda = 1$, $\theta = 1$ and $c_F = 150$.

	$c_Y = 10$				$c_Y = 15$			
	RF		RL		RF		RL	
c_T	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$
15	1.211	87.3	0.551	81.3	1.211	1039	0.763	822.4
17	1.363	87.5	0.600	83.8	1.363	1042	0.817	843.0
19	1.675	87.6	0.712	85.1	1.675	1043	0.875	863.9
21	2.081	57.7	0.825	87.7	2.082	1043	0.955	882.1
23	2.671	57.7	0.949	89.5	2.671	1043	1.052	898.4
25	3.565	57.7	1.092	91.0	3.565	1043	1.169	914.5
27	5.091	57.7	1.245	92.3	5.091	1043	1.303	914.5
29	8.322	57.7	1.419	93.4	8.322	1043	1.452	914.5
31	19.55	57.7	1.615	94.0	19.55	1043	1.631	914.5
35	> 20	57.7	2.052	95.3	> 20	1043	2.076	914.5

Table 5.3: Optimal RF and RL policies with discounting, when $\beta = 0.9$, $\lambda = 1$, $\theta = 1$ and $c_F = 150$.

	$c_Y = 10$				$c_Y = 15$			
	RF		RL		RF		RL	
c_T	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$
15	1.213	52.1	0.656	50.9	1.213	67.2	0.794	51.4
17	1.504	52.9	0.735	52.2	1.504	67.3	0.861	52.6
19	1.892	56.2	0.840	53.4	1.892	67.3	0.947	53.7
21	2.443	56.2	0.966	54.4	2.443	67.4	1.052	54.6
23	3.278	56.2	1.073	55.1	3.278	67.4	1.177	55.4
25	4.110	56.2	1.277	55.9	4.110	67.4	1.326	56.0
27	7.73	56.2	1.465	56.4	7.73	67.4	1.501	56.5
29	18.75	56.2	1.668	56.7	18.75	67.4	1.707	56.9
31	> 20	56.2	1.900	57.0	> 20	67.4	1.948	57.2
35	> 20	56.2	2.552	57.4	> 20	67.4	2.584	57.5

Table 5.4: Optimal RF and RL policies without discounting, when $\lambda = 1$, $\theta = 1$ and $c_F = 150$ in failure-correlated-opportunity case.

		$c_Y = 10$				$c_Y = 15$			
		RF		RL		RF		RL	
c_T	ρ_S	T^*	$EC(T^*)$	T^*	$EC(T^*)$	T^*	$EC(T^*)$	T^*	$EC(T^*)$
15	0.3	1.483	42.5	0	42.9	1.482	52.4	0.349	50.2
15	0.2	1.389	47.2	0	47.6	1.388	57.1	0.479	52.9
15	0.1	1.274	51.9	0	52.5	1.215	61.9	0.603	54.8
15	0.0	0.973	56.6	0	57.5	0.973	66.6	0.711	56.2
15	-0.1	0.777	60.9	0	62.8	0.776	70.9	0.802	57.2
15	-0.2	0.655	64.6	0	68.2	0.653	74.7	0.875	58.0
15	-0.3	0.576	68.0	0	68.0	0.575	78.1	0.935	58.6
20	0.3	1.871	42.8	0	42.9	1.868	52.6	0	57.3
20	0.2	1.838	47.5	0	47.6	1.837	57.4	0.471	57.4
20	0.1	1.728	52.4	0	52.5	1.727	62.3	0.665	59.3
20	0.0	1.469	57.3	0	57.5	1.467	67.3	0.801	60.4
20	-0.1	1.119	62.3	0	62.8	1.116	72.4	0.901	61.2
20	-0.2	0.889	66.9	0	68.2	0.884	77.0	0.978	61.7
20	-0.3	0.744	71.0	0	68.0	0.741	81.2	1.041	62.2
25	0.3	2.381	42.9	0	42.9	2.375	52.7	0	57.3
25	0.2	2.441	47.6	0	47.6	2.440	57.4	0.447	61.7
25	0.1	2.451	52.4	0	52.4	2.450	62.4	0.788	63.4
25	0.0	2.333	57.5	0	57.5	2.330	67.5	0.940	64.2
25	-0.1	1.773	62.7	0	62.7	1.772	72.8	1.041	64.7
25	-0.2	1.206	67.9	0	68.2	1.204	78.0	1.118	65.1
25	-0.3	0.947	72.7	0	68.0	0.941	82.9	1.179	65.4

Table 5.5: Optimal RF and RL policies without discounting, when $\beta = 0.6$, $\lambda = 1$, $\theta = 1$ and $c_F = 150$ in failure-correlated-opportunity case.

		$c_Y = 10$				$c_Y = 15$			
		RF		RL		RF		RL	
c_T	ρ_S	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$
15	0.3	1.669	66.3	∞	96.2	1.668	82.7	0.417	76.1
15	0.2	1.581	73.2	0.325	75.4	1.581	89.7	0.544	79.1
15	0.1	1.404	80.2	0.524	79.0	1.403	96.8	0.663	81.0
15	0.0	1.122	87.3	0.668	81.1	1.121	104	0.761	82.2
15	-0.1	0.867	94.0	0.774	82.4	0.865	111	0.840	83.2
15	-0.2	0.711	100	0.856	83.4	0.710	117	0.905	84.0
15	-0.3	0.616	106	0.922	84.2	0.614	122	0.959	84.6
20	0.3	2.212	66.4	∞	96.2	2.208	82.9	0.110	84.9
20	0.2	2.206	73.3	0	66.4	2.203	89.8	0.672	85.3
20	0.1	2.126	80.4	0.215	87.8	2.125	97.0	0.822	86.5
20	0.0	1.863	87.7	0.169	94.2	1.863	104	0.927	87.3
20	-0.1	1.360	95.1	0.949	87.5	1.358	112	1.006	88.0
20	-0.2	1.001	102	1.028	88.1	0.998	119	1.069	88.3
20	-0.3	0.817	109	1.091	88.5	0.813	125	1.123	88.7
25	0.3	3.057	66.5	∞	96.2	3.048	82.9	0.695	89.4
25	0.2	3.223	73.3	0.726	89.6	3.215	89.8	0.961	90.4
25	0.1	3.404	80.4	0.995	90.5	3.398	97.0	0.151	102
25	0.0	3.563	87.7	1.120	91.0	3.564	104	0.138	110
25	-0.1	3.330	95.2	0.195	106	3.347	112	0.118	117
25	-0.2	1.514	103	0.173	114	1.511	120	0.107	124
25	-0.3	1.076	108	0.156	121	1.090	127	0.099	132

Table 5.6: Optimal RF and RL policies with discounting, when $\beta = 0.9$, $\lambda = 1$, $\theta = 1$ and $c_F = 150$ in failure-correlated-opportunity case.

		$c_Y = 10$				$c_Y = 15$			
		RF		RL		RF		RL	
c_T	ρ_S	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$	T^*	$TC(T^*)$
15	0.3	1.776	43.0	∞	50.9	1.774	54.0	0.456	48.3
15	0.2	1.692	47.9	0.400	48.1	1.692	58.3	0.588	49.8
15	0.1	1.516	51.7	0.579	49.9	1.515	62.7	0.702	50.8
15	0.0	1.213	56.1	0.711	50.9	1.213	67.2	0.794	51.0
15	-0.1	0.920	60.4	0.809	51.5	0.919	71.5	0.867	51.9
15	-0.2	0.745	64.4	0.885	52.0	0.742	75.6	0.930	52.2
15	-0.3	0.639	67.9	0.947	52.4	0.639	79.2	0.981	52.5
20	0.3	2.424	43.0	∞	50.9	2.419	54.0	0.594	52.5
20	0.2	2.418	47.3	0.261	53.0	2.414	58.3	0.787	53.3
20	0.1	2.384	51.7	0.186	56.7	2.382	62.7	0.915	53.8
20	0.0	2.141	56.2	0.937	53.9	2.141	67.3	1.006	54.1
20	-0.1	1.537	60.8	1.028	54.2	1.536	71.9	1.076	54.4
20	-0.2	1.076	65.3	1.097	54.4	1.073	76.6	1.132	54.6
20	-0.3	0.858	69.6	1.153	54.6	0.855	80.9	1.180	54.7
25	0.3	3.561	43.0	∞	50.9	3.549	54.0	1.038	55.6
25	0.2	3.834	47.3	1.072	55.6	3.822	58.3	1.179	55.8
25	0.1	4.192	51.7	1.203	55.8	4.183	62.8	1.267	55.9
25	0.0	4.711	56.2	1.288	55.9	4.711	67.8	1.332	56.0
25	-0.1	5.653	60.9	1.353	56.0	5.677	72.0	1.385	56.1
25	-0.2	7.902	65.8	1.406	56.1	7.997	76.8	1.429	56.1
25	-0.3	7.070	70.4	1.450	51.7	7.030	81.7	1.467	56.2

Chapter 6

Conclusion

This dissertation includes five chapters. We discussed opportunity-based age replacement models and their applications in discrete time and continuous time, respectively. In discrete time setting, the concept of replacement priority was introduced to deal with the case that more than one replacement options occur at same time point. Several discrete time opportunistic age replacement models were formulated, and optimal preventive replacement policies are proposed for each model. A study on pole air switch was presented on optimal preventive replacement policies. For continuous time models, we took the restricted duration consideration in RL model and compare with RF [8] by numerical experiment. We also considered the correlation between lifetime of system and the arrival of opportunity in RF and RL models.

In Chapter 2, two classical age-based replacement models: AR and DD models have been considered in discrete time. For more details, the concept of replacement priority was introduced to address such situations where failure replacement and preventive replacement occur at a given age or opportunity. We explored two priority cases in each replacement model. First, we formulated the optimal preventive replacement policies for each model by the familiar renewal reward argument and NPV method. Next, we studied unified stochastic models incorporating the probabilistic priority of replacement options. Besides, another important contribution is that general framework was proposed to optimize preventive replacement policies in discrete time. The discrete time AR and DD models with/without discounting were reformulated under this framework. To provide practical insights, we present numerical illustrations using real failure data for pole air switches, comparing the performance of these optimal preventive policies. we could find that: (1) When $c_T = c_Y$, the AR policy is better than the DD policy. Additionally, the AR time is larger than the DD time. (2) When $c_T < c_Y$, the DD policy is better than the AR policy in some cases where c_F is relatively smaller. For example, when $c_F = 1.5$, it is easy to confirm that the DD policy is better than the AR policy. In our actual application, under the assumption of $c_T = 2c_Y$, if $c_T < c_F < 1.5c_T$, the decision-maker should consider the opportunity in the preventive replacement. Otherwise, if $c_T > 1.5c_Y$, the decision-maker should consider only the AR policy instead of the DD policy. (3) In terms of the optimal time in AR and DD policies, the optimal replacement time with discounting is longer than that without discounting. This indicates that when the economic environment is unstable, decision-makers will shorten the replacement times for their equipment.

In Chapter 3, we further focused on two important opportunity-based age replacement models in discrete time: RF and RL models, where the expected cost model under each discipline can be further classified into six cases by taking account of the priority of multiple replacement options. We characterize several optimal opportunity-based age replacement policies minimizing the relevant expected costs. Besides, the NPV method was applied to formulate the expected discounted costs over infinite horizon under RF and RL disciplines. In addition, six discrete time opportunity-based age replacement models with/ without discounting were unified with deterministic priorities. In numerical illustrations, we obtain and compare all the optimal scheduled preventive replacement times with RF and RL disciplines. The results indicate that: When c_F is relatively small ($c_F = 1.5, 2.0$), it can be shown in all priority models that RL policies are better than RF policies in both Assumption 1. On the other hand, when c_F is larger and the impact of system failure becomes more remarkable, we find that RF policies are better than RL policies. From the results above, it is confirmed that RL policies can be motivated even in the plausible case of $c_F > c_Y$. However, when the failure impact is remarkable with large c_F, as expected, RF policies always outperform RL policies. In the sensitivity of the cost parameter c_F, as c_F increases, the optimal scheduled preventive replacement time N_0^* and its associated minimum expected cost decreases and increases, respectively.

(2) When the economic environment is unstable, such as $\beta = 0.9$, the optimal preventive replacement times N^* will be delayed. When the economic environment is more unstable, for example, during times of economic uncertainty, the optimal timing for performing preventive equipment replacements will be postponed. This means that in uncertain economic conditions, people may delay replacing equipment or machinery to reduce costs or mitigate risks. (3) The discounted factor cannot affect the structure of optimal preventive replacement policy.

In Chapter 4, we extended RF and RL models with the restricted duration in continuous/discrete time. Firstly, we formulate RF and RL models with the restricted duration in continuous time, where the arrival of opportunities obeys a homogeneous Poisson process. Next, we considered these opportunity-based age replacement models in discrete time, where the inter-arrival times of replacement opportunities obey an independent and identical geometric distribution. The optimal two-phase opportunity-based age replacement policies are characterized by minimizing the long-run average costs. The numerical examples are presented to compare two replacement policies with RF and RL disciplines. The results indicate: (1) In all tables, the optimal prescheduled preventive replacement times for respective priority models often converge to similar values in most cases. This is primarily since these optimal replacement times are discretized as integer values, and the differences in replacement priorities are not particularly significant. (2) When the cost of the corrective replacement c_F is become big, the optimal prescheduled preventive replacement times N_0^* become small. In addition, RL model can degenerate into two special cases in Section 3.1.2. (3) RL policies are only better than RF policies in some limited cases where the failure replacement cost c_F is relatively small. In our example, when $c_F = 1.5$, RL policies are better than RF policies. Conversely, when c_F is large and the impact of system failure becomes more remarkable, we can find that RF policies are better than RL policies.

In Chapter 5, we further generalized RF and RL models in continuous time. The correlation between lifetime and the arrival of opportunity have considered in these models. First, we reformulated two basic opportunity-based age replacement models with RF and RL disciplines, in which the failure time and the arrival time of a replacement opportunity are statistically independent. Next, we take place the NPV analysis for the failure-correlated opportunity-based age replacement models with RF and RL disciplines. We obtained the expected total discounted costs over an infinite time horizon and derived the optimal preventive replacement policies by minimizing them in both cases. The FGM bivariate copula was presented in numerical examples to investigate the dependence of correlation between the failure time and the opportunistic replacement time on the age opportunity-based replacement policies. We could know that: (1) When the opportunistic replacement cost is relatively small, such as $c_Y = 10$, RF policy always outperforms RL policy. When the opportunistic replacement cost is big enough, such as $c_Y = 15$, RF policy is better than RL policy in some specific cases where the preventive replacement cost is very big, such as $c_T = 31, 35$. (2) when the economic environment is not stable, the optimal preventive replacement times T^* are delayed. That is, when the economic environment is unstable, the decision-maker can reduce the frequency of preventive replacements to save money. Moreover, the tendency becomes more remarkable when the economic environment is more unstable. (3) when there is a positive correlation, the expected cost rate decreases in both disciplines. Conversely, a negative correlation indicates the opposite scenario. In such cases, if the unit remains operational for an extended period, the chance for replacement may arise sooner. In the discipline of RL, a positive correlation leads to a longer optimal replacement time, while a negative correlation results a smaller optimal replacement time. However, this trend is observed in the RF discipline only when the cost of preventive replacement c_T is sufficiently low.

Summarily, the main contribution of this thesis are shown as: (1) We introduce the concept of replacement priority to deal with the case that more than one replacement options occur at same time point. (2) The AR, DD, RF, RL models are reformulated in discrete time by renewal reward argument and NPV method. (3) The RF and RF more are generalized into two-phase opportunity-based models. (4) The correlation between lifetime and the arrival of opportunity have considered in RF and RL models.

However, it is noted that there are some limitations in this thesis:

(1) Although we have found that RL policy is only better than RF policy in

some limited cases where the failure replacement cost is relatively small. However, we cannot give that more accurate cost parameter ranges that RF policy equals RL policy in many cases.

(2) In our experiments, we calculated the model parameters using real failure data for pole air switches. In fact, these data is very old (1980s). More detailed statistical properties of the discrete lifetime data should be investigated by checking the goodness-of-fit to the discrete Weibull distribution.

(3) We must acknowledge that although the experiments revealed several trends, real-world decision-making often demands consideration of additional factors such as equipment significance, availability needs, and maintenance schedules. In addition, the discounted factor β is not a constant value.

In the future works, we want to further study opportunity-based models in the following viewpoints:

(1) Section 2 has propose the general framework to optimize preventive replacement policies in discrete time. The discrete time AR and DD models with/without discounting were reformulated under this framework. The results have shown that this method is a very effective tool for discrete time models. In future, we want to study more discrete time models, such as RF and RL, two-phase models by this general framework.

(2) We intend to further study the risk and uncertainties in opportunity-based replacement models, where the lifetime of system and the occurrence of opportunity are correlated. In this model, our findings show that when the economic environment is unstable, the optimal replacement time should be delayed. Other researchers studied this topic from somewhat different viewpoint. Giri and Dohi [46] studied the issue of risk-sensitive preventive maintenance policy with the mean-variance criterion, not just the expected cost criterion in the steady state and reformulated the age and block models under a new framework. Another extended direction of our study is to analyze the correlation between the lifetime of the unit and the occurrence of replacement opportunities in block replacement models.

(3) As Dohi and Okamura [10] were first to recognize the significance of correlation between age and replacement opportunity and proposed opportunity-based models by introducing the bivariate copula with arbitrary marginal distributions, where they consider the correlation between age and the opportunity. So, we will further study the correlation between the life of the system and the arrival of replacement opportunity in discrete-time models.

(4) As we are aware, PH distributions offer high accuracy and are well-suited for modeling failure times. Zheng et al. [1] explored replacement first and replacement last models, where the arrival of opportunities obeys a Markovian arrival process. We will apply the PH/MAP techniques to study the opportunity-based models in discrete time.

Chapter 7

Appendix

7.1 Acronyms

AR	Age replacement
DD	Dekker and Dijkstra's replacement model
IFR	Increasing failure rate
DFR	Decreasing failure rate
NPV	Net present value
RF	Replacement first discipline
RL	Replacement last discipline
PH	Phase-type
MAP	Markovian arrival process
p.d.f.	Probability density function
c.d.f.	Cumulative distribution function
FGM	Farlie-Gumbel-Morgenstern copula

7.2 Symbols

7.3 Lemmas

Lemma 7.1. *The function $R_Y(n)$ ($H_X(n)$) is strictly increasing (decreasing) in n, if and only if $r_Y(n)$ ($h_X(n)$) is strictly increasing (decreasing) in n.*

Proof. We give the proof for only $R_Y(n)$. From the definition in Lemma 7.1, it

C_a	Failure (corrective) replacement
S_c	Preventive
O_p	Opportunistic replacement
c_F	Failure replacement cost
c_P	Preventive replacement cost
c_Y	Opportunistic replacement cost

turns out that

$$\begin{aligned} R_Y(n) &= \frac{f_Y(n)}{\bar{F}_Y(n)} = \frac{\bar{F}_Y(n-1)}{\bar{F}_Y(n)} \cdot \frac{f_Y(n)}{\bar{F}_Y(n-1)} \\ &= \left\{ \frac{\bar{F}_Y(n-1) - f_Y(n)}{\bar{F}_Y(n-1)} \right\}^{-1} r_Y(n) = \{1 - r_Y(n)\}^{-1} r_Y(n). \end{aligned} \tag{7.1}$$

Further difference yields

$$R_Y(n+1) - R_Y(n) = \frac{r_Y(n+1) - r_Y(n)}{\{1 - r_Y(n+1)\}\{1 - r_Y(n)\}}. \tag{7.2}$$

Since $0 < r_Y(n) < 1$ for $n = 1, 2, \ldots$. □

Lemma 7.2. *The function $R_Y(n)$ ($r_X(n)$) is strictly increasing (decreasing) in n, if and only if $H_Y(n)$ ($h_Y(n)$) is strictly increasing (decreasing) in n.*

Proof. If $R_Y(n)$ is a strictly increasing function of n, then we have

$$\frac{f_Y(n)}{\bar{F}_Y(n)} < \frac{f_Y(n+k)}{\bar{F}_Y(n+k)}, \tag{7.3}$$

where, $k \geq 1$ is an arbitrary integer. Sicne

$$f_Y(n)\bar{F}_Y(n+k) < f_Y(n+k)\bar{F}_Y(n), \tag{7.4}$$

we can get the following inequality:

$$f_Y(n)\bar{F}_Y(n+k)(1-p)^k < f_Y(n+k)\bar{F}_Y(n)(1-p)^k. \tag{7.5}$$

Further, we have

$$\begin{aligned} & f_Y(n)\sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k + \sum_{k=1}^{\infty} f_Y(n+k)(1-p)^k \sum_{j=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k \\ & < f_Y(n+k)\bar{F}_Y(n)(1-p)^k + \sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k \sum_{k=1}^{\infty}\bar{F}_Y(n+k)(1-p)^k. \end{aligned} \tag{7.6}$$

X	Arrival times of opportunity for replacement
Y	Failure times
$\Pr\{X = n\} = g_X(n)$	Probability mass function
$\Pr\{X \leq n\} = G_X(n)$	Cumulative distribution function
$\Pr\{X \geq n\} = \bar{G}_X(n-1)$	Survivor function
$h_X(n) = g_X(n)/\bar{G}_X(n-1)$	Hazard rate
$H_X(n) = g_X(n)/\bar{G}_X(n)$	Shifted hazard rate
$\hat{H}_X(n) = g_X(n)/G_X(n)$	Reversed hazard rate
$\Pr\{Y = n\} = f_Y(n)$	Probability mass function
$\Pr\{Y \leq n\} = F_Y(n)$	Cumulative distribution function
$\Pr\{Y \geq n\} = \bar{F}_Y(n-1)$	Reliability function
$r_Y(n) = f_Y(n)/\bar{F}_Y(n-1)$	Failure rate
$R_Y(n) = f_Y(n)/\bar{F}_Y(n)$	Shifted failure rate
N	Preventive replacement time for discrete time model
$A_a(N)$	Expected length of one cycle for AR model
$B_{aj}(N)$	Expected cost of one cycle for AR model
$EC_{aj}(N)$	Expected cost rate for AR model
β	Discounted factor
$A_o(N)$	Expected length of one cycle for DD model
$B_{oj}(N)$	Expected cost of one cycle for DD Model
$EC_{oj}(N)$	Expected cost rate for DD model
$H_Y(n)$	Hazard rate in DD model
$A_a(N, \beta)$	NPV of one unit cost for one cycle for AR model
$B_{aj}(N, \beta)$	Expected total discounted costs of one cycle for AR model
$TC_{aj}(N, \beta)$	Expected total discounted costs over an infinite time horizon for AR model
$A_o(N, \beta)$	NPV of one unit cost for one cycle for DD model
$B_{oj}(N, \beta)$	Expected total discounted costs of one cycle for DD model
$TC_{oj}(N, \beta)$	Expected total discounted costs over an infinite time horizon for DD model
$H_Y(N, \beta)$	Shifted hazard rate in DD model with discounting
$h_Y(N, \beta)$	Hazard rate in DD model with discounting

$A(N)$	Expected length of one cycle for general discrete time model
$B_j(N)$	Expected cost of one cycle for general discrete time model
$EC_j(N)$	Expected cost rate for general discrete time model
$A_f(N)$	Expected length of one cycle for RF model
$B_{fj}(N)$	Expected cost of one cycle for RF model
$EC_{fj}(N)$	Expected cost rate for RF model
$A_l(N)$	Expected length of one cycle for RL model
$B_{lj}(N)$	Expected cost of one cycle for RL model
$EC_{lj}(N)$	Expected cost rate for RL model
$A_f(N,\beta)$	NPV of one unit cost for one cycle for RF model
$B_{fj}(N,\beta)$	Expected total discounted costs of one cycle for RF model
$TC_{fj}(N,\beta)$	Expected total discounted costs over an infinite time horizon for RF model
$A_l(N,\beta)$	NPV of one unit cost for one cycle for RL model
$B_{lj}(N,\beta)$	Expected total discounted costs of one cycle for RL model
$TC_{lj}(N,\beta)$	Expected total discounted costs over an infinite time horizon for RL model
$G(t) = 1 - e^{-\lambda t}$	Inter-arrival time of replacement follows the exponential distribution
$g(t) = dG(t)/dt$	p.d.f. of X in continuous time
$\bar{F}(t)$	Survivor function
$F(t)$	c.d.f of Y in continuous time
$f(t)$	p.d.f. of Y in continuous time
$r(t)$	Failure rate in continnous time
S	Restricted duration in continuous time model
T	Preventive replacement time in continuous time model
$A_l(S,T)$	Expected cycle length per unit for two-phase RL model in continuous time
$B_l(S,T)$	Expected cost per unit time in steady state for two-phase RL model in continuous time
$EC_l(S,T)$	Expected cost per cycle for two-phas RL model in continuous time
$A_f(S,T)$	Expected cycle length per unit for two-phase RF model in continuous time
$B_f(S,T)$	Expected cost per unit time in steady state for two-phase RF model in continuous time

$EC_f(S,T)$	Expected cost per cycle for two-phase RF model in continuous time
N_0	Restricted duration in discrete time model
N_1	Preventive replacement time in discrete time model
$A_f(N_0,N_1)$	Expected cost per unit time in steady state for two-phase RF model in discrete time
$B_{fj}(N_0,N_1)$	Expected cycle length per unit for two-phase RF model in discrete time
$EC_{fj}(N_0,N_1)$	Expected cost per cycle for two-phase RF model in discrete time
$A_l(N_0,N_1)$	Expected cost per unit time in steady state for two-phase RL model in discrete time
$B_{lj}(N_0,N_1)$	Expected cycle length per unit for two-phase RL model in discrete time
$EC_{lj}(N_0,N_1)$	Expected cost per cycle for two-phase RL model in discrete time
$h_Y(N_0,N_1)$	Hazard rate in two-phase RF model in discrete time
$H_Y(N_0,N_1)$	Shifted hazard rate in two-phase RF model in discrete time
$\hat{H}_Y(N_0,N_1)$	Reversed hazard rate in two-phase RF model in discrete time
$h(t)$	Hazard rate in continuous time model
$\hat{H}(t)$	Reversed hazard rate in continuous time model
$A_f(T)$	Expected time length of one cycle with RF in continuous time
$B_f(T)$	Expected cost of one cycle with RF in continuous time
$EC_f(T)$	Expected cost rate with RF in continuous time
$A_l(T)$	Expected time length of one cycle with RL in continuous time
$B_l(T)$	Expected cost of one cycle with RL in continuous time
$EC_l(T)$	Expected cost rate with RL in continuous time
$A_{\beta f}(T,\beta)$	Expected discounted value of one unit cost during one cycle with RF model in continuous time
$B_{\beta f}(T,\beta)$	Expected discounted cost during one cycle with RF model in continuous time
$TC_{\beta f}(T,\beta)$	NPV of the expected total cost with RF model in continuous time
$A_{\beta l}(T,\beta)$	Expected discounted value of one unit cost during one cycle with RL model in continous time

$B_{\beta l}(T,\beta)$	Expected discounted cost during one cycle with RL model in continous time
$TC_{\beta l}(T,\beta)$	NPV of the expected total cost with RL model in continous time
$A_F(T)$	Expected time length of one cycle with failure-correlated opportunity-based RF model
$B_F(T)$	Expected cost of one cycle with failure-correlated opportunity-based RF model
$EC_F(T)$	Expected cost rate with failure-correlated opportunity-based RF model
$A_L(T)$	Expected time length of one cycle with failure-correlated opportunity-based RL model
$B_L(T)$	Expected cost of one cycle with failure-correlated opportunity-based RL model
$EC_L(T)$	Expected cost rate with failure-correlated opportunity-based RL model
$A_{\beta F}(T,\beta)$	Expected discounted value of one unit cost during one cycle with failure-correlated opportunity-based RF model
$B_{\beta F}(T,\beta)$	Expected discounted cost during one cycle with failure-correlated opportunity-based RF model
$TC_{\beta F}(T,\beta)$	NPV of the expected total cost with RF model in continous time with failure-correlated opportunity-based RF model
$A_{\beta L}(T,\beta)$	Expected time length of one cycle with failure-correlated opportunity-based RL model
$B_{\beta L}(T,\beta)$	Expected cost of one cycle with failure-correlated opportunity-based RL model
$TC_{\beta L}(T,\beta)$	NPV of the expected total cost with failure-correlated opportunity-based RL model
$C(y,x)$	Joint c.d.f. of variables Y and X
$c(y,x)$	Bivariate p.d.f of variables Y and X
$M(y,x)$	Bivariate survival function
$\Lambda_Y(t)$	Initial hazard rate function for the bivariate random variable in RF model
$\Lambda_X(t)$	Initial hazard rate function for the bivariate random variable in RF model

$\Phi_Y(t)$	Initial hazard rate function for the bivariate random variable in RL model
$\Phi_X(t)$	Initial hazard rate function for the bivariate random variable in RL model

From above inequality, we can get

$$\frac{\sum_{k=0}^{\infty} f_Y(n+k)(1-p)^k}{\sum_{k=0}^{\infty} \bar{F}_Y(n+k)(1-p)^k} < \frac{\sum_{k=1}^{\infty} f_Y(n+k)(1-p)^k}{\sum_{k=1}^{\infty} \bar{F}_Y(n+k)(1-p)^k}. \tag{7.7}$$

Hence, we have

$$H_Y(n-1) < H_Y(n). \tag{7.8}$$

□

The proof for $h_Y(n)$ is similar to $H_Y(n)$.

From Lemmas 7.1 and 7.2, we can obtain the following lemmas directly.

Lemma 7.3. *The function $H_Y(n)$ is strictly increasing (decreasing) in n, if and only if $h_Y(n)$ is strictly increasing (decreasing) in n.*

Lemma 7.4. *The function $R_Y(n)$ $(r_X(n))$ is strictly increasing (decreasing) in n, if and only if $H_Y(n,\beta)$ $(h_Y(n,\beta))$ is strictly increasing (decreasing) in n.*

Proof . Similar to the proof of Lemma 7.2.

Lemma 7.5. *The function $H_Y(n,\beta)$ is strictly increasing (decreasing) in n, if and only if $h_Y(n,\beta)$ is strictly increasing (decreasing) in n.*

Lemma 7.6. *Suppose that the failure time Y is strictly IFR and that the arrival time of opportunity X is DHR under Assumption 1. Then, additional necessary conditions of strictly increasing $w_{fj}(N)$ $(j = 1, 2, \ldots, 6)$ are given by*

Model 1: $R_Y(N+1)H_X(N+1) \geq R_Y(N)H_X(N)$,

Model 2: *None,*

Model 3: $R_Y(N+1)H_X(N+1) \leq R_Y(N)H_X(N)$,

Model 4: *None,*

Model 5: $r_Y(N+1)h_X(N+1) \geq r_Y(N)h_X(N)$,

Model 6: $r_Y(N+1)h_X(N+1) \leq r_Y(N)h_X(N)$.

Proof. Taking the difference of the functions $w_{fj}(N)$ $(j = 1, 2, \ldots, 6)$, one obtains

$$\begin{aligned} w_{f1}(N+1) - w_{f1}(N) = \Big\{ &(c_F - c_T)\big[R_Y(N+1)\,(1 + H_X(N+1)) \\ &- R_Y(N)\,(1 + H_X(N))\,\big] \\ &- (c_T - c_Y)\big[H_X(N+1) - H_X(N)\big]\Big\} A_f(N+1), \end{aligned} \tag{7.9}$$

$$\begin{aligned} w_{f2}(N+1) - w_{f2}(N) = \Big\{ &(c_F - c_T)\,[r_Y(N+2) - r_Y(N+1)] \\ &- (c_T - c_Y)\,[H_X(N+1) - H_X(N)]\,\Big\} A_f(N+1), \end{aligned} \tag{7.10}$$

$$\begin{aligned} w_{f3}(N+1) - w_{f3}(N) = \Big\{ &(c_F - c_T)\,[R_Y(N+1) - R_Y(N)] \\ &- (c_T - c_Y)\big[\,(1 + R_Y(N+1))\,H_X(N+1) \\ &- (1 + R_Y(N))\,H_X(N)\big]\Big\} A_f(N+1), \end{aligned} \tag{7.11}$$

$$\begin{aligned} w_{f4}(N+1) - w_{f4}(N) = \Big\{ &(c_F - c_T)\,[R_Y(N+1) - R_Y(N)] \\ &- (c_T - c_Y)\,[h_X(N+2) - h_X(N+1)]\,\Big\} A_f(N+1), \end{aligned} \tag{7.12}$$

$$\begin{aligned} w_{f5}(N+1) - w_{f5}(N) = \Big\{ &(c_F - c_T)\,[r_Y(N+2) - r_Y(N+1)] \\ &- (c_T - c_Y)\Big[\,(1 - r_Y(N+2))\,h_X(N+2) \\ &- (1 - r_Y(N+1))\,h_X(N+1)\Big]\Big\} A_f(N+1), \end{aligned} \tag{7.13}$$

$$\begin{aligned} w_{f6}(N+1) - w_{f6}(N) = \Big\{ &(c_F - c_T)\Big[r_Y(N+2)\,(1 - h_X(N+2)) \\ &- r_Y(N+1)\,(1 - h_X(N+1))\,\Big] \\ &- (c_T - c_Y)\,[h_X(N+2) - h_X(N+1)]\,\Big\} A_f(N+1). \end{aligned} \tag{7.14}$$

In Model 1, if $R_Y(N+1) > R_Y(N)$, $R_Y(N+1)H_X(N+1) \geq R_Y(N)H_X(N)$ and $H_X(N+1) \leq H_X(N)$, then $w_{f1}(N+1) > w_{f1}(N)$ for all N under $c_T \geq c_Y$. Similarily, if $r_Y(N+1) > r_Y(N)$ and $H_X(N+1) \leq H_X(N)$, then $w_{f2}(N+1) >$

$w_{f2}(N)$. If $R_Y(N+1) > R_Y(N)$, $R_Y(N+1)H_X(N+1) \leq R_Y(N)H_X(N)$ and $H_X(N+1) \leq H_X(N)$, then $w_{f3}(N+1) > w_{f3}(N)$. If $R_Y(N+1) > R_Y(N)$ and $h_X(N+1) \leq h_X(N)$, then $w_{f4}(N+1) > w_{f4}(N)$. If $r_Y(N+1) > r_Y(N)$, $r_Y(N+1)h_X(N+1) \geq r_Y(N)h_X(N)$ and $h_X(N+1) \leq h_X(N)$, then $w_{f5}(N+1) > w_{f5}(N)$. If $r_Y(N+1) > r_Y(N)$, $r_Y(N+1)h_X(N+1) \leq r_Y(N)h_X(N)$ and $h_X(N+1) \leq h_X(N)$, then $w_{f6}(N+1) > w_{f6}(N)$. From Lemma 7.1, the results hold. □

Lemma 7.7. *Suppose that the failure time Y is strictly IFR under Assumption 1. Then, additional necessary conditions of strictly increasing $w_{lj}(n_0)$ ($j = 1, 2, \ldots, 6$) are given by*

Model 1: $(c_F - c_T)\{R_Y(N)\hat{H}_X(N) - R_Y(N+1)\hat{H}_X(N+1)\} > (c_T - c_Y)\{\hat{H}_X(N) - \hat{H}_X(N+1)\}$,

Model 2 & 5: $(c_F - c_T)\{r_Y(N+2) - r_Y(N+1)\} > (c_T - c_Y)\{\hat{H}_X(N) - \hat{H}_X(N+1)\}$,

Model 3 & 4: $\{1 + R_Y(N+1)\}\hat{H}_X(N+1) \geq \{1 + R_Y(N)\}\hat{H}_X(N)$,

Model 6: $R_Y(N+1)\hat{H}_X(N+1) \geq R_Y(N)\hat{H}_X(N)$, $\{1 + R_Y(N+1)\}\hat{H}_X(N+1) \geq \{1 + R_Y(N)\}\hat{H}_X(N)$.

Lemma 7.8. *Suppose that the failure time Y is strictly IFR and that the arrival time of opportunity X is DHR under Assumption 1. Then, additional necessary conditions of strictly increasing $w_{fj}(N, \beta)$ ($j = 1, 2, \ldots, 6$) are given by*

Model 1: $R_Y(N+1)H_X(N+1) \geq R_Y(N)H_X(N)$,

Model 2: *None,*

Model 3: $R_Y(N+1)H_X(N+1) \leq R_Y(N)H_X(N)$,

Model 4: *None,*

Model 5: $r_Y(N+1)h_X(N+1) \geq r_Y(N)h_X(N)$,

Model 6: $r_Y(N+1)h_X(N+1) \leq r_Y(N)h_X(N)$.

Proof. Taking further difference of $w_{fj}(N \mid \beta)$, we can obtain

$$\begin{aligned} w_{f1}(N+1 \mid \beta) - w_{f1}(N \mid \beta) = \Bigg[& \frac{(c_F - c_T)\Big[R_Y(N+1)\left[1 + H_X(N+1)\right]}{1-\beta} \\ & - \frac{R_Y(N)\left[1 + H_X(N)\right]\Big]}{1-\beta} \\ & - \frac{(c_T - c_Y)\left[H_X(N+1) - H_X(N)\right]}{1-\beta}\Bigg] \\ & \times \left[1 - L_f(N+1)\right], \end{aligned} \tag{7.15}$$

$$\begin{aligned} w_{f2}(N+1 \mid \beta) - w_{f2}(N \mid \beta) = \Bigg[& \frac{\beta\,(c_F - c_T)\left[r_Y(N+2) - r_Y(N+1)\right]}{1-\beta} \\ & - \frac{(c_T - c_Y)\left[H_X(N+1) - H_X(N)\right]}{1-\beta}\Bigg] \\ & \times \left[1 - L_f(N+1)\right], \end{aligned} \tag{7.16}$$

$$\begin{aligned} w_{f3}(N+1 \mid \beta) - w_{f3}(N \mid \beta) = \Bigg[& \frac{(c_F - c_T)\left[R_Y(N+1) - R_Y(N)\right]}{1-\beta} \\ & - \frac{(c_T - c_Y)\Big[H_X(N+1)\left[R_Y(N+1) + 1\right]}{1-\beta} \\ & - \frac{H_X(N)\left[R_Y(N) + 1\right]\Big]}{1-\beta}\Bigg] \\ & \times \left[1 - L_f(N+1)\right], \end{aligned} \tag{7.17}$$

$$\begin{aligned} w_{f4}(N+1 \mid \beta) - w_{f4}(N \mid \beta) = \Bigg[& \frac{(c_F - c_T)\left[r_Y(N+1) - r_Y(N)\right]}{1-\beta} \\ & - \frac{\beta\,(c_T - c_Y)\left[h_X(N+2) - h_X(N+1)\right]}{1-\beta}\Bigg] \\ & \times \left[1 - L_f(N+1)\right], \end{aligned} \tag{7.18}$$

$$
\begin{aligned}
w_{f5}(N+1 \mid \beta) - w_{f5}(N \mid \beta) = &\Bigg[\frac{\beta\,(c_F - c_T)\,[r_Y(N+2) - r_Y(N+1)]}{1-\beta} \\
&- \frac{\beta\,(c_T - c_Y)\,\Big[h_X(N+2)\,[1 - r_Y(N+2)]}{1-\beta} \\
&- \frac{h_X(N+1)\,[1 - r_Y(N+1)]\Big]}{1-\beta}\Bigg] \\
&\times [1 - L_f(N+1)]\,,
\end{aligned} \tag{7.19}
$$

$$
\begin{aligned}
w_{f6}(N+1 \mid \beta) - w_{f6}(N \mid \beta) = &\Bigg[\frac{\beta\,(c_F - c_T)\,\Big[r_Y(N+2)\,[1 - h_X(N+2)]}{1-\beta} \\
&- \frac{r_Y(N+1)\,[1 - h_X(N+1)]\Big]}{1-\beta} \\
&- \frac{\beta\,(c_T - c_Y)\,[h_X(N+2) - h_X(N+1)]}{1-\beta}\Bigg] \\
&\times [1 - L_f(N+1)]\,.
\end{aligned} \tag{7.20}
$$

□

Lemma 7.9. *Suppose that the failure time Y is strictly increases under Assumption 1. Then, additional necessary condition of strictly increasing $w_{lj}(N \mid \beta)(j = 1, \cdots, 6)$ are given in*

Model 1: $(c_F - c_T)\{R_Y(N)\hat{H}_X(N) - R_Y(N+1)\hat{H}_X(N+1)\} > (c_T - c_Y)\{\hat{H}_X(N) - \hat{H}_X(N+1)\}$,

Model 2 & 5: $(c_F - c_T)\{r_Y(N+2) - r_Y(N+1)\} > (c_T - c_Y)\{\hat{H}_X(N) - \hat{H}_X(N+1)\}$,

Model 3 & 4: $\{1 + R_Y(N+1)\}\hat{H}_X(N+1) \geq \{1 + R_Y(N)\}\hat{H}_X(N)$,

Model 6: $R_Y(N+1)\hat{H}_X(N+1) \geq R_Y(N)\hat{H}_X(N)$, $\{1 + R_Y(N+1)\}\hat{H}_X(N+1) \geq \{1 + R_Y(N)\}\hat{H}_X(N)$.

From Lemma 7.2, we can get the following lemma.

Lemma 7.10. *The function $R_Y(N_0)$ $(r_X(N_0))$ is strictly increasing (decreasing) in N_0, if and only if $H_Y(N_0, N_1)$ $(h_Y(N_0, N_1))$ is strictly increasing (decreasing) in N_0.*

Proof. Refer to Lemma 7.2. □

Lemma 7.11. *Suppose that the failure time Y is strictly increases under Assumption 1. Then, additional necessary condition of strictly increasing $w_{fj}(N_0 \mid N_1)(j = 1, \cdots, 6)$ are given in*

Model 1: $\left[r_Y(N_0+1, N_1) - \frac{q(1-q)^{N_1-N_0-3} f_Y(N_1)}{A_{f0}(N_0+1|N_1)}\right]$
$- \left[r_Y(N_0, N_1) - \frac{q(1-q)^{N_1-N_0-2} f_Y(N_1)}{A_{f0}(N_0|N_1)}\right] > 0,$

Model 2: *None,*

Model 3: *None,*

Model 4: *None,*

Model 5: *None,*

Model 6: $\left[H_Y(N_0+2, N_1-1) - \frac{q(1-q)^{N_1-N_0-2} f_Y(N_1)}{A_{f0}(N_0+1|N_1)}\right]$
$- \left[H_Y(N_0+1, N_1-1) - \frac{q(1-q)^{N_1-N_0-1} f_Y(N_1)}{A_{f0}(N_0|N_1)}\right] > 0.$

Proof. Refer to Lemma 7.8. □

Lemma 7.12. *Suppose that the failure time Y is strictly IFR. Then, additional necessary conditions for strictly increasing $w_{lj}(N_1 \mid N_0)$ $(j = 1, 2, \ldots, 6)$ are given by*

Model 1: $(c_F - c_T)\{R_Y(n_1)\hat{H}(N_0, N_1) - R_Y(N_1+1)\hat{H}(N_0, N_1+1)\} > (c_T - c_Y)\{\hat{H}(N_0, N_1) - \hat{H}(N_0, N_1+1)\}$,

Model 2 & 5: $(c_F - c_T)\{r_Y(n_1+2) - r_Y(n_1+1)\} > (c_T - c_Y)\{\hat{H}(N_0, N_1) - \hat{H}(N_0, N_1+1)\}$,

Model 3 & 4: $\{1 + R_Y(n_1+1)\}\hat{H}(N_0, N_1+1) \geq \{1 + R_Y(n_1)\}\hat{H}(N_0, N_1)$,

Model 6: $R_Y(n_1+1)\hat{H}(N_0, N_1+1) \geq R_Y(n_1)\hat{H}(N_0, N_1)$, $\{1+R_Y(n_1+1)\}\hat{H}(N_0, N_1+1) \geq \{1 + R_Y(n_1)\}\hat{H}(N_0, N_1)$.

Proof. Refer to Lemma 7.8. □

7.4 Proofs of Theorem

7.4.1 Proof of Theorem 2.1

Proof. Here, we give the proof for Model 1. From the inequality $EC_{a1}(N+1) - EC_{a1}(N) \geq 0$, we get

$$R_Y(N)\sum_{n=1}^{N}\bar{F}_Y(n-1) - F_Y(N-1) \geq \frac{c_T}{c_F - c_T}. \tag{7.21}$$

Let $w_{a1}(N)$ denote the left-hand side of above equation and further taking the difference yields

$$w_{a1}(N+1) - w_{a1}(N) = [R_Y(N+1) - R_Y(N)]\sum_{n=1}^{N+1}\bar{F}_Y(n-1). \tag{7.22}$$

If the failure time Y is strictly IFR, then the function $w_{a1}(N)$ is strictly increasing in N. If $w_{a1}(\infty) > c_T/(c_F - c_T)$, then there exists at least one (at most two) optimal AR time. If $w_{a1}(\infty) \leq c_T/(c_F - c_T)$, the function $EC_{a1}(N)$ is monotonically decreasing. Then the optimal AR time becomes $N^* \to \infty$. On the other hand, if Y is strictly DFR, then the function $EC_{a1}(N)$ is concave in N. Thus, if $EC_{a1}(1) < EC_{a1}(\infty)$, then $N^* = 1$, otherwise, $N^* \to \infty$. The proof for Model 2 is similar to that for Model 1. □

7.4.2 Proof of Theorem 2.2

Proof. Here, we give the proof for Model 1. From the inequality $EC_{o1}(N+1) - EC_{o1}(N) \geq 0$, we get

$$H_Y(N)A_{o1}(N) - B_{o1}(N) \geq \frac{c_Y}{c_F - c_Y}. \tag{7.23}$$

Let $w_{o1}(N)$ denote the left-hand side of above equation and further taking the difference yields

$$w_{o1}(N+1) - w_{o1}(N) = [H_Y(N+1) - H_Y(N)]\,A_{o1}(N+1). \tag{7.24}$$

If the failure time Y is strictly IFR, then the function $w_{o1}(N)$ is strictly increasing in N. If $w_{o1}(\infty) > c_Y/(c_F - c_Y)$, then there exists at least one (at most two) optimal time limit N^*. If $w_{o1}(\infty) \leq c_Y/(c_F - c_Y)$, the function $EC_{o1}(N)$ is monotonically decreasing. Then the optimal preventive replacement time limit becomes $N^* \to \infty$. On the other hand, if Y is strictly DFR, then the function $EC_{o1}(N)$ is concave in N. Thus, if $EC_{o1}(1) < EC_{o1}(\infty)$, then $N^* = 1$, otherwise, $N^* \to \infty$. The proof for Model 2 is similar to that for Model 1. □

7.4.3 Proof of Theorem 2.13

Proof. (i) Taking the difference of $\Psi_j(N)$, we have

$$\Psi_j(N+1) - \Psi_j(N) = [m_j(N+2) - m_j(N+1)]\, A(N+1) \tag{7.25}$$

If $m_j(N+1)$ is non-increasing, $\Psi_j(N+1) - \Psi_j(N) \le 0$. Since $\Psi_j(N) < b_j$, $EC_j(\underline{N})$ is a decreasing function.

(ii) If $m_j(N+1)$ increases strictly in $\left[\underline{N}, \overline{N}\right]$, then $\Psi_j(N+1)$ is a strictly increasing function. Since $\Psi_j(\underline{N}) < b_j$ and $\Psi_j(\underline{N}) > b_j$, $EC_j(N)$ must have at least one (at most two) minimal value, satisfying Eq. (2.57). Equivalently, we can obtain the Eqs. (2.58) and (2.59).

(iii) If $\Psi_j(N) < b_j$ for all $N > \underline{N}$, then $EC_j(N+1) - EC_j(N) < 0$. Hence, $EC_j(N)$ is decreasing for $N > \underline{N}$.

(iv) It is noted that

$$\begin{aligned}\Psi_j(N) - \Psi_j(\underline{N}) = &\sum_{n=\underline{N}}^{N} [m_j(N+1) - m_j(n+1)]\, a(n) \\ &+ [m_j(N+1) - m_j(\underline{N})]\, A(\underline{N}).\end{aligned} \tag{7.26}$$

Hence, if $m_j(N+1)$ goes to infinity, then $\Psi(N)$ goes to infinity. For a large N, we can get $m_j(N+1) - EC_j(N) > 0$. So, if $\lim_{N\to\infty} m_j(N) = \infty$, then $\lim_{N\to\infty} m_j(N) > \lim_{N\to\infty} EC_j(N)$. Next, we can see that $EC_j(N)$ converges to m from the following inequality

$$b_j - ma + \sum_{n=0}^{\infty} [m - m_j(N)]\, a(N) < 0. \tag{7.27}$$

□

7.4.4 Proof of Theorem 3.1

Proof. In Eq. (3.10), we have

$$w_{fj}(N) = \Delta B_{fj}(N) A_f(N) - B_{fj}(N) \Delta A_f(N), \tag{7.28}$$

where

$$\Delta A_f(N) = \{A_f(N+1) - A_f(N)\}/\{\bar{F}_Y(N)\bar{G}_X(N)\} = 1 \tag{7.29}$$

and

$$\Delta B_{fj}(N) = \{B_{fj}(N+1) - B_{fj}(N)\}/\{\bar{F}_Y(N)\bar{G}_X(N)\}. \tag{7.30}$$

Here we briefly sketch the proof for Model 1. From Eqs (3.2) and (3.3), we get

$$\Delta B_{f1}(N) = (c_F - c_T)R_Y(N)\,(1 + H_X(N)) - (c_T - c_Y)H_X(N). \tag{7.31}$$

Further difference of $w_{f1}(N)$ yields Eq. (7.9). When $w_{f1}(N+1) > q_{f1}(N)$, the function $EC_{f1}(N)$ is strictly quasi-convex in N. Further, if $w_{f1}(1) < 0$ and $q_{f1}(\infty) > 0$, then there exists at least one (at most two) optimal scheduled preventive replacement time N^* which satisfies $w_{f1}(N^*-1) < 0$ and $q_{f1}(N^*) \geq 0$. Conversely, if $w_{f1}(\infty) \leq 0$ and $w_{f1}(1) \geq 0$, then the function $EC_{f1}(N)$ is monotonically decreasing and increasing, respectively, so that the optimal scheduled preventive replacement time for Model 1 is $N^* \to \infty$ or $N^* = 1$. If $w_{f1}(N+1) \leq w_{f1}(N)$ for an arbitrary $N \geq 1$, then the function $EC_{f1}(N)$ is quasi-concave in N. Thus, if $EC_{f1}(1) < EC_{f1}(\infty)$, then $N^* = 1$, otherwise, $N^* \to \infty$. □

7.4.5 Proof of Theorem 3.5

Proof. Here, we only take the example of Model 1. Taking further difference of Eq. (3.49) yields Eq (7.15). When $w_{f1}(N+1 \mid \beta) > w_{f1}(N \mid \beta)$, the function $TC_{f1}(N, \beta)$ is strictly convex in N. if $w_{f1}(N \mid \beta) > 0$, then there exists at least one (at most two) optimal prescheduled preventive replacement time N^* which satisfies $w_{f1}(N^*-1 \mid \beta) < 0$ and $w_{f1}(N^* \mid \beta) \geq 0$. Conversely, if $w_{f1}(N \mid \beta) \leq 0$, then the function is monotonically decreasing. Then the optimal prescheduled preventive replacement time is given by $N^* \to \infty$.

On the other hand, if $w_{f1}(N+1 \mid \beta) \leq w_{f1}(N \mid \beta)$, then the function $TC_{f1}(N \mid \beta)$ is concave in N for a fixed β. Thus, if $TC_{f1}(1 \mid \beta) < TC_{f1}(\infty \mid \beta)$, then $N^* = 1$, otherwise, $N^* \to \infty$. □

7.4.6 Proof of Theorem 4.2

Proof. Taking the further difference of $w(T \mid S)$ with respect to T, we can obtain

$$\frac{dw(T \mid S)}{dT} = \left[(c_F - c_T)r(T)' + (c_T - c_Y)\hat{H}(T-S)'\right] A_l(S,T). \tag{7.32}$$

If $(c_F - c_T)r(T)' + (c_T - c_Y)\hat{H}(T-S)' > 0$, $dw(T \mid S)/dT > 0$. Otherwise, $dw(T \mid S)/dT \leq 0$. If $w(\infty \mid S) > 0$, then exists a finite and unique optimal T^* satisfying $w(T \mid S) = 0$. Otherwise, $w(T \mid S) \leq 0$ and $EC_l(T \mid S)$ is a decreasing function of T. $\square$

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[J-1] J. Wu, C. Qian and T. Dohi, "Optimal opportunity-based age replacement policies in discrete time," Reliability Engineering and System Safety, 241 (2024): 109587.

[J-2] J. Wu, C. Qian and T. Dohi, "Two Discrete-time Age-based Replacement Problems with/without Discounting," International Journal of Mathematical, Engineering and Management Sciences 9(3) (2024), 385-410.

[J-3] J. Wu, C. Qian, T. Dohi and H. Okamura, "An NPV analysis of opportunity-based age replacement model," International Journal of Reliability and Safety. (in Press). 2024, DOI: 10.1504/IJRS.2024.10063333.

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Book Chapter

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