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ORIGINAL ARTICLE

Real and illusionary difficulties in conceptual learning in mathematics: comparison between constructivist and inferentialist perspectives

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Abstract
Due to the learning paradox, students cannot have real difficulty in understanding a mathematical concept that they have not yet understood. There is a gap between real difficulties, directly experienced by students, and illusionary ones, only observed by researchers. This paper aims to offer a critical reflection on our understanding of the term *difficulty* in mathematics education research. We start this paper by arguing that a constructivist perspective, which has often been adopted in researches on mathematical task design, can deal with difficulties in solving a mathematical problem, but it cannot theoretically deal with those in understanding a mathematical concept. Therefore, we need the alternative philosophy of Robert Brandom's inferentialism to capture students' real difficulties in conceptual learning. From an inferentialist perspective, we introduce the idea of illusionary and real difficulties. The former is defined as what students cannot do, but they are not conscious of what they should do, while the latter is defined as what students cannot do despite their consciousness of what they should do. Through an eighth grade classroom episode, we argue that it is important in mathematics education research to focus not only on illusionary difficulties but also on the transition from illusionary to real difficulties. Researchers are encouraged to design a learning environment in which students become conscious of what they cannot do and to observe their mathematics learning in such an environment.

Keywords Difficulty · Conceptual learning · Misconceptions · Inferentialism · Constructivism

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Introduction

Mathematics education research aimed at improving mathematics teaching and learning has often had an interest in educational support for resolving students' difficulty in conceptual understanding (e.g., Simon, 2016, 2017a, b). Researchers have reported difficulties in understanding various mathematical concepts. As a classic example, Nesher (1987) reported a misconception of decimal numbers: Some elementary students overgeneralize the properties of natural numbers (e.g., longer numbers are larger) for decimal numbers. Nesher (1987) remarked that it would be "very difficult for [students] to give up their misconceptions"(p. 36). Like this, we have essentially regarded the understanding of the correct concept as difficult. However, when we discuss such examples of misconceptions, we use the concept of *difficulty* only intuitively. For example, why do we choose to say that *understanding the concept of decimal numbers is difficult for elementary students?* If the misconception Nesher reported stems from the overgeneralization of the properties of natural numbers, we may also say that *real difficulty lies in understanding that the properties hold only within a limited domain*. If we are allowed to choose a theoretical perspective suitable for our own interest (Cobb, 2007), does an arbitrarily chosen theoretical perspective (and thus our own interest itself) determine what is difficult? Or should we always try to grasp multifaceted images of difficulty in understanding mathematical concepts from multiple angles through a networking strategy? (Prediger et al., 2008).

The purpose of this paper is to make a critical argument of our understanding of the term *difficulty* in mathematics education research, from an emerging philosophy of Brandom's (1994, 2000) inferentialism (e.g., Bakker & Derry, 2011; Derry, 2008; Uegatani & Otani, 2021). The remainder of this paper proceeds as follows. First, we point out the *invisibility* of difficulties in understanding mathematical concepts from a constructivist perspective. Second, consideration of consciousness is needed to capture the real difficulty from a student's perspective. Third, we introduce the philosophy of inferentialism, which has received attention in mathematics education research. Fourth, we define real and illusionary difficulties. Fifth, we illustrate the transition from illusionary to real difficulties in our classroom episode through an inferentialist analysis. Sixth, using this episode, we compare constructivist and inferentialist perspectives. Finally, we conclude that the abovementioned misconception is just illusionary difficulty and that the transition from illusionary to real difficulty is essential for conceptual learning in mathematics.

Invisibility of difficulty in learning mathematical concepts from a constructivist perspective

To some extent, difficulties in understanding mathematical concepts seem to stem from the so-called learning paradox (Bereiter, 1985; Cobb et al., 1992). One of the most easily understandable formulations of the learning paradox was proposed by Sfard (1998): "How can we want to acquire a knowledge of something that is not yet known to us? Indeed, if this something does not yet belong to the repertoire of the things we know, then, being unaware of its existence, we cannot possibly inquire about it" (p. 7). From this point of view, especially if we introduce a distinction between a *mathematical concept* (a mathematical idea in a community) and a *mathematical conception* (a subjective counterpart of the concept), like Sfard (1991) and Simon (2017a), it is not rational to assume that a mathematical concept ontologically independent from students directly influences their mental construction of the corresponding mathematical conception. In fact, it is strange that the concept of decimal numbers, which has yet to exist in a complete form in one's head, obstructs one's appropriate mental construction of decimal numbers. For the same reason, it is also strange that the concept of natural numbers tends to encourage one to construct the properties of natural numbers in an easily over-generalizable way. In this sense, if we treat a mathematical concept as an existential entity, it is theoretically impossible that a systematic misconception of a concept emerges, depending on the concept.

As a theoretical means of resolving this paradox, a constructivist tradition of learning mathematics has emphasized the importance of reflection on the process of solving mathematical problems, thereby adopting a problem-solving approach to the acquisition of a new concept. In one of the most famous examples, Sfard (1991) argued that a new algebraic object could emerge from a process of already known calculation. Since any mathematical activity does not necessarily contribute to learning an intended concept, constructivist task design principles are informed by a French didactic assumption that any concept emerges as a rational solution from an appropriate problematic situation (Brousseau, 1997; Brousseau & Warfield, 2014). Such a problematic situation causes perturbation, which makes a particular piece of knowledge meaningful (Harel, 2013). Based on this assumption, Simon and Tzur (2004) illustrated that reflective abstraction on the process of finding equivalent fractions with diagrams leads to noticing a calculating protocol for finding equivalent fractions. Harel (2013) identified five *intellectual needs* (i.e., needs for certainty, causality, computation, communication, and structure) as a key to the development of mathematics as a discipline. In constructivist research, it is a shared view that a way of mathematical thinking is crystalized as a mathematical concept (Tall, 2011). The existence of a problematic situation, thus, functions as the *raison d'être* for a mathematical concept. From this point of view, environmental factors inherent in mathematical activities logically constrain the possible mental construction of mathematical concepts. Therefore, if a misconception emerges in an appropriate mathematical problem-solving activity despite our theoretical expectation, it must originate from the activity rather than from the concept. It is a problem-solving activity, not a solution to the problem, which is accessible from students' perspectives during the mental construction of their own conceptions. However, this view has at least two problems when used to explain students' difficulty in learning mathematical concepts.

First, the effectiveness of the problem-solving approach cannot directly explain what is difficult in learning mathematical concepts. As Simon (2017b) cautions, the problem-solving approach for teaching mathematical concepts assumes that a student is a strong problem-solver and that reflection on problem-solving experiences leads to the mental construction of mathematical concepts. It is a fundamentally necessary condition for constructing mathematical concepts that students can solve given mathematical problems. Hence, it is assumed that students' difficulty in learning mathematical concepts only resides in solving given problems themselves. For example, as a concrete mathematical problem, consider the calculation of the sum of all the natural numbers from 1 to 100. Calculating 101×50 is more efficient than direct addition. There is an intellectual need for calculation in Harel's (2013) sense. Solving this problem can provide students with opportunities to reflect on their efficient use of associative and commutative laws and to deepen their conceptual understanding of some aspects of natural numbers. However, we do not know how students can reach this efficient solution. Success in solving the problem depends heavily on trial and error. It is due to this dependent relationship that Simon (1995) wittily argues "The only thing that is predictable in teaching is that classroom activities will not go as predicted" (p. 133). In addition, supporting students is not easy for teachers. When some students give up solving the problem, if the teacher provides them with some impolitic hints to an efficient method of calculation (for example, questioning "How many 101 s can you make?"), then the mathematical idea of rearranging augends and addends to be taught may be inaccessible from students' points of view. This phenomenon is well known as the Topaze effect in French didactics (Brousseau, 1997). Thus, the problem-solving approach does not offer any explanation of what is difficult for students in learning mathematical concepts except for a trivial one: They have difficulty in finding the key mathematical concepts to solve problems.

Second, as the above example of the sum of the natural numbers from 1 to 100 indicates, the problem-solving view implies that a mathematical activity develops only a limited aspect of a mathematical concept. A mathematical conception develops through the accumulation of various mathematical activities. Although an efficient technique related to the associative and commutative laws is underdeveloped before the current mathematical activity for calculating the sum of 1 to 100, some students become ready to find the technique through the encounters with the two laws in previous mathematical activities, while others do not. This indicates the possibility that the difficulty in solving a particular problem stems not from the problem itself but from the previous activities. However, the problem-solving approach does not provide a theoretical framework for conjecturing how students can develop key mathematical ideas for the current activity through previous activities. In addition, it cannot predict how they develop key ideas for the next activity through the current activity. We should understand the problem-solving approach as a merely theoretical approach that uses various practical cognitive constrains to help students learn a given specific concept. It is less concerned with conceptual development in unconstrained environments.

This problem may stem from the constructivist metaphor of *construction*. This metaphor indicates that understanding is seen as building new mental structures for newly arrived perceptions (e.g., the first encounter with the associative and commutative laws) or restructuring products of previous construction (e.g., new encounters with the two laws in different applicable contexts) (Ernest, 2006). Constructivists assume that knowledge built by students' active construction is tested as viable through environmental constraints (von Glasersfeld & Cobb, 1984). The acceptance of this assumption in mathematics education research is based on findings in research on misconceptions (Confrey & Kazak, 2006): Because students often conceptualize something in idiosyncratic manners, they need some constraints to notice the ways of more rational conceptualization. We acknowledge that this constructivist view has contributed to the research on mathematical conceptions. However, the view itself does not provide any explanation for students' difficulties in learning mathematical concepts. It accepts as natural, rather than as problematic, the fact that students often conceptualize mathematical ideas differently from standard conceptualization in school mathematics. Thus, difficulty shifts from students' learning to teachers' task design for imposing appropriate constraints. The constructivist view fails to explain the difficulties that students have in learning mathematical concepts. In this sense, those difficulties are invisible from the constructivist perspective.

Difficulty and consciousness: impossibility of feeling difficulty in understanding a mathematical concept

The invisibility of the difficulty in constructivism is advantageous in a sense. Constructivists apply their views to their own activities. Constructivist teachers and researchers are also active builders of their own knowledge (Steffe, 1995; Thompson, 2000; Ulrich et al., 2014). Thus, if students do not learn mathematics well, teachers and researchers must try to improve their learning. We can say that this view appropriately reflects the very nature of teachers as reflective practitioners (Jaworski, 2014; Schon, 1984).

The important implication of this constructivist view is that the concept of difficulty is related to consciousness. When one consciously attempts to do something but cannot do it, one can find it difficult. A problem-solving approach in a constructivist tradition is a way of making students conscious that it is difficult for them to solve the problem by using their prior knowledge. This is why the problem functions as a *raison d'être* for the target concept. von Glasersfeld (1995) distinguishes a higher order conscious type of reflective abstraction from the other types, and calls it *reflected* abstraction. However, the difficulty is not in understanding the target concept but in solving the problem. The layer of difficulty is different.

From this perspective, we should be careful about the way we use the term *difficulty*. In everyday discourses, when we observe that one cannot do something, we may say that one has difficulty in doing it. However, this situation needs to be distinguished more precisely. Suppose that even if one is conscious of what one should do, one cannot do it. Then, we can confidently say that one has difficulty in doing it. However, if we only observe that one cannot do it, then it might be inappropriate to say that one has difficulty in doing it. For example, even if we observe that some European people eat Japanese food with forks, we do not immediately conclude that they have difficulty using chopsticks. To conclude so, we must at least know that they *consciously* avoid using chopsticks. If they do not know about chopsticks as a way of eating Japanese food, there is no evidence of their difficulty in using chopsticks. It is possible that one does not precisely and consciously understand what one is expected to do.

The same holds true for learning mathematics. Consider again a mathematical activity for calculating the sum of 1 to 100. At the beginning of the activity, one can become conscious that one should calculate the sum efficiently. However, one cannot become conscious that one should calculate the sum by using the associative and commutative laws in advance. As this example indicates, one's initial consciousness of the goal of a mathematical activity does not match what one should actually do in the activity. The perceived goal is relatively vague at the beginning of the activity. Through trial and error in the activity, one gradually becomes conscious of what one should really do. Therefore, if we observe that one tries to solve a given problem but cannot solve it immediately, then we can roughly say that one has difficulty in solving it; however, we cannot say that one has difficulty in using a particular mathematical idea when one tries to solve the problem. In this sense, students can never find it difficult to conceptualize mathematical ideas.

This argument may be counterintuitive since we all have some experience of feeling difficulty in understanding a mathematical concept in our schooldays, to a greater or lesser extent. However, the above argument is theoretically acceptable. When students find it difficult to understand a mathematical concept, their feeling does not indicate a real difficulty in understanding *the concept*; otherwise, the learning paradox occurs. Rather, we should more precisely say that it is difficult for them to construct a *consistent* conception. What they can actually try is only to construct a consistent conception. Even if they succeed in constructing a consistent one, they can never confirm that it matches the concept as a whole. Owing to the learning paradox, they cannot try to understand the concept precisely in principle.

This theoretical argument does not mean that we, educators and researchers, must stop inquiring about students' difficulties in learning mathematical concepts. The argument does not eliminate the fact that many students intuitively intend to understand a mathematical concept but often cannot succeed in understanding it. Therefore, a different perspective is needed that allows us to speak about the cases where students understand the relevant aspects of the target concepts and yet fail to do what they are expected to do with that understanding. In the following two sections, we introduce a philosophical perspective of inferentialism and, from this perspective, conceptualize a difference between real and illusionary difficulty.

Inferentialism

As we argued in the previous section, one gradually comes to understand what one should do by trial and error in a mathematical activity. In the following way, we can interpret the fact that many students intuitively try to understand a mathematical concept; they can *only seemingly try* to do so because the meaning of understanding the concept is vague for them. In this section, we introduce inferentialism so we can explain the fact that many students try to understand a mathematical concept.

Inferentialism was originally proposed by Brandom (1994) to provide an alternative paradigm to representationalism. Representationalism is a philosophical position that attempts to explain the meaning of words and the nature of cognition based on representational relations between words and their meanings. For example, when one claims, "I try to understand the concept of natural numbers," representationalists (particularly those who are concerned with mathematics learning) may think that "the concept" in the claim represents one's mental image of the concept. However, this view leads to a contradiction, that is, the learning paradox: One only knows the term *natural number*; before understanding what the concept of natural number represents, one cannot try to understand it.

Inferentialism is a contemporary philosophy that builds semantics on pragmatics (Brandom, 1994, 2000) and a critical position against representationalism (Bakker & Derry, 2011; Brandom, 2000). Inferentialism has gradually gained attention in educational research, for example, in history (McCrory, 2021), in science (Causton, 2019), and mathematics (e.g., Bakker & Hußmann, 2017; Nilsson, 2020; Ryan & Chronaki, 2020; Seidouvy et al., 2019; Seidouvy & Schindler, 2020).

In inferentialism, the order of explanation for meaning is the reverse of the traditional order, which representationalists and some old expressivists adopt. Inferentialists do not see one's expression in a public domain as *a sign of one's understanding in one's private domain*. Instead, they see one's expression in a public domain as *what one understands*.

> [W]e might think of the process of expression in the more complex and interesting cases as a matter not of transforming what is inner into what is outer but of making *explicit* what is *implicit*. This can be understood in a pragmatist sense of turning something we can initially only *do* into something we can *say*: codifying some sort of knowing *how* in the form of a knowing *that*.
>
> (Brandom, 2000, p. 8, italics in the original)

In philosophy of language, Brandom introduces a distinction between what is implicit and explicit in social communication. An educational philosophy of inferentialism for mathematics education follows this stance and avoids considering the private mental constructions of learners, turning instead towards their norm-governed activity in applying concepts. Since a private mental state is vague, fragile, and uncertain, we assume that one deepens one's understanding through expressing. This theoretical assumption is characterized as a kind of *sociogenetic* approaches (Radford, 2009; Roth, 2016; Roth & Thom, 2009). Conceptual learning is seen as obtaining the practical ability to judge what can be inferred from what.

Figure 1 illustrates the differences between representationalist and inferentialist accounts for meanings and conceptualizations. Suppose that the two students, Joe and Kei, talk about the triangle ABC: (1) Kei argues, "The triangle ABC is an isosceles triangle"; (2) Joe asks, "Why do you think so?"; and (3) Kei will respond, (A) "Because it has two equal sides" or (B) "Because it looks like isosceles." If her response is A, representationalists may think that she correctly responds to Joe based on her understanding of what the term isosceles triangle means, while if her response is B, they may think that her conceptualization of the isosceles triangle is weak. In a representationalist account, it is theoretically assumed that her expression "Because it has two equal sides" in a public domain is produced based on her understanding of an isosceles triangle in her private domain. The mental state is the meaning of the expression. An expression about an idea *I* is regarded as a fragment of the already determined understandings of *I*. On the other hand, in an inferentialist account

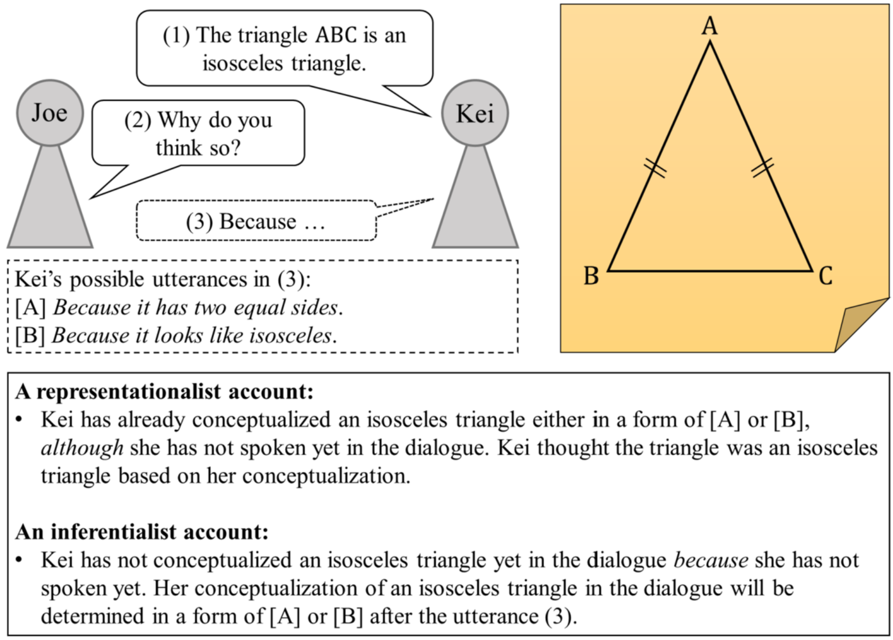

Fig. 1 Comparison between representationalist and inferentialist accounts for meanings and conceptualization

based on the use theory of meaning, an act of explicitly expressing about an idea *I* is theoretically interpreted as an act of determining an aspect of understandings of *I* (Uegatani & Otani, 2021). It is important that inferentialists do not look for the meaning of the expression inside one's private domain. One's private mental state is often vague, and thus, unreliable even from one's own perspective. Inferentialists think that conceptualization progresses in a clear, robust, and conscious form through expressing. Hence, inferentialists think that Kei's conceptualization of the isosceles triangle in a particular situation is determined by expressing a sentence about an isosceles triangle in a public domain. Her way of conceptualizing the isosceles triangle is determined by the response she chooses, A or B. Meanings are developed not in the private domain but in a public domain.

From an inferentialist perspective, such a meaning is essentially social rather than personal. Choosing an expression means building a foundation of social communication rather than creating a suitable representation for a personally intended meaning. Since inferentialist semantics is based on pragmatics, the meaning is the role played by such a foundation in social communication.

> It is at this point that Sellars introduces his central thought: that to have *conceptual* content is just for it to play a role in the *inferential* game of making claims and giving and asking for reasons. To grasp or understand such a concept is to have practical mastery over the inferences it is involved in—to know, in the practical sense of being able to distinguish (a kind of know-how), what follows from the applicability of a concept, and what it follows from.
>
> (Brandom, 2000, p. 48, italics in the original)

Inferentialists understand social communication by using concepts as *a game of giving and asking for reasons*. When one makes a claim *C*, other interlocuters may ask for a reason for *C*, and one may respond to their challenges by giving such a reason, that is, by asserting another claim from which *C* follows.

> expressing something is *conceptualizing* it: putting it into conceptual form. I said at the outset that the goal of the enterprise is a clear account of sapient awareness, of the sense in which being aware of something is bringing it under a concept. On the approach pursued here, doing that is making a claim or judgment about what one is (thereby) aware of, forming a belief about it—in general, addressing it in a form that can serve as and stand in need of reasons, making it *inferentially* significant.
>
> (Brandom, 2000, p. 16, italics in the original)

Any claim could serve as a reason in the future. It can also stand in need of reasons. Expressing such an inferential relationship between claims (i.e., what can be a reason for what) in social communication is conceptualizing particular aspects of concepts included in claims.

From this point of view, we assume that, theoretically, all mental states in private domains are vague, and thus, there is no clear connection between conceptualization and one's mental state. What is worthy of being called *conceptualization of an idea I* is articulating a relevant ascpet of the idea *I* by asserting some sentences about *I* in a public domain so that interlocutors can look back on how the articulator understands the idea. Articulating works to codify and determine it. To investigate conceptualization, we only need to observe what one infers from what.

As a leading educational inferentialist, Derry (2013) pointed out through the review of Spinoza's and Vygotsky's thoughts, "we can only be said to be free when we are guided by adequate knowledge rather than when we are moved by external causes" (p. 92). One can have one's own free will to intend to do something only when adequate knowledge can support one to articulate the intention. "Hence, free action is not a matter of choice or volition but of the mind's activity as opposed to its passivity" (Derry, 2013, p. 96). Based on this argument, talking about one's own action explicitly is a necessary condition for becoming able to choose to take the action freely.

As Derry (2008) argued, the meaning of a concept depends on a system of judgments, explicitly claiming a statement about the concept is a sign of conceptual learning of the concept in inferentialism (Uegatani & Otani, 2021). This view is consistent with a recently developed view in mathematics education research that knowing when to use a concept is a part of knowing the concept (Lavie et al., 2019; Sfard & Lavie, 2005). Although constructivists may see learning as active construction of knowledge (von Glasersfeld, 1995), we as inferentialists require that learning should include not only active construction but also active expression of knowledge in a propositional form.

The constructivist problem-solving approach, which assumes that a solution to a problematic situation is a source of a new concept, provides the *raison d'être* of the concept. From this perspective, we can treat the concept as if it has tangible presence by itself. However, from an inferentialist perspective, pedagogy can be understood "as a process of adjusting the connection of ideas already known but connected differently" (Derry, 2013, p. 96). Although a problematic situation is needed as a motivation to rearrange a connection, it does not necessarily cause perturbation in a constructivist sense, as reviewed in the earlier section. In inferentialism, a problematic situation is always inherent in social communication, that is, a game of giving and asking for reasons. A problematic situation arises when one notices that the other's way of using signs (including terms and symbols) in a propositional form is different from one's own way (for example, when one notices an unusual inferential connection between claims and when one notices that the other's claim includes an inappropriate term). A game of giving and asking for reasons is a process of *deontic scorekeeping*, or a process of keeping track of the deontic statuses of interlocuters (Brandom, 1994; Derry, 2017). This process holds because all the participants have their own system of using signs. Thus, "[i]t was within a system that, for example, sensitivity to contradiction was possible" (Derry, 2013, p. 79).

In addition, inferentialism introduces the ideas of *material inference* and its *nonmonotonicity*.

> The kind of inference whose correctnesses determine the conceptual contents of its premises and conclusions may be called, following Sellars, *material* inferences. As examples, consider the inference from "Pittsburgh is to the west of Princeton" to "Princeton is to the east of Pittsburgh," and that from "Lightning is seen now" to "Thunder will be heard soon." It is the contents of the concepts *west* and *east* that make the first a good inference, and the contents of the concepts *lightning* and *thunder*, as well as the temporal concepts, that make the second appropriate. Endorsing these inferences is part of grasping or mastering those concepts, quite apart from any specifically *logical* competence.
>
> (Brandom, 2000, p. 52, italics in the original)

Then, a material inference is called monotonic "if the fact that the inference from p to q is a good one meant that the inference from p & r to q must be a good one" (Brandom, 2000, p. 87). An example of a nonmonotonic inference is as follows: the fact that a match is struck (p) implies that it will fire (q); however, the fact p and that it is wet (r) does not imply that it will fire (q). People often reach different conclusions (q and $\neg q$) based on the same assumption (p) because they implicitly have different assumptions in their heads. Their incompatible conclusions may allow them to notice the existence of such implicit assumptions. From an educational perspective, implicit unarticulated ideas are seen as conceptualized through conversation, which fits well into the fundamental view that conceptual development in mathematics proceeds in a zigzag manner, as described by Lakatos (1976). Human cognition is always restricted by what we can express in a propositional form.

Real and illusionary difficulties

In this section, we formulate the meaning of difficulty from an inferentialist point of view. First, suppose that someone cannot do something from our point of view and does not actually intend to do it. We then call this *illusionary difficulty*. It includes cases in which one cannot intend to do it in principle due to the learning paradox.

However, suppose one could not do something at a moment from our perspective and that one made explicit that which had to be done at the moment. We find a *real difficulty* in such a situation. When one could not do it and could not articulate that one should do it, the difficulty would be illusionary at that moment. When one becomes conscious of and articulates what one should do, the difficulty becomes real. Since one's intention in a private domain is vague from an inferentialist perspective, it retrospectively becomes clear by expressing one's consciousness in an activity. Therefore, we should theoretically think that real difficulty is formed only when one expresses what one should do.

For example, when we only observe that some people stick their chopsticks into food like forks, their difficulty in picking up food with chopsticks should be considered illusionary. We do not say they stick their chopsticks because they misunderstand how to use them. Instead, they understand a pair of chopsticks as a tool for such usage. When they explicitly claim that they cannot pick up food with chopsticks, their difficulty should be considered real.

These definitions of illusionary and real difficulties have a methodological advantage: They exclude the possibly ambiguous cases where students are conscious of what they should do but do not express it. Until they make their consciousness explicit, they do not encounter real difficulty.

A stable conception, like a misconception, is only a sign of illusionary difficulty by its definition. In the introduction, we mentioned some students' difficulty in understanding the concept of decimal numbers. However, this is not a real difficulty in our view. A misconception is a kind of conception (Confrey, 1991). They are stable for students, and students do not become conscious of other possible conceptualizations. They cannot intend to understand both the concepts of decimal and natural numbers in the same manner as ours. Hence, a problematic situation in social communication is needed in order for one to become conscious of the possibilities of appropriate alternative conceptualization.

By using the two terms *illusionary* and *real difficulties*, we argue for *the importance of research on a shift from illusionary difficulty to real difficulty* rather than research on only either of them. If researchers want to study students' difficulties, they need to design an environment in which the students can shift from illusionary difficulty to real difficulty. If the students only show illusionary difficulty, the researchers cannot determine whether the difficulty originates from didactically avoidable factors or from essentially unavoidable factors, that is, whether they are didactical or epistemological obstacles (Brousseau, 1997).

We acknowledge that educational research often starts from the observation of illusionary difficulties. In this sense, we do not intend to criticize the researchers who treat an illusionary difficulty as if it is a real difficulty, although our word choice of *illusionary* may be aggressive. Focusing on illusionary difficulties at the initial stage of research is valuable as a kind of research heuristics. However, we have a concern that if we only consider illusionary difficulties in later stages, it may mean that we put our efforts into a pseudo problem. As constructivists suggest, even if students do not show any illusionary difficulty in learning mathematics, it is only a sign of their consistency in a particular classroom situation and not a sign of their correct understanding of a given concept beyond the classroom situation. Even if we try to remove illusionary difficulties by making sure that students do not find it difficult to solve a given problem, they may still face another illusionary difficulty in a similar situation. On the other hand, if their difficulty is a real one, they will be able to deal with a similar difficulty since they are conscious of what they should do in such a situation. It is, therefore, important in mathematics education to make students become conscious of their difficulties from an inferentialist perspective.

An example of the transition from illusionary to real difficulties

In this section, from our classroom episode of eighth grade students' mathematical activity, we exemplify the transition from illusionary to real difficulties.

Episode

This episode comes from a lesson conducted by the first author. Forty-one students participated in the lesson (21 males and 20 females). This was an experimental lesson in a school attached to a national university in Japan. All the students and their parents agreed to the experimental curriculum in their regular lessons when they entered the school. The lesson was videotaped, and the students' worksheets were gathered after the lesson.

In the lesson, the students engaged in solving the following three problems:

1. Solve a system of equations $\begin{cases} y = 2x + 3 \\ y = -x + 9 \end{cases}$
2. Suppose that a system of equations $\begin{cases} y = 2x + 3 \\ y = ax + 1 \end{cases}$ has a solution where $1 < x < 4$. Then, determine the range of the constant a.
3. Suppose that a system of equations $\begin{cases} y = 2x + 3 \\ y = ax + b \end{cases}$ has a solution where $1 < x < 4$. Then, determine the relationship between the constants a and b.

Problem [1] was a simple review of how to solve a system of equations. In problem [2], the students were required to find the values of a where $x = 1$ and where $x = 4$. Because a represents the slope of the straight line $y = ax + 1$, the range to be determined is between the two values of a. In problem [3], the students were required to think in a similar way to problem [2]. However, the important difference

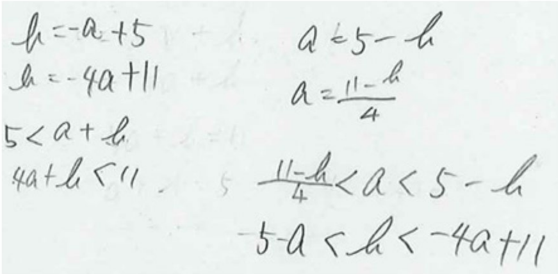

Fig. 2 An example of the students' solutions to problem [3]

between the two problems is the need to compare the two values of a. Given $x = 1$, $a = 5 - b$; given $x = 4$, $a = \frac{11-b}{4}$. However, which value of a is greater depends on the constant b. Thus, the students should conclude that (i) $\frac{11-b}{4} < a < 5 - b$ (where $b < 3$); (ii) $a = 2, b = 3$ (where $b = 3$); and (iii) $5 - b < a < \frac{11-b}{4}$ (where $b > 3$).

It seemed to us that, in the task design phase, the difficulty the students faced in trying to obtain a correct answer to problem [3] appeared to be an illusionary difficulty. In particular, the difficulty of considering all the three cases was illusionary for them as they were yet to realize that they should consider more than one case. Generally speaking, Japanese eighth grade students have only leant to solve mathematical problems with a single arbtirary constant. They are yet to learn how to solve mathematical problems with two constants taking arbitrary real values respectively.

The structure of the lesson was as follows: (1) solving problem [1] individually; (2) checking the solution to it with the entire class; (3) trying to solve problem [2] individually; (4) discussing how to solve it with peers and in the entire class; (5) trying to solve problem [3] individually; and finally, (6) discussing how to solve it with peers and the entire class.

In Sect. (5) of the lesson, the teacher noticed that many students only provided the following formula as a solution to problem [3]:

$$\frac{11-b}{4} < a < 5 - b$$

Based on their worksheets, our research team retrospectively found that at least 10 students[1] provided the same solution. This seemed to be derived from an analogy with the solution to problem [2] because the students did not pay attention to which is greater, $\frac{11-b}{4}$ or $5 - b$. Figure 2 shows an example of the students' writings.

In Sect. (6) of the lesson, the teacher called attention to this solution as follows[2] (T in the transcript represents the teacher).

T: (to the whole class) $a = 5 - b$ and $a = \frac{11-b}{4}$. Many students wrote these in the worksheets. Because we obtained 4 and $\frac{5}{2}$ in the previous problem (Problem [2]), the solution was "from $\frac{5}{2}$ to 4." In a similar way, more and more stu-

[1] Because the other students erased their initial solutions to problem [3] on their worksheets, we did not identify what their initial solutions were.

[2] The language in the lesson is Japanese. The transcript is translated into English by the authors.

dents are providing "from $\frac{11-b}{4}$ to $5-b$" (as a solution to Problem [3]). But is this the only solution?

After a while, the teacher started a class discussion (Chi and Hin in the transcript are pseudonyms of the students, and the formula $5-b$ was written above the formula $\frac{11-b}{4}$ on the blackboard).

T: Did many of you think as follows? If $y = ax + b$ passes through (1, 5), then $a = 5 - b$. If $y = ax + b$ passes through (4, 11), then $a = \frac{11-b}{4}$. In the previous problem (Problem [2]), if (the line $y = ax + 1$) passes through (1, 5), then $a = 4$, and if (the line $y = ax + 1$) passes through (4, 11), then $a = \frac{5}{2}$. So, the range of a was from $\frac{5}{2}$ to 4. Yes. Now, because we obtained $5 - b$ and $\frac{11-b}{4}$, are these the start and the end of the range of a? Thinking in this way, many students provided the inequality ($\frac{11-b}{4} < a < 5 - b$). But, please reconsider. Although 4 was evidently greater than $\frac{5}{2}$ in the previous problem, which is greater in the current problem? Chi, which is greater ($5 - b$ or $\frac{11-b}{4}$ written on the blackboard)?
Chi: The above one? (Pointing to the formula $5 - b$).
T: Oh, the above one is greater? Is the above one always greater?
Chi: Always? Are you saying such things?
T: Yes. I am saying so.
Chi: It depends on the numbers.
T: Aha, it depends on the numbers. That's right. That's right. Depending on what value we assign to b, the order relation (between $5 - b$ and $\frac{11-b}{4}$) may change. So, what should we do?

Immediately after listening to the teacher's final question, Hin, who sat in the first row, looked regretful and sighed.

Hin: Oh, case by case….
T: Yes. Case by case. Depending on the value of b, we must change which the end of the range is ($5 - b$ or $\frac{11-b}{4}$).

After this explanation, the students and the teacher agreed that the order relation between $5 - b$ and $\frac{11-b}{4}$ changed at $b = 3$. At the end of the lesson, the solutions by case were shared with all participants as a final conclusion.

An inferentialist interpretation of the episode

In the episode, Chi and Hin solved problem [3] in their own ways before Sect. (6) of the lesson. However, a new problematic situation occurred when the teacher asked Chi if $5 - b$ was *always* greater than $\frac{11-b}{4}$. Chi realized that the result of the comparison between $5 - b$ and $\frac{11-b}{4}$ depended on the values of b. Hin's regret indicated that the same held for her. Thus, we can interpret the episode above as a transition of an illusionary difficulty into a real one: The students initially could not consider all possible values of the two variables and they did not find it necessary to do so, but with the help of the teacher, they realized what they should do and then find a way to do so—an argument by cases.

Their perception of this problematic situation was caused by the teacher's question, rather than being inherent in problem [3] itself. As Derry (2013) pointed out, the inconsistency between their use of signs (the symbol "$<$" and the expressions "$5 - b$" and "$\frac{11-b}{4}$") was recognized within a whole system of mathematical concepts including numbers, functions, and formulas. They had already been able to use the concept of order relation and that of mathematical expression with two variables separately, but they could not appropriately use the concepts simultaneously before the teacher's question. For the students, the teacher's question caused a problematic situation. The change from illusionary to real difficulty in using the two concepts simultaneously was an opportunity for the students to relocate the two ill-connected concepts into a clearer whole system. Chi's utterance "It depends on the numbers" and Hin's murmur "case by case" were evidence that learning occurs in a game of giving and asking for reasons.

For inferentialists, this moment, where a real difficulty arises, is the very starting point of reconceptualization of the two concepts of order relation and expression with two variables. As we argued, knowing when and how to use these concepts is part of knowing them. The students could not appropriately solve problem [3] in Sect. (5). An aspect of the appropriate way of using the two concepts simultaneously was implicit and unconscious from their perspective when they tackled problem [2]. It became explicit when they reviewed their approach to problem [3] with the teacher's support.

It is not evident whether the students will be able to behave appropriately when they encounter a similar mathematical problem with the two concepts in the future. This lesson is not sufficient for them to master how to use these concepts simultaneously. However, it is important from an inferentialist perspective that they consciously experienced the difficulty in using the two concepts simultaneously. Although using these concepts simultaneously is still difficult for them, the conscious experience of this difficulty makes this difficulty manageable in the future as long as they do not completely forget the experience.

We do not intend to indicate that the two concepts of order relation and expression with two variables should always be used only in a particular way, although we used the term "appropriate" in the above paragraph for the purpose of brevity. For example, if the students consider the order relation in a polynomial ring at the tertiary level in the future, they will be required to use the two concepts in a different way than the way demonstrated in the episode. Ways of using concepts are sensitive to context, and there is no absolute way to use them. Unlike representationalists, we do not think that the same term can represent different concepts, depending on the context. For example, if the term *order relation* can represent two different concepts of order relation in the episode and in the context of a polynomial ring, we abandon the value of recognizing two different structures with the same ordinal structure through mathematical abstraction. Rather, we take an inferentialist stance that the same concept can be used differently, depending on the context: Knowing a variety of ways to use a concept is a part of conceptual development.

Discussion: comparison between the constructivist and inferentialist views

In the above episode, conceptual learning also can occur twice in a constructivist sense. First, reflective abstraction in the process of solving problem [2] can lead to the initial solution to problem [3] in Simon and Tzur's (2004) sense. Second, the teacher's question can cause an intellectual need for communication in Harel's (2013) sense because it prompted the students to formalize the order relation between expressions with two variables, and to formulate the three solutions by cases based on the formalized meaning of the order relation between expressions with two variables (i.e., in the sense that the order relation depends on values to be assigned). However, this perspective clips this episode only as the development of a new *static* understanding of the two concepts. It does not provide any explanation of how the students' conceptualization of the two concepts can continue to proceed *dynamically* after the lesson.

It is now widely accepted that constructivism as a post-epistemology does not imply any practical prescription, although a constructivist philosophy often used to be connected with a prescription that direct teaching prevents individual active construction of knowledge (Cobb, 2002; Noddings, 1990). Hence, a constructivist view does not explain what might happen if, for example, the students engage in drill and practice using the two concepts after the lesson. By contrast, if we take an inferentialist point of view, such drill and practice can continuously contribute to the adjustment of the inferential connection between the two concepts as a kind of conceptual development. From this perspective, meaningless learning never arises even in drills and practice, as long as some inferential connections are strengthened or updated.

In addition, from a constructivist perspective, it is difficult to answer why Chin and Hin did not solve problem [3] in an appropriate way without the teacher's question, though they seem to have the two concepts of order relation and expressions with two variables at least separately. However, inferentialists can simply say that the two concepts were unconnected before engaging in problem [3] because finding the solution to Problem [2] did not connect them.

Learning from Sects. (3) to (6) in the lesson can be explained in terms of non-monotonic inference. In problem [2], the students inferred the following: when $x = 1$ (p_1), the end of the range of a can be obtained (q_1); when $x = 4$ (p_2), the start of the range of a can be obtained (q_2). In problem [3], the students repeated this inference. However, the teacher's question challenged it; there is an overlooked condition that the value of b determines which the end is (r). The consciousness of this condition (r) changed the students' conclusions. This is evidence that an implicit idea on the inferential relation between the two concepts of order relation and expression with two variables is conceptualized by making it explicit in the zigzag process of a mathematical activity.

As our example shows, the students did not notice, by themselves, that depending on the value of b, they must change what the end of the range is, $5 - b$ or $\frac{11-b}{4}$. As inferentialists suggest but constructivists do not explicitly state, the students behaved in an invalid way influenced by their previous experience until they obtained an opportunity to express this fact (e.g., Chi's utterance "It depends on the numbers" and Hin's utterance "[C]ase by case"). Note that after the students obtained the opportunity, they correctly recognized what they should and could infer from their commited assumption. This is why we argue for the importance of active expressions by themselves.

Conclusion

This paper discusses the issue of our intuitive understanding of the term *difficulty*: If we simply observe that students do not do something, we cannot immediately conclude that they have difficulty in doing so. Since the learning paradox makes it impossible to explain the difficulty in learning a concept simply as a difficulty in trying to know the concept itself without knowing what it is, we invoked an inferentialist perspective to formulate the distinction between illusionary and real difficulties and illustrated the transition from illusionary to real difficulties in our classroom episode. Conceptual learning does not occur when an implicit idea appears in one's head; rather, it occurs when it is made explicit in a propositional form. The transition from illusionary difficulty to real difficulty in a classroom is expected to create students' future manageable difficulties in new situations. Therefore, we as inferentialists put a stronger emphasis on learners' active expressions than on their active construction of knowledge. While a constructivist view seems to understand conceptualization as a static achievement, our inferentialist view can capture conceptualization as a dynamic continuous process not only through problem-solving but also through drill and practice.

Returning to the initial questions posed in the introduction, we argue that a misconception of decimal number indicates only an illusionary difficulty. Real difficulties never arise without consciousness. When one articulates that some properties of natural numbers are not extendable to decimal numbers and that one cannot appropriately judge what properties are extendable, the difficulty in extending the properties becomes real.

This inferentialist view differs from the constructivist one because it focuses on articulation, not unobservable subjective consciousness. The inferentialist view enables us to avoid the learning paradox without appealing to some unobservable factors. Before one wants to learn a concept, one does not need to know it. The key to learning the concept is the observable transition from illusionary difficulty to real difficulty in using a particular set of material inferences involving the term for the concept.

This paper sheds light on the advantage of the inferentialist perspective for capturing the real difficulty in conceptual learning in mathematics. However, as a limitation of this paper, we can point out that our episode focuses on the development of relatively small mathematical ideas. Mathematics education research has often focused on big ideas, such as numbers, functions, and geometrical figures. Since an inferentialist perspective views conceptual development as dynamic, it cannot describe a mathematical concept as large at a particular moment. Thus, we acknowledge that the different perspectives work better for different purposes. In fact, the constructivist perspective is still useful, for example, in designing a mathematical task.

While the constructivist perspective emphasizes the active role of learners' knowledge construction, the inferentialist perspective can focus on the active role of teachers' intervention in mathematics education and the essential role of social environments in which learners express their ideas. Adopting a networking strategy (Prediger et al., 2008), the two perspectives will play different roles in a future research project. A mathematical task is designed from the constructivist perspective, and its implementation in a classroom is analyzed from the inferentialist perspective.

Author contribution Yusuke Uegatani and Hiroki Otani first contributed to the study conception and design and made a poster presentation for the third research meeting of the Japan Society for Science Education in 2018. Material preparation and data collection were performed by Yusuke Uegatani. As researchers of the philosophy of inferentialism, Shintaro Shirakawa and Ryo Ito were invited to analyze data and all authors participated in reanalysis of the empirical classroom episode presented at the abovementioned research meeting. The first draft of the manuscript was written by Yusuke Uegatani and all authors commented on previous versions of the manuscript. All authors have read and approved the final manuscript.

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Declarations

Ethics approval This study has been approved by the Research Ethics Review Board of the Graduate School of Humanities and Social Sciences, Hiroshima University, Registration number HR-ES-000286.

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