広島大学学位請求論文

Tropical lifting problem for the intersection of plane curves
(トロピカル平面曲線の交わりの実現問題)

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主論文

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Tropical lifting problem for the intersection of plane curves

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Abstract
Given a tropical divisor D in the intersection of two tropical plane curves, we study when it can be realized as the tropicalization of the intersection of two algebraic curves, and give a sufficient condition. It is shown that under a certain condition involving a graph determined by these tropical curves, we can algorithmically find algebraic curves such that the tropicalization of their intersection is D.

Keywords Tropical geometry · Intersection theory · Lifting problem · Divisor theory

Mathematics Subject Classification Primary 14T05; Secondary 14H50

1 Introduction

In this paper, let k be a fixed algebraically closed field with a nontrivial valuation $\mathrm{val} : k \to \mathbb{R} \cup \{+\infty\}$. A tropical plane curve is obtained by the tropicalization of an algebraic curve in $(k^*)^2$. Here, the tropicalization is defined using the following map:

$$\begin{aligned} \mathrm{trop} : (k^*)^2 &\to \mathbb{R}^2 \\ (x, y) &\mapsto (-\mathrm{val}(x), -\mathrm{val}(y)). \end{aligned}$$

Let $f = \sum_{ij} c_{ij} x^i y^j \in k[x^{\pm 1}, y^{\pm 1}]$ be given. For a given tropical divisor D on the tropical plane curve $\mathrm{Trop}(V(f))$, it has been considered whether D can be obtained by the tropicalization of the intersection of two algebraic curves (Brugalle 2012; Len and Satriano 2020; Morrison 2015; Osserman and Payne 2013; Osserman and Rabinoff 2013). This kind of problem is called a tropical lifting problem or a tropical realization problem. In this paper, we give a sufficient condition involving a graph determined by given tropical curves for the lifting problem for the intersection of curves.

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1.1 Tropical lifting problem

First, we explain what is known about tropical lifting problems for the intersection of two tropical plane curves. Let $\mathcal{F}$ and $\mathcal{G}$ be bivariate tropical polynomials. They define the tropical plane curves $V(\mathcal{F})$ and $V(\mathcal{G})$ (see Sect. 2).

Definition 1.1 We say that two tropical plane curves Γ_1 and Γ_2 meet *properly* at a point p if p is an isolated point in $\Gamma_1 \cap \Gamma_2$. We define $\mathcal{PI}(\mathcal{F}, \mathcal{G})$ as the multiset of the points p at which $V(\mathcal{F})$ and $V(\mathcal{G})$ meet properly, with the local intersection numbers as multiplicities. We also write $\mathcal{PI}(\mathrm{trop}(f), \mathrm{trop}(g))$ as $\mathcal{PI}(f, g)$ (for the tropicalization of a Laurent polynomial, see Definition 2.5).

Proper intersections are the simplest intersections of tropical plane curves. Tropical lifting problems of proper intersections are studied in Osserman and Rabinoff (2013) (see Theorem 2.21). For algebraic curves $C_1, C_2 \subset (k^*)^2$, if the tropical curves $\mathrm{Trop}(C_1)$ and $\mathrm{Trop}(C_2)$ meet properly, then $\mathrm{trop}(C_1 \cap C_2)$ is equal to the intersection $\mathrm{Trop}(C_1) \cap \mathrm{Trop}(C_2)$, considered with multiplicities. Thus, we have to consider the case where $\mathrm{Trop}(C_1) \cap \mathrm{Trop}(C_2)$ does not consist of isolated points, i.e., contains 1-dimensional components.

Definition 1.2 A (tropical) *divisor* on a tropical curve Γ is a finite sum $D = \sum n_i P_i$, where $P_i \in \Gamma$ and $n_i \in \mathbb{Z}$.

Definition 1.3 A *tropical rational function* on a tropical curve Γ is a continuous function $\psi : \Gamma \to \mathbb{R}$ such that its restriction to any edge of Γ is a piecewise linear function with integer slopes, i.e., piecewise $\mathbb{Z}$-affine, and with only finitely many pieces. The divisor of ψ is $\sum_{P\in\Gamma} \mathrm{ord}_P(\psi)P$, where $\mathrm{ord}_P(\psi)$ is (-1) times the sum of the outgoing slopes of ψ at P. We write (ψ) for the divisor of ψ. If D and E are divisors such that $D - E = (\psi)$ for some tropical rational function ψ, we say that D and E are linearly equivalent. We define the support of ψ as $\mathrm{Supp}(\psi) = \overline{\{P \in \Gamma \mid \psi(P) \neq 0\}}$.

Morrison showed the following necessary condition for the realizability of a tropical divisor as the intersection of curves.

Theorem 1.4 (Morrison 2015, Theorem 1.2) *Let Γ_1 and Γ_2 be tropical plane curves such that Γ_1 is smooth (Definition 2.16). Let E be the stable intersection divisor (Definition 2.20) of Γ_1 and Γ_2, and let $D = \sum n_i P_i$ ($n_i \in \mathbb{Z}_{\geq 0}$) be a divisor on $\Gamma_1 \cap \Gamma_2$. Assume that there exist algebraic curves $C_1, C_2 \subset (k^*)^2$ without common irreducible components such that* $\mathrm{Trop}(C_1) = \Gamma_1$, $\mathrm{Trop}(C_2) = \Gamma_2$, *and* $\mathrm{trop}(C_1 \cap C_2) = D$ *as multisets. Then, there exists a tropical rational function ψ on Γ_1 such that $(\psi) = D - E$ and* $\mathrm{Supp}(\psi) \subset \Gamma_1 \cap \Gamma_2$.

In Morrison (2015), a conjecture on the converse is also presented.

Problem 1.5 (Morrison 2015, *Conjecture 3.3*) Let ψ be a tropical rational function on a tropical curve $\mathrm{Trop}(V(f))$ such that $\mathrm{Supp}(\psi) \subset \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ and $(\psi) = D - E$, where E is the stable intersection divisor and $D = \sum n_i P_i$ ($n_i \in \mathbb{Z}_{\geq 0}$) is a divisor on $\Gamma_1 \cap \Gamma_2$ such that each coordinate of P_i is in the value group of k. Then is it possible to find $f', g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that $\mathrm{Trop}(V(f')) = \mathrm{Trop}(V(f))$, $\mathrm{Trop}(V(g')) = \mathrm{Trop}(V(g))$ and $\mathrm{trop}(V(f', g')) = D$?

This was answered in the negative. See Len and Satriano (2020, Theorem 5.2) for a tropical self-intersection case, and Brugalle (2012, Lemma 3.15) for a non self-intersection case. On the other hand, it would be useful to find sufficient conditions for the realizability. The purpose of this paper is to give a sufficient condition involving a certain graph. We introduce several notations before explaining the setting of the main problem.

Definition 1.6 Let Γ_1 and Γ_2 be tropical plane curves. Let $\mathfrak{K}$ be a connected component of $\Gamma_1 \cap \Gamma_2$. The *intersection multiplicity* of $\Gamma_1 \cap \Gamma_2$ on $\mathfrak{K}$ is defined as the sum of the multiplicities of the stable intersection points on $\mathfrak{K}$ (Definitions 2.19 and 2.20).

Let us introduce notations on the second simplest components of the intersection of tropical curves.

Definition 1.7 We define $\mathcal{R}_1(\mathcal{F}, \mathcal{G})$ as the set of rays L satisfying the following:

- L is a connected component of the intersection $V(\mathcal{F}) \cap V(\mathcal{G})$.
- The intersection multiplicity of $V(\mathcal{F})$ and $V(\mathcal{G})$ on L is 1.
- Each 1-dimensional cell of $V(\mathcal{F})$ or $V(\mathcal{G})$ which has a 1-dimensional intersection with L and contains the endpoint of L as its vertex has weight 1.

Also, we define $\mathcal{LS}_2(\mathcal{F}, \mathcal{G})$ as the set of (bounded) line segments L satisfying the following:

- L is a connected component of the intersection $V(\mathcal{F}) \cap V(\mathcal{G})$.
- The intersection multiplicity of $V(\mathcal{F})$ and $V(\mathcal{G})$ on L is 2.
- Each 1-dimensional cell of $V(\mathcal{F})$ or $V(\mathcal{G})$ which has a 1-dimensional intersection with L and contains an endpoint of L as its vertex has weight 1.

We write $\mathcal{L}_s(\mathcal{F}, \mathcal{G}) := \mathcal{R}_1(\mathcal{F}, \mathcal{G}) \cup \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$. It turns out that any edge of $V(\mathcal{F})$ or $V(\mathcal{G})$ that meets $L \in \mathcal{L}_s(\mathcal{F}, \mathcal{G})$ has weight 1 and any vertex contained in L is smooth (see Lemma 3.3). For Laurent polynomials $f, g \in k[x^{\pm 1}, y^{\pm 1}]$, we also write $\mathcal{R}_1(f, g)$, $\mathcal{LS}_2(f, g)$ and $\mathcal{L}_s(f, g)$ for $\mathcal{R}_1(\mathrm{trop}(f), \mathrm{trop}(g))$, $\mathcal{LS}_2(\mathrm{trop}(f), \mathrm{trop}(g))$ and $\mathcal{L}_s(\mathrm{trop}(f), \mathrm{trop}(g))$, respectively.

Thus, the connected components of $V(\mathcal{F}) \cap V(\mathcal{G})$ are points in $\mathcal{PI}(\mathcal{F}, \mathcal{G})$, elements of $\mathcal{L}_s(\mathcal{F}, \mathcal{G})$, and possibly a number of other 1-dimensional sets.

We will see that, if $L \in \mathcal{R}_1(f, g)$, then there are at most one point in the intersection $\mathrm{trop}(V(f, g)) \cap L$ (Corollary 4.2). Thus, in this paper, we will consider the following condition.

Definition 1.8 The condition $(*)$ on a divisor D on $\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ is the following:

- $D = \sum n_i P_i$ $(n_i \geq 0)$.
- Each coordinate of P_i is in the value group of k.
- There exists a tropical rational function ψ on the tropical curve $\mathrm{Trop}(V(f))$ such that $\mathrm{Supp}(\psi) \subset \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ and $(\psi) = D - E$, where E is the stable intersection divisor of $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$.
- For $L \in \mathcal{R}_1(f, g)$, $\deg(D|_L) = 1$.

Note that this condition is natural in view of Theorem 1.4.

Notation 1.9 For a tropical plane curve Γ, we write $\Sigma^{(n)}(\Gamma)$ for the set of the n-dimensional cells of Γ (see Theorem 2.8). For a tropical polynomial $\mathcal{F} \in \mathbb{T}[x^{\pm 1}, y^{\pm 1}]$, we write $\Delta_{\mathcal{F}}^{(n)}$ for the set of the n-dimensional cells of $\Delta_{\mathcal{F}}$, where $\Delta_{\mathcal{F}}$ is the dual subdivision of the Newton polygon of $\mathcal{F}$ (see Definition 2.8). For a Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$, we write Δ_f and $\Delta_f^{(n)}$ for $\Delta_{\mathrm{trop}(f)}$ and $\Delta_{\mathrm{trop}(f)}^{(n)}$, respectively.

Note that the intersection multiplicity at an endpoint of a ray or a line segment $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ must be at least 1, and hence the tropical curves $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$ have no vertices in the interior of L (see Lemma 3.3 for details). Thus, we can define the following maps.

Definition 1.10 We define maps ϕ_i $(i = 1, 2)$ as follows:

$$\begin{aligned}
\phi_1 : \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G}) &\to \Sigma^{(1)}(V(\mathcal{F})) \\
L &\mapsto \text{the 1-dimensional cell of } V(\mathcal{F}) \text{ containing } L, \\
\phi_2 : \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G}) &\to \Sigma^{(1)}(V(\mathcal{G})) \\
L &\mapsto \text{the 1-dimensional cell of } V(\mathcal{G}) \text{ containing } L,
\end{aligned}$$

and we define maps Φ_i $(i = 1, 2)$ as follows:

$$\begin{aligned}
\Phi_1 : \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G}) &\to \Delta_{\mathcal{F}}^{(1)} \\
L &\mapsto \text{the 1-simplex of } \Delta_{\mathcal{F}} \text{ corresponding to } \phi_1(L), \\
\Phi_2 : \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G}) &\to \Delta_{\mathcal{G}}^{(1)} \\
L &\mapsto \text{the 1-simplex of } \Delta_{\mathcal{G}} \text{ corresponding to } \phi_2(L).
\end{aligned}$$

Notation 1.11 Let $a, b \in \mathbb{R}^2$ $(a \neq b)$ be points such that the line segment $\overline{ab}$ has a rational slope. Then, there is a primitive integer vector $v \in \mathbb{Z}^2$ which has the same slope as $\overline{ab}$. The *lattice length* of $\overline{ab}$ is the ordinary length of $\overline{ab}$ divided by the ordinary length of v. When $a = b$, we define the lattice length of $\overline{ab}$ as 0. We write $\mathrm{dist}(a, b)$ for the lattice length of $\overline{ab}$. We note that dist does not satisfy the metric inequality.

On a line segment $L \in \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$, a divisor D satisfying $(*)$ can be described as follows.

Lemma 1.12 *Let $L \in \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$ be a line segment. Let D be a divisor satisfying $(*)$. Then, $D|_L = P_1 + P_2$ for some $P_1, P_2 \in L$, and we have* $\mathrm{dist}(P_+, P_1) = \mathrm{dist}(P_-, P_2)$ *and* $\mathrm{dist}(P_+, P_2) = \mathrm{dist}(P_-, P_1)$, *where P_+ and P_- are the endpoints of L.*

Proof Straightforward from the fact that a tropical rational function ψ on $V(\mathcal{F})$ as in $(*)$ takes 0 at P_+ and P_-. □

Notation 1.13 Let a tropical divisor D satisfy $(*)$. For a line segment $L \in \mathcal{LS}_2(f, g)$, we define $\mathrm{dist}(D|_L, E|_L) = \min\{\mathrm{dist}(P_+, P_1), \mathrm{dist}(P_+, P_2)\}$, where $D|_L = P_1 + P_2$ and $E|_L = P_+ + P_-$. This is well-defined by Lemma 1.12. Also, when $L \in \mathcal{R}_1(f, g)$, we write $\mathrm{dist}(D|_L, E|_L)$ for the lattice length of the distance of the point in $D|_L$ and the endpoint of L.

By analogy with plane algebraic curves, it is a natural setting to fix f and change g in realizing D, i.e., the zeros of ψ. For example, in Len and Satriano (2020), a tropical curve Γ and an algebraic curve C satisfying $\mathrm{Trop}(C) = \Gamma$ are fixed and $\mathrm{trop}(C \cap C')$ are studied for curves C' with $\mathrm{Trop}(C') = \Gamma$. Also, it would be useful to study whether it is possible to realize a certain part of D. Let $\mathcal{L}'_{\mathrm{s}}$ be a subset of $\mathcal{L}_{\mathrm{s}}(f, g)$ and $\mathcal{PI} := \mathcal{PI}(f, g)$ the proper intersections. Let $D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}$ denote the restriction of D to the union of the elements of $\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}$. Then, when is it possible to realize $D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}$, i.e., does there exist a Laurent polynomial $g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that $\mathrm{Trop}(V(g')) = \mathrm{Trop}(V(g))$ and $\mathrm{trop}(V(f, g'))|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}} = D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}$?

1.2 Main result

As a partial answer to the above question, our main theorems give sufficient conditions for the realizability. To state the main theorems, we introduce terminologies on trees.

Notation 1.14 It is well known that any two vertices of a tree T are connected by a unique simple path in T (see Diestel 2017, Theorem 1.5.1). We write pTq for the simple path between two vertices p and q in T.

Definition 1.15 Let T be a tree and $\leq$ a total ordering on the set of its vertices. Let p_0 denote the smallest vertex for $\leq$. The order $\leq$ is called *normal* if $p \in p_0Tq$ implies $p \leq q$.

Definition 1.16 For lattice points $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^2$ such that $\mathbf{j} - \mathbf{i}$ is primitive and a tropical polynomial $\mathcal{F} = \bigoplus_{\mathbf{i} \in \mathbb{Z}^2} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ with $\alpha_{\mathbf{i}}, \alpha_{\mathbf{j}} \neq -\infty$, where $\mathbf{x}^{(i_1, i_2)}$ denotes $x^{i_1} y^{i_2}$, we define $\mu_n(\mathcal{F}; \overline{\mathbf{ij}})$ ($n \in \mathbb{Z}$) and $\mu(\mathcal{F}; \overline{\mathbf{ij}})$ by

$$\mu_n(\mathcal{F}; \overline{\mathbf{ij}}) := -\alpha_{\mathbf{i}+n(\mathbf{j}-\mathbf{i})} + \alpha_{\mathbf{i}} + n(\alpha_{\mathbf{j}} - \alpha_{\mathbf{i}}),$$
$$\mu(\mathcal{F}; \overline{\mathbf{ij}}) := \min\{\mu_n(\mathcal{F}; \overline{\mathbf{ij}}) \mid n \in \mathbb{Z} \setminus \{0, 1\}\}.$$

For $f = \sum_{\mathbf{i}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in k[x^{\pm 1}, y^{\pm 1}]$ with $c_{\mathbf{i}}, c_{\mathbf{j}} \neq 0$, we write $\mu_n(f; \overline{\mathbf{ij}})$ and $\mu(f; \overline{\mathbf{ij}})$ for $\mu_n(\mathrm{trop}(f); \overline{\mathbf{ij}})$ and $\mu(\mathrm{trop}(f); \overline{\mathbf{ij}})$, respectively (Fig. 1).

Note that μ_n depends on the orientation of $\overline{\mathbf{ij}}$ but μ does not, and that $\mu_0(\mathcal{F}; \overline{\mathbf{ij}}) = \mu_1(\mathcal{F}; \overline{\mathbf{ij}}) = 0$.

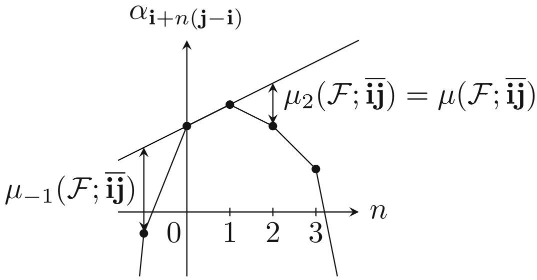


Fig. 1 $\mu_n(\mathcal{F}; \overline{\mathbf{ij}})$ and $\mu(\mathcal{F}; \overline{\mathbf{ij}})$

Remark 1.17 Let $\mathcal{F} = \bigoplus_{\mathbf{i} \in \mathbb{Z}^2} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$ be a tropical polynomial, $\overline{\mathbf{ij}}$ a 1-simplex of $\Delta_{\mathcal{F}}$ with $\mathbf{j} - \mathbf{i}$ primitive, and $\overline{L}$ the corresponding edge of $V(\mathcal{F})$ (see Theorem 2.9). Then, for any $P \in \overline{L}$ and $n \in \mathbb{Z} \backslash \{0, 1\}$, we have the following (see Remark 2.12):

$$\alpha_{\mathbf{i}} + \mathbf{i} \cdot P = \alpha_{\mathbf{j}} + \mathbf{j} \cdot P > \alpha_{\mathbf{i}+n(\mathbf{j}-\mathbf{i})} + (\mathbf{i} + n(\mathbf{j} - \mathbf{i})) \cdot P,$$

and hence,

$$-\alpha_{\mathbf{i}+n(\mathbf{j}-\mathbf{i})} + \alpha_{\mathbf{i}} + n(\alpha_{\mathbf{j}} - \alpha_{\mathbf{i}}) > 0,$$

i.e.,

$$\mu_n(\mathcal{F}; \overline{\mathbf{ij}}) > 0.$$

Thus, in this case, we have $\mu(\mathcal{F}; \overline{\mathbf{ij}}) > 0$. In particular, if $L \in \mathcal{L}_{\mathrm{s}}(f, g)$, $P \in L$ and $\overline{\mathbf{ij}} = \Phi_2(L)$, then $\mu(g; \overline{\mathbf{ij}}) > 0$.

The value $\mu(\mathcal{F}; \overline{\mathbf{ij}})$ measures the margin for $\overline{\mathbf{ij}}$ to be a 1-simplex of $\Delta_{\mathcal{F}}$, in a sense.

Now, let us state the main theorems. We consider the following graph theoretic condition which will be crucial in our sufficient conditions.

Definition 1.18 We say that $\mathcal{L}'_{\mathrm{s}}$ is *acyclic* with respect to Φ_2 if the map $\Phi_2|_{\mathcal{L}'_{\mathrm{s}}}$ is injective (see Definition 1.10), i.e., there is no duplication in $\Delta' := \Phi_2(\mathcal{L}'_{\mathrm{s}})$, and the union of the elements of Δ' is a forest.

Remark 1.19 The acyclicity of $\mathcal{L}'_{\mathrm{s}}$ is not directly correlated with acyclicity in $\mathrm{Trop}(V(g))$. Even if $\mathcal{L}'_{\mathrm{s}}$ is acyclic with respect to Φ_2, the union of the corresponding edges of $\mathrm{Trop}(V(g))$ may have cycles (cf. Example 5.1).

The following theorem implies that D can be realized on $\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}$ if $\mathcal{L}'_{\mathrm{s}}$ is acyclic with respect to Φ_2 and D is sufficiently close to E.

Theorem 1.20 (=Theorem 4.6) *Let a divisor D satisfy the condition* $(*)$ *in Definition 1.8. Assume that $\mathcal{L}'_{\mathrm{s}}$ is acyclic with respect to Φ_2 and that for each $L \in \mathcal{L}'_{\mathrm{s}}$, we have* $\mathrm{dist}(D|_L, E|_L) < \mu(g; \Phi_2(L))$. *Then, there exists $g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that* $\mathrm{trop}(g') = \mathrm{trop}(g)$ *and*

$$\mathrm{trop}(V(f, g'))|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}} = D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}.$$

Remark 1.21 In the above theorem, we assume the distance condition that for each $L \in \mathcal{L}'_{\mathrm{s}}$, we have $\mathrm{dist}(D|_L, E|_L) < \mu(g; \Phi_2(L))$. However, we do not know whether this condition is absolutely necessary or not. Therefore, there may be room for omitting this condition.

Imposing a further assumption on $\mathcal{L}'_{\mathrm{s}}$, we may drop the restriction on the distance.

Notation 1.22 For a given set $S \subset \mathbb{R}^n$, we write $\mathrm{Aff}(S)$ for the affine span of S.

Theorem 1.23 (=Theorem 4.7) *Let a divisor D satisfy the condition $(*)$ in Definition 1.8. Assume that $\mathcal{L}'_{\mathrm{s}}$ is acyclic with respect to Φ_2 and that we can number and order the endpoints of the elements of $\Delta' := \Phi_2(\mathcal{L}'_{\mathrm{s}})$ as $p_1 < \cdots < p_n$ so that this order is normal on each tree of the forest and that for each element $\overline{p_i p_j}$ of Δ', its affine span* $\mathrm{Aff}(\overline{p_i p_j})$ *does not contain a point p_l with $l > i, j$. Then, there exists $g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that* $\mathrm{trop}(g') = \mathrm{trop}(g)$ *and*

$$\mathrm{trop}(V(f, g'))|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}} = D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}.$$

The proofs of the theorems proceed as follows. For an element $L \in \mathcal{L}_{\mathrm{s}}(f, g)$, we will give an algorithm to determine $\mathrm{trop}(V(f, g)) \cap L$ (see Lemma 3.8, Definition 3.10 and Proposition 4.1). This algorithm proceeds by constructing a suitable Laurent polynomial in the ideal (f, g) and tells us how to modify g in order to realize D on L. Using this, we will determine the coefficients of g' one by one. In the setting of Theorem 4.7, we use the given ordering. We need the acyclicity condition to maintain the consistency.

Remark 1.24 Let $L \in \mathcal{L}'_{\mathrm{s}}$ and $\Phi_2(L) = \overline{p_i p_j}$. In determining $\mathrm{trop}(V(f, g)) \cap L$ and the coefficient d'_{p_i} of g', the coefficients $d_{\mathbf{i}}$ for $\mathbf{i} \in \mathrm{Aff}(\overline{p_i p_j})$ are essential. This is why Theorem 4.6 (resp. 4.7) requires the condition about the coefficients $d_{\mathbf{i}}$ for $\mathbf{i} \in \mathrm{Aff}(\overline{p_i p_j})$ (resp. about $\mathrm{Aff}(\overline{p_i p_j})$).

Remark 1.25 The condition $p_l \notin \mathrm{Aff}(\overline{p_i p_j})$ $(l > i, j)$ in Theorem 4.7 depends on the ordering, not just on $\mathcal{L}'_{\mathrm{s}}$. For example, the order on the left in Fig. 2 satisfies the condition, but the one on the right does not.

The rest of this paper is organized as follows. Section 2 gives fundamental definitions and facts about tropical curves. In Sect. 3, we show several lemmas concerning properties of $V(\mathcal{F})$ and $V(\mathcal{G})$ in a neighborhood of $L \in \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G})$ and introduce a kind of division procedure for Laurent polynomials over a valuation field. In Sect. 4, we explain how to determine $\mathrm{trop}(V(f, g)) \cap L$ for $L \in \mathcal{L}_{\mathrm{s}}(f, g)$, and prove the main theorems. In the last section, we give several examples concerning the main theorems to illustrate the necessity of the acyclicity condition.

2 Tropical curves

In this section, we recall the basics about tropical plane curves. For details, see Maclagan and Sturmfels (2015). First, we give the definition of the tropical algebra which is essential for studying tropical geometry.

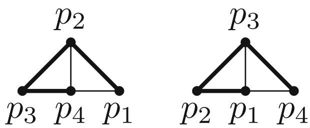


Fig. 2 Two orderings of the endpoints of the elements of $\Delta' = \Phi_2(\mathcal{L}'_{\mathrm{s}})$

Definition 2.1 *(Tropical algebra)* We define $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$. The *tropical algebra* is the triple $(\mathbb{T}, \oplus, \odot)$, where the addition $\oplus$ is defined as the operation that takes the maximum of two numbers and the multiplication $\odot$ is defined as the ordinary addition. We can easily check that $(\mathbb{T}, \oplus, \odot)$ is a semifield.

To define tropical plane curves in terms of tropical algebra, we define tropical polynomials.

Definition 2.2 *(Tropical polynomials)* A *tropical polynomial* $\mathcal{F}$ is an expression of the form

$$\mathcal{F} = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} x_1^{i_1} \dots x_n^{i_n},$$

where $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$ and $\alpha_{\mathbf{i}} \in \mathbb{T}$, and only finitely many of the coefficients $\alpha_{\mathbf{i}}$ are not $-\infty$. We may drop terms with coefficients $-\infty$. A tropical polynomial defines a map from $\mathbb{R}^n$ to $\mathbb{R} \cup \{-\infty\}$ in a natural way:

$$\mathcal{F}(t_1, \dots, t_n) = \max_{\mathbf{i}} (\alpha_{\mathbf{i}} + i_1 t_1 + \dots + i_n t_n).$$

We write $\mathbb{T}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ for the set of all n-variate tropical polynomials, and define the addition and the multiplication in a natural way.

Definition 2.3 *(Tropical hypersurfaces)* Let $\mathcal{F} = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} x_1^{i_1} \dots x_n^{i_n} \neq -\infty$ be a tropical polynomial. The *tropical hypersurface* $V(\mathcal{F})$ defined by $\mathcal{F}$ is the set

$$V(\mathcal{F}) = \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists \mathbf{i} = (i_1, \dots, i_n), \mathbf{j} = (j_1, \dots, j_n) \in \mathbb{Z}^n (\mathbf{i} \neq \mathbf{j}) \text{ s.t.} \\ \alpha_{\mathbf{i}} + i_1 t_1 + \dots + i_n t_n = \alpha_{\mathbf{j}} + j_1 t_1 + \dots + j_n t_n \\ = \mathcal{F}(t_1, \dots, t_n) \end{array} \right\}.$$

If $\mathcal{F} = -\infty$, i.e. all the coefficients of $\mathcal{F}$ are $-\infty$, we define $V(-\infty) = \mathbb{R}^n$. When $n = 2$ and $\mathcal{F} \neq -\infty$, we call $V(\mathcal{F})$ a *tropical plane curve*. Later, we will consider a tropical plane curve as a polyhedral complex endowed with weights on its 1-dimensional cells (see Definition 2.11).

The following map is a bridge between algebraic geometry and tropical geometry.

Definition 2.4 *(Tropicalization map)* We define the *tropicalization map* as follows:

$$\begin{array}{rcl} \mathrm{trop}: & (k^*)^n & \to \mathbb{R}^n \\ & (x_1, \dots, x_n) & \mapsto (-\operatorname{val}(x_1), \dots, -\operatorname{val}(x_n)). \end{array}$$

Definition 2.5 *(Tropicalization of Laurent polynomials)* Let $f = \sum_{\mathbf{i}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. We define the *tropicalization* of f as

$$\mathrm{trop}(f) = \bigoplus_{\mathbf{i}} \mathrm{trop}(c_{\mathbf{i}}) \mathbf{x}^{\mathbf{i}} \left(= \bigoplus_{\mathbf{i}} \left((-\operatorname{val}(c_{\mathbf{i}})) \mathbf{x}^{\mathbf{i}} \right) \right).$$

Notation 2.6 For $A \subset (k^*)^n$, we write $\mathrm{Trop}(A)$ for the closure of $\mathrm{trop}(A)$ in $\mathbb{R}^n$.

Recall that k is an algebraically closed field with a nontrivial valuation.

Theorem 2.7 (Kapranov's Theorem, Einsiedler et al. (2006), Theorem 2.1.1) *Let* $f \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ *be a Laurent polynomial. Then, we have*

$$V(\mathrm{trop}(f)) = \mathrm{Trop}(V(f)).$$

Definition 2.8 (*Dual subdivisions*, (Mikhalkin 2005, Definition 3.10)) Let $\mathcal{F} = \bigoplus_{i,j} \alpha_{ij} x^i y^j$ be a tropical polynomial. We write $\mathrm{Newt}(\mathcal{F}) \subset \mathbb{R}^2$ for the convex hull of the set $\{(i, j) \in \mathbb{Z}^2 \mid \alpha_{ij} \neq -\infty\}$. Let $A_{\mathcal{F}} \subset \mathbb{R}^3$ be the convex hull of the set

$$\{(i, j, \alpha) \in \mathbb{Z}^2 \times \mathbb{R} \mid \alpha \leq \alpha_{ij}\}.$$

Then, the projections of the bounded faces of $A_{\mathcal{F}}$ form a lattice subdivision of $\mathrm{Newt}(\mathcal{F})$. This naturally has a structure of a polyhedral complex. The *dual subdivision* of $\mathcal{F}$ is this polyhedral complex and we denote it by $\Delta_{\mathcal{F}}$. For a Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$, we also write Δ_f for the dual subdivision of $\mathrm{trop}(f)$.

Theorem 2.9 (The Duality Theorem, (Mikhalkin 2005, Proposition 3.11)) *Let* $\Gamma = V(\mathcal{F})$ *be a tropical plane curve. Then,* Γ *is the support of a finite* 1*-dimensional polyhedral complex* $\Sigma_{\mathcal{F}}$ *(possibly with noncompact cells) in* $\mathbb{R}^2$*. It is dual to the subdivision* $\Delta_{\mathcal{F}}$ *in the following sense:*

- *(Closures of) domains of* $\mathbb{R}^2 \setminus \Gamma$ *correspond to lattice points in* $\Delta_{\mathcal{F}}$*.*
- 1*-dimensional cells in* $\Sigma_{\mathcal{F}}$ *correspond to* 1*-simplexes in* $\Delta_{\mathcal{F}}$*.*
- 0*-dimensional cells in* $\Sigma_{\mathcal{F}}$ *correspond to* 2*-dimensional cells in* $\Delta_{\mathcal{F}}$*.*
- *This correspondence is inclusion-reversing.*
- *A* 1*-dimensional cell in* $\Sigma_{\mathcal{F}}$ *is orthogonal to the corresponding* 1*-simplex in* $\Delta_{\mathcal{F}}$ *(see* Fig. 3*).*

For a cell $\sigma \in \Delta_{\mathcal{F}}$*, the corresponding cell in* $\Sigma_{\mathcal{F}}$ *is given by* $\{P \in \mathbb{R}^2 \mid \mathcal{F}(P) = \alpha_{\mathbf{i}} + \mathbf{i} \cdot P$ *for any vertex* $\mathbf{i}$ *of* $\sigma\}$*. In particular,* 1*-dimensional cells in* $\Sigma_{\mathcal{F}}$ *have rational slopes.*

Notation 2.10 Let Γ be a tropical plane curve. We call a 0-dimensional cell of Γ a vertex of Γ and a 1-dimensional cell of Γ an edge of Γ.

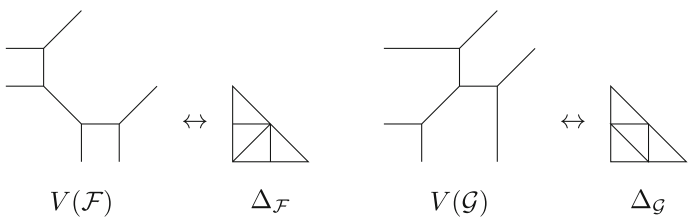


Fig. 3 (Smooth) tropical plane curves and their dual subdivisions

We define the weight of an edge of a tropical plane curve using the dual subdivision.

Definition 2.11 Let $\Gamma = V(\mathcal{F})$ be a tropical plane curve and $\sigma \in \Sigma^{(1)}(\Gamma)$ an edge of Γ. The *weight* w_σ of σ in Γ is the lattice length of the corresponding 1-simplex of $\Delta_{\mathcal{F}}$.

From now on, a "tropical curve" will refer to the polyhedral set Γ together with weights on its edges.

Remark 2.12 Let $\mathcal{F} = \bigoplus_{i,j} \alpha_{ij} x^i y^j$ be a tropical polynomial. Let σ be an edge of $\Sigma_{\mathcal{F}}$ with weight 1 and $\overline{\mathbf{ij}}$ the corresponding 1-simplex of $\Delta_{\mathcal{F}}$. Assume that $\alpha_{\mathbf{k}} + \mathbf{k} \cdot P = \mathcal{F}(P)$ for $P \in \sigma$ and $\mathbf{k} \in \mathbb{Z}^2 \backslash \{\mathbf{i}, \mathbf{j}\}$. Then $\mathbf{k}$ is one of the vertices of a 2-dimensional cell in $\Delta_{\mathcal{F}}$, corresponding to a vertex of σ, containing $\overline{\mathbf{ij}}$ as its face. In particular, for any $\mathbf{k} \in (\mathrm{Aff}(\overline{\mathbf{ij}}) \cap \mathbb{Z}^2) \backslash \{\mathbf{i}, \mathbf{j}\}$, we have

$$\alpha_{\mathbf{k}} + \mathbf{k} \cdot P < \mathcal{F}(P).$$

Notation 2.13 Let Γ be a tropical plane curve, P a vertex of Γ and L an edge of Γ containing P. Let R be the ray which contains L such that P is its endpoint. We denote by $\mathbf{v}_{P,L}$ the primitive vector that have the same direction as R.

Tropical plane curves satisfy the following balancing condition.

Theorem 2.14 (Maclagan and Sturmfels 2015, Theorem 3.3.2) *Let Γ be a tropical plane curve and P a vertex of Γ and $L_1, \ldots, L_n$ the edges of Γ containing P with weights w_{L_i}. Then, we have*

$$\sum_i w_{L_i} \mathbf{v}_{P,L_i} = \mathbf{0}.$$

Definition 2.15 Let $\Gamma = V(\mathcal{F})$ be a tropical plane curve. A vertex $P \in \Gamma$ is called *smooth* if the area of the corresponding cell in $\Delta_{\mathcal{F}}$ is $1/2$. We see that this is equivalent to the condition that it is trivalent and all the weights of the three edges L_1, L_2 and L_3 containing P are 1, and for some (or any) pair (i, j) $(i, j \in \{1, 2, 3\}, i \neq j)$, we have $|\det(\mathbf{v}_{P,L_i}, \mathbf{v}_{P,L_j})| = 1$.

Definition 2.16 *(Smooth tropical plane curves)* A tropical plane curve $\Gamma = V(\mathcal{F})$ is called *smooth* if all the lattice lengths of the 1-simplexes of $\Delta_{\mathcal{F}}$ are 1 and all the areas of the 2-dimensional cells of $\Delta_{\mathcal{F}}$ are $1/2$ (see Fig. 3). In other words, Γ is smooth if all the vertices are smooth and all the weights of the edges are 1.

Notation 2.17 Let Γ be a tropical plane curve and σ an edge of Γ. We denote by $\mathbf{v}_\sigma$ primitive vector that have the same direction as $\mathrm{Aff}(\sigma)$. This is well-defined up to sign.

Definition 2.18 *(Transverse intersection points)* Let Γ_1 and Γ_2 be tropical plane curves. A point P is a *transverse intersection point* of Γ_1 and Γ_2 if it is a proper intersection point of them and is a vertex of neither of them. For a transverse intersection point P, there exist unique edges $L_i \in \Sigma^{(1)}(\Gamma_i)$ $(i = 1, 2)$ containing P in their interiors. In this case, we say that L_1 and L_2 intersect *transversely* at P, and we define the *intersection multiplicity* at P as

$$i(P; \Gamma_1 \cdot \Gamma_2) := w_{L_1} w_{L_2} |\det(\mathbf{v}_{L_1}, \mathbf{v}_{L_2})|.$$

Tropical plane curves Γ_1 and Γ_2 intersect *transversely* if all the points in $\Gamma_1 \cap \Gamma_2$ are transverse intersection points.

Note that for a tropical plane curve Γ and a vector $\mathbf{v} \in \mathbb{R}^2$, $\Gamma + \mathbf{v}$ is a tropical plane curve. For tropical plane curves Γ_1 and Γ_2, it is known that for a generic vector $\mathbf{v} \in \mathbb{R}^2$ and a nonzero real number $\epsilon \in \mathbb{R}$ with sufficiently small absolute value, $\Gamma_1 + \epsilon\mathbf{v}$ and Γ_2 intersect transversely, and the following sum is well-defined (see Osserman and Rabinoff 2013, Sect. 6).

Definition 2.19 *(Intersection multiplicities)* Let Γ_1 and Γ_2 be tropical plane curves. We define the *intersection multiplicity* at a point $P \in \Gamma_1 \cap \Gamma_2$ as

$$i(P; \Gamma_1 \cdot \Gamma_2) := \sum_{L_1 \ni P, L_2 \ni P} \left(\sum_{Q \in (L_1 + \epsilon\mathbf{v}) \cap L_2} i(Q; (\Gamma_1 + \epsilon\mathbf{v}) \cdot \Gamma_2) \right),$$

where L_1 and L_2 are edges of Γ_1 and Γ_2, $\mathbf{v} \in \mathbb{R}^2$ is a generic vector, $\epsilon \in \mathbb{R}$ is a sufficiently small nonzero real number.

It is easy to see that $i(P; \Gamma_1 \cdot \Gamma_2) = i(P; \Gamma_2 \cdot \Gamma_1)$.

Definition 2.20 *(Stable intersection divisor)* Let Γ_1 and Γ_2 be tropical plane curves. The *stable intersection divisor* of Γ_1 and Γ_2 is defined as

$$\sum_{P \in \Gamma_1 \cap \Gamma_2} i(P; \Gamma_1 \cdot \Gamma_2) P.$$

In an appropriate sense, this is equal to the limit of $(\Gamma_1 + \epsilon\mathbf{v}) \cap \Gamma_2$ as $\epsilon \to 0$, where $\mathbf{v} \in \mathbb{R}^2$ is a generic vector and ϵ is a sufficiently small nonzero real number.

The following theorem says that the tropicalization conserves the intersection number in a certain sense.

Theorem 2.21 (Osserman and Rabinoff 2013, Corollary 6.13) *Let $X_1, \ldots, X_m \in (k^*)^n$ be pure dimensional closed subschemes of $(k^*)^n$ with $\sum_i \mathrm{codim}(X_i) = n$. Let $\mathfrak{K}$ be a connected component of $\bigcap_i \mathrm{Trop}(X_i)$, and suppose that $\mathfrak{K}$ is bounded. Then there are only finitely many k-valued points $x \in \left(\bigcap_i X_i\right)(k)$ with $\mathrm{trop}(x) \in \mathfrak{K}$, and*

$$\sum_{\substack{x \in (\bigcap_i X_i)(k) \\ \mathrm{trop}(x) \in \mathfrak{K}}} i(x; X_1 \ldots X_m) = \sum_{P \in \mathfrak{K}} i(P; \mathrm{Trop}(X_1) \cdots \mathrm{Trop}(X_m)).$$

3 Preparations for the main theorems

We will provide additional explanations for some facts from Section 1 and make preparations for the next section. Since we are going to compare the valuations of different terms in polynomials, we make the following definition.

Definition 3.1 We define a map τ as follows:

$$\begin{aligned} \tau : \mathbb{T}[x^{\pm 1}, y^{\pm 1}] \times \mathbb{Z}^2 \times \mathbb{R}^2 &\to \mathbb{R} \cup \{-\infty\} \\ \left(\bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}; \mathbf{j}; P \right) &\mapsto \alpha_{\mathbf{j}} + \mathbf{j} \cdot P. \end{aligned}$$

For a Laurent polynomial $f \in k[x^{\pm 1}, y^{\pm 1}]$, we write $\tau(f; \mathbf{j}; P)$ for $\tau(\mathrm{trop}(f); \mathbf{j}; P)$.

The map τ satisfies the following.

Lemma 3.2 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials. Then, for all $\mathbf{i} \in \mathbb{Z}^2$ and $P \in \mathbb{R}^2$, we have*

$$\tau(f + g; \mathbf{i}; P) \le \max\{\tau(f; \mathbf{i}; P), \tau(g; \mathbf{i}; P)\}.$$

Moreover, the equality holds if $\tau(f; \mathbf{i}; P) \neq \tau(g; \mathbf{i}; P)$.

Proof This is clear from the ultrametric inequality for the valuation. $\square$

For $A = (a_{ij}) \in \mathrm{GL}_2(\mathbb{Z})$, $\mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ and $\mathbf{t} = (t_1, t_2) \in (k^*)^2$ such that $\mathrm{trop}(\mathbf{t}) = \mathbf{b}$, we define the following automorphisms and an affine transformation.

$$\begin{aligned} \phi : (k^*)^2 &\to (k^*)^2 \\ (a, b) &\mapsto (a^{a_{11}} b^{a_{12}} t_1, a^{a_{21}} b^{a_{22}} t_2), \\ \phi^* : k[x^{\pm 1}, y^{\pm 1}] &\to k[x^{\pm 1}, y^{\pm 1}] \\ x &\mapsto x^{a_{11}} y^{a_{12}} t_1, \quad y \mapsto x^{a_{21}} y^{a_{22}} t_2, \\ \mathrm{trop}(\phi) = \Phi : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{v} &\mapsto A\mathbf{v} + \mathbf{b}, \\ \Phi^* : \mathbb{T}[x^{\pm 1}, y^{\pm 1}] &\to \mathbb{T}[x^{\pm 1}, y^{\pm 1}] \\ x &\mapsto b_1 x^{a_{11}} y^{a_{12}}, \quad y \mapsto b_2 x^{a_{21}} y^{a_{22}}, \\ {}^t\Phi^- : \mathbb{R}^2 &\to \mathbb{R}^2 \\ \mathbf{v} &\mapsto {}^t A^{-1} (\mathbf{v} - \mathbf{b}). \end{aligned}$$

Then, the following can be verified by direct calculations.

- $\forall P \in (k^*)^2,\ \mathrm{trop}(\phi(P)) = \Phi(\mathrm{trop}(P))$.
- $\forall f \in k[x^{\pm 1}, y^{\pm 1}],\ \mathrm{Trop}(V(\phi^*(f))) = V(\Phi^*(\mathrm{trop}(f)))$.
- $\forall f \in k[x^{\pm 1}, y^{\pm 1}],\ \forall P \in (k^*)^2,\ f(\phi(P)) = (\phi^*(f))(P)$.

- $\forall \mathcal{F} \in \mathbb{T}[x^{\pm 1}, y^{\pm 1}]$, ${}^t\Phi^-(V(\mathcal{F})) = V(\Phi^*(\mathcal{F}))$.

Thus, if L is an edge of $\mathrm{Trop}(V(f))$, we can find an automorphism ϕ of $(k^*)^2$ such that $\mathrm{trop}(\phi)(L)$ is contained in the y-axis, for example.

Recall that $\mathcal{R}_1(\mathcal{F}, \mathcal{G})$ and $\mathcal{LS}_2(\mathcal{F}, \mathcal{G})$ were the sets of rays and line segments contained in $V(\mathcal{F}) \cap V(\mathcal{G})$, defined in Definition 1.7, and that $\mathcal{L}_s(\mathcal{F}, \mathcal{G}) = \mathcal{R}_1(\mathcal{F}, \mathcal{G}) \cup \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$.

Lemma 3.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be bivariate tropical polynomials.*

(1) *Let $L \in \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$, and P_+ and P_- the endpoints of L. Then, P_* ("$* = +$ or $-$") is a smooth vertex in one of $V(\mathcal{F})$ and $V(\mathcal{G})$, and is in the interior of an edge of weight* 1 *in the other. Furthermore, the interior of L contains no vertices of $V(\mathcal{F})$ and $V(\mathcal{G})$. In other words, for a neighborhood U of L, the restrictions of $V(\mathcal{F})$ and $V(\mathcal{G})$ to U are as in* Fig. 4, *where each vertex is smooth and each edge has weight* 1.
In particular, the stable intersection points of $V(\mathcal{F})$ and $V(\mathcal{G})$ on L are the endpoints of L, each with weight 1.

(2) *For $L \in \mathcal{R}_1(\mathcal{F}, \mathcal{G})$, the endpoint of L is a smooth vertex in one of $V(\mathcal{F})$ and $V(\mathcal{G})$, and is in the interior of an edge of weight* 1 *in the other. Furthermore, the interior of L contains no vertices of $V(\mathcal{F})$ and $V(\mathcal{G})$.*

Proof First, note that each endpoint of L is a vertex of at least one of $V(\mathcal{F})$ and $V(\mathcal{G})$, and that if $P \in L$ is a vertex of $V(\mathcal{F})$ or $V(\mathcal{G})$, then we have $i(P; V(\mathcal{F}) \cdot V(\mathcal{G})) \geq 1$. It follows that $V(\mathcal{F})$ and $V(\mathcal{G})$ intersect with multiplicity 1 at each endpoint of L, and that $V(\mathcal{F})$ and $V(\mathcal{G})$ do not have a vertex in the interior of L. Hence, an edge of $V(\mathcal{F})$ (resp. $V(\mathcal{G})$) intersecting the interior of L contains L.

Let us prove (1). The proof of (2) is similar. We can assume that L is contained in the y-axis and $P_+ = (0, a_1)$ and $P_- = (0, a_2)$ with $a_1 > a_2$ by applying an affine transformation with a unimodular integral coefficient matrix. Let U_+ be a sufficiently small neighborhood of P_+ and $D_R := \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1 > 0\}$. Let $L_1, \ldots, L_s$ be the edges of $V(\mathcal{F})$ which intersect U_+, and $L'_1, \ldots, L'_t$ the edges of $V(\mathcal{G})$ which intersect U_+, with $L_1 \supset L$ and $L'_1 \supset L$. Assume that P_+ is a vertex in both of $V(\mathcal{F})$ and $V(\mathcal{G})$, i.e. $s \geq 3$ and $t \geq 3$. Then, by the balancing condition at P_+, there are i and j such that

$$D_R \cap L_i \neq \emptyset \text{ and } D_R \cap L'_j \neq \emptyset.$$

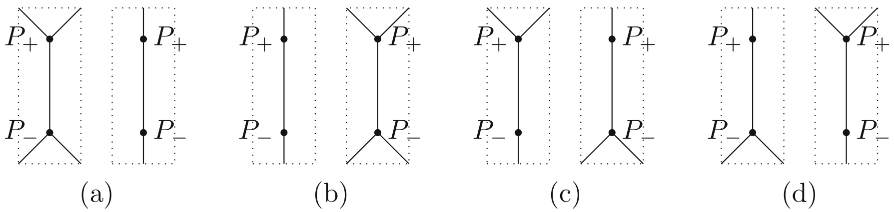


Fig. 4 $V(\mathcal{F})$ and $V(\mathcal{G})$ in a neighborhood of $L \in \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$

Note that by the assumption that $L \in \mathcal{LS}_2(\mathcal{F}, \mathcal{G})$ is a connected component of $V(\mathcal{F}) \cap V(\mathcal{G})$, we have $L_i \cap L'_j = \{P_+\}$. By symmetry, we assume that the slope of L_i is larger than that of L'_j. Let $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ be a general vector such that $v_1, v_2 > 0$ and v_2/v_1 is sufficiently large, and $\epsilon > 0$ a sufficiently small positive number. We will consider $V(\mathcal{F}) \cap (\epsilon\mathbf{v} + V(\mathcal{G}))$. Then, we have $L_i \cap (\epsilon\mathbf{v} + L'_j) \neq \emptyset$ and $L_i \cap (\epsilon\mathbf{v} + L'_1) \neq \emptyset$. Hence, we have $i(P_+; V(\mathcal{F}) \cdot V(\mathcal{G})) \geq 2$, contradicting to what we saw at the beginning. Thus P_+ is a vertex of exactly one of $V(\mathcal{F})$ and $V(\mathcal{G})$.

Assume that P_+ is a vertex of $V(\mathcal{F})$. Then, by the definition of $\mathcal{LS}_2(\mathcal{F}, \mathcal{G})$ (see Definition 1.7), the multiplicity of L_1 is 1. Since the intersection multiplicity at P_+ is 1, it is cleat that exactly one of $\{L_2, \ldots, L_s\}$, say L_2, intersects D_R, that the weights of L'_1 and L_2 are both 1 and that $|\det(\mathbf{v}_{L_2}, \mathbf{v}_{L'_1})| = 1$. Similarly, there is a unique edge of $V(\mathcal{F})$ intersecting $U_+ \cap \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1 < 0\}$, and $V(\mathcal{F})$ is trivalent at P_+ (note that $V(\mathcal{F})$ contains no edge intersecting $U_+ \cap \{(p_1, p_2) \in \mathbb{R}^2 \mid p_1 = 0,\ p_2 > a_1\}$ since P_+ is an endpoint of L). By the balancing condition, P_+ is a smooth vertex of $V(\mathcal{F})$. The same holds at the point P_-. □

Corollary 3.4 *Let $\mathcal{F}, \mathcal{G} \in \mathbb{T}[x^{\pm 1}, y^{\pm 1}]$, $L \in \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G})$, $\Phi_1(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$ and $\Phi_2(L) = \overline{\mathbf{j}_0\mathbf{j}_1}$. Then $\mathbf{i}_1 - \mathbf{i}_0 = \pm(\mathbf{j}_1 - \mathbf{j}_0)$.*

Proof It is clear that $\mathrm{Aff}(\Phi_1(L)) = \mathrm{Aff}(\Phi_2(L))$, and hence, it is sufficient to show that $\Phi_1(L)$ and $\Phi_2(L)$ have the same lattice length. By Lemma 3.3, each edge of $V(\mathcal{F})$ and $V(\mathcal{G})$ intersecting L has weight 1. Then, by the definition of the weight of an edge of a tropical plane curve (see Definition 2.11), the lattice lengths of $\Phi_1(L)$ and $\Phi_2(L)$ are 1. □

Lemma 3.5 *Let $\mathcal{F} = \bigoplus_{\mathbf{i}} \alpha_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, $\mathcal{G} = \bigoplus_{\mathbf{i}} \beta_{\mathbf{i}}\mathbf{x}^{\mathbf{i}} \in \mathbb{T}[x^{\pm 1}, y^{\pm 1}]$ be tropical polynomials, $L \in \mathcal{L}_{\mathrm{s}}(\mathcal{F}, \mathcal{G})$, and P_+ an endpoint of L. Assume that $\Phi_1(L) = \Phi_2(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$, $\alpha_{\mathbf{i}_0} = \beta_{\mathbf{i}_0}$ and $\alpha_{\mathbf{i}_1} = \beta_{\mathbf{i}_1}$. Let U_+ be a sufficiently small neighborhood of P_+ and $\overline{L} = \mathrm{Aff}(L)$ (see Notation 1.22). By Lemma 3.3, for either $(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{F}, \mathcal{G})$ or $(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{G}, \mathcal{F})$, the point P_+ is a smooth vertex of $V(\mathcal{F}_1)$ and is in the interior of an edge of multiplicity 1 in $V(\mathcal{F}_2)$. Let σ_+ be the 2-simplex of $\Delta_{\mathcal{F}_1}$ corresponding to P_+ and $\mathbf{i}_+$ the vertex of σ_+ other than $\mathbf{i}_0$ and $\mathbf{i}_1$. Then, for all $P \in U_+ \cap (\overline{L} \setminus L)$, we have*

$$\tau(\mathcal{F}_1; \mathbf{i}_+; P) > \tau(\mathcal{F}; \mathbf{i}; P), \tau(\mathcal{G}; \mathbf{i}; P), \tau(\mathcal{F}_2; \mathbf{i}_+; P) \ (\mathbf{i} \in \mathbb{Z}^2 \setminus \{\mathbf{i}_+\}),$$

and

$$\begin{aligned}\tau(\mathcal{F}_1; \mathbf{i}_+; P_+) &= \tau(\mathcal{F}; \mathbf{i}; P_+) = \tau(\mathcal{G}; \mathbf{i}; P_+) \ (\mathbf{i} = \mathbf{i}_0 \textit{ or } \mathbf{i}_1) \\ &> \tau(\mathcal{F}; \mathbf{j}; P_+), \tau(\mathcal{G}; \mathbf{j}; P_+), \tau(\mathcal{F}_2; \mathbf{i}_+; P_+) \ (\mathbf{j} \in \mathbb{Z}^2 \setminus \{\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_+\}).\end{aligned}$$

Proof By symmetry, we may assume that $\mathcal{F}_1 = \mathcal{F}$. Let $P \in U_+ \cap (\overline{L} \setminus L)$. Then, the restrictions of the two tropical plane curves to a neighborhood of P_+ are as in Fig. 5. We have

$$\tau(\mathcal{F}; \mathbf{i}_+; P) > \tau(\mathcal{F}; \mathbf{i}; P) \ \ (\mathbf{i} \in \mathbb{Z}^2 \setminus \{\mathbf{i}_+\}), \tag{1}$$

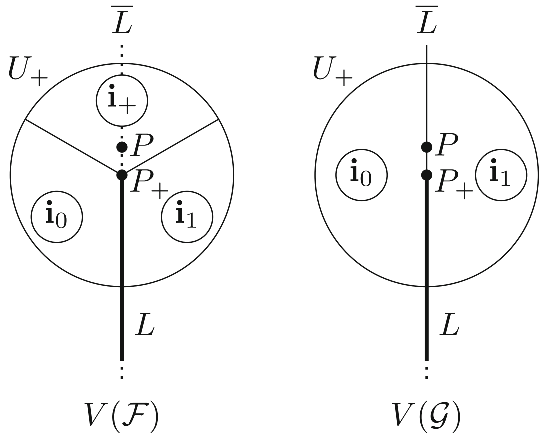


Fig. 5 $V(\mathcal{F})$ and $V(\mathcal{G})$ in a neighborhood U_+ of P_+ ($\mathcal{F} = \mathcal{F}_1$)

since $\mathbf{i}_+$ is the vertex corresponding to the domain containing P, and for all $\mathbf{j} \in \mathbb{Z}^2 \backslash \{\mathbf{i}_0, \mathbf{i}_1, \mathbf{i}_+\}$, we have

$$\tau(\mathcal{F}; \mathbf{i}_+; P_+) = \tau(\mathcal{F}; \mathbf{i}_0; P_+) = \tau(\mathcal{F}; \mathbf{i}_1; P_+) > \tau(\mathcal{F}; \mathbf{j}; P_+). \tag{2}$$

For all $\mathbf{i} \in \mathbb{Z}^2 \setminus \{\mathbf{i}_0, \mathbf{i}_1\}$, we have

$$\tau(\mathcal{G}; \mathbf{i}_0; P) = \tau(\mathcal{G}; \mathbf{i}_1; P) > \tau(\mathcal{G}; \mathbf{i}; P), \tag{3}$$

$$\tau(\mathcal{G}; \mathbf{i}_0; P_+) = \tau(\mathcal{G}; \mathbf{i}_1; P_+) > \tau(\mathcal{G}; \mathbf{i}; P_+). \tag{4}$$

By the assumption that $\alpha_{\mathbf{i}_0} = \beta_{\mathbf{i}_0}$ and $\alpha_{\mathbf{i}_1} = \beta_{\mathbf{i}_1}$, we have

$$\tau(\mathcal{F}; \mathbf{i}; P) = \tau(\mathcal{G}; \mathbf{i}; P),\ \ \tau(\mathcal{F}; \mathbf{i}; P_+) = \tau(\mathcal{G}; \mathbf{i}; P_+)\ \ (\mathbf{i} = \mathbf{i}_0 \text{ or } \mathbf{i}_1). \tag{5}$$

By the inequalities (1), (3) and (5), we have

$$\tau(\mathcal{F}; \mathbf{i}_+; P) > \tau(\mathcal{F}; \mathbf{i}_0; P) = \tau(\mathcal{G}; \mathbf{i}_0; P) \geq \tau(\mathcal{G}; \mathbf{i}; P)\ \ (\mathbf{i} \in \mathbb{Z}^2).$$

The second inequalities follow from (2), (4) and (5). $\square$

Notation 3.6 For a Laurent polynomial $f = \sum_{\mathbf{i}} c_{\mathbf{i}} \mathbf{x}^{\mathbf{i}} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we define $\mathrm{coeff}_{\mathbf{i}}(f) = c_{\mathbf{i}}$ and $v_{\mathbf{i}}(f) = \mathrm{val}(c_{\mathbf{i}})$.

Notation 3.7 Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ and $L \in \mathcal{L}_{\mathrm{s}}(f, g) = \mathcal{R}_1(f, g) \cup \mathcal{LS}_2(f, g)$. In the rest of this paper, we use the following notation.

- $\Phi_1(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$ and $\Phi_2(L) = \overline{\mathbf{j}_0\mathbf{j}_1}$, where $\mathbf{i}_1 - \mathbf{i}_0 = \mathbf{j}_1 - \mathbf{j}_0$ (see Corollary 3.4).

- An endpoint $P_+ \in L$ is a vertex of $\mathrm{Trop}(V(f_1))$ ($f_1 \in \{f, g\}$). Let $(\mathbf{l}_0, \mathbf{l}_1)$ be $(\mathbf{i}_0, \mathbf{i}_1)$ (resp. $(\mathbf{j}_0, \mathbf{j}_1)$) if $f_1 = f$ (resp. $f_1 = g$).
- The vertex of the 2-simplex of Δ_{f_1} corresponding to P_+ are $\mathbf{l}_0$, $\mathbf{l}_1$ and $\mathbf{l}_+$. Let $\mathbf{i}_+ := \mathbf{i}_0 + (\mathbf{l}_+ - \mathbf{l}_0)$ and $\mathbf{j}_+ := \mathbf{j}_0 + (\mathbf{l}_+ - \mathbf{l}_0)$.
- If $L \in \mathcal{LS}_2(f, g)$, the other endpoint $P_- \in L$ is a vertex of $\mathrm{Trop}(V(f_1'))$ ($f_1' \in \{f, g\}$). Let $(\mathbf{l}_0', \mathbf{l}_1')$ be $(\mathbf{i}_0, \mathbf{i}_1)$ (resp. $(\mathbf{j}_0, \mathbf{j}_1)$) if $f_1' = f$ (resp. $f_1' = g$).
- The vertex of the 2-simplex of $\Delta_{f_1'}$ corresponding to P_- are $\mathbf{l}_0'$, $\mathbf{l}_1'$ and $\mathbf{l}_-'$. Let $\mathbf{i}_- := \mathbf{i}_0 + (\mathbf{l}_-' - \mathbf{l}_0')$ and $\mathbf{j}_- := \mathbf{j}_0 + (\mathbf{l}_-' - \mathbf{l}_0')$.

By multiplying a unit, we may assume that f, g and L further satisfy the following condition (¶):

- $\Phi_1(L) = \Phi_2(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$.
- $v_{\mathbf{i}_0}(f) = v_{\mathbf{i}_0}(g)$.
- $v_{\mathbf{i}_1}(f) = v_{\mathbf{i}_1}(g)$.

Furthermore, by applying an affine transformation, multiplying units and changing the variable x to $\mathrm{coeff}_{10}(f)x$, we may assume that f, g and L satisfy the following condition (¶′):

- $\Phi_1(L) = \Phi_2(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$.
- $\mathbf{i}_0 = (0, 0)$, $\mathbf{i}_1 = (1, 0)$ and $\mathbf{i}_+ = (0, 1)$.
- $v_{\mathbf{i}_0}(f) = v_{\mathbf{i}_0}(g) = v_{\mathbf{i}_1}(f) = v_{\mathbf{i}_1}(g) = 0$.
- $P_+ = (0, y_+)$ and $P_- = (0, y_-)$.

Now we are going to find an element of the ideal (f, g) that is useful in studying $\mathrm{trop}(V(f) \cap V(g))$. This will be of the form $G = g + h(\mathbf{x}^{\mathbf{v}})f$, where $h \in k[t^{\pm 1}]$ is a univariate Laurent polynomial. The proof of the following lemma gives an algorithm to find this element.

Lemma 3.8 *Let $\lambda > 0$ be a positive number. Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials satisfying the following:*

- $v_{\mathbf{i}_0}(f) = v_{\mathbf{i}_0}(g) \neq \infty$, $v_{\mathbf{i}_1}(f) = v_{\mathbf{i}_1}(g) \neq \infty$.
- $\mathbf{i}_1 - \mathbf{i}_0$ *is primitive.*
- $\mu(f; \overline{\mathbf{i}_0\mathbf{i}_1}) > 0$, $\mu(g; \overline{\mathbf{i}_0\mathbf{i}_1}) > 0$ *(see Definition 1.16).*

Then, there exists a Laurent polynomial $h \in k[t^{\pm 1}]$ satisfying the following conditions:

- *For all $i \in \mathbb{Z}$, we have $v_i(h) > i(v_{\mathbf{i}_1}(f) - v_{\mathbf{i}_0}(f))$.*
- *For the Laurent polynomial $g' := g + h(\mathbf{x}^{\mathbf{i}_1 - \mathbf{i}_0})f$, we have*

$$v_{\mathbf{i}_0}(g') = v_{\mathbf{i}_0}(g'),$$
$$v_{\mathbf{i}_1}(g') = v_{\mathbf{i}_1}(g'),$$
$$\mu(g'; \overline{\mathbf{i}_0\mathbf{i}_1}) > \lambda.$$

Proof We can assume that $\mathbf{i}_0 = (0, 0)$ and $\mathbf{i}_1 = (1, 0)$ by applying an affine transformation. Then, the statements are only about the coefficients of x^i, and we can assume that $f = \sum_i c_i x^i$, $g = \sum_i d_i x^i \in k[x^{\pm 1}]$. We can also assume that $c_0 = 1$ by multiplying a unit. By changing the variable x to $c_1 x$, we may also assume $c_1 = 1$. Given a Laurent polynomial $F = \sum_i \alpha_i x^i \in k[x^{\pm 1}]$, we define

$$\begin{aligned} v(F) &= \min\{\mathrm{val}(\alpha_i)\}, \\ v'(F) &= \min\{\mathrm{val}(\alpha_i) \mid i \neq 0, 1\}, \\ v'_+(F) &= \min\{\mathrm{val}(\alpha_i) \mid i > 1\}, \\ v'_-(F) &= \min\{\mathrm{val}(\alpha_i) \mid i < 0\}. \end{aligned}$$

Then, we have $v'(f) > 0$ and $v'(g) > 0$. It is sufficient to show that there exists a Laurent polynomial $h \in k[x^{\pm 1}]$ with $v(h) > 0$ such that for the Laurent polynomial $g' := g + hf$, we have $v_0(g') = v_1(g') = 0$ and $v'(g') \geq \lambda$. Let $\lambda_0 = v'(f)$ and $\lambda_1 = v'(g)$. Given a Laurent polynomial $F \in k[x^{\pm 1}]$, we define

$$\begin{aligned} \mathrm{M}_-(F) &= \min\{n \in \mathbb{Z} \mid n \leq 0,\ v_n(F) < \lambda_0 + \lambda_1\}, \\ \mathrm{M}_+(F) &= \max\{n \in \mathbb{Z} \mid n \geq 1,\ v_n(F) < \lambda_0 + \lambda_1\}. \end{aligned}$$

Note that $\mathrm{M}_-(F) \leq 0$, $\mathrm{M}_+(F) \geq 1$ and that for Laurent polynomials $F_1, F_2 \in k[x^{\pm 1}]$, we have

$$\begin{aligned} \mathrm{M}_-(F_1 + F_2) &\geq \min\{\mathrm{M}_-(F_1), \mathrm{M}_-(F_2)\}, \\ \mathrm{M}_+(F_1 + F_2) &\leq \max\{\mathrm{M}_+(F_1), \mathrm{M}_+(F_2)\}. \end{aligned}$$

Claim 1 *The following hold.*

- *If* $\mathrm{M}_+(g) > 1$, *then there exist* $a \in k$ *and* $i \in \mathbb{Z}$ *such that* $\mathrm{val}(a) > 0$, $\mathrm{M}_+(g - ax^i f) < \mathrm{M}_+(g)$ *and* $\mathrm{M}_-(g - ax^i f) \geq \mathrm{M}_-(g)$.
- *If* $\mathrm{M}_-(g) < 0$, *then there exist* $a \in k$ *and* $i \in \mathbb{Z}$ *such that* $\mathrm{val}(a) > 0$, $\mathrm{M}_+(g - ax^i f) \leq \mathrm{M}_+(g)$ *and* $\mathrm{M}_-(g - ax^i f) > \mathrm{M}_-(g)$.

Proof We show the case where $n := \mathrm{M}_+(g) > 1$. The proof in the case where $\mathrm{M}_-(g) < 0$ is similar. Let $a = d_n$. Then, we have $\mathrm{val}(a) \geq \lambda_1 (> 0)$. Let $-ax^{n-1} f = \sum_i \alpha_i x^i$ and $g - ax^{n-1} f = \sum_i \beta_i x^i$. Then, we have $\mathrm{val}(\beta_n) = \mathrm{val}(0) = \infty$ and

$$\begin{aligned} i < n - 1 &\Rightarrow \mathrm{val}(\alpha_i) \geq \lambda_0 + \lambda_1, \\ i = n - 1, n &\Rightarrow \mathrm{val}(\alpha_i) = \mathrm{val}(a) \geq \lambda_1, \\ n < i &\Rightarrow \mathrm{val}(\alpha_i) \geq \lambda_0 + \lambda_1. \end{aligned}$$

Thus, we have

$$n \leq i \Rightarrow \mathrm{val}(\beta_i) \geq \lambda_0 + \lambda_1,$$

and $\mathrm{M}_-(-ax^{n-1} f) = 0$. Hence, we have

$$\mathrm{M}_+(g - ax^{n-1} f) < n = \mathrm{M}_+(g),$$

and

$$\mathrm{M}_-(g - ax^{n-1}f) \geq \min\{\mathrm{M}_-(g), \mathrm{M}_-(-ax^{n-1}f)\} \geq \mathrm{M}_-(g).$$

$\square$

Note that in the proof of the above claim, we have $\mathrm{val}(\beta_0) = \mathrm{val}(\beta_1) = 0$. From the first bullet in the above claim, we can show by induction on $n = \mathrm{M}_+(g)$ that there exists a Laurent polynomial $h_0 \in k[x^{\pm 1}]$ with $v(h_0) > 0$ such that for the Laurent polynomial $g_1 := g + h_0 f$, we have $v_0(g_1) = v_1(g_1) = 0$ and $v'_+(g_1) \geq \lambda_0 + \lambda_1$. Then, from the second bullet in the above claim, we may ensure that there exists $h_1 \in k[x^{\pm 1}]$ with $v(h_1) > 0$ such that for $g_2 := g_1 + h_1 f$, we have $v_0(g_2) = v_1(g_2) = 0$ and $v'_-(g_2) > \lambda_0 + \lambda_1$. It follows that $g_2 = g + (h_0 + h_1)f$, $v(h_0 + h_1) > 0$ and $v'(g_2) > \lambda_0 + \lambda_1$.

Then, by induction on $\max\left\{0, \left[(\lambda - v'(g))/v'(f)\right] + 1\right\}$, we may ensure that there exists $h \in k[x^{\pm 1}]$ with $v(h) > 0$ such that for $g' := g + hf$, we have $v_0(g') = v_1(g') = 0$ and $v'(g') > \lambda$. $\square$

Definition 3.9 Let $\lambda > 0$ be a positive number and $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ Laurent polynomials satisfying the assumption of Lemma 3.8. We define $h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1}) \in k[t^{\pm 1}]$ to be the Laurent polynomial $h \in k[t^{\pm 1}]$ obtained by the algorithm in the proof of Lemma 3.8. We also define $G(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1}) \in k[x^{\pm 1}, y^{\pm 1}]$ by

$$G(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1}) := g_\lambda - \frac{\mathrm{coeff}_{\mathbf{i}_1}(g_\lambda)}{\mathrm{coeff}_{\mathbf{i}_1}(f_\lambda)} f_\lambda,$$

where

$$\begin{aligned} g_\lambda &:= g + h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1})(\mathbf{x}^{\mathbf{i}_1 - \mathbf{i}_0}) f, \\ f_\lambda &:= f + h(\lambda; f, f; \overline{\mathbf{i}_0\mathbf{i}_1})(\mathbf{x}^{\mathbf{i}_1 - \mathbf{i}_0}) f. \end{aligned}$$

More generally, we define the following set.

Definition 3.10 Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials, $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ a ray or a line segment and $\lambda > 0$ a positive number. Then, we define $\mathrm{H}_4(\lambda; f, g; L) \subset k[x^{\pm 1}, y^{\pm 1}]^4$ and $\mathrm{Elim}(\lambda; f, g; L) \subset k[x^{\pm 1}, y^{\pm 1}]$ by

$$\mathrm{H}_4(\lambda; f, g; L) = \left\{ (h_1, h_2, h_3, h_4) \;\middle|\; \begin{array}{l} \mu\left(f + h_1 f + h_2 g; \Phi_1(L)\right) > \lambda, \\ \mu\left(g + h_3 f + h_4 g; \Phi_2(L)\right) > \lambda, \\ \text{and, for any } P \in L, \\ \mathrm{trop}(h_1)(P) < 0, \\ \mathrm{trop}\left(h_2 \mathbf{x}^{\mathbf{j}_0 - \mathbf{i}_0}\right)(P) < v_{\mathbf{i}_0}(f) - v_{\mathbf{j}_0}(g), \\ \mathrm{trop}\left(h_3 \mathbf{x}^{\mathbf{i}_0 - \mathbf{j}_0}\right)(P) < v_{\mathbf{j}_0}(g) - v_{\mathbf{i}_0}(f), \\ \mathrm{trop}(h_4)(P) < 0 \end{array} \right\},$$

where $\Phi_1(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$ and $\Phi_2(L) = \overline{\mathbf{j}_0\mathbf{j}_1}$ are endowed with the same orientation, and

$$\mathrm{Elim}(\lambda; f, g; L) = \left\{ G \;\middle|\; \begin{array}{l} \exists (h_1, h_2, h_3, h_4) \in \mathrm{H}_4(\lambda; f, g; L) \text{ s.t.} \\ G = g' - \frac{\mathrm{coeff}_{\mathbf{j}_1}(g')}{\mathrm{coeff}_{\mathbf{i}_1}(f')} \mathbf{x}^{\mathbf{j}_1 - \mathbf{i}_1} f', \\ \text{where } f' = f + h_1 f + h_2 g \text{ and } g' = g + h_3 f + h_4 g \end{array} \right\}.$$

Remark 3.11 Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ a ray or a line segment satisfying the condition (¶). Let $\lambda > 0$ be a positive number. Then, we have $(h(\lambda; f, f; \overline{\mathbf{i}_0\mathbf{i}_1})(x), 0, h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1})(x), 0) \in \mathrm{H}_4(\lambda; f, g; L)$ and $G(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1}) \in \mathrm{Elim}(\lambda; f, g; L)$.

To compare the tropicalizations of $V(f)$, $V(g)$ and $V(G)$ for $G \in \mathrm{Elim}(\lambda; f, g; L)$, we use the following lemma.

Lemma 3.12 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ a ray or a line segment. Let $\lambda > 0$ be a positive number and $(h_1, h_2, h_3, h_4) \in \mathrm{H}_4(\lambda; f, g; L)$. Then, the following hold.*

(1) *For any $\mathbf{i} \in \mathbb{Z}^2$ and $P \in L$, we have*

$$\tau\,(h_1 f + h_2 g; \mathbf{i}; P) < \tau(f; \mathbf{i}_0; P),$$
$$\tau\,(h_3 f + h_4 g; \mathbf{i}; P) < \tau(g; \mathbf{j}_0; P).$$

(2) *We have*

$$v_{\mathbf{i}_0}\,(f + h_1 f + h_2 g) = v_{\mathbf{i}_0}(f),$$
$$v_{\mathbf{i}_1}\,(f + h_1 f + h_2 g) = v_{\mathbf{i}_1}(f),$$
$$v_{\mathbf{j}_0}\,(g + h_3 f + h_4 g) = v_{\mathbf{j}_0}(g),$$
$$v_{\mathbf{j}_1}\,(g + h_3 f + h_4 g) = v_{\mathbf{j}_1}(g).$$

Proof By replacing g by $(\mathrm{coeff}_{\mathbf{i}_0}(f)/\,\mathrm{coeff}_{\mathbf{j}_0}(g))\mathbf{x}^{\mathbf{i}_0 - \mathbf{j}_0} g$, we may assume that f, g and L satisfy (¶). Since inequalities about τ does not change by coordinate change, we may further assume that f, g and L satisfy the condition (¶′). Then, the statements (1) and (2) clearly hold. □

Lemma 3.13 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ a ray or a line segment. For any $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$, the following hold.*

$$V(f, G) \cap \mathrm{trop}^{-1}(L) = V(g, G) \cap \mathrm{trop}^{-1}(L) = V(f, g) \cap \mathrm{trop}^{-1}(L).$$

Proof We may assume that f, g and L satisfy the condition (¶′). Let $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$. We show $V(f, G) \cap \mathrm{trop}^{-1}(L) = V(f, g) \cap \mathrm{trop}^{-1}(L)$. We can show $V(g, G) \cap \mathrm{trop}^{-1}(L) = V(f, g) \cap \mathrm{trop}^{-1}(L)$ in the same way. There exists $(h_1, h_2, h_3, h_4) \in \mathrm{H}_4(\lambda; f, g; L)$ such that $G = g' - (d'/c')f'$, where $f' =$

$f + h_1 f + h_2 g$, $g' = g + h_3 f + h_4 g$, $c' = \mathrm{coeff}_{\mathbf{i}_1}(f')$ and $d' = \mathrm{coeff}_{\mathbf{i}_1}(g')$. Thus, we have

$$V(f, G) = V\left(f, \left(1 + h_4 - \frac{d'}{c'}h_2\right) g\right).$$

Here, by Lemma 3.12 (2), we have $\mathrm{val}(d'/c') = 0$. Combined with $\mathrm{trop}(h_2)(P) < 0 = \mathrm{trop}(1)(P)$ and $\mathrm{trop}(h_4)(P) < 0 = \mathrm{trop}(1)(P)$ for all $P \in L$, it follows that

$$V\left(1 + h_4 - \frac{d'}{c'}h_2\right) \cap \mathrm{trop}^{-1}(L) = \emptyset.$$

Therefore, we have

$$\begin{aligned} V(f, G) \cap \mathrm{trop}^{-1}(L) &= V(f) \cap V\left(\left(1 + h_4 - \frac{d'}{c'}h_2\right) g\right) \cap \mathrm{trop}^{-1}(L) \\ &= V(f) \cap V(g) \cap \mathrm{trop}^{-1}(L) \\ &= V(f, g) \cap \mathrm{trop}^{-1}(L). \end{aligned}$$

□

Notation 3.14 We write $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1) \in \mathbb{R}^2$ for the standard basis.

Lemma 3.15 *Let $h \in k[x^{\pm 1}, y^{\pm 1}]$, $\mathbf{j} \in \mathbb{Z}^2$, $P' \in \mathbb{R}^2$, $\mathbf{v}_1 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ and $\mathbf{v}_2 \in \mathbb{R}^2 \setminus \mathrm{Aff}(\mathbf{v}_1)$. Let $\mathbf{w} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ a normal vector of $\mathbf{v}_1$ such that $\mathbf{w} \cdot \mathbf{v}_2 < 0$. Assume that for a lattice point $\mathbf{i} \neq \mathbf{j}$ in the half plane $\mathbf{j} + \mathbb{R}\mathbf{v}_1 + \mathbb{R}_{\geq 0}\mathbf{v}_2$, we have $\tau(h; \mathbf{j}; P') > \tau(h; \mathbf{i}; P')$. Then, for all $P \in P' + \mathbb{R}_{\geq 0}\mathbf{w}$, we have*

$$\tau(h; \mathbf{j}; P) > \tau(h; \mathbf{i}; P).$$

Proof Let $P \in P' + \mathbb{R}_{\geq 0}\mathbf{w}$. Then, there exists a non-negative number $r \geq 0$ such that $P = P' + r\mathbf{w}$, and hence, we have

$$\tau(h; \mathbf{j}; P) - \tau(h; \mathbf{i}; P) = (\tau(h; \mathbf{j}; P') - \tau(h; \mathbf{i}; P')) + r(\mathbf{j} - \mathbf{i}) \cdot \mathbf{w} > 0.$$

□

Remark 3.16 Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ satisfy (¶). Then, for an endpoint $P_+ \in L$, by Lemma 3.5, we have

$$\tau(f_1; \mathbf{i}_+; P_+) > \tau(f_2; \mathbf{i}_+; P_+),$$

where $\{f_1, f_2\} = \{f, g\}$ and $\mathrm{trop}(V(f_1))$ has a vertex at P_+. Therefore, either $v_{\mathbf{i}_+}(f) < v_{\mathbf{i}_+}(g)$ and $f_1 = f$ or $v_{\mathbf{i}_+}(f) > v_{\mathbf{i}_+}(g)$ and $f_1 = g$, and hence,

$$v_{\mathbf{i}_+}(f_1) = \min\{v_{\mathbf{i}_+}(f), v_{\mathbf{i}_+}(g)\}.$$

Lemma 3.17 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$. Let $\lambda > 0$ be a positive number and $G \in \mathrm{Elim}(\lambda; f, g; L)$. Then, the following hold.*

(1) *If f, g and L satisfy (¶), then $v_{\mathbf{i}_+}(G) = v_{\mathbf{i}_+}(f_1)$ $(= \min\{v_{\mathbf{i}_+}(f), v_{\mathbf{i}_+}(g)\})$.*
(2) *Assume that a lattice point $\mathbf{i} \neq \mathbf{j}_+$ is in the half plane $\mathbf{j}_+ + \mathbb{R}(\mathbf{j}_1 - \mathbf{j}_0) + \mathbb{R}_{\geq 0}(\mathbf{j}_+ - \mathbf{j}_0)$. Then, for all $P \in L$, we have*

$$\tau(G; \mathbf{i}; P) < \tau(G; \mathbf{j}_+; P).$$

(3) *Assume that $L \in \mathcal{LS}_2(f, g)$ and $\mathbf{i} \neq \mathbf{j}_-$ is a lattice point in the half plane $\mathbf{j}_- + \mathbb{R}(\mathbf{j}_1 - \mathbf{j}_0) + \mathbb{R}_{\geq 0}(\mathbf{j}_- - \mathbf{j}_0)$. Then, for any point $P \in L$, we have*

$$\tau(G; \mathbf{i}; P) < \tau(G; \mathbf{j}_-; P).$$

Proof Let $G = g' - \frac{\mathrm{coeff}_{\mathbf{i}_1}(g')}{\mathrm{coeff}_{\mathbf{i}_1}(f')} f'$, where $f' = f + h_1 f + h_2 g$ and $g' = g + h_3 f + h_4 g$ with $(h_1, h_2, h_3, h_4) \in \mathrm{H}_4(\lambda; f, g; L)$. To show (1), first note that, by Lemma 3.12, we have

$$\tau(f; \mathbf{i}_0; P_+) > \tau(h_1 f + h_2 g; \mathbf{i}_+; P_+),\ \tau(h_3 f + h_4 g; \mathbf{i}_+; P_+).$$

Let U_+ be a sufficiently small neighborhood of P_+ and $P \in U_+ \cap (\mathrm{Aff}(L) \setminus L)$ a point. Then, we have

$$\tau(f; \mathbf{i}_0; P) > \tau(h_1 f + h_2 g; \mathbf{i}_+; P),\ \tau(h_3 f + h_4 g; \mathbf{i}_+; P).$$

Combined with Lemma 3.5, this implies

$$\tau(f_1; \mathbf{i}_+; P) > \tau(h_1 f + h_2 g; \mathbf{i}_+; P),\ \tau(h_3 f + h_4 g; \mathbf{i}_+; P). \tag{6}$$

Now, by Lemma 3.12 (2), we have $\mathrm{val}(\mathrm{coeff}_{\mathbf{i}_1}(g') / \mathrm{coeff}_{\mathbf{i}_1}(f')) = 0$ and

$$G = g - \frac{\mathrm{coeff}_{\mathbf{i}_1}(g')}{\mathrm{coeff}_{\mathbf{i}_1}(f')} f + h_3 f + h_4 g - \frac{\mathrm{coeff}_{\mathbf{i}_1}(g')}{\mathrm{coeff}_{\mathbf{i}_1}(f')} (h_1 f + h_2 g).$$

Therefore, we have $\tau(G; \mathbf{i}_+; P) = \tau(f_1; \mathbf{i}_+; P)$ by (6) and Lemmas 3.2 and 3.5. Thus, we have $v_{\mathbf{i}_+}(G) = v_{\mathbf{i}_+}(f_1)$.

Next, let us show (2). (3) follows from (2) by symmetry. We may assume that f, g and L satisfy the condition (¶′). Let $P' \in L$. By Lemmas 3.12 and 3.15, we have

$$\max\{\tau\left(h_1 f + h_2 g; \mathbf{i}; P'\right), \tau\left(h_3 f + h_4 g; \mathbf{i}; P'\right)\} < \tau(f_1; \mathbf{i}_+; P').$$

Since $\mathbf{i} \neq \mathbf{i}_+$, by Lemmas 3.5 and 3.15, we have

$$\max\{\tau(f; \mathbf{i}; P'), \tau\left(g; \mathbf{i}; P'\right)\} < \tau(f_1; \mathbf{i}_+; P').$$

Therefore, by Lemma 3.2, we have

$$\tau(G; \mathbf{i}; P') < \tau(f_1; \mathbf{i}_+; P').$$

Since $v_{\mathbf{i}_+}(f_1) = v_{\mathbf{i}_+}(G)$, we have

$$\tau(f_1; \mathbf{i}_+; P') = \tau(G; \mathbf{i}_+; P'),$$

and hence, we have

$$\tau(G; \mathbf{i}; P') < \tau(G; \mathbf{i}_+; P').$$

□

Corollary 3.18 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$. Let $\lambda > 0$ be a positive number and $G \in \mathrm{Elim}(\lambda; f, g; L)$. Then, for a point $P_1 = P_+ + r_1 \mathbf{v}_{P_+, L}$ (see Notation 2.13), where $r_1 \in \mathbb{R}$, we have*

$$\tau(G; \mathbf{j}_+; P) = \tau(g; \mathbf{j}_0; P_+) - r_1.$$

If $L \in \mathcal{LS}_2(f, g)$, then for a point $P_2 = P_- + r_2 \mathbf{v}_{P_-, L}$ ($r_2 \in \mathbb{R}$), we have

$$\tau(G; \mathbf{j}_-; P) = \tau(g; \mathbf{j}_0; P_-) - r_2.$$

Proof We may assume that f, g and L satisfy the condition (¶′). By Lemmas 3.5 and 3.17 (1), for the point $P_1 = (0, y_+ - r_1) \in \mathbb{R}^2$, we have

$$\begin{aligned}
\tau(G; \mathbf{i}_+; P_1) &= \tau(f_1; \mathbf{i}_+; P_1) \\
&= \tau(f_1; \mathbf{i}_+; P_+) + r_1 \mathbf{i}_+ \cdot (-\mathbf{e}_2) \\
&= \tau(g; (0, 0); P_+) - r_1.
\end{aligned}$$

Similarly, if $L \in \mathcal{LS}_2(f, g)$, we have $\tau(G; \mathbf{i}_-; P_2) = \tau(g; (0, 0); P_-) - r_2$. □

Notation 3.19 For $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ and $\lambda > 0$, we define

$$L_+^\lambda = L \cap \{P_+ + r \mathbf{v}_{P_+, L} \mid 0 \leq r < \lambda\}.$$

If $L \in \mathcal{LS}_2(f, g)$, we also define

$$L_-^\lambda = L \cap \{P_- + r \mathbf{v}_{P_-, L} \mid 0 \leq r < \lambda\}.$$

Lemma 3.20 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ and $L \in \mathcal{L}_{\mathrm{s}}(f, g)$. Let $\lambda > 0$ be a positive number and $G \in \mathrm{Elim}(\lambda; f, g; L)$. Then, the following hold.*

(1) $\forall n \in \mathbb{Z} \backslash \{0\},\ \forall P \in L,\ \tau(G; \mathbf{j}_0 + n(\mathbf{j}_1 - \mathbf{j}_0); P) < \tau(g; \mathbf{j}_0; P) - \lambda.$
(2) *If $L \in \mathcal{R}_1(f, g)$, then for a sufficiently small neighborhood U_+ of L_+^λ, we have $\{\mathbf{i} \in \mathbb{Z}^2 \mid \exists P \in U_+ \text{ s.t. } \tau(G; \mathbf{i}; P) = \mathrm{trop}(G)(P)\} \subset \{\mathbf{j}_0, \mathbf{j}_+\}$.*

(3) *If $L \in \mathcal{LS}_2(f, g)$, then for sufficiently small neighborhoods U_+ of L_+^λ and U_- of L_-^λ, we have $\{\mathbf{i} \in \mathbb{Z}^2 \mid \exists P \in U_+ \cup U_- \text{ s.t. } \tau(G; \mathbf{i}; P) = \operatorname{trop}(G)(P)\} \subset \{\mathbf{j}_0, \mathbf{j}_+, \mathbf{j}_-\}$.*

Proof We may assume that f, g and L satisfy the condition (¶′). Let us show (1). Let $\lambda > 0$ and $P \in L$. Since we have

$$\forall n \in \mathbb{Z} \setminus \{0, 1\},\ \ \mu(f'; \overline{\mathbf{i}_0\mathbf{i}_1}) > \lambda \ \text{ and } \ \mu(g'; \overline{\mathbf{i}_0\mathbf{i}_1}) > \lambda,$$

and $G = g' - (\operatorname{coeff}_{\mathbf{i}_1}(g')/\operatorname{coeff}_{\mathbf{i}_1}(f'))f'$, we have

$$\forall n \in \mathbb{Z} \setminus \{0, 1\},\ \ v_{n0}(G) > \lambda,$$

i.e.,

$$\forall n \in \mathbb{Z} \setminus \{0, 1\},\ \ \tau(G; (n, 0); P) = -v_{n0}(G) < -\lambda.$$

Noting that

$$\tau(G; (1, 0); P) = -v_{10}(G) = -\infty < -\lambda,$$

we see that

$$\forall n \in \mathbb{Z} \setminus \{0\},\ \ \tau(G; (n, 0); P) < -\lambda.$$

Let us show (2). (3) follows from (2) symmetry. Since the number of the terms of G is finite and each term of $\operatorname{trop}(G)$ is a continuous and piecewise linear map, it is sufficient to show that $\{\mathbf{i} \in \mathbb{Z}^2 \mid \exists P \in L_+^\lambda \text{ s.t. } \tau(G; \mathbf{i}; P) = \operatorname{trop}(G)(P)\} \subset \{\mathbf{i}_0, \mathbf{i}_+\}$. By the assumption that the three vertices of the corresponding 2-simplex of Δ_{f_1} are $\mathbf{i}_0$, $\mathbf{i}_1$ and $\mathbf{i}_+ = (0, 1)$, we have $L = P_+ + \mathbb{R}_{\geq 0}(-\mathbf{e}_2)$, and the condition $\Phi_1(L) = \Phi_2(L) = \overline{\mathbf{i}_0\mathbf{i}_1}$ implies

$$\forall (i, j) \in \mathbb{Z}^2, j < 0 \Rightarrow c_{ij} = d_{ij} = 0.$$

Combined with Lemma 3.17 (2), it follows that

$$\{\mathbf{i} \in \mathbb{Z}^2 \mid \exists P \in L \text{ s.t. } \tau(G; \mathbf{i}; P) = \operatorname{trop}(G)(P)\} \subset \mathbb{Z}\mathbf{e}_1 \cup \{\mathbf{i}_+\}.$$

Then, by (1) and Corollary 3.18, for any $n \in \mathbb{Z}\setminus\{0\}$ and any $P \in L_+^\lambda$, if we write $P = P_+ + r(-\mathbf{e}_2)$ $(0 \leq r < \lambda)$, we have

$$\tau(G; (n, 0); P) < \tau(g; (0, 0); P) - \lambda = \tau(g; (0, 0); P_+) - \lambda < \tau(G; \mathbf{i}_+; P),$$

and hence, the assertion holds. □

4 Proofs of the main theorems

The following proposition gives us a way of determining $\mathrm{trop}(V(f,g)) \cap L$ for $L \in \mathcal{L}_{\mathrm{s}}(f,g)$, and will be the main tool in finding a polynomial that realizes the desired intersection. Note that the points in $\mathrm{trop}(V(f,g))$ are equipped with the multiplicities coming from the intersection multiplicities of $V(f) \cap V(g)$.

Proposition 4.1 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials.*

(1) *Let $L \in \mathcal{R}_1(f,g)$ be a ray. Then, for $\lambda > 0$ and some (or any) $G \in \mathrm{Elim}(\lambda; f, g; L)$, we have*

$$\mathrm{trop}(V(f,g)) \cap L_+^\lambda = \begin{cases} \{P_+ + (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g))\mathbf{v}_{P_+,L}\} & (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) < \lambda), \\ \emptyset & (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) \geq \lambda). \end{cases}$$

In particular, $\mathrm{trop}(V(f,g)) \cap L = \emptyset$ *if and only if for any $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$, we have $v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) \geq \lambda$.*

(2) *Let $L \in \mathcal{LS}_2(f,g)$ be a line segment. Let $l = \mathrm{dist}(P_+, P_-)$. Then, for $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$, we have*

$$\begin{aligned} &\mathrm{trop}(V(f,g)) \cap (L_+^\lambda \cup L_-^\lambda) \\ &= \begin{cases} \left\{ \begin{array}{l} P_+ + (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g))\mathbf{v}_{P_+,L}, \\ P_- + (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g))\mathbf{v}_{P_-,L} \end{array} \right\} & \left(v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) < \min\left\{\frac{l}{2}, \lambda\right\}\right), \\ \left\{\frac{P_+ + P_-}{2}\right\} \ (\textit{multiplicity} = 2) & \left(\frac{l}{2} \leq v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) \textit{ and } \frac{l}{2} < \lambda\right), \\ \emptyset & \left(\lambda \leq \min\left\{\frac{l}{2}, v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g)\right\}\right). \end{cases} \end{aligned}$$

In particular, if $\lambda > l/2$, then we have

$$\mathrm{trop}(V(f,g)) \cap L = \begin{cases} \left\{ \begin{array}{l} P_+ + (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g))\mathbf{v}_{P_+,L}, \\ P_- + (v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g))\mathbf{v}_{P_-,L} \end{array} \right\} & \left(v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g) < \frac{l}{2}\right), \\ \left\{\frac{P_+ + P_-}{2}\right\} \ (\textit{multiplicity} = 2) & \left(\frac{l}{2} \leq v_{\mathbf{j}_0}(G) - v_{\mathbf{j}_0}(g)\right). \end{cases}$$

Proof We may assume that f, g and L satisfy the condition (¶′). Let us show (1). Let $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$. Let U_+ be a sufficiently small neighborhood of L_+^λ. By Corollary 3.18, for a point $(0, y) \in \mathbb{R}^2$, we have $\tau(G; \mathbf{i}_+; (0, y)) = y - y_+$.

Assume that $v_{00}(G) < \lambda$. Then, noting that $\tau(G; \mathbf{i}_0; (0, y)) = -v_{00}(G)$, we have

$$\begin{aligned} y_+ - v_{00}(G) < y &\Rightarrow \tau(G; \mathbf{i}_+; (0, y)) > \tau(G; \mathbf{i}_0; (0, y)), \\ y = y_+ - v_{00}(G) &\Rightarrow \tau(G; \mathbf{i}_+; (0, y)) = \tau(G; \mathbf{i}_0; (0, y)), \\ y < y_+ - v_{00}(G) &\Rightarrow \tau(G; \mathbf{i}_+; (0, y)) < \tau(G; \mathbf{i}_0; (0, y)). \end{aligned}$$

Combined with Lemma 3.20 (2), it follows that $\mathrm{Trop}(V(G)) \cap U_+ \cap (x = 0) = \{(0, y_+ - v_{00}(G))\}$. Note that we consider U_+ to deal with the case where $v_{00}(G) = 0$.

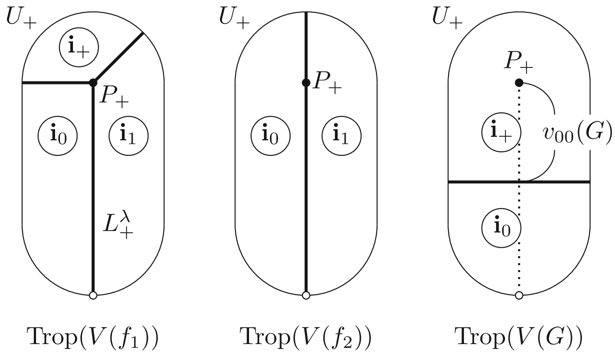


Fig. 6 $\mathrm{Trop}(V(f_1))$, $\mathrm{Trop}(V(f_2))$ and $\mathrm{Trop}(V(G))$ in a neighborhood U_+ of L_+^λ

Then, for $f_2 = f$ or g, we have

$$\mathrm{Trop}(V(f_2)) \cap \mathrm{Trop}(V(G)) \cap U_+ = \{(0, y_+ - v_{00}(G))\} \ \text{(see Fig. 6)}.$$

Hence, $\{(0, y_+ - v_{00}(G))\}$ is an isolated point of $\mathrm{Trop}(V(f_2)) \cap \mathrm{Trop}(V(G))$. Note that the intersection multiplicity of $\mathrm{Trop}(V(f_2))$ and $\mathrm{Trop}(V(G))$ at $(0, y_+ - v_{00}(G))$ is 1 (see Fig. 6). Hence, by Theorem 2.21, there exists a unique point $\mathbf{x} \in V(f_2, G)$ such that $\mathrm{trop}(\mathbf{x}) = (0, y_+ - v_{00}(G))$. Thus, by Lemma 3.13, we have

$$\mathrm{trop}(V(f, g)) \cap L_+^\lambda = \mathrm{trop}(V(f_2, G)) \cap L_+^\lambda = \{(0, y_+ - v_{00}(G))\}.$$

If $v_{00}(G) \geq \lambda$ and $y \in (y_+ - \lambda, y_+]$, then $\{\tau(G; \mathbf{i}; (0, y)) \mid \mathbf{i} \in \mathbb{Z}^2\}$ takes the maximal value only at $\mathbf{i} = \mathbf{i}_+$, and we have $\mathrm{Trop}(V(G)) \cap L_+^\lambda = \emptyset$. In this case, by Lemma 3.13, it follows that $\mathrm{trop}(V(f, g)) \cap L_+^\lambda \subset \mathrm{Trop}(V(G)) \cap L_+^\lambda = \emptyset$.

Let us show (2). Let $\lambda > 0$ and $G \in \mathrm{Elim}(\lambda; f, g; L)$. Note that $L = \{(0, y) \mid y_- \leq y \leq y_+\}$, $l = y_+ - y_-$ and that by Corollary 3.18, we have

$$\tau(G; \mathbf{i}_+; (0, y)) = y - y_+, \quad \tau(G; \mathbf{i}_-; (0, y)) = y_- - y.$$

First, consider the case where $v_{00}(G) < \min\{l/2, \lambda\}$. Let $y_1 := y_+ - v_{00}(G)$. Then, we have

$$\begin{aligned}
y_1 < y &\Rightarrow \tau(G; \mathbf{i}_+; (0, y)) > \tau(G; \mathbf{i}_0; (0, y)) > \tau(G; \mathbf{i}_-; (0, y)),\\
y = y_1 &\Rightarrow \tau(G; \mathbf{i}_+; (0, y)) = \tau(G; \mathbf{i}_0; (0, y)) > \tau(G; \mathbf{i}_-; (0, y)),\\
y_+ - \frac{l}{2} < y < y_1 &\Rightarrow \tau(G; \mathbf{i}_0; (0, y)) > \tau(G; \mathbf{i}_+; (0, y)) > \tau(G; \mathbf{i}_-; (0, y)).
\end{aligned}$$

Combined with Lemma 3.20 (3), it follows that $V(\mathrm{trop}(G)) \cap L_+^\lambda \cap L_+^{\frac{l}{2}} = \{(0, y_1)\}$, and in the same way as in (1), we have

$$\mathrm{trop}(V(f, g)) \cap L_+^\lambda \cap L_+^{\frac{l}{2}} = \{(0, y_+ - v_{00}(G))\}.$$

Similarly, we have

$$\operatorname{trop}(V(f,g)) \cap L_-^{\lambda} \cap L_-^{\frac{l}{2}} = \{(0, y_- + v_{00}(G))\}.$$

By considering the intersection multiplicity, we have

$$\operatorname{trop}(V(f,g)) \cap (L_+^{\lambda} \cup L_-^{\lambda}) = \{(0, y_+ - v_{00}(G)), (0, y_- + v_{00}(G))\}$$

(in fact, this is equal to $\operatorname{trop}(V(f,g)) \cap L$).

Next, consider the case where $l/2 \leq v_{00}(G)$ and $l/2 < \lambda$. Then, we have

$$\begin{aligned} \frac{y_+ + y_-}{2} < y &\Rightarrow \tau(G; \mathbf{i}_+; (0,y)) > \tau(G; \mathbf{i}_-; (0,y)),\ \tau(G; \mathbf{i}_0; (0,y)), \\ y = \frac{y_+ + y_-}{2} &\Rightarrow \tau(G; \mathbf{i}_+; (0,y)) = \tau(G; \mathbf{i}_-; (0,y)) \geq \tau(G; \mathbf{i}_0; (0,y)), \\ y < \frac{y_+ + y_-}{2} &\Rightarrow \tau(G; \mathbf{i}_-; (0,y)) > \tau(G; \mathbf{i}_+; (0,y)),\ \tau(G; \mathbf{i}_0; (0,y)). \end{aligned}$$

Combined with Lemma 3.20 (3), it follows that

$$V(\operatorname{trop}(G)) \cap (L_+^{\lambda} \cup L_-^{\lambda}) = V(\operatorname{trop}(G)) \cap L = \left\{\left(0, \frac{y_+ + y_-}{2}\right)\right\}.$$

and in the same way as in (1), we have

$$\operatorname{trop}(V(f,g)) \cap (L_+^{\lambda} \cup L_-^{\lambda}) = \left\{\left(0, \frac{y_+ + y_-}{2}\right)\right\}.$$

By Theorem 2.21, the multiplicity is 2.

Finally, consider the case where $\lambda \leq \min\{l/2, v_{00}(G)\}$. Here, we have

$$\begin{aligned} y_+ - \lambda < y &\Rightarrow \tau(G; \mathbf{i}_+; (0,y)) > \tau(G; \mathbf{i}_-; (0,y)),\ \tau(G; \mathbf{i}_0; (0,y)), \\ y < y_- + \lambda &\Rightarrow \tau(G; \mathbf{i}_-; (0,y)) > \tau(G; \mathbf{i}_+; (0,y)),\ \tau(G; \mathbf{i}_0; (0,y)). \end{aligned}$$

Combined with Lemma 3.20 (3), it follows that

$$V(\operatorname{trop}(G)) \cap (L_+^{\lambda} \cup L_-^{\lambda}) = \emptyset,$$

and hence, $\operatorname{trop}(V(f,g)) \cap (L_+^{\lambda} \cup L_-^{\lambda}) = \emptyset$.

Thus, we conclude the proof of Proposition 4.1. ◻

The following corollary is immediate.

Corollary 4.2 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials and $L \in \mathcal{R}_1(f,g)$ a ray. Then, there is at most one point, counted with multiplicity, in the intersection $\operatorname{trop}(V(f,g)) \cap L$.*

The following corollary shows a special case of the main theorems where $\mathcal{L}'_{\mathrm{s}}$ consists of one element.

Corollary 4.3 *Let f and $g = \sum_{i,j} d_{ij} x^i y^j$ be Laurent polynomials in $k[x^{\pm 1}, y^{\pm 1}]$ and D a divisor satisfying the condition $(*)$ in Definition 1.8. Let $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ be a ray or a line segment and let $\Phi_2(L) = \overline{\mathbf{j}_0\mathbf{j}_1}$. Then there exists an element $\tilde{d}_{\mathbf{j}_0} \in k$ such that if we set $g' := g - d_{\mathbf{j}_0}\mathbf{x}^{\mathbf{j}_0} + \tilde{d}_{\mathbf{j}_0}\mathbf{x}^{\mathbf{j}_0}$, we have $\mathrm{trop}(g) = \mathrm{trop}(g')$ and $\mathrm{trop}(V(f, g'))|_L = D|_L$.*

Proof We will show the statement in the case where $L \in \mathcal{R}_1(f, g)$, and the proof in the case where $L \in \mathcal{LS}_2(f, g)$ is similar. We may assume that f, g, L and an endpoint $P_+ := (0, y_+) \in L$ satisfy the condition (¶′). Let $P_1 = (0, y_+ - \kappa)$ $(\kappa \geq 0)$ be the intersection point of D and L. Recall that we are using Notation 3.7 and P_+ is a vertex of $\mathrm{Trop}(V(f_1))$. Since we have $\tau(f_1; \mathbf{i}_+; P_+) = \tau(f_1; \mathbf{i}_0; P_+) = 0$ and $P_1 = P_+ - \kappa\mathbf{e}_2$, we have

$$\tau(f_1; \mathbf{i}_+; P_1) = \tau(f_1; \mathbf{i}_+; P_+) - \kappa(\mathbf{i}_+ \cdot \mathbf{e}_2) = -\kappa.$$

Thus, we have

$$\kappa = -\tau(f_1; \mathbf{i}_+; P_1) = v_{\mathbf{i}_+}(f_1) - \mathbf{i}_+ \cdot P_1.$$

Since the coordinates of P_1 are assumed to belong to the value group, there exists an $\alpha \in k^*$ such that $\mathrm{val}(\alpha) = v_{\mathbf{i}_+}(f_1) - \mathbf{i}_+ \cdot P_1 = \kappa$. Let $\lambda > \kappa$, $h_\lambda := h(\lambda; f, f; \overline{\mathbf{i}_0\mathbf{i}_1})$, $h'_\lambda := h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1})$, $f_\lambda := f + h_\lambda(x) f = \sum_{i,j} c'_{ij} x^i y^j$, $g_\lambda := g + h'_\lambda(x) f = \sum_{i,j} d'_{ij} x^i y^j$ and $G_\lambda := g_\lambda - (d'_{10}/c'_{10}) f_\lambda = \sum_{i,j} e_{ij} x^i y^j$. Then $G_\lambda \in \mathrm{Elim}(\lambda; f, g; L)$ (see Definitions 3.9 and 3.10), and by the construction of g_λ, the term $\beta := d'_{00} - d_{00} \in k$ satisfies $\mathrm{val}(\beta) > 0$. We set

$$\tilde{d}_{00} = \alpha - \beta + \frac{d'_{10}}{c'_{10}} c'_{00} = \alpha + d_{00} - d'_{00} + \frac{d'_{10}}{c'_{10}} c'_{00} = \alpha + d_{00} - e_{00}.$$

Since we have

$$\mathrm{val}(\alpha) = \kappa \geq 0, \ \mathrm{val}(\beta) > 0, \ \mathrm{val}\left(\frac{d'_{10}}{c'_{10}} c'_{00}\right) = 0,$$

we have $\mathrm{val}(\tilde{d}_{00}) = 0 = \mathrm{val}(d_{00})$ if $\kappa > 0$. If $\kappa = 0$, we may assume the same by replacing α if neccesary. Let $g' := g - d_{00} + \tilde{d}_{00}$. Then, we have $\mathrm{trop}(g') = \mathrm{trop}(g)$. Note that

$$h(\lambda; g', f; \overline{\mathbf{i}_0\mathbf{i}_1}) = h(\lambda; g - d_{00} + \tilde{d}_{00}, f; \overline{\mathbf{i}_0\mathbf{i}_1}) = h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1}) = h'_\lambda,$$

since in the algorithm of Lemma 3.8, the coefficient of g at $\mathbf{i}_0$ is not used. For the Laurent polynomial

$$G'_\lambda := g' + h'_\lambda(x) f - \frac{d'_{10}}{c'_{10}} f_\lambda = G_\lambda - d_{00} + \tilde{d}_{00} = \sum_{i,j} e'_{ij} x^i y^j,$$

we have $e'_{00} = \alpha$ and $e'_{\mathbf{i}} = e_{\mathbf{i}}$ ($\mathbf{i} \neq (0, 0)$). Here, we have $\mathrm{val}(e'_{00}) = \mathrm{val}(\alpha) = \kappa$, and hence, by Proposition 4.1, we have $\mathrm{trop}(V(f, g'))|_L = D|_L$. □

Remark 4.4 In Corollary 4.3, we change the coefficient $d_{\mathbf{i}_0}$. By symmetry, we may change the coefficient $d_{\mathbf{i}_1}$ instead.

Corollary 4.5 *Let f and g be Laurent polynomials in $k[x^{\pm 1}, y^{\pm 1}]$, $L \in \mathcal{L}_{\mathrm{s}}(f, g)$ a ray or a line segment and $\Phi_2(L) = \overline{\mathbf{j}_0\mathbf{j}_1} \in \Delta_g$. Let $D := \mathrm{trop}(V(f, g))|_L$, and assume that $D \neq 0$ if L is a ray. Let $g' \in k[x^{\pm 1}, y^{\pm 1}]$ be a Laurent polynomial such that* $\mathrm{trop}(g) = \mathrm{trop}(g')$ *and*

$$v_{\mathbf{j}_0+n(\mathbf{j}_1-\mathbf{j}_0)}(g' - g) > v_{\mathbf{j}_0}(g) + n(v_{\mathbf{j}_1}(g) - v_{\mathbf{j}_0}(g)) + \mathrm{dist}(D, E|_L) \ (n \in \mathbb{Z}),$$

where E is the stable intersection divisor of $\mathrm{Trop}(V(f))$ *and* $\mathrm{Trop}(V(g))$. *Then, we have*

$$\mathrm{trop}(V(f, g'))|_L = \mathrm{trop}(V(f, g))|_L = D.$$

Proof We may assume that f, g and L satisfy the condition (¶′). Note that since $\mathrm{trop}(g) = \mathrm{trop}(g')$, we have $L \in \mathcal{L}_{\mathrm{s}}(f, g')$ and L is contained in the edge of $V(\mathrm{trop}(g'))$ corresponding to $\overline{\mathbf{i}_0\mathbf{i}_1} \in \Delta_{g'}$, and hence, f, g' and L also satisfy the condition (¶′). Since $\min(v_{\mathbf{j}_0+n(\mathbf{j}_1-\mathbf{j}_0)}(g' - g)) > \mathrm{dist}(D, E|_L)$, we can take λ such that $\lambda > \mathrm{dist}(D, E|_L)$ and $\lambda < v_{\mathbf{j}_0+n(\mathbf{j}_1-\mathbf{j}_0)}(g' - g)$ for any $n \in \mathbb{Z}$. Let $h_\lambda := h(\lambda; f, f; \overline{\mathbf{i}_0\mathbf{i}_1})$ and $h'_\lambda := h(\lambda; g, f; \overline{\mathbf{i}_0\mathbf{i}_1})$. Let $f_\lambda := f + h_\lambda(x)f = \sum_{i,j} c'_{ij}x^i y^j$, $g_\lambda := g + h'_\lambda(x)f = \sum_{i,j} d'_{ij}x^i y^j$ and $g'_\lambda := g' + h'_\lambda(x)f = \sum_{i,j} d''_{ij}x^i y^j$. Then, we have $g'_\lambda = g' + h'_\lambda(x)f = (g' - g) + g_\lambda$. Here, by the assumption, we have

$$v_{\mathbf{i}_0+n(\mathbf{i}_1-\mathbf{i}_0)}(g' - g) > \lambda \ (n \in \mathbb{Z}).$$

Combined with Lemma 3.12 (2), this implies that $\mu\left(g' + h'_\lambda(x)f; \overline{\mathbf{i}_0\mathbf{i}_1}\right) > \lambda$, and hence, $(h_\lambda, 0, h'_\lambda, 0) \in H_4(\lambda; f, g'; L)$. Let $G_\lambda := g_\lambda - (d'_{10}/c'_{10})f_\lambda$ and $G'_\lambda := g'_\lambda - (d''_{10}/c'_{10})f_\lambda$. Then, we have

$$\begin{aligned} G'_\lambda &= (g' - g) + g_\lambda - \frac{d'_{10} + \mathrm{coeff}_{00}(g' - g)}{c'_{10}} f_\lambda \\ &= (g' - g) + G_\lambda - \frac{\mathrm{coeff}_{00}(g' - g)}{c'_{10}} f_\lambda, \end{aligned}$$

and hence, $v_{00}(G'_\lambda) = v_{00}(G_\lambda) = \mathrm{dist}(D, E|_L)$. Thus, by Proposition 4.1, we have

$$\mathrm{trop}(V(f, g'))|_L = \mathrm{trop}(V(f, g))|_L = D.$$

□

Theorem 4.6 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials and D a divisor satisfying the condition $(*)$ in Definition 1.8. Let $\mathcal{L}'_{\mathrm{s}}$ be a subset of $\mathcal{L}_{\mathrm{s}}(f, g)$ and write*

$\mathcal{PI} := \mathcal{PI}(f, g)$. *Assume that* $\mathcal{L}'_{\mathrm{s}}$ *is acyclic with respect to* Φ_2 *and that for each* $L \in \mathcal{L}'_{\mathrm{s}}$, *we have* $\mathrm{dist}(D|_L, E|_L) < \mu(g; \Phi_2(L))$. *Then, there exists* $g' \in k[x^{\pm 1}, y^{\pm 1}]$ *such that* $\mathrm{trop}(g') = \mathrm{trop}(g)$ *and*

$$\mathrm{trop}(V(f, g'))|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}} = D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}.$$

Proof Let $g = \sum_{i,j} d_{ij} x^i y^j$ and $\mathcal{C}$ the union of the elements of $\Delta' := \Phi_2(\mathcal{L}'_{\mathrm{s}})$. We number and order the endpoints of the elements of Δ' as $p_1 < \cdots < p_n$ so that this ordering is normal on each tree of the forest. We write $L_{ij} \in \mathcal{L}'_{\mathrm{s}}$ for the ray or the line segment corresponding to $\overline{p_i p_j} \in \Delta'$. We will construct $g' = g - \sum_{i=1}^n d_{p_i} \mathbf{x}^{p_i} + \sum_{i=1}^n \tilde{d}_{p_i} \mathbf{x}^{p_i}$ by determining $g_j := g - \sum_{i=1}^j d_{p_i} \mathbf{x}^{p_i} + \sum_{i=1}^j \tilde{d}_{p_i} \mathbf{x}^{p_i}$ $(j = 1, \ldots, n)$ inductively. Assume that we have determined g_{t-1} with $\mathrm{trop}(g) = \mathrm{trop}(g_{t-1})$ and so that $\mathrm{trop}(V(f, g_{t-1}))|_L = D|_L$ holds for $L \in \mathcal{L}'_{\mathrm{s}}$ if both vertices of $\Phi_2(L)$ belong to $\{p_1, \ldots, p_{t-1}\}$. Let T be the connected component of $\mathcal{C}$ containing p_t, and $m = \min\{i \in \mathbb{Z} \mid p_i \in T\}$. If $t = m$, we set $\tilde{d}_{p_t} = d_{p_t}$. If $t > m$, there is a unique s such that the path $p_m T p_t$ contains $\overline{p_s p_t}$. By the normality of the ordering, $s < t$ holds, and $\tilde{d}_{p_s}$ is already determined. By the assumption, we have $\mathrm{dist}(D|_{L_{st}}, E|_{L_{st}}) < \mu(g; \overline{p_s p_t}) = \mu(g_{t-1}; \overline{p_s p_t})$. By Corollary 4.3 and Remark 4.4, we determine an element $\tilde{d}_{p_t} \in k$ such that, if we set $g_t = g_{t-1} - d_{p_t} \mathbf{x}^{p_t} + \tilde{d}_{p_t} \mathbf{x}^{p_t}$, then we have $\mathrm{val}(\tilde{d}_{p_t}) = \mathrm{val}(d_{p_t})$ and $\mathrm{trop}(V(f, g_t))|_{L_{st}} = D|_{L_{st}}$. Note that p_t might be contained in $\mathrm{Aff}(\overline{p_q p_r})$ $(q < r < t$, $\overline{p_q p_r} \in \Delta')$. To show that $\mathrm{trop}(V(f, g_t))|_{L_{qr}} = \mathrm{trop}(V(f, g_{t-1}))|_{L_{qr}}$, we check the inequality

$$v_{p_q + n(p_r - p_q)}(g_t - g_{t-1}) > v_{p_q}(g_{t-1}) + n(v_{p_r}(g_{t-1}) - v_{p_q}(g_{t-1})) + \kappa_{qr},$$

where $\kappa_{qr} := \mathrm{dist}(D|_{L_{qr}}, E|_{L_{qr}})$, and apply Corollary 4.5. This clearly holds for $n = 0$ and 1. For $n \neq 0, 1$, this follows from

$$\begin{aligned}
&v_{p_q + n(p_r - p_q)}(g_t - g_{t-1}) - v_{p_q}(g_{t-1}) - n(v_{p_r}(g_{t-1}) + v_{p_q}(g_{t-1})) - \kappa_{qr} \\
&\quad \geq v_{p_q + n(p_r - p_q)}(g_{t-1}) - v_{p_q}(g_{t-1}) - n(v_{p_r}(g_{t-1}) + v_{p_q}(g_{t-1})) - \kappa_{qr} \\
&\quad \geq \mu(g_{t-1}; \overline{p_q p_r}) - \kappa_{qr} \\
&\quad > 0.
\end{aligned}$$

By repeating this process, we get a Laurent polynomial $g' = g - \sum_{i=1}^n d_{p_i} \mathbf{x}^{p_i} + \sum_{i=1}^n \tilde{d}_{p_i} \mathbf{x}^{p_i}$ such that for all $L \in \mathcal{L}'_{\mathrm{s}}$, we have

$$\mathrm{trop}(V(f, g'))|_L = D|_L.$$

Since we have $\mathrm{trop}(g') = \mathrm{trop}(g)$, we have $\mathcal{PI}(f, g) = \mathcal{PI}(f, g') \subset \mathrm{trop}(V(f, g'))$ with the multiplicities taken into account by Theorem 2.21. This concludes the proof of Theorem 4.6. □

Theorem 4.7 *Let* $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ *be Laurent polynomials and* D *a divisor satisfying the condition* $(*)$ *in Definition 1.8. Let* $\mathcal{L}'_{\mathrm{s}}$ *be a subset of* $\mathcal{L}_{\mathrm{s}}(f, g)$ *and write*

$\mathcal{PI} := \mathcal{PI}(f, g)$. Assume that $\mathcal{L}'_{\mathrm{s}}$ is acyclic with respect to Φ_2 and that we can number and order the endpoints of the elements of $\Delta' := \Phi_2(\mathcal{L}'_{\mathrm{s}})$ as $p_1 < \cdots < p_n$ so that this order is normal on each tree of the forest and that for each element $\overline{p_i p_j}$ of Δ', its affine span $\mathrm{Aff}(\overline{p_i p_j})$ does not contain a point p_l with $l > i, j$. Then, there exists $g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that $\mathrm{trop}(g') = \mathrm{trop}(g)$ and

$$\mathrm{trop}(V(f, g'))|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}} = D|_{\mathcal{L}'_{\mathrm{s}} \cup \mathcal{PI}}.$$

Proof Let $g = \sum_{i,j} d_{ij} x^i y^j$ and $\mathcal{C}$ the union of the elements of Δ'. We write $L_{ij} \in \mathcal{L}'_{\mathrm{s}}$ for the ray or the line segment corresponding to $\overline{p_i p_j} \in \Delta'$. Let us construct $g' = g - \sum_{i=1}^{n} d_{p_i} \mathbf{x}^{p_i} + \sum_{i=1}^{n} \tilde{d}_{p_i} \mathbf{x}^{p_i}$ by determining $g_t := g - \sum_{i=1}^{t} d_{p_i} \mathbf{x}^{p_i} + \sum_{i=1}^{t} \tilde{d}_{p_i} \mathbf{x}^{p_i}$ $(t = 1, \ldots, n)$ inductively, as in the proof of Theorem 4.6. By the assumption, p_t is not contained in $\mathrm{Aff}(\overline{p_q p_r})$ $(q < r < t, \overline{p_q p_r} \in \Delta')$. Combined with Corollary 4.5, it follows that $\mathrm{trop}(V(f, g_t))|_{L_{qr}} = \mathrm{trop}(V(f, g_r))|_{L_{qr}}$. Thus, for all $L \in \mathcal{L}'_{\mathrm{s}}$, we have $\mathrm{trop}(V(f, g'))|_L = D|_L$.

Since we have $\mathrm{trop}(g') = \mathrm{trop}(g)$, we have $\mathcal{PI}(f, g) = \mathcal{PI}(f, g') \subset \mathrm{trop}(V(f, g'))$ with the multiplicities taken into account by Theorem 2.21. Thus, we conclude the proof of Theorem 4.7. $\square$

As an example of applications of Theorem 4.7, we have the following corollary, which deals with the case where a tropical line and a smooth tropical plane curve intersect.

Corollary 4.8 *Let $f, g \in k[x^{\pm 1}, y^{\pm 1}]$ be Laurent polynomials such that $\mathrm{trop}(f) = x \oplus y \oplus 0$ and $\mathrm{Trop}(V(g))$ is smooth. Let a divisor D satisfy the condition $(*)$ in Definition 1.8. Assume that the origin $(0, 0)$ is not a vertex of $\mathrm{Trop}(V(g))$. Then, there exists a Laurent polynomial $g' \in k[x^{\pm 1}, y^{\pm 1}]$ such that $\mathrm{trop}(g') = \mathrm{trop}(g)$ and $\mathrm{trop}(V(f, g')) = D$.*

Proof First, we show that all the connected components of $\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ are in $\mathcal{L}_{\mathrm{s}}(f, g) \cup \mathcal{PI}(f, g)$. Let A be a connected component of $(\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))) \setminus \mathcal{PI}(f, g)$. Since the origin $(0, 0)$ is not a vertex of $\mathrm{Trop}(V(g))$, it is clear that A is either a ray or a line segment. If the origin is an endpoind of A, the origin is a smooth vertex in $\mathrm{Trop}(V(f))$ and is contained in the interior of an edge of $\mathrm{Trop}(V(g))$. An endpoint $P \neq (0, 0)$ of A is a smooth vertex of $\mathrm{Trop}(V(g))$ and is contained in the interior of an edge of $\mathrm{Trop}(V(f))$. Therefore, all the multiplicities of the endpoints of A are 1. Since $\mathrm{Trop}(V(g))$ is smooth, it is clear that the interior of A does not contain a vertex of $\mathrm{Trop}(g)$. Therefore, we have $A \in \mathcal{L}_{\mathrm{s}}(f, g)$.

Next, let us show that the map Φ_2 is injective and the union of $\Delta' := \Phi_2(\mathcal{L}_{\mathrm{s}}(f, g))$ is a forest. First, note that $\Sigma^{(1)}(\mathrm{Trop}(V(f)))$ consists of three rays and they have different slopes and that each region of $\mathbb{R}^2 \setminus \mathrm{Trop}(V(g))$ is a convex polyhedral set. Thus, if $\Phi_2(L) = \Phi_2(L')$, then we have $L = L'$. Thus, the map Φ_2 is injective. Next, we show that the union of Δ' is a forest. Assume that the union of Δ' is not a forest, i.e., it contains a cycle C. Let $q_1, \ldots, q_m$ $(m \geq 3)$ be the vertices of C such that $\overline{q_i q_{i+1}} \in \Delta'$ for all $i = 1, \ldots, m$ (we regard $m + 1 = 1$). Let

$$D_x := \{(x, y) \in \mathbb{R}^2 \mid x > y,\ x > 0\},$$

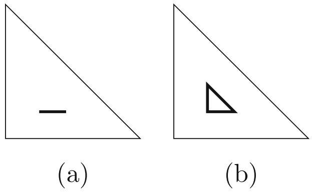


Fig. 7 Elements of $\Phi_2(\mathcal{L}_s(f,g))$

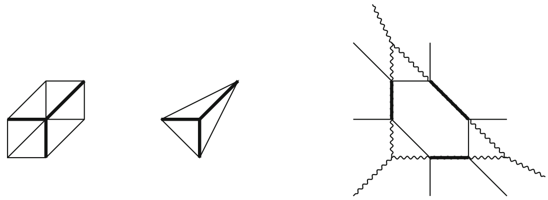
$$D_y := \{(x,y) \in \mathbb{R}^2 \mid y > x,\ y > 0\},$$
$$D_0 := \{(x,y) \in \mathbb{R}^2 \mid 0 > x,\ 0 > y\}.$$

Let $\overline{D_i}$ $(i = 1, \ldots, m)$ be the closures of the domains of $\mathbb{R}^2 \backslash \mathrm{Trop}(V(g))$ corresponding to q_i. Then, $\overline{D_i}$ are convex polyhedral sets, and hence, each intersection $\overline{D_i} \cap \overline{D_{i+1}}$ is contained in exactly one of the edges of $\mathrm{Trop}(V(f))$. Assume that $\overline{D_i} \cap \overline{D_{i+1}}$ $(i \geq 2)$ is contained in a ray $Y_- := \{(0,y) \in \mathbb{R}^2 \mid y \leq 0\}$ in $\mathrm{Trop}(V(f))$ (we can handle the cases where it is contained in other rays in a similar way). Then, we have $\overline{D_i} \cap D_0 \neq \emptyset$ or $\overline{D_{i+1}} \cap D_0 \neq \emptyset$. By renumbering if necessary, assume that $\overline{D_i} \cap D_0 \neq \emptyset$. Then, we have $\overline{D_i} \cap D_x = \emptyset$, $\overline{D_{i+1}} \cap D_x \neq \emptyset$ and $\overline{D_{i+1}} \cap D_0 = \emptyset$. Here, since $\overline{D_{i+1}}$ is a convex set and intersects D_x, the intersection $\overline{D_{i+1}} \cap \overline{D_{i+2}}$ must be contained in the ray $XY := \{(x,y) \in \mathbb{R}^2 \mid x = y \geq 0\}$. By similar arguments, we have $\overline{D_{i-1}} \cap \overline{D_i} \subset X_- := \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\}$, and so on. Thus, if $\overline{q_i q_{i+1}} \in \Phi_2(\mathcal{L}_s(f,g))$ is the bold line segment in (a) of Fig. 7, $\Phi_2(\mathcal{L}_s(f,g))$ must contain the bold line segments in (b) of Fig. 7. Here, the 2-dimensional cell of Δ_g enclosed by the bold line segments in (b) of Fig. 7 corresponds to a vertex of $\mathrm{Trop}(V(g))$. Since the edges of $\mathrm{Trop}(V(g))$ corresponding to the three 1-simplices are contained in the three edges of $\mathrm{Trop}(V(f))$, this vertex must be the origin, and this contradicts the assumption. Thus, the union of $\Phi_2(\mathcal{L}_s(f,g))$ is a forest.

To prove the statement, it is sufficient to show that we can number and order the endpoints of the elements of Δ' as $p_1 < \cdots < p_n$ so that this order is normal on each tree of the forest and that for each element $\overline{p_i p_j}$ of Δ', its affine span $\mathrm{Aff}(\overline{p_i p_j})$ does not contain a point p_l with $l > i, j$. Note that for each $\overline{\mathbf{ij}} \in \Delta'$, we have $\mathrm{val}(d_{\mathbf{i}}) = \mathrm{val}(d_{\mathbf{j}}) < \mathrm{val}(d_{\mathbf{l}})$ $(\mathbf{l} \in (\mathrm{Aff}(\overline{\mathbf{ij}}) \cap \mathbb{Z}^2) \backslash \{\mathbf{i}, \mathbf{j}\})$. Hence, if $\mathbf{i}'$ and $\mathbf{j}'$ are contained in the same connected component of the union of Δ', then $\mathrm{val}(d_{\mathbf{i}'}) = \mathrm{val}(d_{\mathbf{j}'})$. Let $p_1, \ldots, p_n$ be the endpoints of the elements of Δ' such that $\mathrm{val}(d_{p_1}) \geq \mathrm{val}(d_{p_2}) \geq \cdots \geq \mathrm{val}(d_{p_n})$ and the order $p_1 < \cdots < p_n$ is normal on each tree of the forest. For an element $\overline{p_i p_j}$ of Δ', if its affine span $\mathrm{Aff}(\overline{p_i p_j})$ contains a point p_l, then $\mathrm{val}(d_{p_l}) > \mathrm{val}(d_{p_i}) = \mathrm{val}(d_{p_j})$, and hence, by the condition of the numbering of the endpoints $p_1, \ldots, p_n$, we have $l < i, j$. $\square$

5 Examples

In the following, let $k = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with coefficients in the complex numbers with the usual valuation.



Δ_f Δ_g Trop($V(f)$) and Trop($V(g)$)

Fig. 8 The tropical curves and dual subdivisions in Example 5.1

Example 5.1 Let

$$f = t^3x^2y^2 + t^2x^2y + t^2xy^2 + xy + x + y + t^{-1} \in k[x^{\pm 1}, y^{\pm 1}],$$
$$g = t^3x^2y^2 + xy + x + y \in k[x^{\pm 1}, y^{\pm 1}].$$

Then, the tropical curves $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$ are as in Fig. 8, and hence the intersection $\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ is the union of the elements of $\mathcal{LS}_2(f, g)$. If we set $\mathcal{L}'_s = \mathcal{LS}_2(f, g)$, then it is acyclic with respect to Φ_2 and satisfies the condition in Theorem 4.7. Therefore, a divisor D satisfying the condition $(*)$ in Definition 1.8 can be realized. Here, the edges of $\mathrm{Trop}(V(g))$ corresponding to $\Phi_2(\mathcal{L}'_s)$ forms a loop, but this is irrelevant to our condition.

Now we will give two examples to show that we need the acyclicity condition.

Example 5.2 Let

$$f = xy^3 + t^2xy^2 + y^3 + t^5xy + ty^2 + t^5y + t^{10} \in k[x^{\pm 1}, y^{\pm 1}],$$
$$g = ax + by + 1 \in k[x^{\pm 1}, y^{\pm 1}] \ (\mathrm{val}(a) = \mathrm{val}(b) = 0).$$

Then, the tropical curves $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$ are as in Fig. 9, and hence the intersection $\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ is the union of the elements of $\mathcal{LS}_2(f, g)$, and the stable intersection divisor is

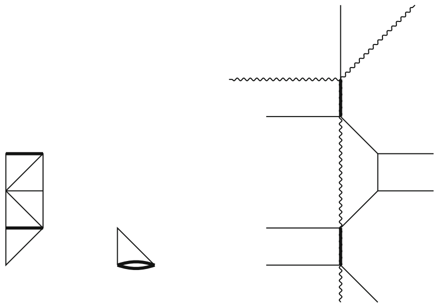
$$E = (0, 0) + (0, -1) + (0, -4) + (0, -5).$$

Let

$$D = \left(0, -\frac{1}{4}\right) + \left(0, -\frac{3}{4}\right) + \left(0, -\frac{13}{3}\right) + \left(0, -\frac{14}{3}\right).$$

Then, it is easy to see that there exists a tropical rational function ψ on $\mathrm{Trop}(V(f))$ satisfying $\mathrm{Supp}(\psi) \subset \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ and $(\psi) = D - E$. Let $L_1 = \overline{(0, 0)(0, -1)}$, $L_2 = \overline{(0, -4)(0, -5)}$ and $\mathcal{L}'_s = \mathcal{LS}_2(f, g) = \{L_1, L_2\}$. Note that the map $\Phi_2|_{\mathcal{L}'_s}$ is not injective. Assume that $\mathrm{trop}(V(f, g))|_{\mathcal{L}'_s} = D$.

First, we consider $\mathrm{trop}(V(f, g))|_{L_1}$. Noting that $\Phi_1(L_1) = \overline{(0, 3)(1, 3)}$ and $\Phi_2(L_1) = \overline{(0, 0)(1, 0)}$, we easily see that $(0, 0, 0, 0) \in \mathrm{H}_4(1; f, g; L_1)$ and that



Δ_f Δ_g $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$

Fig. 9 The tropical curves and dual subdivisions in Example 5.2

$\sum_{i,j} e_{ij}x^i y^j = g - ay^{-3}f$ belongs to $\mathrm{Elim}(1; f, g; L_1)$, we have $e_{00} = 1 - a$ and, by Proposition 4.1, $\mathrm{val}(1 - a) = 1/4$.

Next, let us consider $\mathrm{trop}(V(f, g))|_{L_2}$. For $\sum_{i,j} e'_{ij}x^i y^j := g - (a/t^5)y^{-1}f \in \mathrm{Elim}(1; f, g; L_2)$, we have $e'_{00} = 1-a$ and $\mathrm{val}(1-a) = 1/3$. This is a contradiction. Therefore, there does not exist $g \in k[x^{\pm1}, y^{\pm1}]$ such that $\mathrm{trop}(g) = x \oplus y \oplus 0$ and $\mathrm{trop}(V(f, g))|_{\mathcal{L}'_{\mathrm{s}}} = D$.

Example 5.2 explains why we need the assumption that the map $\Phi_2|_{\mathcal{L}'_{\mathrm{s}}}$ is injective.

Remark 5.3 If we regard the two bold line segments in Δ_g as different things as in Fig. 9, they form a cycle. Thus, we can regard the assumption that the map $\Phi_2|_{\mathcal{L}'_{\mathrm{s}}}$ is injective is a part of the assumption that the union of the elements of $\Phi_2(\mathcal{L}'_{\mathrm{s}})$, regarded as a multiset, is a forest.

Example 5.4 Let

$$\begin{aligned} &f = t^3x^3y^3 + tx^3y^2 + tx^2y^3 + x^2y^2 + tx^2y + txy^2 + txy + t^3 \in k[x^{\pm1}, y^{\pm1}],\\ &g = ax + by + 1 \in k[x^{\pm1}, y^{\pm1}]\ (\mathrm{val}(a) = \mathrm{val}(b) = 0). \end{aligned}$$

Then, the tropical curves $\mathrm{Trop}(V(f))$ and $\mathrm{Trop}(V(g))$ are as in Fig. 10, and hence the intersection $\mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ is the union of the elements of $\mathcal{LS}_2(f, g)$, and the stable intersection divisor is

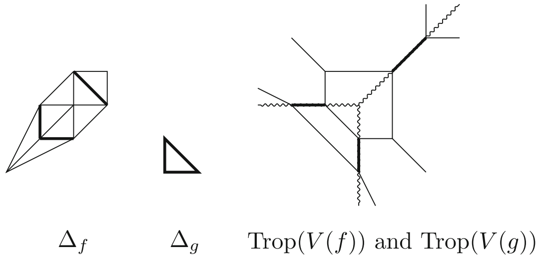
$$E = (-2, 0) + (-1, 0) + (0, -2) + (0, -1) + (1, 1) + (2, 2).$$

Let

$$D = \left(-\frac{7}{4}, 0\right) + \left(-\frac{5}{4}, 0\right) + \left(0, -\frac{5}{3}\right) + \left(0, -\frac{4}{3}\right) + \left(\frac{4}{3}, \frac{4}{3}\right) + \left(\frac{5}{3}, \frac{5}{3}\right).$$

It is easy to see that there exists a tropical rational function ψ on $\mathrm{Trop}(V(f))$ satisfying $\mathrm{Supp}(\psi) \subset \mathrm{Trop}(V(f)) \cap \mathrm{Trop}(V(g))$ and $(\psi) = D - E$. Let $L_1 = \overline{(1, 1)(2, 2)}$, $L_2 = \overline{(-1, 0)(-2, 0)}$, $L_3 = \overline{(0, -1)(0, -2)}$ and $\mathcal{L}'_{\mathrm{s}} = \mathcal{LS}_2(f, g) = \{L_1, L_2, L_3\}$. Note that

Fig. 10 The tropical curves and dual subdivisions in Example 5.4



the union of the elements of $\Phi_2(\mathcal{LS}'_2)$ is not a forest. Assume that $\mathrm{trop}(V(f,g))|_{\mathcal{LS}'_2} = D$.

First, we consider $\mathrm{trop}(V(f,g))|_{L_1}$. We have $\Phi_2(L_1) = \overline{(0,1)(1,0)}$. We regard $(0,1)$ as $\mathbf{j}_0$, and then for $\sum_{i,j} e_{ij}x^i y^j := g - (a/t)x^{-2}y^{-2}f \in \mathrm{Elim}(1; f, g; L_1)$, we have $e_{01} = b - a$. Then, by Proposition 4.1, we have $\mathrm{val}(b-a) = 1/3$.

Next, let us consider $\mathrm{trop}(V(f,g))|_{L_2}$ and $\mathrm{trop}(V(f,g))|_{L_3}$. For $\sum_{i,j} e'_{ij}x^i y^j := g - (b/t)x^{-1}y^{-1}f \in \mathrm{Elim}(1; f, g; L_2)$, we have $e'_{00} = 1 - b$ and $\mathrm{val}(1-b) = 1/4$. For $\sum_{i,j} e''_{ij}x^i y^j := g - (a/t)x^{-1}y^{-1}f \in \mathrm{Elim}(1; f, g; L_3)$, we have $e''_{00} = 1 - a$ and $\mathrm{val}(1-a) = 1/3$. Thus, we have

$$\mathrm{val}(1-a) = \mathrm{val}(b-a) = \frac{1}{3},$$
$$\mathrm{val}(1-b) = \frac{1}{4}.$$

Then, we would have

$$\frac{1}{3} = \mathrm{val}(1-a) = \mathrm{val}((1-b)+(b-a)) = \frac{1}{4}.$$

This is a contradiction. Therefore, there does not exist $g \in k[x^{\pm 1}, y^{\pm 1}]$ such that $\mathrm{trop}(g) = x \oplus y \oplus 0$ and $\mathrm{trop}(V(f,g))|_{\mathcal{L}'_{\mathrm{s}}} = D$.

Example 5.4 explains why we need the assumption that the union of the elements of $\Phi_2(\mathcal{L}'_{\mathrm{s}})$ is a forest.

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