Challenging Default Assumptions in Learning Mathematical Modeling Through Word Problems

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1. Introduction

There are two controversial opinions on the relationship between mathematical word problem-solving and mathematical modeling: *The ability of mathematical modeling can be cultivated through word problem-solving* (e.g., Czocher et al., 2020) vs. *mathematical modeling is essentially different and more complex tasks than word problem solving* (e.g., Maaß, 2006). To understand both opinions consistently, Verschaffel et al. (2020) hypothesize that mathematical word problems are oversimplified versions of mathematical modeling tasks and that word problems and modeling tasks are two poles of a continuum. In this paper, we call this *the continuum hypothesis*. However, there is little evidence that they make a continuum. In other words, we have not yet built an empirically supported theory for designing intermediate tasks between traditional word problems and mathematical modeling tasks.

The purpose of this paper is, thus, to propose a key feature of intermediate tasks between traditional mathematical word problem-solving and mathematical modeling. Following a design research methodological framework, the study presented in the present paper will empirically support our proposal.

2. Theoretical issue

First of all, the authors believe the continuum hypothesis is valid because the difference between traditional word problem-solving and mathematical modeling is the degree of their authenticity. Vos (2018) contrasts the natures of mathematical modeling in real and school contexts: Professional responsibility requires mathematical modelers in real societies to provide appropriate results through their work, while educational purposes allow students to make mistakes in classroom mathematical modeling. In this sense, mathematical modeling in classrooms cannot be completely real mathematical modeling in societies. Educational purposes rationalize to remove some authentic aspects of mathematical modeling. Vos (2018) call this *de-authentication*.

If we take this perspective, the continuum hypothesis implies that we can create intermediate tasks between traditional word problems and mathematical modeling tasks by controlling the degree of authenticity. Previous studies support this argument. We show some exemplary studies in this paper. First, Degrande, van Hoof, Verschaffel, and van Dooren (2018) propose the Snake Problem. Students can interpret this problem in multiple ways (e.g., some students regarded a snake's growth as a difference, and others as a rate). It is an authentic condition of mathematical modeling that there are multiple possible interpretations of the given situation. Second, Ishibashi and Uegatani (2022) propose the Sushi Problem, which requires students to validate their solutions based on their cultural standard of eating sushi. Cultural relevance is not given by the problem itself and is assumed by its interpreters. In this sense, students interpret such an ill-defined problem in multiple ways and need additional information to determine its solution.

Ishibashi and Uegatani (2022) report that the Sushi Problem can prompt the modeling activity to become a cyclic process. Third, Maaß (2006) proposes an authentic modeling task, which requires students to estimate the height of a statue from its photo, and Stillman and Brown (2021) offer a data-rich task. These modeling activities use non-text materials or rich data. Such information drastically increases the complexity of the modeling tasks like those in real societies. Table 1, thus, summarizes four authentic aspects presented in the existing literature. Traditional word problem-solving does not contain any authentic element (Level 1). Ill-defined word problem-solving allows students to interpret the problems in multiple ways but does not necessarily require additional information to determine their solutions (Level 2). Cyclic modeling prompts students to use additional information to validate their solutions repeatedly by its nature (Level 3). Authentic modeling is as complex as real modeling in society, except that professional responsibility is removed (Level 4).

Based on Table 1, we can describe the vital mission of this study as follows: Building a design theory for building cyclic modeling tasks. Cyclic modeling tasks (Level 3) are simple but contain minimum and essential aspects of mathematical modeling activities. To validate the continuum hypothesis, we must identify the borderline between word problem-solving and mathematical modeling. However, existing literature must provide more information on how such tasks can be designed.

3. An empirical study

We adopt a design research methodological framework (Bakker, 2018). We ask: *What task design contributes to classroom cyclic modeling?*

(1) Task design, participants, and method

The Sushi Problem created by Ishibashi and Uegatani (2022) is an example of a potential task design for cyclic modeling. They utilize cultural prejudice for task design. In their experimental lesson, the students first built a model based on their biased belief that plates are countable only by natural numbers. However, the students noticed that people share two pieces of sushi on one plate and that half of a plate can be a unit of analysis. It is suggested that the need to rebuild a model must take place so that a modeling activity becomes a cyclic process.

From this perspective, we propose the Popcorn Problem (Figure 1). In Task 1, students may offer four possibilities (Table 2). If we assume that Ken's brothers and sister did not share a cup of popcorn, only these four options are options. However, Task 2 suggests that this naïve assumption can be false. If we consider the additional information (Ken's sister had eaten one Size L cup of popcorn with caramel flavor), Ken's brothers must share one cup. The critical difference between the Sushi Problem and the Popcorn Problem is the potential way of rebuilding mathematical models. The former only requires students to reconstruct a model with a new unit of analysis, while the latter requires them to rebuild a model by using an essentially different idea. Because the Popcorn Problem suggests infinitely many possibilities about what proportion of popcorn Ken's brothers shared, the students must give up using a cup as a unit

Table 1: Authentic aspects of mathematical tasks in each level of authenticity

Levels \ Possible aspects	Multiple interpretations	Additional information	Non-text materials or rich data	Professional responsibility
1 Traditional word problem solving				
2 Ill-defined word problem solving	X			
3 *Cyclic modeling in classrooms*	X	X		
4 Authentic modeling in classrooms	X	X	X	
5 Real modeling in societies	X	X	X	X

Ken went to see the movie with his brother and his sister. Popcorn with salt and caramel flavors was sold at the following price at the theater.

		Size S	Size M	Size L
Salt	**Weight per one cup**	60g	120g	150g
	Price per one cup	400yen	500yen	550yen
Caramel	**Weight per one cup**	150g	300g	375g
	Price per one cup	400yen	500yen	550yen

Ken's brother bought much popcorn. When they measured the weight of the popcorn he purchased, it totaled 810g. Ken received 270g of popcorn from his brother and ate it. Then, his brother said, "Please pay me for the popcorn you ate." Ken obeyed with reluctance.

Task 1 What flavor and how much popcorn did Ken eat? How much should he pay his brother? Explain what we can know about these questions.

Taks 2 Ken's sister had eaten one *Size L* cup of popcorn with caramel flavor. What flavor and how much popcorn did Ken eat? How much should he pay his brother? Explain what we can know about these questions.

(*Note*: Task 2 should be presented after the students solve Task 1.)

Figure 1: The Popcorn Problem

Table 2: Four possibilities students may propose when engaging with Task 1

Salt	Caramel	Total weight	Total price
Two Size S cups	One Size S cup	270g	1200yen
Two Size S and one Size L cups		270g	1350yen
One Size M cup	One Size S cup	270g	900yen
One Size M and one Size L cups		270g	1050yen

of analysis and introduce a variable to describe infinitely many possibilities simultaneously.

We conducted an experimental lesson with the Popcorn Problem at a junior high school attached to a Japanese national university. The teacher of this lesson was the first author, who sometimes teaches this class. The participants were 19 eighth-grade students. We video-recorded the whole lesson from the rear of the classroom. Using document cameras, we also video-recorded the two students' real-time problem-solving processes. We gathered all the worksheets the students used.

(2) Results

When engaging with Task 1 in the first half of the lesson, the students noticed the four possibilities listed in Table 2. All of them agreed that they knew the existence of the four possibilities in a *mathematical* sense. However, interestingly, when the teacher asked what the best solution to the problem was, some students did not regard the four possibilities as equally likely. For example, they argued that Ken did not need two Size S cups of popcorn with salt flavor because one Size M cup was more reasonable. As another example, they also questioned whether Ken liked too much popcorn only with salt flavor; he would probably want to eat popcorn with caramel flavor, too. The students distinguished between mathematical and likely solutions to the word problem.

Through engaging with Task 2 in the second half of the lesson, the students noticed that none of the four possibilities was satisfied with the given additional condition that Ken's sister had eaten one Size L cup of popcorn with caramel flavor. In that case, Ken's brother had to eat 165g of popcorn, but they could not make 165g with the given cups of popcorn. When the students noticed this strange fact, they started looking around. The teacher considered their behaviors a sign that they wanted to discuss their ideas with their neighbors. He allowed them to discuss, and they started to assume that Ken's brothers share cups of popcorn. They let *X*g be the amount of popcorn Ken's brother ate and tried to describe how Ken should pay his brother. This episode indicates that they rebuilt their model for the situation.

4. Discussion and conclusion

As we expected, when the students started to engage with Task 2, they noticed the possibility that Ken's brother shared popcorn. The students had a default assumption that a cup of popcorn was a unit of analysis but doubted its validity when they engaged with Task 1. Let us introduce subtly different four axes for describing this experimental lesson. First, essential information (popcorn is sharable) was *experiential* rather than *given* by the problem. Second, the problem situation was *inauthentic* in Vos's (2018) sense because the information was not certificated as socially originated. Third, the problem was *fictional* rather than *real*. Forth, the problem situation was *unrealistic* rather than *realistic* because the students considered the possibility that Ken bought too much popcorn. From this perspective, we can argue that a modeling activity can be cyclic even though the problem is inauthentic, fictional, and unrealistic. Students must rebuild their model by considering the validity of their default assumption because the word problem requires them to refer to the given information and their *experiential* knowledge. Therefore, we propose a crucial feature of intermediate tasks: Essential reference to experiential knowledge for solving the problems and contrast between their default assumptions to determine which is better. Our empirical data succeeded in corroborating the continuum hypothesis.

Acknowledgments

This work was supported by the JSPS KAKENHI (Grant Number 21K13587 and 23K12847). The empirical study reported in this paper is a part of studies approved by the Research Ethics Review Board of the Graduate School of Humanities and Social Sciences, Hiroshima University (Registration Number HR-ES-000286).

References

Bakker, A. (2018). What is design research in education? In A. Bakker (Ed.), *Design Research in Education: A Practical Guide for Early Career Researchers* (pp. 3–22). Routledge.

Czocher, J. A., Moss, D. L., & Maldonado, L. A. (2020). Revitalizing and repurposing conventional word problems. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(5), 404–410. https://doi.org/10.5951/MTLT.2019.0031

Degrande, T., van Hoof, J., Verschaffel, L., & van Dooren, W. (2018). Open word problems: taking the additive or the multiplicative road?. *ZDM*, *50*, 91-102. https://doi.org/10.1007/s11858-017-0900-6

Ishibashi, I., & Uegatani, Y. (2022). Cultural relevance of validation during mathematical modeling and word problem-solving: Reconceptualizing validation as an integration of possible fictional worlds. *The Journal of Mathematical Behavior*, *66*, 100934. https://doi.org/10.1016/j.jmathb.2022.100934

Maaß, K. (2006). What are modelling competencies?, *ZDM*, *38*(2), 113–142. https://doi.org/10.1007/BF02655885

Stillman, G. A., & Brown, J. P. (2021). Modeling the phenomenon versus modeling the data set. *Mathematical Thinking and Learning*. https://doi.org/10.1080/10986065.2021.2013144

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. (2020). Word problems in mathematics education: A survey. *ZDM*, *52*(1), 1–16. https://doi.org/10.1007/s11858-020-01130-4

Vos, P. (2018). "How real people really need mathematics in the real world"—Authenticity in mathematics education. *Education Sciences*, *8*(4), 195. https://doi.org/10.3390/educsci8040195