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Some new examples of nonorientable maximal surfaces in the Lorentz-Minkowski 3-space

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Abstract. We study nonorientable maximal surfaces in the Lorentz-Minkowski 3-space $\mathbb{L}^3$. We construct nonorientable maximal surfaces containing hypocycloid and maximal surfaces in $\mathbb{L}^3$ which are homeomorphic to the real projective plane $\mathbb{RP}^2$ minus two points with degree of the Gauss map being equal to 4.

1. Introduction

The first example of a complete nonorientable minimal surface in the Euclidean 3-space $\mathbb{R}^3$ was found by W. H. Meeks III [16] in 1981. He constructed a minimal Möbius strip with total curvature -6π (see Figure 1.1: left). After that, M. Elisa G. G. de Oliveira [18] generalized Meeks' example to the surface with total curvature $-2\pi(2n+1)$, where $n \in \mathbb{Z}_{>0}$ (see Figure 1.1: middle).

The Björling problem asks for the existence of a minimal surface containing a given curve in $\mathbb{R}^3$ and a given unit normal coincides with its Gauss map. H. A. Schwarz gave the representation formula of the solution of this problem. Minimal Möbius strips constructed by Meeks and Oliveira contain a circle. So we can construct them as solutions of the Björling problem by giving a circle and suitable unit normals. Meeks and M. Weber [17] constructed minimal Möbius strips by using Schwarz' formula (see Figure 1.1: right).

In [16], Meeks showed the only complete nonorientable minimal surface with total curvature -6π is the minimal Möbius strip he constructed. Therefore, there does not exist a complete minimal immersion of the real projective plane minus two points $\mathbb{RP}^2\backslash\{p_1, p_2\}$ into $\mathbb{R}^3$ with total curvature -6π.

A maximal surface in Lorentz-Minkowski 3-space $\mathbb{L}^3$ is a spacelike surface with zero mean curvature. Maximal surfaces share many local properties with minimal surfaces in $\mathbb{R}^3$. However, their global properties are quite different.

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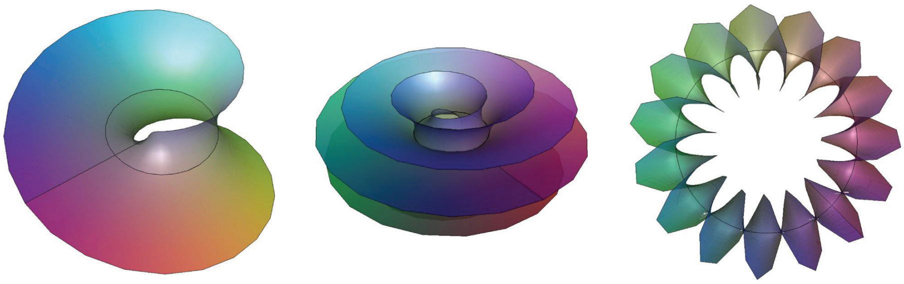

Fig. 1.1. Minimal Möbius strips

In fact, the only complete maximal surface in $\mathbb{L}^3$ is the plane [2, 3]. Maximal surfaces with singularities are investigated in [4], [5], [6], [7], [11], [19] and so on. M. Kokubu and M. Umehara investigated co-orientability of maximal surfaces in [13]. The Björling problem for the maximal surfaces is studied in [1], [15], and so on.

S. Fujimori and F. J. López [10] studied some basic aspects of the global theory of nonorientable maximal surfaces with singularities. They constructed one-ended maximal Klein bottles, and higher genus nonorientable maximal surfaces are constructed by S. Fujimori and the author [8]. S. Fujimori and F. J. López [10] also constructed two maximal Möbius strips, one contains an epicycloid and the other contains a hypocycloid.

In this paper, we construct some new examples of nonorientable maximal surfaces. In Section 2, we recall the definition of nonorientable maximal surface with singularities and introduce some basic properties of these surfaces. In Section 3, we construct maximal Möbius strips containing epicycloids and hypocycloids. In Section 4, we construct a nonorientable maximal surface homeomorphic to $\mathbb{RP}^2 \setminus \{p_1, p_2\}$ and show its uniqueness under some assumptions (Theorem 4.1).

2. Preliminaries

Let $\mathbb{L}^3$ be the three dimensional Lorentz-Minkowski space with the metric $\langle\, ,\rangle := dx_1^2 + dx_2^2 - dx_3^2$. In $\mathbb{L}^3$, we can define the cross-product $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{L}^3$, given by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} -(a_2b_3 - a_3b_2) \\ -(a_3b_1 - a_1b_3) \\ a_1b_2 - a_2b_1 \end{pmatrix},$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{L}^3$.

An immersion $f : M \to \mathbb{L}^3$ of a 2-manifold M into $\mathbb{L}^3$ is called *spacelike* if the induced metric $\mathrm{I} = f^*\langle\,,\rangle$ is positive definite. Let (u, v) be a local coordinate system of $U \subset M$. We can define a unit normal vector field as

$$\nu = \frac{f_u \times f_v}{|f_u \times f_v|} : U \to H^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 \mid x_1^2 + x_2^2 - x_3^2 = -1\}.$$

We call this vector the *Gauss map* of f. We define the first and the second fundamental forms I II of f by

$$\mathrm{I} = \langle df, df\rangle, \qquad \mathrm{II} = -\langle df, d\nu\rangle.$$

We then define the Gaussian curvature K and the mean curvature H by

$$K = -\det(\mathrm{I}^{-1}\mathrm{II}), \qquad H = \frac{1}{2}\,\mathrm{tr}(\mathrm{I}^{-1}\mathrm{II}).$$

A spacelike immersion $f : M \to \mathbb{L}^3$ is said to be *maximal* if the mean curvature vanishes identically.

Definition 2.1 (Maxfaces [19], see also [9]). Let M be a Riemann surface and $f : M \to \mathbb{L}^3$ a C^∞-map. f is called a maxface if there exists an open dense subset W of M so that the restriction $f|_W : W \to \mathbb{L}^3$ is a conformal maximal immersion and the rank of df at p is positive for all $p \in M$. A point $p \in M$ is said to be singular if $\mathrm{rank}(df) < 2$ at p.

Theorem 2.2 (Weierstrass-type representation [12, 19]). *Let (g, η) be a pair consisting of a meromorphic function g and a holomorphic differential η on a Riemann surface M such that*

$$(1 + |g|^2)^2 \eta\bar{\eta} \tag{2.1}$$

gives a Riemannian metric on M. Set

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} (1 + g^2)\eta \\ i(1 - g^2)\eta \\ 2g\eta \end{pmatrix}. \tag{2.2}$$

Suppose that

$$\mathrm{Re} \oint_\gamma \Phi = \mathbf{0} \tag{2.3}$$

holds for any $\gamma \in H_1(M, \mathbb{Z})$. Then

$$f = \mathrm{Re}\int_{z_0}^{z} \Phi : M \to \mathbb{L}^3, \qquad (z_0 \in M) \tag{2.4}$$

is a maxface. We call (M, g, η) *(or* (g, η)*) the Weierstrass data of* $f : M \to \mathbb{L}^3$ *(we define the Weierstrass data of a nonorientable maxface later).*

Remark 2.3. (i) *The first fundamental form* I *and the second fundamental form* II *of the surface* (2.4) *are given by*

$$\mathrm{I} = (1 - |g|^2)^2 \eta\bar{\eta}, \qquad \mathrm{II} = \eta\, dg + \overline{\eta\, dg}.$$

The singular set corresponds to $\{p \in M \mid |g(p)| = 1\}$.

(ii) *Let* (g, η) *be the Weierstrass data of* f. (g, η) *can be written as*

$$g = \frac{\phi_3}{\phi_1 - i\phi_2}, \qquad \eta = \frac{1}{2}(\phi_1 - i\phi_2).$$

Moreover, g *coincides with the composition of the unit normal* ν *of* f *and the Lorentzian stereographic projection* σ*, that is,* $g = \sigma \circ \nu$. *Thus we call* g *the Gauss map of* f.

(iii) *The condition* (2.3) *is called the period problem, which guaranties the well-defindness of* f. *It is equivalent to*

$$\oint_\gamma g^2\eta + \overline{\oint_\gamma \eta} = 0 \tag{2.5}$$

and

$$\mathrm{Re} \oint_\gamma g\eta = 0 \tag{2.6}$$

for any $\gamma \in H_1(M, \mathbb{Z})$.

Definition 2.4 (Complete maxfaces [19]). A maxface $f : M \to \mathbb{L}^3$ is said to be *complete* if there exists a compact set C and a symmetric $(0,2)$-tensor T on M such that T vanishes on $M \backslash C$ and $\mathrm{I} + T$ is a complete Riemannian metric.

Proposition 2.5 ([19, Proposition 4.5]). *Let* $f : M \to \mathbb{L}^3$ *be a complete maxface with the Weierstrass data* (M, g, η). *Then the Riemann surface* M *is biholomorphic to a compact Riemann surface* $\overline{M}$ *minus a finite number of points* $\{p_1, \ldots, p_n\}$. *Moreover,* g *and* η *extend meromorphically to* $\overline{M}$.

We say that a complete maxface $f : \overline{M} \backslash \{p_1, \ldots, p_n\} \to \mathbb{L}^3$ is of genus k if $\overline{M}$ is a compact Riemann surface of genus k.

Let μ_i denote the winding number of the multigraph f around p_i. It is easy to check that

$$\mu_i = \max\{\mathrm{Ord}_{p_i}(\phi_j) \mid j = 1, 2, 3\} - 1,$$

where $\mathrm{Ord}_{p_i}(\phi_j)$ is the pole order of ϕ_j at p_i.

Theorem 2.6 ([19, 5]). *Let $\overline{M}$ be a compact Riemann surface and $f : M = \overline{M} \setminus \{p_1, \dots, p_n\} \to \mathbb{L}^3$ a complete maxface with the Weierstrass data (g, η). Then $g : \overline{M} \to \mathbb{C} \cup \{\infty\}$ satisfies*

$$2 \deg g = -\chi(\overline{M}) + \sum_{i=1}^{n} (\mu_i + 1) \tag{2.7}$$

where $\chi(\overline{M})$ denotes the Euler characteristic of $\overline{M}$. In particular,

$$2 \deg g \geq -\chi(\overline{M}) + 2n, \tag{2.8}$$

and equality holds if and only if all ends are properly embedded.

Let M' be a nonorientable surface with conformal coordinates. We denote $\pi : M \to M'$ the orientable conformal double cover of M'.

Definition 2.7 (Nonorientable maxfaces [10]). A conformal map $f' : M' \to \mathbb{L}^3$ is said to be a *nonorientable maxface* if the composition

$$f = f' \circ \pi : M \to \mathbb{L}^3$$

is a maxface. In addition, f' is said to be complete if f is complete.

We say that a complete nonorientable maxface $f' : M' \to \mathbb{L}^3$ is of genus k $(k \geq 1)$ if the double cover M of M' is biholomorphic to $\overline{M} \setminus \{p_1, \dots, p_n\}$, where $\overline{M}$ is a compact Riemann surface of genus $k - 1$.

Let $f' : M' \to \mathbb{L}^3$ be a nonorientable maxface, and let $I : M \to M$ denote the antiholomorphic order two deck transformation associated with the orientable double cover $\pi : M \to M'$. Since $f \circ I = f$, we have

$$g \circ I = \frac{1}{\bar{g}} \qquad \text{and} \qquad I^*\eta = \overline{g^2 \eta}. \tag{2.9}$$

Conversely, if (g, η) is the Weierstrass data of an orientable maxface $f : M \to \mathbb{L}^3$ and I is an antiholomorphic involution without fixed points in M satisfying (2.9), then the unique map $f' : M' = M/\langle I \rangle \to \mathbb{L}^3$ satisfying that $f = f' \circ \pi$ is a nonorientable maxface. We call (M, I, g, η) the Weierstrass data of $f' : M' \to \mathbb{L}^3$.

Theorem 2.8 (Björling representation [1, Theorem 3.1]). *Let $f : U \subset \mathbb{C} \to \mathbb{L}^3$ be a maximal surface, and define $c(s) = f(s, 0)$, $n(s) = v(s, 0)$ on a real interval $I \subset U$. Choose any simply connected open set $\Delta \subset U$ containing I over which we can define holomorphic extensions $c(z)$, $n(z)$ of c, n. Then for all $z \in \Delta$ it holds*

$$f(z) = \operatorname{Re} \int_{s_0}^{z} (c' - in \times c') dz.$$

Here s_0 is a point fixed but arbitrary of I and the integral is taken along an arbitrary path in Δ joining s_0 and z.

3. Cycloids and maximal Möbius strips

3.1. Epicycloids and maximal Möbius strips. Let $c(t) = (x(t), y(t))$ be an epicycloid. Then it is written by

$$\begin{cases} x(t) = (r_c + r_m)\cos t - r_m \cos\left(\dfrac{r_c + r_m}{r_m} t\right), \\ y(t) = (r_c + r_m)\sin t - r_m \sin\left(\dfrac{r_c + r_m}{r_m} t\right), \end{cases} \tag{3.1}$$

where r_c and r_m are radii of a fixed circle and a rolling circle around outside of a fixed circle respectively [14]. See Figure 3.1.

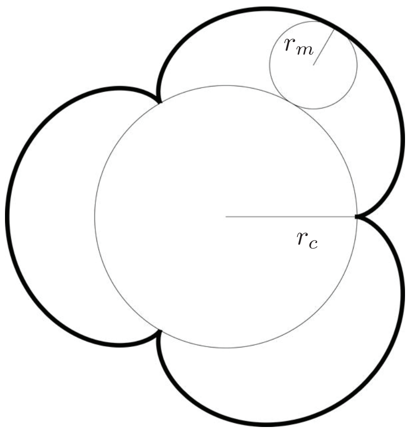

Fig. 3.1. The thick curve indicates an epicycloid, the big circle is a fixed circle, and the small circle is a rolling circle.

In the case $r_m = 1/(k+1)$, $r_c = 1/(k(k+1))$, $k \in \mathbb{Z}_{>0}$, (3.1) becomes

$$\begin{cases} x(t) = \dfrac{\cos t}{k} - \dfrac{\cos(((k+1)/k)t)}{k+1}, \\ y(t) = \dfrac{\sin t}{k} - \dfrac{\sin(((k+1)/k)t)}{k+1}. \end{cases} \tag{3.2}$$

After the substitution $t = 2k(\theta + \pi/2)$ and multiplying the right-hand side by $(-1)^k$, we have

$$\begin{cases} x(\theta) = \dfrac{\cos 2k\theta}{k} + \dfrac{\cos(2(k+1)\theta)}{k+1}, \\ y(\theta) = \dfrac{\sin 2k\theta}{k} + \dfrac{\sin(2(k+1)\theta)}{k+1}. \end{cases} \tag{3.3}$$

See Figure 3.2.

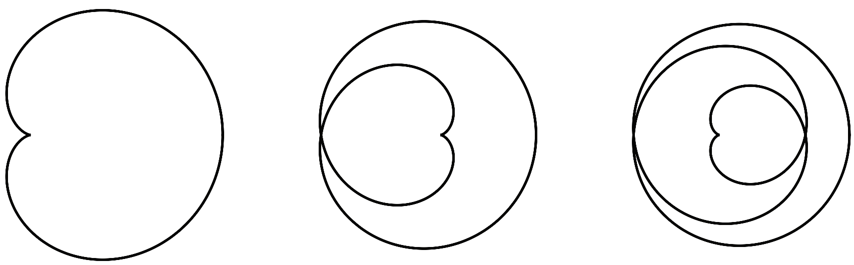

Fig. 3.2. Epicycloids (left: $k = 1$, middle: $k = 2$, right: $k = 3$)

We define the Weierstrass data (M, I, g, η) as

$$\begin{aligned} M &= \mathbb{C}\backslash\{0\}, & I(z) &= -\frac{1}{\bar{z}}, \\ g &= \frac{z^{2k+1}(z+1)}{z-1}, & \eta &= i\frac{(z-1)^2}{z^{2k+3}}\,dz, \end{aligned} \tag{3.4}$$

where $k \in \mathbb{Z}_{>0}$. (3.4) gives a complete nonorientable maxface with $\deg g = 2(k+1)$. These data are given by [10, Remark 3.2]. See Figure 3.3.

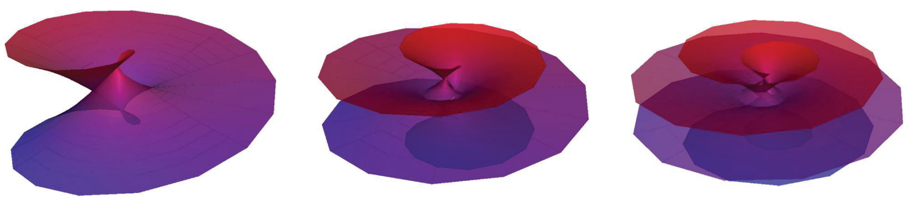

Fig. 3.3. Maximal Möbius strips given by (3.4) (left: $k = 1$, middle: $k = 2$, right: $k = 3$)

When $z = e^{it}$, we have

$$\mathrm{Re}\int \phi_3 = 2\,\mathrm{Re}\int i\frac{z^2-1}{z^2}\,dz = -4\,\mathrm{Re}\int i \sin t\,dt = 4\,\mathrm{Re}(i\cos t + C_3) = 4\,\mathrm{Re}\,C_3,$$

where C_3 is a complex constant. So we can assume the image of $z = e^{it}$ by f is a planar curve on the x_1x_2-plane. A straightforward computation gives

$$\begin{cases} \mathrm{Re} \displaystyle\int \phi_1 = -\frac{\sin(2(k+1)t)}{k+1} - \frac{\sin(2kt)}{k} + \mathrm{Re}\ C_1, \\ \mathrm{Re} \displaystyle\int \phi_2 = \frac{\cos(2(k+1)t)}{k+1} + \frac{\cos(2kt)}{k} + \mathrm{Re}\ C_2, \end{cases}$$

where C_1 and C_2 are complex constants. Define a curve $\tilde{c}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$ as follows:

$$\begin{cases} \tilde{x}_1(t) = -\dfrac{\sin(2(k+1)t)}{k+1} - \dfrac{\sin(2kt)}{k}, \\ \tilde{x}_2(t) = \dfrac{\cos(2(k+1)t)}{k+1} + \dfrac{\cos(2kt)}{k}. \end{cases} \tag{3.5}$$

After the rotating the curve $\tilde{c}(t)$ in $\mathbb{L}^3$ by $-\pi/2$ about the x_3-axis, we obtain $c(t) = (x_1(t), x_2(t))$ as follows:

$$\begin{cases} x_1(t) = \dfrac{\cos(2(k+1)t)}{k+1} + \dfrac{\cos(2kt)}{k}, \\ x_2(t) = \dfrac{\sin(2(k+1)t)}{k+1} + \dfrac{\sin(2kt)}{k}. \end{cases} \tag{3.6}$$

(3.6) coincides with (3.3). So we see the surface given by (3.4) contains the epicycloid.

Remark 3.1. *We set a curve* $c(t) = (\tilde{x}_1(t), \tilde{x}_2(t), 0)$ *as* (3.5) *and a unit normal* $n(t)$ *of* $c(t)$ *as*

$$n(t) = \left(-\sin((2k+1)t)\ \tan t, \cos((2k+1)t)\ \tan t, \frac{1}{\cos t}\right).$$

Applying Theorem 2.8 *to* $c(t)$ *and* $n(t)$, *we obtain the same surface defined by* (3.4).

3.2. Hypocycloids and maximal Möbius strips. Let $c(t) = (x(t), y(t))$ be a hypocycloid. Then it is written by

$$\begin{cases} x(t) = (r_c - r_m)\cos t + r_m \cos\left(\dfrac{r_c - r_m}{r_m} t\right), \\ y(t) = (r_c - r_m)\sin t - r_m \sin\left(\dfrac{r_c - r_m}{r_m} t\right), \end{cases} \tag{3.7}$$

where r_c and r_m are radii of a fixed circle and a rolling circle around inside of a fixed circle respectively [14]. See Figure 3.4.

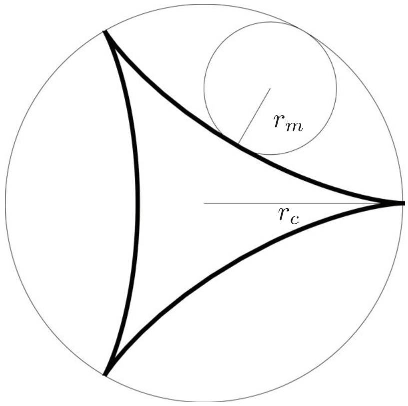

Fig. 3.4. The thick curve indicates a hypocycloid, the big circle is a fixed circle, and the small circle is a rolling circle.

In the case $r_m = k/(2k+1)$, $r_c = 1$, $k \in \mathbb{Z}_{>0}$, (3.7) becomes

$$\begin{cases} x(t) = \left(\dfrac{k+1}{2k+1}\right) \cos t + \left(\dfrac{k}{2k+1}\right) \cos\left(\dfrac{k+1}{k} t\right), \\ y(t) = \left(\dfrac{k+1}{2k+1}\right) \sin t - \left(\dfrac{k}{2k+1}\right) \sin\left(\dfrac{k+1}{k} t\right). \end{cases} \tag{3.8}$$

After the substitution $t = 2k\theta + k\pi/(2k+1)$, we have

$$\begin{cases} x(\theta) = \left(\dfrac{k+1}{2k+1}\right) \cos\left(2k\theta + \dfrac{k\pi}{2k+1}\right) + \left(\dfrac{k}{2k+1}\right) \cos\left(2(k+1)\theta + \dfrac{k\pi}{2k+1}\right), \\ y(\theta) = \left(\dfrac{k+1}{2k+1}\right) \sin\left(2k\theta + \dfrac{k\pi}{2k+1}\right) - \left(\dfrac{k}{2k+1}\right) \sin\left(2(k+1)\theta + \dfrac{k\pi}{2k+1}\right). \end{cases}$$

Rotating the curve in $\mathbb{L}^3$ by $\pi/(2(2k+1))$ about the x_3-axis and scaling the curve by multiplying $(2k+1)/(k(k+1))$, we have

$$\begin{cases} x(\theta) = -\dfrac{\sin(2k\theta)}{k} - \dfrac{\sin(2(k+1)\theta)}{k+1}, \\ y(\theta) = -\dfrac{\cos(2k\theta)}{k} + \dfrac{\cos(2(k+1)\theta)}{k+1}. \end{cases} \tag{3.9}$$

See Figure 3.5.

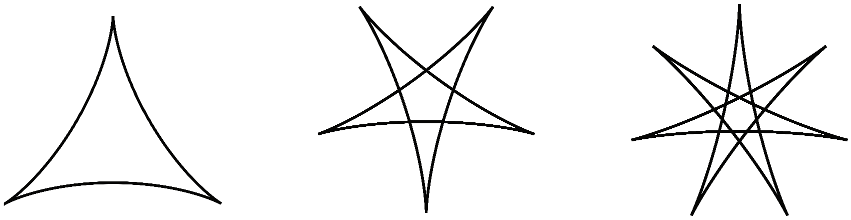

Fig. 3.5. Hypocycloids (left: $k = 1$, middle: $k = 2$, right: $k = 3$)

We define the Weierstrass data (M, I, g, η)

$$\begin{aligned} M &= \mathbb{C}\setminus\{0\}, \qquad I(z) = -\frac{1}{\bar{z}}, \\ g &= \frac{z(z^{2k+1}-1)}{z^{2k+1}+1}, \qquad \eta = i\frac{(z^{2k+1}+1)^2}{z^{2k+3}}\,dz, \end{aligned} \tag{3.10}$$

where $k \in \mathbb{Z}_{>0}$. (3.10) gives a complete nonorientable maxface with $\deg g = 2(k+1)$. See Figure 3.6.

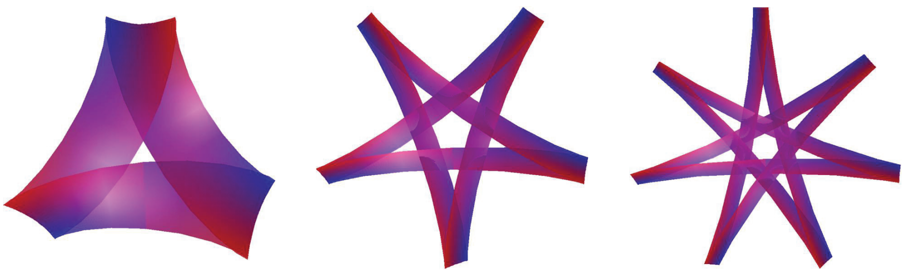

Fig. 3.6. Maximal Möbius strips given by (3.10) (left: $k = 1$, middle: $k = 2$, right: $k = 3$)

Remark 3.2. *In* [10], *Fujimori and López constructed a maximal Möbius strip defined by*

$$\begin{aligned} M &= \mathbb{C}\setminus\{0\}, \qquad I(z) = -\frac{1}{\bar{z}}, \\ g &= \frac{z(z-r)(z-s)(z-t)}{(z+r)(z+\bar{s})(z+\bar{t})}, \qquad \eta = i\frac{(z+r)^2(z+\bar{s})^2(z+\bar{t})^2}{z^5}\,dz, \end{aligned} \tag{3.11}$$

where $r \in \mathbb{R}_{>0}$, $s, t \in \mathbb{C}\setminus\{0\}$. *In the case* $(r, s, t) = (1, e^{2\pi i/3}, e^{-2\pi i/3})$, (3.11) *coincides with* (3.10) *with* $k = 1$.

When $z = e^{it}$, we have

$$\mathrm{Re}\int \phi_3 = 2\ \mathrm{Re}\int i\frac{z^{2(2k+1)}-1}{z^{2(k+1)}}\ dz = 4\ \mathrm{Re}\left(\frac{i\cos((2k+1)t)}{2k+1} + C_3\right) = 4\ \mathrm{Re}\ C_3,$$

where C_3 is a complex constant. So we can assume the image of $z = e^{it}$ by f is a planar curve on the x_1x_2-plane. A straight computation gives

$$\begin{cases} \mathrm{Re}\int \phi_1 = -\dfrac{\sin(2(k+1)t)}{k+1} - \dfrac{\sin(2kt)}{k} + \mathrm{Re}\ C_1, \\ \mathrm{Re}\int \phi_2 = \dfrac{\cos(2(k+1)t)}{k+1} - \dfrac{\cos(2kt)}{k} + \mathrm{Re}\ C_2, \end{cases}$$

where C_1 and C_2 are complex constants. Define a curve $\tilde{c}(t) = (\tilde{x}_1(t), \tilde{x}_2(t))$ as follows:

$$\begin{cases} \tilde{x}_1(t) = -\dfrac{\sin(2(k+1)t)}{k+1} - \dfrac{\sin(2kt)}{k}, \\ \tilde{x}_2(t) = \dfrac{\cos(2(k+1)t)}{k+1} - \dfrac{\cos(2kt)}{k}. \end{cases} \tag{3.12}$$

After the rotating the curve $\tilde{c}(t)$ in $\mathbb{L}^3$ by $-\pi/2$ about the x_3-axis, we obtain $c(t) = (x_1(t), x_2(t))$ as follows:

$$\begin{cases} x_1(t) = -\dfrac{\cos(2(k+1)t)}{k+1} - \dfrac{\cos(2kt)}{k}, \\ x_2(t) = \dfrac{\sin(2(k+1)t)}{k+1} - \dfrac{\sin(2kt)}{k}. \end{cases} \tag{3.13}$$

(3.13) coincides with (3.9). So we see the surface given by (3.10) contains the hypocycloid.

Remark 3.3. *We set a curve* $c(t) = (\tilde{x}_1(t), \tilde{x}_2(t), 0)$ *as* (3.12) *and a unit normal* $n(t)$ *of* $c(t)$ *as*

$$n(t) = \left(-\sin t \tan((2k+1)t), \cos t \tan((2k+1)t), -\frac{1}{\cos((2k+1)t)}\right).$$

Applying Theorem 2.8 *to* $c(t)$ *and* $n(t)$, *we obtain the same surface defined by* (3.10).

4. Nonorientable maximal surfaces with two ends

The degree of the Gauss map of complete nonorientable maxfaces is greater than or equal 4. In [10], Fujimori and López constructed maximal

Möbius strips and maximal Klein bottles with $\deg g = 4$ and showed their uniqueness. In this section, we prove the following theorem.

Theorem 4.1. *There exists a unique complete nonorientable maxface* $f : \mathbb{RP}^2 \setminus \{p_1, p_2\} \to \mathbb{L}^3$ *with* $\deg g = 4$.

We set the Riemann surface M as

$$M = \mathbb{S}^2 \setminus \{p_1, I(p_1), p_2, I(p_2)\}, \qquad (p_1, p_2 \in \overline{M} = \mathbb{S}^2)$$

where $I = I(z) = -1/\bar{z}$ is an antiholomorphic involution without fixed points in $\overline{M}$. Let $\pi : M \to M' = M/\langle I \rangle$ be a double cover of M'. Then M' can be written as

$$M' = \mathbb{RP}^2 \setminus \{\pi(p_1), \pi(p_2)\}.$$

Let $f' : M' \to \mathbb{L}^3$ be a complete nonorientable maximal surface with $\deg g = 4$. When we set $(q_1, q_2, q_3, q_4) = (p_1, I(p_1), p_2, I(p_2))$, by (2.7), we have

$$\sum_{j=1}^{4} (\mu_j + 1) = 10,$$

where μ_j is a winding number of $f = f' \circ \pi : M \to \mathbb{L}^3$ around q_j. Without loss of generality, we may assume

$$(q_1, q_2, q_3, q_4) = (0, \infty, 1, -1), \qquad (\mu_1, \mu_2, \mu_3, \mu_4) = (1, 1, 2, 2),$$

and $|g(0)| < 1$.

Case 1. The map g has a branch point of order three at $z = 0$.

After a suitable rotation $f(M)$ in $\mathbb{L}^3$ we can take g as

$$g = az^4,$$

where $a \in \mathbb{R} \setminus \{0\}$. Since g and η satisfy (2.9) and (2.1) gives a Riemannian metric on M, we can assume

$$g = z^4, \qquad \eta = \frac{i}{z^2 (z-1)^3 (z+1)^3} \, dz. \tag{4.1}$$

See Table 4.1.

z	-1	0	1	∞
g	–	0^4	–	∞^4
η	∞^3	∞^2	∞^3	0^6

Table 4.1. Orders of zeros and poles of g and η

It is clear that $|g| = 1$ at the two ends $z = \pm 1$. So the maximal surface given by (4.1) is not a complete maxface.

Case 2. The map g has a branch point of order two at $z = 0$.

After a rotation $f(M)$ in $\mathbb{L}^3$ we can take g as

$$g = a\frac{z^3(z-b)}{\bar{b}z+1},$$

where $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{C}\backslash\{0\}$. Since g and η satisfy (2.9) and (2.1) gives a Riemannian metric on M, we can take g as

$$g = \frac{z^3(z-b)}{\bar{b}z+1}, \qquad \eta = i\frac{(\bar{b}z+1)^2}{z^2(z-1)^3(z+1)^3}\,dz. \tag{4.2}$$

See Table 4.2, 4.3, and 4.4.

z	0	∞	1	-1	b	$-1/\bar{b}$
g	0^3	∞^3	–	–	0^1	∞^1
η	∞^2	0^4	∞^3	∞^3	–	0^2

Table 4.2. Orders of zeros and poles of g and η ($b \neq \pm 1$)

z	0	∞	1	-1
g	0^3	∞^3	0^1	∞^1
η	∞^2	0^4	∞^3	∞^1

Table 4.3. Orders of zeros and poles of g and η ($b = 1$)

z	0	∞	1	-1
g	0^3	∞^3	∞^1	0^1
η	∞^2	0^4	∞^1	∞^3

Table 4.4. Orders of zeros and poles of g and η ($b = -1$)

We now check the period problem. A straightforward computation gives

$$\mathrm{Res}_{z=0}(\phi_1) = -2i\bar{b}, \qquad \mathrm{Res}_{z=0}(\phi_2) = 2\bar{b}, \qquad \mathrm{Res}_{z=0}(\phi_3) = 0.$$

So (4.2) satisfies (2.3) for a loop around $z = 0$ if and only if $b = 0$. This contradicts $b \neq 0$.

Case 3. The map g has a branch point of order one at $z = 0$.

After a rotation $f(M)$ in $\mathbb{L}^3$ we can take g as

$$g = a\frac{z^2(z-b)(z-c)}{(\bar{b}z+1)(\bar{c}z+1)},$$

where $a \in \mathbb{R}\backslash\{0\}$ and $b, c \in \mathbb{C}\backslash\{0\}$. Since g and η satisfy (2.9) and (2.1) gives a Riemannian metric on M, we can assume

$$g = \frac{z^2(z-b)(z-c)}{(\bar{b}z+1)(\bar{c}z+1)}, \qquad \eta = i\frac{(\bar{b}z+1)^2(\bar{c}z+1)^2}{z^2(z-1)^3(z+1)^3}\,dz. \tag{4.3}$$

See Table 4.5.

z	0	∞	1	-1	b	$-1/\bar{b}$	c	$-1/\bar{c}$
g	0^2	∞^2	–	–	0^1	∞^1	0^1	∞^1
η	∞^2	0^2	∞^3	∞^3	–	0^2	–	0^2

Table 4.5. Orders of zeros and poles of g and η

We check the period condition. A straightforward computation gives

$$\mathrm{Res}_{z=0}(\phi_1) = -2i(\bar{b}+\bar{c}), \qquad \mathrm{Res}_{z=0}(\phi_2) = 2(b+c), \qquad \mathrm{Res}_{z=0}(\phi_3) = 0,$$
$$\mathrm{Res}_{z=\infty}(\phi_1) = 2i(b+c), \qquad \mathrm{Res}_{z=\infty}(\phi_2) = 2(b+c), \qquad \mathrm{Res}_{z=\infty}(\phi_3) = 0.$$

So we have $b = -c$. Moreover,

$$\mathrm{Res}_{z=1}(\phi_1) = -\frac{i}{8}(\mathrm{Re}\ c^4 + 6\ \mathrm{Re}\ c^2 - 15),$$
$$\mathrm{Res}_{z=-1}(\phi_1) = \frac{i}{8}(\mathrm{Re}\ c^4 + 6\ \mathrm{Re}\ c^2 - 15), \tag{4.4}$$
$$\mathrm{Res}_{z=1}(\phi_2) = -\frac{i}{8}(\mathrm{Im}\ c^4 + 6\ \mathrm{Im}\ c^2),$$
$$\mathrm{Res}_{z=-1}(\phi_2) = \frac{i}{8}(\mathrm{Im}\ c^4 + 6\ \mathrm{Im}\ c^2), \tag{4.5}$$
$$\mathrm{Res}_{z=1}(\phi_3) = -\frac{i}{8}(|c|^4 + 6\ \mathrm{Re}\ c^2 + 1),$$
$$\mathrm{Res}_{z=-1}(\phi_3) = \frac{i}{8}(|c|^4 + 6\ \mathrm{Re}\ c^2 + 1). \tag{4.6}$$

Setting $c = c_1 + ic_2$, where $c_1, c_2 \in \mathbb{R}$, we have

$$\mathrm{Im}\ c^4 + 6\ \mathrm{Im}\ c^2 = 4c_1c_2(c_1^2 - c_2^2 + 3).$$

So $\mathrm{Res}_{z=1}(\phi_2)$ is real if and only if $c_1 = 0$, $c_2 = 0$, or $c_1^2 = c_2^2 - 3$. Moreover, we obtain

$$\mathrm{Re}\, c^4 + 6\,\mathrm{Re}\, c^2 - 15 = (c_1^2 - c_2^2)^2 - 4c_1^2c_2^2 + 6(c_1^2 - c_2^2) - 15, \tag{4.7}$$

$$|c|^4 + 6\,\mathrm{Re}\, c^2 + 1 = (c_1^2 + c_2^2)^2 + 6(c_1^2 - c_2^2) + 1. \tag{4.8}$$

In the case $c_1 = 0$, g and η satisfy (2.3) if and only if

$$\begin{cases} c_2^4 - 6c_2^2 - 15 = 0, \\ c_2^4 - 6c_2^2 + 1 = 0. \end{cases} \tag{4.9}$$

The simultaneous equations (4.9) have no solution.

In the case $c_2 = 0$, g and η satisfy (2.3) if and only if

$$\begin{cases} c_1^4 + 6c_1^2 - 15 = 0, \\ c_1^4 + 6c_2^2 + 1 = 0. \end{cases} \tag{4.10}$$

The simultaneous equations (4.10) have no solution.

In the case $c_1^2 = c_2^2 - 3$, g and η satisfy (2.3) if and only if

$$\begin{cases} c_2^4 - 3c_2^2 + 6 = 0, \\ c_2^4 - 3c_2^2 - 2 = 0. \end{cases} \tag{4.11}$$

The simultaneous equations (4.11) have no solution.

Case 4. The map g has no branch point at $z = 0$.

After a rotation $f(M)$ in $\mathbb{L}^3$ we can take g as

$$g = a\frac{z(z-b)(z-c)(z-d)}{(\bar{b}z+1)(\bar{c}z+1)(\bar{d}z+1)},$$

where $a \in \mathbb{R}\backslash\{0\}$ and $b, c, d \in \mathbb{C}\backslash\{0\}$. Since g and η satisfy (2.9) and (2.1) gives a Riemannian metric on M, we can assume

$$g = \frac{z(z-b)(z-c)(z-d)}{(\bar{b}z+1)(\bar{c}z+1)(\bar{d}z+1)}, \qquad \eta = i\frac{(\bar{b}z+1)^2(\bar{c}z+1)^2(\bar{d}z+1)^2}{z^2(z-1)^3(z+1)^3}\,dz. \tag{4.12}$$

See Table 4.6.

z	0	∞	1	-1	b	$-1/\bar{b}$	c	$-1/\bar{c}$	d	$-1/\bar{d}$
g	0^1	∞^1	–	–	0^1	∞^1	0^1	∞^1	0^1	∞^1
η	∞^2	–	∞^3	∞^3	–	0^2	–	0^2	–	0^2

Table 4.6. Orders of zeros and poles of g and η

Let γ_j $(j = 1, 2, 3, 4)$ be loops around q_j respectively. Note that $(q_1, q_2, q_3, q_4) = (0, \infty, 1, -1)$. Then a direct calculation gives that

$$\oint_{\gamma_j} g^2\eta + \overline{\oint_{\gamma_j} \eta} = 2\pi i(\mathrm{Res}_{z=q_j}(g^2\eta) - \overline{\mathrm{Res}_{z=q_j}(\eta)}).$$

For loops γ_j, (2.5) is equivalent to

$$\mathrm{Res}_{z=q_j}(g^2\eta) - \overline{\mathrm{Res}_{z=q_j}(\eta)} = 0. \tag{4.13}$$

A straightforward computation gives

$$\mathrm{Res}_{z=0}(g^2\eta) - \overline{\mathrm{Res}_{z=0}(\eta)} = -2i(b + c + d), \tag{4.14}$$

$$\mathrm{Res}_{z=\infty}(g^2\eta) - \overline{\mathrm{Res}_{z=\infty}(\eta)} = 2i(b + c + d). \tag{4.15}$$

So g and η satisfy (4.13) at $z = 0, \infty$ if and only if $d = -b - c$. In the same way, we have

$$\begin{aligned} &\mathrm{Res}_{z=1}(g^2\eta) - \overline{\mathrm{Res}_{z=1}(\eta)} \\ &\quad = \frac{i}{8}((3c^2 - 1)b^4 + 2c(3c^2 - 1)b^3 + 3(c^4 - c^2 - 2)b^2) \\ &\qquad + \frac{i}{8}(-2c(c^2 + 3)b - (c^4 + 6c^2 - 15)), \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\mathrm{Res}_{z=-1}(g^2\eta) - \overline{\mathrm{Res}_{z=-1}(\eta)} \\ &\quad = -\frac{i}{8}((3c^2 - 1)b^4 + 2c(3c^2 - 1)b^3 + 3(c^4 - c^2 - 2)b^2) \\ &\qquad - \frac{i}{8}(-2c(c^2 + 3)b - (c^4 + 6c^2 - 15)). \end{aligned} \tag{4.17}$$

So g and η satisfy (4.13) at $z = 1, -1$ if and only if

$$\begin{aligned} &(3c^2 - 1)b^4 + 2c(3c^2 - 1)b^3 + 3(c^4 - c^2 - 2)b^2 - 2c(c^2 + 3)b \\ &\quad - (c^4 + 6c^2 - 15) = 0. \end{aligned} \tag{4.18}$$

Moreover, we have

$$\mathrm{Res}_{z=0}(g\eta) = -ibc(b + c), \qquad \mathrm{Res}_{z=\infty}(g\eta) = i\overline{bc(b + c)}. \tag{4.19}$$

By (4.19), g and η satisfy (2.6) at $z = 0, \infty$ if and only if

$$\mathrm{Re}(bc(b + c)) = 0. \tag{4.20}$$

We also have

$$\mathrm{Res}_{z=1}(g\eta) = \frac{i}{16}(|bc(b+c)|^2 - |b^2+bc+c^2+3|^2 + 8bc(b+c) - 8\overline{bc(b+c)} + 8), \tag{4.21}$$

$$\mathrm{Res}_{z=-1}(g\eta) = \frac{i}{16}(|bc(b+c)|^2 - |b^2+bc+c^2+3|^2 - 8bc(b+c) + 8\overline{bc(b+c)} + 8). \tag{4.22}$$

By (4.20), (4.21) and (4.22) become

$$\mathrm{Res}_{z=1}(g\eta) = \frac{i}{16}(|bc(b+c)|^2 - |b^2+bc+c^2+3|^2 + 8 + 16bc(b+c)), \tag{4.23}$$

$$\mathrm{Res}_{z=-1}(g\eta) = \frac{i}{16}(|bc(b+c)|^2 - |b^2+bc+c^2+3|^2 + 8 - 16bc(b+c)), \tag{4.24}$$

respectively. So g and η satisfy (2.6) at $z = 1, -1$ if and only if

$$|bc(b+c)|^2 - |b^2+bc+c^2+3|^2 + 8 = 0. \tag{4.25}$$

Setting $\alpha = b + c$ and $\beta = bc$, (4.18), (4.20), and (4.25) are equivalent to

$$\alpha^4 - (3\beta^2 + 2\beta - 6)\alpha^2 + \beta^2 - 6\beta - 15 = 0, \tag{4.26}$$

$$\alpha\beta + \overline{\alpha\beta} = 0, \tag{4.27}$$

$$|\alpha\beta|^2 - |\alpha^2 - \beta + 3|^2 + 8 = 0, \tag{4.28}$$

respectively.

Remark 4.2. (i) *If* (α, β) *is a solution of* (4.26)–(4.28), $(-\alpha, \beta)$, $(\bar{\alpha}, \bar{\beta})$, *and* $(-\bar{\alpha}, \bar{\beta})$ *are also solutions of them. They give the same surface.*
(ii) *If* $(\alpha, \beta) = (b + c, bc)$ *is a solution of* (4.26)–(4.28), $(b + d, bd)$ *and* $(c + d, cd)$ *are also solutions of them. They give the same surface.*

We determine the number of the solutions (α, β) which give complete maxfaces.

Lemma 4.3. *The maxface given by* (4.12) *satisfying* (4.26)–(4.28) *is not complete if and only if* $\alpha^2 - \beta \in \mathbb{R}$.

Proof. The maxface given by (4.12) is not complete if and only if $|g(1)| = |g(-1)| = 1$. So it is not complete if and only if

$$|(1-b)(1-c)(1-d)|^2 = |(\bar{b}+1)(\bar{c}+1)(\bar{d}+1)|^2. \tag{4.29}$$

Note that $\alpha = b + c$, $\beta = bc$. A straightforward computation shows that (4.29) holds if and only if

$$\overline{\alpha\beta}(-\alpha^2 + \beta + 1) + \alpha\beta\overline{(-\alpha^2 + \beta + 1)} = 0.$$

By (4.27) and $\alpha\beta \neq 0$, we have

$$\alpha^2 - \beta = \overline{\alpha^2 - \beta}.$$

This completes the proof.

Setting $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$, $(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R})$, (4.26)–(4.28) are written as

$$\begin{aligned} &\alpha_1^4 + \alpha_1^2(-6\alpha_2^2 - 3\beta_1^2 - 2\beta_1 + 3\beta_2^2 + 6) + 4\alpha_1\alpha_2(3\beta_1 + 1)\beta_2 \\ &\qquad + \alpha_2^4 + \alpha_2^2(3\beta_1^2 + 2\beta_1 - 3\beta_2^2 - 6) + \beta_1^2 - 6\beta_1 - \beta_2^2 - 15 = 0, \end{aligned} \tag{4.30}$$

$$\begin{aligned} &2\alpha_1^3\alpha_2 - \alpha_1^2(3\beta_1\beta_2 + \beta_2) + \alpha_1\alpha_2(-2\alpha_2^2 - 3\beta_1^2 - 2\beta_1 + 3\beta_2^2 + 6) \\ &\qquad + \beta_2(\alpha_2^2(3\beta_1 + 1) + \beta_1 - 3) = 0, \end{aligned} \tag{4.31}$$

$$2\alpha_1\beta_1 - 2\alpha_2\beta_2 = 0, \tag{4.32}$$

$$\begin{aligned} &-\alpha_1^4 + (-2\alpha_2^2 + \beta_1^2 + 2\beta_1 + \beta_2^2 - 6)\alpha_1^2 + 4\alpha_2\beta_2\alpha_1 \\ &\qquad - \alpha_2^4 + (\beta_1^2 - 2\beta_1 + \beta_2^2 + 6)\alpha_2^2 - \beta_1^2 + 6\beta_1 - \beta_2^2 - 1 = 0. \end{aligned} \tag{4.33}$$

Note that

$$\alpha^2 - \beta = \alpha_1^2 - \alpha_2^2 - \beta_1 + i(2\alpha_1\alpha_2 - \beta_2). \tag{4.34}$$

In the case that the maxfaces given by (4.12) satisfying (4.30)–(4.33) are complete, we have $\beta_2 \neq 0$ by (4.27) and (4.34). If $\alpha_2 = 0$, we have $\alpha_1 = 0$ or $\beta_1 = 0$ by (4.32). In the case $\alpha_1 = 0$, we have $d = -(b + c) = 0$. This contradicts $d \neq 0$. So we have $\alpha_1 \neq 0$. In the case $\beta_1 = 0$, (4.31) becomes

$$(3 + \alpha_1^2)\beta_2 = 0.$$

It contradicts $\beta_2 \neq 0$. Therefore we have $\alpha_2 \neq 0$. Then by (4.32), we have

$$\beta_2 = \frac{\alpha_1\beta_1}{\alpha_2}. \tag{4.35}$$

Substitute this β_2 into (4.30), (4.31), and (4.33), we have

$$\begin{aligned} &(\alpha_2^2 + 3\beta_1^2)\alpha_1^4 + (-6\alpha_2^4 + 2(3\beta_1^2 + \beta_1 + 3)\alpha_2^2 - \beta_1^2)\alpha_1^2 \\ &\qquad + \alpha_2^2(\alpha_2^4 + (3\beta_1^2 + 2\beta_1 - 6)\alpha_2^2 + \beta_1^2 - 6\beta_1 - 15) = 0, \end{aligned} \tag{4.36}$$

$$\alpha_1(-\beta_1+\alpha_1^2-\alpha_2^2+3)(2\alpha_2^2-\beta_1)=0, \tag{4.37}$$

$$(\alpha_2^2-\beta_1^2)\alpha_1^4+(2\alpha_2^4-2(\beta_1^2+3\beta_1-3)\alpha_2^2+\beta_1^2)\alpha_1^2$$
$$+\alpha_2^2(\alpha_2^4-(\beta_1^2-2\beta_1+6)\alpha_2^2+\beta_1^2-6\beta_1+1)=0. \tag{4.38}$$

By (4.37), we have $\beta_1=2\alpha_2^2$ or $\alpha_1^2-\alpha_2^2+3$.

In the case $\beta_1=2\alpha_2^2$, we have $\beta=2\alpha_2^2+2i\alpha_1\alpha_2$. A straightforward computation gives

$$\alpha^2-\beta=\alpha_1^2-3\alpha_2^2\in\mathbb{R}.$$

By Lemma 4.3, this case is excluded.

In the case $\beta_1=\alpha_1^2-\alpha_2^2+3$, we have

$$\alpha^2-\beta=-3+i\frac{\alpha_1(-\alpha_1^2+3\alpha_2^2-3)}{\alpha_2}. \tag{4.39}$$

We see that if a maxface with the Weierstrass data (4.12) is not complete, then it does not satisfy the period problem (4.36)–(4.38). In fact, (4.39) implies

$$\alpha_2^2=\frac{\alpha_1^2+3}{3}.$$

Then (4.36) and (4.38) are rewritten as

$$16\alpha_1^6+72\alpha_1^4+81\alpha_1^2-27=0, \tag{4.40}$$

$$16\alpha_1^6+72\alpha_1^4+81\alpha_1^2+81=0, \tag{4.41}$$

and it is clear that no α_1 satisfies (4.40) and (4.41) simultaneously. Hence, in this case, $\beta_1=\alpha_1^2-\alpha_2^2+3$, the solutions of (4.36)–(4.38) always give complete maxfaces.

After the substitution $\beta_1=\alpha_1^2-\alpha_2^2+3$, (4.36) and (4.38) become

$$3\alpha_2^8-18\alpha_2^6-3(2\alpha_1^4+9\alpha_1^2-9)\alpha_2^4$$
$$+24(\alpha_1^4+3\alpha_1^2-1)\alpha_2^2+\alpha_1^2(\alpha_1^2+3)^2(3\alpha_1^2-1)=0, \tag{4.42}$$

$$\alpha_2^8-6\alpha_2^6+(-2\alpha_1^4-15\alpha_1^2+9)\alpha_2^4$$
$$+4(3\alpha_1^4+9\alpha_1^2+2)\alpha_2^2+\alpha_1^2(\alpha_1^2+3)^2(\alpha_1^2-1)=0. \tag{4.43}$$

Define

$$p_1(A_1,A_2):=3A_2^4-18A_2^3-3(2A_1^2+9A_1-9)A_2^2$$
$$+24(A_1^2+3A_1-1)A_2+A_1(A_1+3)^2(3A_1-1),$$

$$p_2(A_1, A_2) := A_2^4 - 6A_2^3 + (-2A_1^2 - 15A_1 + 9)A_2^2 + 4(3A_1^2 + 9A_1 + 2)A_2 + A_1(A_1+3)^2(A_1 - 1).$$

Lemma 4.4. *p_1 and p_2 have at least one common positive real root (A_1, A_2).*

Proof. A straightforward computation gives

$$p_1 - 3p_2 = -2(9A_1A_2^2 - 6(A_1^2 + 3A_1 + 4)A_2 + A_1(A_1+3)^2).$$

So the common roots (A_1, A_2) of p_1 and p_2 satisfy

$$A_2 = \frac{A_1^2 + 3A_1 + 4 \pm 2\sqrt{2(A_1^2 + 3A_1 + 2)}}{3A_1}.$$

We set

$$\begin{aligned} A_{2+} &= \frac{A_1^2 + 3A_1 + 4 + 2\sqrt{2(A_1^2 + 3A_1 + 2)}}{3A_1}, \\ A_{2-} &= \frac{A_1^2 + 3A_1 + 4 - 2\sqrt{2(A_1^2 + 3A_1 + 2)}}{3A_1}. \end{aligned} \tag{4.44}$$

On the other hand, the resultant of p_1 and p_2 about A_2 is

$$\begin{aligned} &\mathrm{Resultant}(p_1, p_2; A_2) \\ &\quad = 65536A_1^2(A_1+3)^4(-13824 + 39204A_1 + 19116A_1^2 + 19845A_1^3 - 62370A_1^4 \\ &\qquad - 28287A_1^5 - 3600A_1^6 + 5472A_1^7 + 2304A_1^8 + 256A_1^9)^2. \end{aligned}$$

Define

$$\begin{aligned} R(A_1) = {} & -13824 + 39204A_1 + 19116A_1^2 + 19845A_1^3 - 62370A_1^4 \\ & - 28287A_1^5 - 3600A_1^6 + 5472A_1^7 + 2304A_1^8 + 256A_1^9. \end{aligned}$$

Note that p_1 and p_2 have common real roots if and only if there exists a real number A_1 satisfying $R(A_1) = 0$. A straightforward computation gives

$$R(0) = -13824 < 0, \qquad R(3) = 18384192 > 0.$$

By the intermediate value theorem, R has at least one root. It is easy to check that

$$(A_1^2 + 3A_1 + 4)^2 - 8(A_1^2 + 3A_1 + 2) = A_1^2(A_1+3)^2 > 0.$$

So we have $A_{2-} > 0$. Therefore we see p_1 and p_2 have at least one positive real root (A_1, A_2).

The equation (4.44) is a necessary condition that A_2 becomes a common root of p_1 and p_2. The equations $p_1(A_1, A_{2+}) = p_1(A_1, A_{2-}) = p_2(A_1, A_{2+}) = p_2(A_1, A_{2-}) = 0$ yield that

$$\begin{aligned} &256A_1^9 + 2304A_1^8 + 5472A_1^7 - 3600A_1^6 - 28287A_1^5 \\ &\quad - 62370A_1^4 + 19845A_1^3 + 19115A_1^2 + 39204A_1 - 13824 = 0. \end{aligned} \tag{4.45}$$

The real solutions of this equation are candidates for the common real roots of p_1 and p_2. By the Descartes's rule of signs (cf. [20]), we see that (4.45) has at most 3 real positive solutions. Therefore we obtain at most 6 pairs $(A_1, A_{2\pm})$ that are common real roots of p_1 and p_2. Straightforward computations and the intermediate value theorem give

$$A_1 \approx 0.307, \quad 0.863, \quad 2.203, \tag{4.46}$$

are real roots of (4.45).

Lemma 4.5. *If (A_1, A_{2+}) is a common real positive root of p_1 and p_2, (A_1, A_{2-}) is not a common real positive root of p_1 and p_2.*

Proof. We assume that (A_1, A_{2+}) and (A_1, A_{2-}) are common roots of p_1 and p_2. It is easy to check that

$$\begin{aligned} p_1(A_1, A_{2+}) &= 3p_2(A_1, A_{2+}) \\ &= \frac{4}{27A_1^4}(16A_1^8 + 120A_1^7 + 241A_1^6 - 126A_1^5 - 605A_1^4 - 822A_1^3 + 8A_1^2 \\ &\quad + 384A_1 + 512 + 2\sqrt{2(A_1^2 + 3A_1 + 2)}(-8A_1^6 - 18A_1^5 + 11A_1^4 \\ &\quad - 213A_1^3 + 6A_1^2 + 128)), \\ p_1(A_1, A_{2-}) &= 3p_2(A_1, A_{2-}) \\ &= \frac{4}{27A_1^4}(16A_1^8 + 120A_1^7 + 241A_1^6 - 126A_1^5 - 605A_1^4 - 822A_1^3 + 8A_1^2 \\ &\quad + 384A_1 + 512 - 2\sqrt{2(A_1^2 + 3A_1 + 2)}(-8A_1^6 - 18A_1^5 + 11A_1^4 \\ &\quad - 213A_1^3 + 6A_1^2 + 128)). \end{aligned}$$

We have

$$p_1(A_1, A_{2+}) - p_1(A_1, A_{2-})$$
$$= 4\sqrt{2(A_1^2 + 3A_1 + 2)}(-8A_1^6 - 18A_1^5 + 11A_1^4 - 213A_1^3 + 6A_1^2 + 128). \quad (4.47)$$

Define

$$P(A_1) := -8A_1^6 - 18A_1^5 + 11A_1^4 - 213A_1^3 + 6A_1^2 + 128.$$

The common real positive roots of $p_1(A_1, A_{2+})$ and $p_1(A_1, A_{2-})$ satisfy $P(A_1) = 0$. Moreover, we have

$$P'(A_1) = -48A_1^5 - 90A_1^4 + 44A_1^3 - 639A_1^3 + 12A_1.$$

It is clear that $P(1) < 0$ and $P'(A_1) < 0$ if $A_1 \geq 1$. So $P(A_1)$ does not have real roots on $[1, \infty)$. If there exists $A_0 \in (0,1)$ satisfying $P(A_0) = 0$, we have

$$44A_0^3 = 32A_0^5 + 72A_0^4 + 852A_0^2 - 24A_0 - \frac{512}{A_0}.$$

It is clear that

$$P'(A_0) = -16A_0^5 - 18A_0^4 + 213A_0^2 - 12A_0 - \frac{512}{A_0}.$$

So we see that $P'(A_0) < 0$ on $(0,1)$.

A direct computation shows $P(0) = 128 > 0$. Hence the intermediate value theorem yields the existence of the exactly one positive real root on $(0,1)$, and this root is

$$A_0 \approx 0.84. \quad (4.48)$$

However, (4.48) is not in the list (4.46). So we conclude that $A_0 \approx 0.84$ is not a common real root of p_1 and p_2.

By (4.46) and Lemma 4.5, we have at most 3 positive roots (A_1, A_2) of p_1 and p_2. So we have at most 12 common real solutions (α_1, α_2) of (4.42) and (4.43).

On the other hand, by Remark 4.2, we have 12 solutions of (4.26)–(4.28) giving the same surface. Therefore we conclude that there exists the unique nonorientable complete maxface homeomorphic to $\mathbb{RP}^2 \backslash \{p_1, p_2\}$ with $\deg g = 4$. See Figure 4.1. Summing up, we have proven Theorem 4.1.

$(\alpha, \beta) \approx (-0.554922 - 0.336273i, 3.19486 + 5.27219i)$

Fig. 4.1. Nonorientable complete maxface defined by (4.12)

Remark 4.6. *There exists the* 12 *solutions of* (4.26)–(4.28) *satisfying* $\alpha^2 - \beta \in \mathbb{R}$. *So we have non-complete maxfaces. See Figure* 4.2.

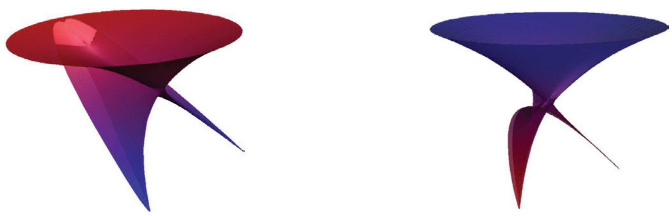

Fig. 4.2. Nonorientable non-complete maxfaces (left: $(\alpha, \beta) \approx (2.684i, -0.744)$, right: $(\alpha, \beta) \approx (1.197 + 0.5687i, 0.647 + 1.362i)$)

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