広島大学学位請求論文

Distance and the Goeritz groups of Heegaard splittings

（Heegaard 分解の距離と Goeritz 群）

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   S. Hirose, D. Iguchi, E. Kin and Y. Koda, Goeritz groups of bridge decompositions, International Mathematics Research Notices IMRN **2022** (2022), no. 12, 9308–9356.

主論文

Distance and the Goeritz groups of Heegaard splittings

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Abstract

Any closed orientable 3-manifold is obtained by gluing two handlebodies of the same genus along their boundary. Such a decomposition of a 3-manifold is called a Heegaard splitting, and the common boundary of the two handlebodies, which is a closed orientable surface, is called a Heegaard surface. The distance of a Heegaard splitting is defined to be the distance in the the curve graph for the Heegaard surface between the two sets of meridian disks corresponding to the two handlebodies. The Goeritz group of a Heegaard splitting is the group of isotopy classes of orientation preserving self-homeomorphisms of the ambient 3-manifold that leave the splitting invariant. In this thesis we investigate the Goeritz groups of Heegaard splittings with the distance greater than one. The thesis consists of two parts. In the first part, we study Heegaard splittings obtained from twisted book decompositions of 3-manifolds, and give explicit computations for their Goeritz groups. In the second part, we consider the Goeritz groups for bridge decompositions of links in 3-manifolds. We prove that if the distance of a bridge decomposition of a link is at least 6, then its Goeritz group is a finite group. This is a generalization of a result of Johnson [**24**].

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Contents

Notations and conventions

We will use the following notations:

$\mathbb{R}^n$	The n dimensional Euclidean space.
$A - B$ or A_B	The relative complement of B in A, where A and B are sets.
$\#X$ or $\|X\|$	The number of path-components of a topological space X.
$\mathrm{Cl}(Y; X)$ or $\mathrm{Cl}(Y)$	The closure of Y in X, where $Y \subset X$.
$\mathrm{Int}(Y)$	The interior of Y in X.
∂M	The boundary of a manifold M.

We will not distinguish curves, surfaces, maps, etc. from their isotopy classes in their notation when there is no possibility of confusion.

CHAPTER 1

Twisted book decompositions and the Goeritz groups

1. Introduction

It is well known that every closed orientable 3-manifold M is the result of taking two copies H_1, H_2 of a handlebody and gluing them along their boundaries. Such a decomposition $M = H_1 \cup_\Sigma H_2$ is called a *Heegaard splitting* for M. The surface Σ here is called the *Heegaard surface* of the splitting, and the genus of Σ is called its *genus*. In [**19**], Hempel introduced a measure of the complexity of a Heegaard splitting called the *distance* of the splitting. Roughly speaking, this is the distance between the sets of meridian disks of H_1 and H_2 in the *curve graph* $\mathcal{C}(\Sigma)$ of the Heegaard surface Σ.

The *Goeritz group* (or the *mapping class group*) of a Heegaard splitting for a 3-manifold is the group of isotopy classes of orientation-preserving automorphisms (self-homeomorphisms) of the manifold that preserve each of the two handlebodies of the splitting setwise. We note that the Goeritz group of a Heegaard splitting is a subgroup of the mapping class group of the Heegaard surface.

Concerning the structure of the Goeritz groups, Minsky asked in [**16**] when the Goeritz group of a Heegaard splitting is finite, finitely generated, or finitely presented, respectively. The distance of Heegaard splittings gives a nice way to describe those properties of the Goeritz groups. In [**38**], Namazi showed that the Goeritz group is a finite group if a Heegaard splitting has a sufficiently high distance. This result was improved by Johnson [**24**] showing the same consequence when the distance of the splitting is at least 4. On the contrary, it is an easy fact that the Goeritz group is always an infinite group when the distance of the Heegaard splitting is at most one (see e.g. Johnson-Rubinstein [**30**] or Namazi [**38**]). In this case, there have been many efforts to find finite generating sets or presentations of the Goeritz groups. For example, the sequence of works [**15**, **44**, **1**, **5**, **6**, **7**, **8**, **9**, **10**] by many authors completed to give a finite presentation of the Goeritz group of every genus-2 Heegaard splitting of distance 0. Recently, Freedman-Scharlemann [**14**] gave a finite generating set of the genus-3 Heegaard splitting of the 3-sphere. For the higher genus Heegaard splittings of the 3-sphere, the problem of existence of finite generating sets of the Goeritz groups still remains open. For other works on finite generating sets of Goeritz groups, see [**26**, **27**, **46**, **11**].

In this chapter, we concern the Goeritz groups of *strongly-irreducible* (that is, distance at least 2) Heegaard splittings. There are few isolated examples that are known. First, we think of a natural question: how can the Goeritz group be “small” fixing the genus and the distance of the splitting. In Section 3, we consider finiteness properties of the Goeritz groups of *keen* Heegaard splittings (see Proposition 1.9). As a direct corollary, we get the following:

Corollary 1.10. For any $g \geq 3$ and $n \geq 2$, there exists a genus-g Heegaard splitting of distance n whose Goeritz group is either a finite cyclic group or a finite dihedral group.

Roughly speaking, it is believed that the "majority" of the Heegaard splittings of distance 2 or 3 have the Goeritz groups of at most finite orders. One typical example of a "minority" here is constructed by using an open book decomposition with a monodromy of infinite order, see for instance the preprint Johnson [**29**]. In fact, this construction gives a distance-2 Heegaard splitting whose Goeritz group is an infinite groups. Since the Heegaard splitting induced from an open book decomposition admits the "accidental" symmetry coming from the rotation around the binding, we might wonder wether this type of Heegaard splittings is the only "minority".

In the main part of the chapter, we focus on the Heegaard splittings induced from *twisted book decompositions*, which are first studied in Johnson-Rubinstein [**30**]. Here is a brief construction (see Sections 4–6 for the detailed definitions). Let F be a compact non-orientable surface of negative Euler characteristic with a single boundary component, let $\pi : H \to F$ be the orientable I-bundle with the *binding* $b \subset \partial H =: \Sigma$. Let $M = H_1 \cup_\Sigma H_2$ be the Heegaard splitting obtained by gluing H to a copy of itself via an automorphism φ of Σ that preserves b. It is easy to see that the distance of such a Heeegaard splitting is at most 4. We compute the Goeritz group of $M = H_1 \cup_\Sigma H_2$ in the following two cases.

The first case is that the gluing map φ is particularly "simple".

Theorem 1.16. Suppose that the gluing map φ is a k-th power of the Dehn twist about the binding b, where $|k| \geq 5$. For the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ as above, we have the following.

(1) The splitting $M = H_1 \cup_\Sigma H_2$ is not induced from an open book decomposition.
(2) The Goeritz group of $M = H_1 \cup_\Sigma H_2$ is isomorphic to the mapping class group of F. In particular, it is an infinite group.

Note that it follows directly from Yoshizawa [**51**] that the distance of the splitting $M = H_1 \cup_\Sigma H_2$ in the above theorem is exactly 2. Theorem 1.16 indicates that the "minorities" is not as minor as we wondered in the previous paragraph. Further, it is remarkable that the above theorem gives the first explicit computation of the infinite-order Goeritz groups of strongly-irreducible Heegaard splittings.

The second case is, on the contrary, that the gluing map φ is complicated in the sense that the distance in the curve graph $\mathcal{C}(\Sigma_b)$ between the images of the *subsurface projection* π_{Σ_b} of the sets of meridian disks $\mathcal{D}(H_1)$ of H_1 and $\mathcal{D}(H_2)$ of H_2 is sufficiently large, where $\Sigma_b := \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$. In this case, we can show that the distance of the splitting $M = H_1 \cup_\Sigma H_2$ is exactly 4 and we can compute the Goeritz group as follows, where the definition of the group $G(S, \iota_0, \iota_1)$ is given in Section 6.2:

THEOREM 1.18. Suppose that the distance in $\mathcal{C}(\Sigma_b)$ between $\pi_{\Sigma_b}(\mathcal{D}(H_1))$ and $\pi_{\Sigma_b}(\mathcal{D}(H_2))$ is greater than 10. For the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ as above, we have the following.

(1) The distance of the splitting $M = H_1 \cup_\Sigma H_2$ is exactly 4.
(2) The Goeritz group of $M = H_1 \cup_\Sigma H_2$ is isomorphic to the group $G(S, \iota_0, \iota_1)$.

We will see that there actually exist (generiacally, in some sense) the Heegaard splittings satisfying the condition in Theorem 1.18. The Goeritz group in Theorem 1.18 is of course a finite group since the distance of the Heegaard splitting is 4. The existence of a Heegaard splitting of distance 3 having the infinite-order Goeritz group still remains open.

2. Preliminaries

For abbriviation, throughout this chapter, we will assume the following : Any curves on a surface, or surfaces in a 3-manifold are always assumed to be properly embedded, and their intersection is transverse and minimal up to isotopy.

2.1. Curve graphs. Let Σ be a compact surface. A simple closed curve on Σ is said to be *essential* if it is not homotopic to a point or a loop around a boundary component of Σ. An arc on Σ is said to be *essential* if it is not homotopic (rel. endpoints) to a subarc of a boundary component of Σ.

Let Σ be a compact orientable surface of genus g with p boundary components. We say that Σ is *sporadic* if $3g + p \leq 4$. Otherwise, Σ is said to be *non-sporadic*. Suppose that Σ is non-sporadic. The *curve graph* $\mathcal{C}(\Sigma)$ of Σ is the 1-dimensional simplicial complex whose vertices are the isotopy classes of essential simple closed curves on Σ such that a pair of distinct vertices spans an edge if and only if they admit disjoint representatives. Similarly, the *arc and curve graph* $\mathcal{AC}(\Sigma)$ of Σ is defined to be the 1-dimensional simplicial complex whose vertices are the isotopy classes of essential arcs and simple closed curves on Σ such that a pair of distinct vertices spans an edge if and only if they admit disjoint representatives. The sets of vertices of $\mathcal{C}(\Sigma)$ and $\mathcal{AC}(\Sigma)$ are denoted by $\mathcal{C}^{(0)}(\Sigma)$ and $\mathcal{AC}^{(0)}(\Sigma)$, respectively. We equip the curve graph $\mathcal{C}(\Sigma)$ (resp. the arc and curve graph $\mathcal{AC}(\Sigma)$) with the simplicial distance $d_{\mathcal{C}(\Sigma)}$ (resp. $d_{\mathcal{AC}(\Sigma)}$). Note that both $\mathcal{C}(\Sigma)$ and $\mathcal{AC}(\Sigma)$ are geodesic metric spaces.

A subsurfcae Y in Σ is said to be *essential* if each component of ∂Y is not contractible, and if Y is not an annulus parallel to $\partial\Sigma$. Let Y be an essential, non-sporadic subsurface of Σ. The *subsurface projection* $\pi_Y : \mathcal{C}^{(0)}(\Sigma) \to P(\mathcal{C}^{(0)}(Y))$, where $P(\cdot)$ denotes the power set, is defined as follows. First, define $\kappa_Y : \mathcal{C}^{(0)}(\Sigma) \to P(\mathcal{AC}^{(0)}(Y))$ to be the map that takes $\alpha \in \mathcal{C}^{(0)}(S)$ to $\alpha \cap Y$. Further, define the map $\sigma_Y : \mathcal{AC}^{(0)}(Y) \to P(\mathcal{C}^{(0)}(Y))$ by taking $\alpha \in \mathcal{AC}^{(0)}(Y)$ to the set of simple closed curves on Y consisting of the components of the boundary of $\mathrm{Nbd}(\alpha \cup \partial Y; Y)$ that are essential in Y. The map σ_Y naturally extends to a map $\sigma_Y : P(\mathcal{AC}^{(0)}(Y)) \to P(\mathcal{C}^{(0)}(Y))$. The subsurface projection $\pi_Y : \mathcal{C}^{(0)}(\Sigma) \to P(\mathcal{C}^{(0)}(Y))$ is then defined by $\pi_Y = \sigma_Y \circ \kappa_Y$. See for example Masur-Minsky [**36**] and Masur-Schleimer [**37**] for details. The following lemma is straightforward from the definition.

LEMMA 1.1. *Let* $(\alpha_0, \ldots, \alpha_n)$ *be a geodesic segment in* $\mathcal{C}(\Sigma)$. *If* $\alpha_j \cap Y \neq \emptyset$ *for each* $j \in \{0, \ldots, n\}$, *then it holds* $d_{\mathcal{C}(Y)}(\pi_Y(\alpha_0), \pi_Y(\alpha_n)) \leq 2n$.

2.2. Distance of a Heegaard splitting. Let H be a handlebody of genus at lest 2. We denote by $\mathcal{D}(H)$ the subset of $\mathcal{C}^{(0)}(\partial H)$ consisting of simple closed curves that bound disks in H. Given a Heegaard splitting $M = H_1 \cup_\Sigma H_2$, the *distance* $d(M, \Sigma)$ of the splitting is defined by $d(M, \Sigma) = d_{\mathcal{C}(\Sigma)}(\mathcal{D}(H_1), \mathcal{D}(H_2))$. We say that a Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is *strongly irreducible* if $d(M, \Sigma) \geq 2$.

A Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is said to be *keen* if there exists a unique pair of $\alpha \in \mathcal{D}(H_1)$ and $\alpha' \in \mathcal{D}(H_2)$ satisfying $d_{\mathcal{C}(\Sigma)}(\alpha, \alpha') = d(M, \Sigma)$. In particular, $M = H_1 \cup_\Sigma H_2$ is said to be *strongly keen* if there exists a unique geodesic segment $(\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n = \alpha')$, where $n = d(M, \Sigma)$, such that $\alpha \in \mathcal{D}(H_1)$ and $\alpha' \in \mathcal{D}(H_2)$. We say that a Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is *weakly keen* if there exist only finitely many pairs of $\alpha \in \mathcal{D}(H_1)$ and $\alpha' \in \mathcal{D}(H_2)$ satisfying $d_{\mathcal{C}(\Sigma)}(\alpha, \alpha') = d(M, \Sigma)$. The notion of a keen (and a strongly keen) Heegaard splitting was first introduced by Ido-Jang-Kobayashi [**22**], who showed the following theorem.

THEOREM 1.2 (Ido-Jang-Kobayashi [**22**]). *For any* $g \geq 3$ *and* $n \geq 2$, *there exists a genus-*g *strongly keen Heegaard splitting* $M = H_1 \cup_\Sigma H_2$ *with* $d(M, \Sigma) = n$.

2.3. Mapping class groups. Let $Y_1, \ldots, Y_n$ be possibly empty subspaces of a compact manifold X. We denote by $\mathrm{Aut}(X, Y_1, \ldots, Y_n)$ the group of automorphisms of X which map Y_i onto Y_i for any $i = 1, \ldots, n$. The *mapping class group* of $(X, Y_1, \ldots, Y_n)$, denoted by $\mathrm{MCG}(X, Y_1, \ldots, Y_n)$, is defined to be the group of connected components of $\mathrm{Aut}(X, Y_1, \ldots, Y_n)$. The equivalence class in $\mathrm{MCG}(X, Y_1, \ldots, Y_n)$ of a map in $\mathrm{Aut}(X, Y_1, \ldots, Y_n)$ is called its *mapping class*. As mentioned in the introduction, we usually will not distinguish a map and its mapping class. This should not cause any confusion since it will usually be clear from the context in which equivalence relation we consider for the maps in question. When X is orientable, the "plus" subscripts, for instance in $\mathrm{Aut}_+(X, Y_1, \ldots, Y_n)$ and $\mathrm{MCG}_+(X, Y_1, \ldots, Y_n)$, indicate the subgroups of $\mathrm{Aut}(X, Y_1, \ldots, Y_n)$ and $\mathrm{MCG}(X, Y_1, \ldots, Y_n)$, respectively, consisting of orientation-preserving automorphisms (or their mapping classes) of X.

Let $M = H_1 \cup_\Sigma H_2$ be a Heegaard splitting. The group $\mathcal{G}(M, \Sigma)$ is called the *mapping class group*, or the *Goeritz group*, of the splitting. Note that the natural map $\mathcal{G}(M, \Sigma) \to \mathrm{MCG}_+(\Sigma)$ that takes (the mapping class of) $\varphi \in \mathcal{G}(M, \Sigma)$ to (that of) $\varphi|_\Sigma \in \mathrm{MCG}_+(\Sigma)$ is injective. In this way $\mathcal{G}(M, \Sigma)$ can be naturally regarded as a subgroup of $\mathrm{MCG}_+(\Sigma)$. In general, an automorphism ψ of a submanifold Y of a manifold X is said to be *extendable over* X if ψ extends to an automorphism of the pair (X, Y). We can say that the Goeritz group for the splitting $M = H_1 \cup_\Sigma H_2$ is the subgroup of the mapping class group $\mathrm{MCG}_+(\Sigma)$ of the Heegaard surface Σ consisting of elements that are extendable over M.

In this paper, the mapping class groups of non-orientable surfaces will also be particularly important. Let F be a compact non-orientable surface with nonempty boundary. Let $p : \Sigma \to F$ be the orientation double-cover. Since the set of two-sided loops are preserved by any automorphism of F, any map $\varphi \in \mathrm{Aut}(F)$ lifts to a unique orientation-preserving automorphism of Σ. (The other lift of φ is orientation-reversing.) This gives a well-defined homomorphism $L : \mathrm{MCG}(F) \to \mathrm{MCG}_+(\Sigma)$. We use the following easy but important lemma in Section 6.1.

LEMMA 1.3. *The above map $L : \mathrm{MCG}(F) \to \mathrm{MCG}_+(\Sigma)$ is injective.*

PROOF. Let $F\tilde{\times}I$ be the orientable twisted product, which is a handlebody, and $\pi : F\tilde{\times}I \to F$ the natural projection. We identify Σ with $F\tilde{\times}\partial I \subset F\tilde{\times}I$, and F with $F\tilde{\times}\{1/2\} \subset F\tilde{\times}I$. Note that $\pi|_\Sigma$ is nothing but the orientation double cover $p : \Sigma \to F$.

Let φ_F be an automorphism of F whose mapping class belongs to the kernel of $L : \mathrm{MCG}(F) \to \mathrm{MCG}_+(\Sigma)$. The map φ_F extends to a fiber-preserving homeomorphism $\Phi \in \mathrm{Aut}_+(F\tilde{\times}I)$ with $\varphi := \Phi|_\Sigma = L(\varphi_F)$. The map φ is isotopic to the identity id_Σ, thus, $\Phi|_{\partial(F\tilde{\times}I)}$ can be described as

$$\Phi|_{\partial(F\tilde{\times}I)} = \tau_{c_1}^{k_1} \circ \cdots \circ \tau_{c_n}^{k_n},$$

where $c_1, \ldots, c_n$ are the connected components of ∂F, $\tau_{c_i}^{k_i}$ is the Dehn twist about the simple closed curve c_i ($i = 1, \ldots, n$), and $k_1, \ldots, k_n$ are integers. Since each c_i does not bound a disk in $F\tilde{\times}I$, and each pair of c_i and c_j ($1 \le i < j \le n$) does not cobound an annulus, we have $k_1 = \cdots = k_n = 0$ due to Oertel [**43**] or McCullough [**34**]. Therefore, Φ is isotopic to the identity $\mathrm{id}_{\partial(F\tilde{\times}I)}$, so Φ is isotopic to the identity $\mathrm{id}_{F\tilde{\times}I}$. Since the inclusion $\iota : F \to F\tilde{\times}I$ is a homotopy equivalence with π a homotopy inverse, the composition $\pi \circ \Phi \circ \iota$ is homotopic to id_F. It follows that φ_F is homotopic to the identity. Now by Epstein [**12**], φ_F is isotopic to id_F. □

2.4. Pants decompositions and twisting numbers. Let Σ be a closed orientable surface of genus g, where $g \ge 2$. The set of $3g-3$ mutually disjoint, mutually non-isotopic, essential simple closed curves on Σ is called a *pants decomposition* of Σ. Let $\mathscr{P}$ be a pants decomposition of Σ. Let C be the union of the simple closed curves of $\mathscr{P}$. Let α be an essential arc on a component P, which is a pair of pants, of $\mathrm{Cl}(\Sigma - \mathrm{Nbd}(C))$. We call α a *wave for* $\mathscr{P}$ if the both endpoints of α lie on the same component of ∂P. Otherwise, α is called a *seam for* $\mathscr{P}$. Let $k > 0$. An essential simple closed curve β on Σ (that intersects C minimally up to isotopy) is said to be *k-seamed with respect to* $\mathscr{P}$ if for each component P of $\mathrm{Cl}(\Sigma - \mathrm{Nbd}(C))$, there exist at least k arcs of $\beta \cap P$ representing each of the three distinct isotopy classes of seams for $\mathscr{P}$.

Let l be a simple closed curve on a closed oriented surface Σ of genus at least 2. We denote by τ_l the (left-handed) *Dehn twist* about l. Let $\mathscr{P}$ be a pants decomposition of Σ. Let C be the union of the simple closed curves of $\mathscr{P}$. Set $N := \mathrm{Nbd}(l)$. Fix an identification of N with the product $l \times I$, where l corresponds to $l \times \{1/2\}$. We may assume that each component of $N \cap C$ is an I-fiber of N. Let α be an essential simple arc on N with the endpoints disjoint from $N \cap C$ that intersects each I-fiber of N transversely. Then the *twisting number* of α in N with respect to C is defined as follows. Let p be an endpoint of α. Let v_α be the inward-pointing tangent vector of α based at p. Likewise, let v_I be the inward-pointing tangent vector based at p of the I-fiber of N with p an endpoint. If the pair (v_α, v_I) is compatible with the orientation of Σ, the twisting number is defined to be $\#(\alpha \cap C)/\#(N \cap C) \in \mathbb{Q}$. Otherwise, it is defined to be $-\#(\alpha \cap C)/\#(N \cap C) \in \mathbb{Q}$. See Figure 1. We refer the reader to Yoshizawa [**51**] for more details on the twisting numbers.

Let Σ, l, N, $\mathscr{P}$ and C be as above. Let β be a simple closed curve on Σ. We say that β is in *efficient position* with respect to (N, C) if

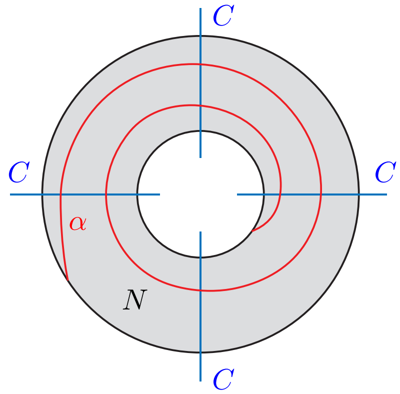


FIGURE 1. The twisting number of the arc α in N with respect to C is $7/4$.

- β intersects ∂N and C minimally (up to isotopy);
- β intersects each I-fiber of N transversely; and
- $\beta \cap C \cap \partial N = \emptyset$.

Suppose that β is in efficient position with respect to (N, C). A disk E in $\Sigma - \mathrm{Int}(N)$ is called an *outer triangle* of N with respect to $(N, \mathscr{P}, \beta)$ if $\partial E \subset \partial N \cup C \cup \beta$ and each of $\partial E \cap \partial N$, $\partial E \cap C$, $\partial E \cap \beta$ is a single arc. See Figure 2. Note that we can

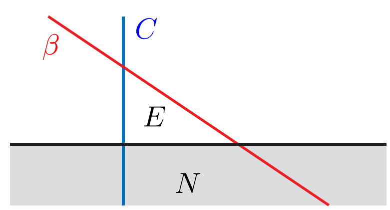


FIGURE 2. An outer trianble E of N with respect to $(N, \mathscr{P}, \beta)$.

perform an isotopy of β keeping that β is in efficient position with respect to (N, C) so that $(N, \mathscr{P}, \beta)$ admits no outer triangles.

LEMMA 1.4 (Yoshizawa [**51**]). *Let Σ, l, N, $\mathscr{P}$ and C be as above. Let β be a simple closed curve on Σ in efficient position with respect to (N, C) such that $(N, \mathscr{P}, \beta)$ admits no outer triangles. Let $\alpha_1, \ldots, \alpha_r$ be the components of $\beta \cap N$, and t_j ($j \in \{1, \ldots, r\}$) the twisting number of α_j in N with respect to C. Let k be an integer such that either $k + t_j \geq 0$ (for all j) or $k + t_j \leq 0$ (for all j). Then $\tau_l^k(\beta)$ remains to be in efficient position with respect to (N, C), and the twisting number of $\tau_l^k(\alpha_j)$ in N with respect to C is $k + t_j$.*

The following lemma, which we will use in Section 6.1, is straightforward from the definitions.

LEMMA 1.5. *Let Σ, l, N, $\mathscr{P}$, β, α_j and t_j $(j \in \{1, \dots, r\})$ be as in Lemma* 1.4. *If l is* 1*-seamed with respect to the pants decomposition $\mathscr{P}$ and there exists j with $|t_j| > k$, then β is k-seamed with respect to $\mathscr{P}$.*

2.5. Measured laminations. In this subsection, Σ denotes a compact (possible non-orientable) surface with $\chi(\Sigma) < 0$, where $\chi(\cdot)$ denotes the Euler characteristic. We fix a hyperbolic metric on Int Σ. The main references for this subsection are Thurston [**49**], Fathi-Laudenbach-Poénaru [**13**] and Penner-Harer [**40**].

Recall that a *geodesic lamination* on Σ is a foliation of a nonempty closed subset of Σ by geodesics. A *transverse measure* m for a geodesic lamination λ is a function that assigns a positive real number to each smooth compact arc transverse to λ so that m is invariant under isotopy respecting the leaves of λ. A geodesic lamination equipped with a transverse measure is called a *measured geodesic lamination*. The set $\mathcal{ML}(\Sigma)$ of measured geodesic laminations on Σ can be equipped with the weak-* topology, for which two measured geodesic laminations are close if they induce approximately the same measures on any finitely many arcs transverse to them. The quotient $\mathbb{P}\mathcal{ML}(\Sigma)$ of $\mathcal{ML}(\Sigma)$ under the natural action of the multiplicative group $\mathbb{R}_+ := (0, \infty)$ is called the *projective measured geodesic lamination*.

THEOREM 1.6 (Thurston [**49**]). *Suppose that Σ is a compact surface with $\chi(\Sigma) <$ 0.*

(1) The space $\mathcal{ML}(\Sigma)$ (*resp.* $\mathbb{P}\mathcal{ML}(\Sigma)$) *admits a natural piecewise linear* (*resp. piecewise projective*) *structure.*
(2) There exists a piecewise linear (*resp. piecewise projective*) *homeomorphism between $\mathcal{ML}(\Sigma)$* (*resp.* $\mathbb{P}\mathcal{ML}(\Sigma)$) *and* $\mathbb{R}^{6g+3h+2n-6} - \{0\}$ (*resp.* $S^{6g+3h+2n-7}$), *where $\Sigma \cong (\#_g T^2)\#(\#_h \mathbb{RP}^2) - \sqcup_n \operatorname{Int} D^2$.*

A multiset of pairwise disjoint, pairwise non-isotopic, closed geodesics on Σ is called a *weighted multicurve*. The set of multicurves on Σ is denoted by $\mathcal{S}(\Sigma)$. Using the Dirac mass, we regard $\mathcal{S}(\Sigma)$ as a subset of $\mathbb{P}\mathcal{ML}(\Sigma)$. We will use the following theorem in Section 6.2.

THEOREM 1.7 (see Penner-Harer [**40**]). *The set $\mathcal{S}(\Sigma)$ is dense in* $\mathbb{P}\mathcal{ML}(\Sigma)$.

We regard closed geodesics on Σ as points in $\mathcal{ML}(\Sigma)$. For simple closed geodesics α and β on Σ, $i(\alpha, \beta)$ denotes the geometric intersection number. For $(\lambda, m) \in \mathcal{ML}(\Sigma)$, $i((\lambda, m), \alpha)$ is defined to be the minimal transverse length with respect to the measure m for λ.

THEOREM 1.8 (Rees [**41**]). *The above $i(\cdot, \cdot)$ extends to a continuous function $\mathcal{ML}(\Sigma) \times \mathcal{ML}(\Sigma) \to \mathbb{R}$ that is bilinear and invariant under the action of* $\mathrm{MCG}(\Sigma)$.

3. The Goeritz groups of keen Heegaard splittings

In this section, we discuss the finiteness of the Goeritz groups of keen Heegaard splittings.

PROPOSITION 1.9. *Let $M = H_1 \cup_\Sigma H_2$ be a Heegaard splitting of genus at least* 2.

(1) *If the splitting $M = H_1 \cup_\Sigma H_2$ is strongly keen and the distance $d(M, \Sigma)$ is 2, the Goeritz group $\mathcal{G}(M, \Sigma)$ is either a finite cyclic group or a finite dihedral group.*
(2) *If the splitting $M = H_1 \cup_\Sigma H_2$ is keen and the distance $d(M, \Sigma)$ is at least 3, the Goeritz group $\mathcal{G}(M, \Sigma)$ is either a finite cyclic group or a finite dihedral group.*
(3) *If the splitting $M = H_1 \cup_\Sigma H_2$ is weakly keen and the distance $d(M, \Sigma)$ is at least 3, the Goeritz group $\mathcal{G}(M, \Sigma)$ is a finite group.*

Proof. (1) Suppose that $M = H_1 \cup_\Sigma H_2$ is strongly keen and $d(M, \Sigma) = 2$. There exists a unique geodesic segment $(\alpha_0, \alpha_1, \alpha_2)$ such that $\alpha_0 \in \mathcal{D}(H_1)$ and $\alpha_2 \in \mathcal{D}(H_2)$. Let $\varphi \in \mathcal{G}(M, \Sigma)$. By the uniqueness of the geodesic segment $(\alpha_0, \alpha_1, \alpha_2)$, we have $\varphi(\alpha_j) = \alpha_j$ for each $j \in \{0, 1, 2\}$. Thus the group $\mathrm{MCG}_+(M, H_1)$ acts on the pair $(\alpha_0, \alpha_0 \cap \alpha_2)$ in a natural way. It suffices to show that the action of $\mathrm{MCG}_+(M, H_1)$ on $(\alpha_0, \alpha_0 \cap \alpha_2)$ is faithful, which in turn implies that $\mathrm{MCG}_+(M, H_1)$ is either a finite cyclic group or a finite dihedral group. Let ψ be an element of $\mathcal{G}(M, \Sigma)$ that acts on $(\alpha_0, \alpha_0 \cap \alpha_2)$ trivially. Since ψ is orientation-preserving, ψ preserves an orientation of α_2. Therefore, we can assume that $\psi|_{\alpha_0 \cup \alpha_2}$ is the identity on $\alpha_0 \cup \alpha_2$. Since the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is strongly keen and $d(M, \Sigma) = 2$, $\mathrm{Cl}(\Sigma - \mathrm{Nbd}(\alpha_0 \cup \alpha_2))$ consists of finitely many disks and a single annulus, and α_1 is the core of that annulus. By the Alexander trick, we can assume that ψ is the identity outside of the annulus $\mathrm{Nbd}(\alpha_1)$. Thus ψ is a power $\tau_{\alpha_1}^n$ of the Dehn twist τ_{α_1}. If $n \neq 0$, the circle α_1 bounds disks both in H_1 and H_2 due to Oertel [**43**] or McCullough [**34**], which is a contradiction. Therefore, ψ is the identity.
(2) Suppose that $M = H_1 \cup_\Sigma H_2$ is keen and $d(M, \Sigma) \geq 3$. This case is easier than (1). Since $M = H_1 \cup_\Sigma H_2$ is keen, there exists a unique pair of $\alpha \in \mathcal{D}(H_1)$ and $\alpha' \in \mathcal{D}(H_2)$ satisfying $d_{\mathcal{C}(\Sigma)}(\alpha, \alpha') = d(M, \Sigma)$. Thus any $\varphi \in \mathcal{G}(M, \Sigma)$ preserves both α and α'. Since $d(M, \Sigma) \geq 3$, $\mathrm{Cl}(\Sigma \setminus \mathrm{Nbd}(\alpha \cup \alpha'))$ consists only of disks. Thus, the same argument as in the proof of (1) shows that $\mathcal{G}(M, \Sigma)$ is either a finite cyclic group or a finite dihedral group.
(3) Suppose that $M = H_1 \cup_\Sigma H_2$ is weakly keen and $d(M, \Sigma) \geq 3$. In this case, we can show that the order of any $\varphi \in \mathcal{G}(M, \Sigma)$ is finite as follows. Let $\varphi \in \mathcal{G}(M, \Sigma)$. Choose $\alpha \in \mathcal{D}(H_1)$ and $\alpha' \in \mathcal{D}(H_2)$ such that $d_{\mathcal{C}(\Sigma)}(\alpha, \alpha') = d(M, \Sigma)$. Since the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is weakly keen, there exists an integer n such that $\varphi^n(\alpha) = \alpha$ and $\varphi^n(\alpha') = \alpha'$. Since $d(M, \Sigma) \geq 3$, $\mathrm{Cl}(\Sigma - \mathrm{Nbd}(\alpha \cup \alpha'))$ consists only of disks. Thus, the same argument as above shows that the order of the element φ^n is finite in $\mathcal{G}(M, \Sigma)$. Due to Serre [**47**], any torsion subgroup of $\mathrm{MCG}_+(\Sigma)$ is a finite group. The above argument therefore immediately implies that the Goeritz group $\mathcal{G}(M, \Sigma)$ is a finite group. □

As a direct corollary of Proposition 1.9 and Theorem 1.2, we get the following:

Corollary 1.10. *For any $g \geq 3$ and $n \geq 2$, there exists a genus-g Heegaard splitting $M = H_1 \cup_\Sigma H_2$ with $d(M, \Sigma) = n$ such that the Goeritz group $\mathcal{G}(M, \Sigma)$ is either a finite cyclic group or a finite dihedral group.*

4. Handlebodies as interval bundles

Let F be a compact (possibly non-orientable) surface with nonempty boundary. Let $\pi : H \to F$ be the orientable I-bundle. Note that H is a handlebody and $\pi^{-1}(\partial F)$ consists of annuli on ∂H. We call the union of the core curves of $\pi^{-1}(\partial F)$ the *binding* of this I-bundle. In this paper, we often identify F with the image $F \times \{1/2\}$ of a section of the I-bundle $H \to F$, and under this identification, we regard that $b = \partial F$. The union of disjoint simple closed curves on the boundary ∂H of a handlebody H is called a *binding* of H if it is the binding of an I-bundle structure $H \to F$.

In the following, let H be a handlebody of genus g, where $g \geq 2$.

Lemma 1.11. *If a simple closed curve b on ∂H is a binding, then we have $d_{\mathcal{C}(\Sigma)}(b, \mathcal{D}(H)) = 2$.*

Proof. Since b is connected and $\partial H - b$ is incompressible in H, the distance $d_{\mathcal{C}(\Sigma)}(b, \mathcal{D}(H))$ is at least 2. Let $\pi : H \to F$ be the I-bundle such that b is its binding. Let α be an essential arc on F. Then $D := \pi^{-1}(\alpha)$ is an essential disk in H. Since the Euler characteristic of F is negative, there exists a null-homotopic simple closed curve β on F disjoint from α. Then $A := \pi^{-1}(\beta)$ is an annulus or a Möbius band in H that satisfies $\partial D \cap \partial A = \emptyset$ and $\partial A \cap b = \emptyset$. Thus we have $d_{\mathcal{C}(\Sigma)}(b, \mathcal{D}(H)) = 2$. See Figure 3. □

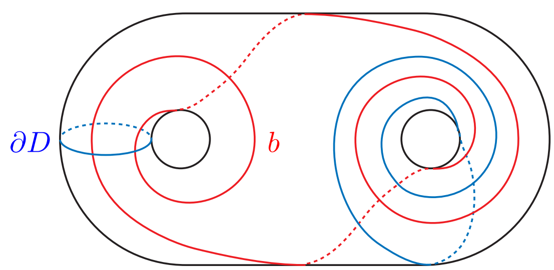


Figure 3. This figure depicts the case where the genus of H is two and F is non-orientable. The distance between ∂D and b in $\mathcal{C}(\Sigma)$ is two.

The set of $3g-3$ mutually disjoint, mutually non-isotopic, essential disks in H is called a *solid pants decomposition* of H. Let $\mathscr{S} = \{D_1, \ldots, D_{3g-3}\}$ be a solid pants decomposition of H. An essential arc α on a component P of $\mathrm{Cl}(\partial H - \mathrm{Nbd}(\bigcup_{i=1}^{3g-3} \partial D_i))$ is called a *wave* (resp. *seam*) *for $\mathscr{S}$* if it is a wave (resp. seam) for the pants decomposition $\mathscr{P} = \{\partial D_1, \ldots, \partial D_{3g-3}\}$ of the surface ∂H. An essential simple closed curve β on ∂H is said to be *k-seamed with respect to $\mathscr{S}$* if β is k-seamed with respect to the pants decomposition $\mathscr{P}$ of ∂H.

The proof of the following lemma is straightforward.

Lemma 1.12. *Let $\mathscr{S}$ be a solid pants decomposition of H. Then the boundary of each essential disk D in H with $D \notin \mathscr{S}$ contains at least two waves for $\mathscr{S}$.*

LEMMA 1.13. *Each binding b of H admits a solid pants decomposition $\mathscr{S}$ of H such that b is* 1*-seamed with respect to $\mathscr{S}$.*

PROOF. Let $\pi : H \to F$ be the I-bundle such that b is its binding. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a maximal collection of mutually disjoint, mutually non-isotopic, essential arcs on F. Then $\{\pi^{-1}(\alpha_1), \ldots, \pi^{-1}(\alpha_n)\}$ forms the required solid pants decomposition of H. □

LEMMA 1.14. *Let β be an essential simple closed curve on ∂H. If β is* 2*-seamed with respect to a solid pants decomposition $\mathscr{S}$ of H, then β is not a binding of H.*

PROOF. Suppose that β is 2-seamed with respect to a solid pants decomposition $\mathscr{S}$ of H. Let D be an essential disk in H. If D is a member of $\mathscr{S}$, we have

$$i(\beta, \partial D) \geq 4,$$

where $i(\cdot, \cdot)$ is the geometric intersection number. Otherwise, by Lemma 1.12, ∂D contains at least two waves α_1, α_2 with respect to $\mathscr{S}$. Thus, in this case, we have

$$i(\beta, \partial D) \geq \#(\beta \cap \alpha_1) + \#(\beta \cap \alpha_2) \geq 4.$$

See Figure 4. Consequently, for any essential disk D in H we have $i(\beta, \partial D) > 2$.

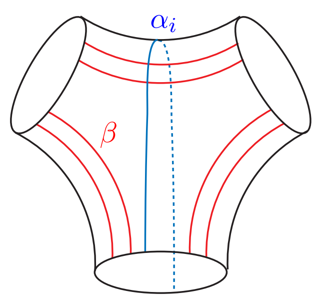


FIGURE 4. Each wave α_i intersects β in at least two points.

On the other hand, it is easily seen that for any binding b of H, there exists an essential disk D in H with $i(b, \partial D) = 2$. This implies that β is not a binding of H. □

5. Open and twisted book decompositions

In this section, we consider two analogous structures on a closed orientable 3-manifold, *open* and *twisted book decompositions*. Both decompositions naturally induce Heegaard splittings, where each handlebody of the splittings inherits the structure of an I-bundle.

Let S be a compact orientable surface with nonempty boundary. Let h be an orientation preserving automorphism of S that fixes ∂S. Consider the mapping torus $S(h)$, which is the result of taking $S \times I$ and gluing $S \times \{1\}$ to $S \times \{0\}$ according to h. The boundary of $S(h)$ can naturally be identified with $\partial S \times S^1$. By shrinking each circle $\{x\} \times S^1$, where $x \in \partial S$, to a point, we obtain a closed orientable 3-manifold

M. In this paper we shall call such a pair (S, h) an *open book decomposition* of M. The image b of $\partial S \times I$ under the quotient map $q : S \times I \to M$ forms a link in M. We call b the *binding* of the open book decomposition (S, h). The images H_1 and H_2 of $S \times [0, 1/2]$ and $S \times [1/2, 1]$, respectively, under the quotient map q give a Heegaard splitting for M, that is, H_1 and H_2 are handlebodies in M satisfying $H_1 \cup H_2 = M$ and $H_1 \cap H_2 = \partial H_1 = \partial H_2$. We call this one *the Heegaard splitting of M induced from the open book decomposition* (S, h). Note that the Heegaard surface of the splitting is homeomorphic to the double of S.

Again, let S be a compact orientable surface with nonempty boundary. Let ι_0 and ι_1 be orientation reversing, fixted-point-free involutions of S satisfying $\iota_0|_{\partial S} = \iota_1|_{\partial S}$. Remark that, here, the number of the boundary components of S must be even. Let $C_1, \ldots, C_{2n}$ be the boundary components of ∂S such that $\iota_0(C_i) = C_{i+n}$ (subscripts $\pmod n$). Consider the resulting space $S(\iota_0, \iota_1)$ of taking $S \times I$ and gluing $S \times \{0\}$ to itself according to ι_0 and $S \times \{1\}$ to itself according to ι_1. The boundary of $S(\iota_0, \iota_1)$ consists of n copies of the torus. For each point x in $\cup_{i=1}^{n} C_i$, the image of the union $(\{x\} \times [0, 1]) \cup (\{\iota_0(x)\} \times [0, 1])$ under the quotient map $S \times [0, 1] \to S(\iota_0, \iota_1)$ is a circle on the boundary tori. By shrinking each such circle to a point, we obtain a closed orientable 3-manifold M. We call such a triple (S, ι_0, ι_1) a *twisted book decomposition* of M. The image b of $\partial S \times I$ under the quotient map $q : S \times I \to M$ forms a link in M. We call b the *binding* of the twisted book decomposition (S, ι_0, ι_1). The images H_1 and H_2 of $S \times [0, 1/2]$ and $S \times [1/2, 1]$, respectively, under the quotient map q gives a Heegaard splitting for M. We call this one *the Heegaard splitting of M induced from the twisted book decomposition* (S, ι_0, ι_1). Since $\Sigma := q(S \times \{1/2\})$ is the Heegaard surface of the splitting, the surface $\Sigma_b := \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$ is homeomorphic to S.

Note that if (S, h) is an open book decomposition of M with the binding b, $\mathrm{Cl}(M - \mathrm{Nbd}(b))$ admits a natural foliation with all leaves (called *pages*) homeomorphic to S. Similarly, if (S, ι_0, ι_1) is a twisted book decomposition of M with the binding b, $\mathrm{Cl}(M - \mathrm{Nbd}(b))$ admits a natural foliation with all but two leaves (called *pages*) homeomorphic to S, where the two exceptional leaves are homeomorphic to the non-orientable surface S/ι_0 $(\cong S/\iota_1)$.

LEMMA 1.15. *Let F be a compact surface with nonempty boundary. Let $H \to F$ be the orientable I-bundle with the binding b. Let H_1 and H_2 be copies of H. Let M be a closed orientable 3-manifold obtained by gluing H_1 to H_2 according to an automorphism of ∂H preserving b. Then we have the following:*

(1) If F is orientable, the resulting Heegaard splitting $M = H_1 \cup H_2$ is induced from an open book decomposition where b is the binding.

(2) If F is non-orientable, $M = H_1 \cup H_2$ is induced from a twisted book decomposition where b is the binding.

PROOF. The first assertion is clear from the definition. Suppose that F is non-orientable. For each $i \in \{1, 2\}$, let F_i be the surface in H_i corresponding to the section $F \times \{1/2\}$ of the twisted I-bundle $H \to F$. Set $\Sigma_b := \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$, where Σ is the Heegaard surface of the splitting $M = H_1 \cup H_2$. Then $\mathrm{Cl}(M - \mathrm{Nbd}(F_1 \cup F_2))$ is homeomorphic to $\Sigma_b \times I$, which gives the structure of a twisted book decomposition of M. The assertion is now clear from the construction. $\square$

6. The Goeritz groups of the Heegaard splittings induced from twisted book decompositions

In this section, let F denote a compact non-orientable surface of negative Euler characteristic with a single boundary component, let $\pi : H \to F$ denote the orientable I-bundle with the binding b. We set $\Sigma := \partial H$, and $M = H_1 \cup_\Sigma H_2$ always denotes the Heegaard splitting, where H_1 and H_2 are copies of H, and M is obtained by gluing H_1 to H_2 according to an automorphism $\varphi \in \mathrm{Aut}_+(\Sigma, b)$. Note that by Lemma 1.15, $M = H_1 \cup_\Sigma H_2$ is induced from a twisted book decomposition where b is the binding. By Lemma 3.1, the distance $d(H_1, H_2)$ is at most 4. In this section, we will compute the Goeritz group of $M = H_1 \cup_\Sigma H_2$ in two cases. The first case, where we will consider in Subsection 6.1, is that the gluing map φ is particularly simple in the sense that φ is a power of the Dehn twist about the binding b. The second case, where we will consider in Subsection 6.2, is, on the contrary, that the gluing map φ is complicated in the sense that the distance in $\mathcal{C}(\Sigma_b)$ between the images of subsurface projection π_{Σ_b} of $\mathcal{D}(H_1)$ and $\mathcal{D}(H_2)$ is sufficiently large, where $\Sigma_b := \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$.

6.1. The Goeritz groups of distance-2 Heegaard splittings. Let k be an integer. Suppose that the gluing map $\partial H_1 \to \partial H_2$ is the k-th power τ_b^k of the Dehn twist τ_b. Note that by Lemma 3.1 and Yoshizawa [**51**, Theorem 1.3], if $|k| \geq 2$ the distance $d(M, \Sigma)$ of this splitting is exactly 2. The aim of this subsection is to prove the following theorem.

THEOREM 1.16. *Suppose that $|k| \geq 5$. For the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ as above, we have the following.*

(1) The splitting $M = H_1 \cup_\Sigma H_2$ is not induced from an open book decomposition.

(2) The Goeritz group $\mathcal{G}(M, \Sigma)$ is isomorphic to the group $\mathrm{MCG}(F)$*. In particular, $\mathcal{G}(M, \Sigma)$ is an infinite group.*

PROOF OF THEOREM 1.16 (1). We suppose for a contradiction that the Heegaard splitting $M = H_1 \cup_\Sigma H_2$ is induced from an open book decomposition. Let b' be the binding of the open book decomposition. Using the identification of H_1 with H, we regard b and b' as bindings of H. Since τ_b^k is the gluing map for the Heegaard splitting, $\tau_b^k(b')$ is a binding of H as well. By Lemma 1.13 there exists a solid pants decomposition $\mathscr{S}$ of H such that b is 1-seamed with respect to $\mathscr{S}$. Since $\Sigma - b$ is connected whereas $\Sigma - b'$ consists of two components, b and b' are not isotopic on Σ.

Suppose first that $b \cap b' = \emptyset$. Let $\pi' : H \to F'$ be the I-bundle with b' the binding. Needless to say, this is the trivial bundle. Hence, $\pi'(b)$ is a simple closed curve on F'. Since b and b' are not parallel, and F' is orientable, there exists an essential simple arc α on F' disjoint from $\pi'(b)$. Then ${\pi'}^{-1}(\alpha)$ is an essential disk in H disjoint from b. It follows that $d_{\mathcal{C}(\Sigma)}(b, \mathcal{D}(H)) \leq 1$. This contradicts Lemma 1.11.

Suppose that $b \cap b' \neq \emptyset$. By Lemma 1.14 the binding b' cannot be 2-seamed with respect to $\mathscr{S}$. Let $\mathscr{P}$ be the set of the boundaries of the disks in $\mathscr{S}$. Let C be the union of the simple closed curves of $\mathscr{P}$. Set $N := \mathrm{Nbd}(b; \Sigma)$. We may isotope b' so that b' in efficient position with respect to (N, C) and $(N, \mathscr{P}, b')$ admits no outer triangles. Let $\alpha_1, \dots, \alpha_r$ be the components of $b' \cap N$, and t_j ($j \in \{1, \dots, r\}$) the twisting number of α_j in N with respect to C. By Lemma 1.5 we have $|t_j| \leq 2$ for all j. Since $|k| \geq 5$ by the assumption, this implies that either $k + t_j \geq 0$ (for all j) or $k + t_j \leq 0$ (for all j). It then follows from Lemma 1.4 that $\tau_b^k(b')$ remains to be in efficient position with respect to (N, C), and the twisting number of $\tau_b^k(\alpha_j)$ in N with respect to N is $k + t_j$. In particular, we have $|k + t_j| \geq |k| - |t_j| \geq |k| - 2 > 2$. Again by Lemma 1.5, the binding $\tau_b^k(b')$ is 2-seamed with respect to $\mathscr{S}$. This contradicts Lemma 1.14. □

To prove Theorem 1.16 (2), we need the following lemma.

Lemma 1.17. *Let φ be an automorphism of Σ that is extendable over H_1. If φ preserves the binding b, φ is extendable over H_2 as well. Thus, φ can be regarded as an element of $\mathcal{G}(M, \Sigma)$.*

Proof. We will first show that φ commutes with τ_b^k up to isotopy. We identify $\mathrm{Nbd}(b; \Sigma)$ with $S^1 \times I$. Let R and T_k be the automorphisms of $S^1 \times I$ defined by $R(\theta, r) = (-\theta, 1 - r)$ and $T_k(\theta, r) = (\theta + 2\pi k r, r)$. Clearly R commutes with T_k. Up to isotopy, we can assume that φ preserves $\mathrm{Nbd}(b; \Sigma)$ and $\varphi|_{\mathrm{Nbd}(b;\Sigma)}$ is the identity or R. We can also assume that the support of τ_b^k is $\mathrm{Nbd}(b; \Sigma)$ and $\tau_b^k|_{\mathrm{Nbd}(b;\Sigma)} = T_k$. Therefore φ commutes with τ_b^k up to isotopy.

To prove that φ is extendable over H_2, it suffices to see that $\varphi(\mathcal{D}(H_2)) = \mathcal{D}(H_2)$. This is equivalent to say that $\varphi(\tau_b^k(\mathcal{D}(H_1))) = \tau_b^k(\mathcal{D}(H_1))$. Since φ is extendable over H_1, it holds $\varphi(\mathcal{D}(H_1)) = \mathcal{D}(H_1)$. Therefore it follows that $\varphi(\tau_b^k(\mathcal{D}(H_1))) = \tau_b^k(\varphi(\mathcal{D}(H_1))) = \tau_b^k(\mathcal{D}(H_1))$. □

Recall that F is a compact non-orientable surface with $\chi(F) < 0$ and $\#\partial F = 1$, and $\pi : H \to F$ is the orientable I-bundle with the binding b. We regard that $F \subset H$ with $\partial F = b$. The annulus $\pi^{-1}(\partial F) = \mathrm{Nbd}(b)$ is equipped with the structure of a subbundle of $\pi : H \to F$. The restriction of π to Σ_b ($= \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$) is the orientation double cover of F. Using the identification of H_1 with H, we regard F as a surface in H_1. By Lemma 1.3, each element $\varphi_F \in \mathrm{MCG}(F)$ lifts to a unique element of $\mathrm{Aut}_+(\Sigma_b)$. Using the I-bundle structure of $\mathrm{Nbd}(b)$, this element extends to an automorphism of Σ in a unique way. Clearly, this is extendable over H_1, and further, extendable over H_2 as well by Lemma 1.17. In this way we get a map $L : \mathrm{MCG}(F) \to \mathcal{G}(M, \Sigma)$.

Proof of Theorem 1.16 (2). We will show below that the above map $L : \mathrm{MCG}(F) \to \mathcal{G}(M, \Sigma)$ is an isomorphism. The injectivity immediately follows from Lemma 1.3. To prove the surjectivity of L, it suffices to see that any map $\varphi \in \mathcal{G}(M, \Sigma)$ preserves the binding b (up to isotopy). Indeed, there exists a unique I-bundle structure of H with b the binding. Thus, if φ preserves b (up to isotopy), it preserves F (up to isotopy). Putting $\varphi_F := \varphi|_F$, we have $\varphi = L(\varphi_F)$. Suppose for a contradiction that there exists a map $\varphi \in \mathcal{G}(M, \Sigma)$ that does not preserve b.

First we will show that we can replace φ with another one, if necessary, so that $b \cap \varphi(b) \neq \emptyset$. Suppose that $b \cap \varphi(b) = \emptyset$. Then $\varphi(b)$ is a simple closed curve on $\Sigma_b := \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$. Let α and β be two-sided simple closed curves on F satisfying $d_{\mathcal{C}(F)}(\alpha, \beta) \geq 3$. Due to Penner [**39**], the composition $\tau_\alpha \circ \tau_\beta$ of Dehn twists is pseudo-Anosov. Let ψ be the element of $\mathrm{Aut}_+(\Sigma)$ defined by taking an orientation-preserving lift of $\tau_\alpha \circ \tau_\beta$ to $\mathrm{Aut}_+(\Sigma_b)$, and then extending it to the automorphism of the whole Σ as explained right before the proof. Note that $\psi|_{\Sigma_b}$ is also a pseudo-Anosov map. Thus, for a sufficiently large integer n, we have $\psi^n(\varphi(b)) \cap \varphi(b) \neq \emptyset$. By Lemma 1.17, ψ can be regarded as an element of the Goeritz group $\mathcal{G}(M, \Sigma)$. Therefore, $\varphi^{-1} \circ \psi^n \circ \varphi$ is an element of $\mathcal{G}(M, \Sigma)$ that satisfies $(\varphi^{-1} \circ \psi^n \circ \varphi)(b) \cap b \neq \emptyset$.

In the following, we assume that $b \cap \varphi(b) \neq \emptyset$. Set $b' := \varphi(b)$. Since b is a binding of a twisted book decomposition of M, so is b' of another twisted book decomposition of M that induces that same Heegaard splitting $M = H_1 \cup_\Sigma H_2$. As explained in the proof of Theorem 1.16 (1), it follows that both b' and $\tau_b^k(b')$ are bindings of H_1. The same argument as in the proof of Theorem 1.16 (1) shows that at least one of b' and $\tau_b^k(b')$ is 2-seamed with respect to a solid pants decomposition $\mathscr{S}$ of H_1. Thus, by Lemma 1.14 at least one of b' and $\tau_b^k(b')$ is not a binding of H_1. This is a contradiction. $\square$

6.2. The Goeritz groups of distance-4 Heegaard splittings. Recall that H_1 and H_2 are copies of H, and $M = H_1 \cup_\Sigma H_2$ is the Heegaard splitting with the gluing map $\varphi \in \mathrm{Aut}_+(\Sigma, b)$. Let (S, ι_0, ι_1) be the twisted book decomposition of M that induces $M = H_1 \cup_\Sigma H_2$. Set $G := \mathrm{MCG}(S)$ and $G_+ := \mathrm{MCG}_+(S)$. Let $G(S, \iota_0, \iota_1)$ denote the intersection of the centralizers $C_G(\iota_0)$, $C_G(\iota_1)$, and the subgroup G_+ of G, that is, $G(S, \iota_0, \iota_1) = C_G(\iota_0) \cap C_G(\iota_1) \cap G_+$. Set $\mathcal{D}_{\Sigma_b} := \pi_{\Sigma_b}(\mathcal{D}(H))$. Also, recall that $\Sigma_b = \mathrm{Cl}(\Sigma - \mathrm{Nbd}(b))$ and $\pi_{\Sigma_b} : \mathcal{C}^{(0)}(\partial H) \to P(\mathcal{C}^{(0)}(\Sigma_b))$ is a subsurface projection.

The following is the main theorem of this subsection:

THEOREM 1.18. *Suppose* $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \varphi(\mathcal{D}_{\Sigma_b})) > 10$. *For the Heegaard splitting* $M = H_1 \cup_\Sigma H_2$ *as above, we have the following.*

(1) *The distance* $d(M, \Sigma)$ *is exactly* 4.
(2) *The Goeritz group* $\mathcal{G}(M, \Sigma)$ *is isomorphic to the group* $G(S, \iota_0, \iota_1)$.

In Lemma 1.22, we will see that there actually exists a Heegaard splitting satisfying the condition in Theorem 1.18.

Recall that F is a compact non-orientable surface with $\chi(F) < 0$ and $\#\partial F = 1$, and $\pi : H \to F$ is the orientable I-bundle with the binding b. We equip with $\mathrm{Int}\, F$ and $\mathrm{Int}\, \Sigma_b$ hyperbolic metrics so that the covering map $p := \pi|_{\mathrm{Int}\, \Sigma_b}$ is a local isometry. Consider the pull-back $p^* : \mathcal{ML}(F) \to \mathcal{ML}(\Sigma_b)$ defined by $p^*(\lambda, m) = (p^{-1}(\lambda), m \circ p)$ for $(\lambda, m) \in \mathcal{ML}(F)$. Clearly, this is a well-defined, injective piecewise linear map that is equivariant under the action of $\mathbb{R}_+$. Thus, this map induces an injective piecewise projective map $c : \mathbb{P}\mathcal{ML}(F) \to \mathbb{P}\mathcal{ML}(\Sigma_b)$. Let $\mathcal{F} \subset \mathbb{P}\mathcal{ML}(\Sigma_b)$ denote the image of the set $\mathcal{S}(F)$ of weighted multicurves on F by the map c.

LEMMA 1.19 (Johnson [**28**]). *The set* $\mathcal{F}$ *is nowhere dense in* $\mathbb{P}\mathcal{ML}(\Sigma_b)$.

In the unpublished paper [**28**], Johnson gave a sketch of the proof of this lemma. The following proof is essentially due to his idea.

PROOF OF LEMMA 1.19. By Lemma 1.7, the set $\mathcal{S}(F)$ is dense in $\mathbb{P}\mathcal{ML}(F)$. Since c is a continuous map between spheres, which are compact and Hausdorff, we have

$$c(\mathbb{P}\mathcal{ML}(F)) = c(\mathrm{Cl}(\mathcal{S}(F))) = \mathrm{Cl}(c(\mathcal{S}(F))) = \mathrm{Cl}(\mathcal{F}).$$

Let $F = \#_h \mathbb{R}P^2 - \mathrm{Int}(D^2)$ (thus, $\Sigma_b = \#_{h-1} T^2 - \sqcup_2 \mathrm{Int}(D^2)$). By Theorem 1.6, we have $\mathbb{P}\mathcal{ML}(F) \cong S^{3h-5}$ and $\mathbb{P}\mathcal{ML}(\Sigma_b) \cong S^{6h-9}$. Thus, c is a piecewise projective embedding. Noting that $3h - 5 < 6h - 9$ for $h \geq 2$, we conclude that $\mathrm{Im}\ c = \mathrm{Cl}(\mathcal{F})$ is nowhere dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$. □

Let $\mathcal{I}$ denote the set of projectivizations of stable and unstable laminations of pseudo-Anosov automorphisms of Σ_b. In the following, by abuse of notation we simply write λ to mean a projective geodesic measured lamination $[(\lambda, m)] \in \mathbb{P}\mathcal{ML}(\Sigma_b)$. This will not cause any confusion.

LEMMA 1.20. *(1) The set $\mathcal{I}$ is dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$.*
(2) Let λ be a point of $\mathcal{I}$, and λ' a point of $\mathbb{P}\mathcal{ML}(\Sigma_b)$. If the intersection number of any representatives of λ and λ' in $\mathcal{ML}(\Sigma_b)$ is zero, then $\lambda = \lambda'$.

PROOF. (1) follows from Long [**33**, Lemma 2.6]. (2) follows from a well-known fact that the stable and unstable laminations for a pseudo-Anosov automorphism are minimal, uniquely ergodic, and fill up the surface. □

LEMMA 1.21. *The set $\mathcal{D}_{\Sigma_b}$ is nowhere dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$.*

REMARK 1.1. It it worth noting that in [**35**] Masur proved that $\mathcal{D}(H)$ is nowhere dense in $\mathbb{P}\mathcal{ML}(\Sigma)$.

PROOF. Suppose for a contradiction that $\mathcal{D}_{\Sigma_b}$ is not nowhere dense, that is, there exists an open set U of $\mathbb{P}\mathcal{ML}(\Sigma_b)$ contained in $\mathrm{Cl}(\mathcal{D}_{\Sigma_b})$. We will prove that this implies that U is also contained in $\mathrm{Cl}(\mathcal{F})$, which contradicts Lemma 1.19.

To prove that, we show that the set $U \cap \mathcal{I}$ is contained in $\mathrm{Cl}(\mathcal{F})$. Let $\lambda \in U \cap \mathcal{I}$. Since U is contained in $\mathrm{Cl}(\mathcal{D}_{\Sigma_b})$, there exists a sequence (α_n) in $\mathrm{Cl}(\mathcal{D}_{\Sigma_b})$ such that α_n converges to λ as n tends to ∞. For each α_n, we have $d_{\mathcal{C}(\Sigma_b)}(\alpha_n, \mathcal{F}) \leq 3$ due to Masur-Schleimer [**37**, Lemma 12.20]. Thus, for each n there exists a path $(\beta_n^0, \beta_n^1, \beta_n^2, \beta_n^3)$ such that $\beta_n^0 = \alpha_n$ and $\beta_n^3 \in \mathcal{F}$. By Theorem 1.6, $\mathbb{P}\mathcal{ML}(\Sigma_b)$ is sequentially compact. After passing to a subsequence if necessary, which we still write (β_n^j), we can assume that the sequence (β_n^j) converges to a point λ^j in $\mathbb{P}\mathcal{ML}(\Sigma_b)$ (as $n \to \infty$) for all $j \in \{0, 1, 2, 3\}$. Note that $\lambda^0 = \lambda \in \mathcal{I}$ and $\lambda^3 \in \mathrm{Cl}(\mathcal{F})$. Since the intersection number of any representatives of β_n^j and β_n^{j+1} in $\mathcal{ML}(\Sigma_b)$ is zero for all n and j, that of any representatives of λ^j and λ^{j+1} in $\mathcal{ML}(\Sigma_b)$ is zero for all $j \in \{0, 1, 2\}$. Since $\lambda^0 \in \mathcal{I}$, we have $\lambda^0 = \lambda^1$ by Lemma 1.20 (2). Applying the same argument repeatedly, we finally get $\lambda^0 = \cdots = \lambda^3$. Therefore, λ is contained in $\mathrm{Cl}(\mathcal{F})$.

By Lemma 1.20 (1), the set $\mathcal{I}$ is dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$. Thus, we conclude that $U \subset \mathrm{Cl}(U \cap \mathcal{I}) = \mathrm{Cl}(\mathcal{F})$. □

The following lemma shows the existence of a Heegaard splitting satisfying the condition in Theorem 1.18.

LEMMA 1.22. *There exists an automorphism ψ of Σ_b such that $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \psi^n(\mathcal{D}_{\Sigma_b}))$ tends to ∞ as n tends to ∞.*

PROOF. By Lemma 1.21, $\mathcal{D}_{\Sigma_b}$ is nowhere dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$. Since $\mathcal{I}$ is dense in $\mathbb{P}\mathcal{ML}(\Sigma_b)$ by Lemma 1.20, there exists a pseudo-Anosov automorphism $\psi : \Sigma_b \to \Sigma_b$ such that none of its invariant laminations λ^+, λ^- is contained in $\mathrm{Cl}(\mathcal{D}_{\Sigma_b})$. We will show that ψ is the required automorphism in the assertion.

Suppose for a contradiction that there exists $N > 0$ such that $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \psi^n(\mathcal{D}_{\Sigma_b})) \leq N$ for any $n > 0$. Thus, for each n there exists a path $(\alpha_n^0, \ldots, \alpha_n^N)$ such that $\alpha_n^0 \in \mathcal{D}_{\Sigma_b}$ and $\alpha_n^N \in \psi^n(\mathcal{D}_{\Sigma_b})$. Recall that $\mathbb{P}\mathcal{ML}(\Sigma_b)$ is sequentially compact by Theorem 1.6. After passing to a subsequence if necessary, which we still write (α_n^j), we can assume that the sequence (α_n^j) converges to a point λ^j in $\mathbb{P}\mathcal{ML}(\Sigma_b)$ (as $n \to \infty$) for all $j \in \{0, \ldots, N\}$. Note that $\lambda^0 \in \mathrm{Cl}(\mathcal{D}_{\Sigma_b})$ and $\lambda^N = \lambda^-$. Since the intersection number of any representatives of α_n^j and α_n^{j+1} in $\mathcal{ML}(\Sigma_b)$ is zero for all n and j, that of any representatives of λ^j and λ^{j+1} in $\mathcal{ML}(\Sigma_b)$ is zero for all $j \in \{0, \ldots, N-1\}$. Since $\lambda^N \in \mathcal{I}$, we have $\lambda^{N-1} = \lambda^N$ by Lemma 1.20 (2). Applying the same argument repeatedly, we finally get $\lambda^0 = \cdots = \lambda^N$. This is impossible because $\lambda^0 \in \mathrm{Cl}(\mathcal{D}_{\Sigma_b})$ and $\lambda^N = \lambda^- \notin \mathrm{Cl}(\mathcal{D}_{\Sigma_b})$. □

LEMMA 1.23. *Let ψ be an automorphism of Σ that preserve the binding b. If the distance $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \psi(\mathcal{D}_{\Sigma_b}))$ is greater than* 6, *the distance $d_{\mathcal{C}(\Sigma)}(\mathcal{D}(H), \psi(\mathcal{D}(H)))$ is exactly* 4.

PROOF. By Lemma 1.11, the distance $d_{\mathcal{C}(\Sigma)}(\mathcal{D}(H), \psi(\mathcal{D}(H)))$ is at most 4 for any ψ. Suppose that $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \psi(\mathcal{D}_{\Sigma_b})) > 6$. Suppose for a contradiction that the distance $d_{\mathcal{C}(\Sigma)}(\mathcal{D}(H), \psi(\mathcal{D}(H)))$ is less than 4. Then there exists an integer $k \in \{0, 1, 2, 3\}$ and a geodesic segment $(\alpha_0, \ldots, \alpha_k)$ in $\mathcal{C}(\Sigma)$ with $\alpha_0 \in \mathcal{D}(H)$ and $\alpha_k \in \psi(\mathcal{D}(H))$. If there exists j such that $\alpha_j = b$, we have either $d_{C(\Sigma)}(\mathcal{D}(H), b) < 2$ or $d_{\mathcal{C}(\Sigma)}(b, \psi(\mathcal{D}(H))) < 2$. Since b is a binding of the handlebody whose disk sets corresponds to $\psi(\mathcal{D}(H))$, this is impossible by Lemma 1.11. Suppose that $\alpha_j \neq b$ for all j. Then by Lemma 1.1, we have $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}, \psi(\mathcal{D}_{\Sigma_b})) \leq 6$, which is a contradiction. □

We remark that in the proof of Lemma 1.23 we have used the assumption that F has a single boundary component. Indeed, Lemma 1.11, which is used to get a contradiction in the argument, is valid only when the binding b on ∂H is a single simple closed curve. In the case where b has more than one components, we have $d_{\mathcal{C}(\Sigma)}(b, \mathcal{D}(H)) = 1$, which cannot lead to a contradiction.

PROOF OF THEOREM 1.18. The first assertion is a direct consequence of Lemma 1.23. By identifying Σ_b with $S \times \{1/2\}$, we get a natural injective homomorphism $\eta : G(S, \iota_0, \iota_1) \to \mathcal{G}(M, \Sigma)$. We will show the surjectivity of η. Suppose that there exists an element $\varphi \in \mathcal{G}(M, \Sigma)$ such that $\varphi(b) \neq b$. Set $b' := \varphi(b)$. Since b' is also a binding of a twisted book decomposition of M, we have $d_{\mathcal{C}(\Sigma)}(b', \mathcal{D}(H_j)) = 2$ for $j \in \{1, 2\}$ by Lemma 1.11. Set $\mathcal{D}_{\Sigma_b}^j := \pi_{\Sigma_b}(\mathcal{D}(H_j))$. By Lemma 1.1, we have $d_{\mathcal{C}(\Sigma_b)}(\pi_{\Sigma_b}(b'), \mathcal{D}_{\Sigma_b}^j) \leq 4$. This together with the fact that the diameter of $\pi_{\Sigma_b}(b')$ is at most 2 implies that $d_{\mathcal{C}(\Sigma_b)}(\mathcal{D}_{\Sigma_b}^1, \mathcal{D}_{\Sigma_b}^2)$ is at most 10. This contradicts the assumption on φ. In Consequence, any element of $\mathrm{MCG}_+(M, H_+)$ preserves the binding b.

Let $\varphi \in \mathcal{G}(M, \Sigma)$. Let $q : S \times I \to M$ be the quotient map. Set $S_t := q(S \times \{t\})$ for $t \in [0, 1]$. Since the I-bundle structure of H with the binding b is unique, φ preserves each S_t. In particular, φ restricts to an orientation preserving automorphism of $S_{1/2} = \Sigma$. Thus φ is contained in the image of η. $\square$

CHAPTER 2

Goeritz groups of bridge decompositions of links

1. Introduction

Let L be a link in a closed orientable 3-manifold M. A *bridge decomposition* of L is a Heegaard splitting $M = V^+ \cup_\Sigma V^-$ such that L intersects each of V^+ and V^- in properly embedded trivial arcs. When the genus of the surface Σ, called a *bridge surface*, is g, and the number of components of $V^\pm \cap L$ is n, we particularly call such a decomposition a (g,n)*-decomposition* of L. The distance is also defined for a bridge decomposition in the same way as in the case of a Heegaard splitting, and many results about Heegaard splittings have been extended to bridge decompositions. For example, Bachman-Schleimer [**2**] showed that the distance of a bridge decomposition of a knot is bounded above by the Euler characteristic of an essential surface in the complement of the knot, which is a generalization of a result in Hartshorn [**18**]. The arguments in [**2**] apply to the case of links as well, and their results, in particular, imply that if the distance of a link in a 3-manifold is at least 5, then the complement of the link admits a complete hyperbolic structure of finite volume. (The definition of the distance in this paper is slightly different from that in [**2**]; see Section 2.3.) As another example, Tomova [**50**] gave a sufficient condition for uniqueness of bridge decompositions of knots in terms of the distances and the Euler characteristics of bridge surfaces, which is a generalization of a result of Sharlemann-Tomova [**45**].

In this chaper, we are interested in the Goeritz group of a bridge decomposition. The concept of Goeritz group has been extended for bridge decompositions in [**20**]. In that paper, an analogous result with Namazi [**38**] and Johnson [**24**] was obtained: it was shown that there exists a constant C such that if the distance of a bridge decomposition of a link is at least C, then its Goeritz group is finite. Furthermore, such a constant C can be taken uniformly to be at most 3796. The goal of this chapter is the following, which improves the above mentioned result of [**20**].

Theorem 2.1. *Let $g \geq 0$, $n > 0$ and $(g,n) \neq (0,1),(0,2),(1,1)$. Let $(M, L; \Sigma)$ be a (g,n)-decomposition of a link L in a 3-manifold M. If the distance of $(M, L; \Sigma)$ is at least 6, then the Goeritz group $\mathcal{G}(M, L; \Sigma)$ is a finite group. Further, for a $(0,n)$-decomposition $(S^3, L; \Sigma)$ of a link L in the 3-sphere S^3, where $n \geq 3$, if the distance of $(S^3, L; \Sigma)$ is at least 5, then the Goeritz group $\mathcal{G}(S^3, L; \Sigma)$ is a finite group.*

Theorem 2.1 is proved by extending the argument of [**24**] to the case of bridge decompositions. In fact, the major part of the proof is devoted to show the following.

Theorem 2.8. Let L be a link in a 3-manifold M and $(M, L) = (V^+, V^+ \cap L) \cup_\Sigma (V^-, V^- \cap L)$ a bridge decomposition. If the distance between the sets of disks and once-punctured disks in $V^+ - L$ and $V^- - L$ in the curve graph of $\Sigma - L$ are at least 4, then the natural homomorphism $\eta : \mathcal{G}(M, L; \Sigma) \to \mathrm{MCG}_+(M, L)$ is injective.

The key tool for the proof is the *double sweep-out* technique involving the theory of graphics introduced by Rubinstein-Scharlemann [**42**]. As noted above, if the distance of the bridge decomposition $(M, L; \Sigma)$ is at least 6, then the complement $M - L$ admits a hyperbolic structure, and hence the mapping class $\mathrm{MCG}(M, L)$ is a finite group. Theorem 2.1 thus follows from Theorem 2.8 and these facts.

In Section 2, we review basic definitions and properties of the distance and the Goeritz group of a bridge decomposition. In Section 3, we review the theory of sweep-outs, which is the main tool of the paper. In Section 4, we give the proof of Theorem 2.8. Finally, in Section 5, we give the proof of Theorem 2.1.

2. Definitions of the distance and the Goeritz group for a bridge decomposition

2.1. Bridge decompositions. Let $g \geq 0$ and $n > 0$. Let V be a handlebody of genus g. The union of n properly embedded, mutually disjoint arcs in V is called an *n-tangle*. An n-tangle in V is said to be *trivial* if the arcs can be isotoped into ∂V simultaneously. Let L be a link in a closed orientable 3-manifold M. Let $M = V^+ \cup_\Sigma V^-$ be a genus-g Heegaard splitting of M. A decomposition $(M, L) = (V^+, V^+ \cap L) \cup_\Sigma (V^-, V^- \cap L)$ is called a *(g, n)-decomposition* of L if $V^+ \cap L$ and $V^- \cap L$ are trivial n-tangles in V^+ and V^-, respectively. We sometimes denote such a decomposition by $(M, L; \Sigma)$. The surface Σ here is called the *bridge surface* of L. Two bridge decompositions of L are said to be *equivalent* if their bridge surfaces are isotopic through bridge surfaces of L.

2.2. Curve graphs. In the previous chapter, we have considered the curve complex (or the arc and curve complex) for a compat surface. In fact, these complexes can also be defined for a punctured surface in a similar way. The following is a more detailed description.

Let $g \geq 0$ and $k > 0$. Let Σ be a closed orientable surface of genus g with k marked points $p_1, p_2, \ldots, p_k$. Set $\Sigma' := \Sigma - \{p_1, p_2, \ldots, p_k\}$. A simple closed curve in Σ' is said to be *essential* if it does not bound a disk or a once-punctured disk in Σ'. We say that an open arc α in Σ' is *essential* if it satisfies:

- $\mathrm{Cl}(\alpha; \Sigma) - \alpha \subset \{p_1, p_2, \ldots, p_k\}$; and
- If $\mathrm{Cl}(\alpha; \Sigma)$ is a simple closed curve bounding a disk D in Σ, then the interior of D contains at least one point of $\{p_1, p_2, \ldots, p_k\}$.

The *curve graph $\mathcal{C}(\Sigma')$ of Σ'* is the graph whose vertices are isotopy classes of essential simple closed curves in Σ', and the edges are pairs of vertices $\{\alpha, \beta\}$ with $\alpha \cap \beta = \emptyset$. Similarly, the *arc and curve graph $\mathcal{AC}(\Sigma')$ of Σ'* is the graph whose vertices are isotopy classes of essential simple closed curves and essential open arcs in Σ', and the edges are pairs of vertices $\{\alpha, \beta\}$ with $\alpha \cap \beta = \emptyset$. By abuse of notation, we denote the underlying space of the curve graph (the arc and curve graph, respectively) by the same symbol $\mathcal{C}(\Sigma')$ ($\mathcal{AC}(\Sigma')$, respectively). The graph $\mathcal{C}(\Sigma')$ ($\mathcal{AC}(\Sigma')$, respectively) can be viewed as a geodesic metric space with the simplicial metric $d_{\mathcal{C}(\Sigma')}$ ($d_{\mathcal{AC}(\Sigma')}$, respectively). We note that the curve graph $\mathcal{C}(\Sigma')$ is nonempty and connected if and only if $3g - 4 + k > 0$.

2.3. Distances. The concept of the distance can be defined not only for a Heegaard splitting but also for a bridge decomposition. In fact, there are three different ways of generalizing the definiton of the distance to this case. In this subsection, we give the definition of the distance of a bridge decomposition and its variations, and summarize their basic properties.

Let $(M, L) = (V^+, V^+ \cap L) \cup_\Sigma (V^-, V^- \cap L)$ be a (g, n)-decomposition of a link L in a closed orientable 3-manifold M, where $3g - 4 + 2n > 0$. We denote by $\mathcal{D}(V_L^\pm)$ the set of vertices of $\mathcal{C}(\Sigma_L)$ that are represented by simple closed curves bounding disks in $V_L^\pm$.

DEFINITION. The *distance* $d(M, L; \Sigma)$ of the bridge decomposition $(M, L; \Sigma)$ is defined by $d(M, L; \Sigma) := d_{\mathcal{C}(\Sigma_L)}(\mathcal{D}(V_L^+), \mathcal{D}(V_L^-))$.

There are other variations, $d_{\mathcal{PD}}(M, L; \Sigma)$ and $d_{BS}(M, L; \Sigma)$, of the distance. The first one, $d_{\mathcal{PD}}(M, L; \Sigma)$, is defined as follows. Let $\mathcal{PD}(V_L^\pm)$ denote the set of all vertices of $\mathcal{C}(\Sigma_L)$ that are represented by simple closed curves bounding disks in $V^\pm$ that intersect L at most once. Then $d_{\mathcal{PD}}(M, L; \Sigma)$ is defined by $d_{\mathcal{PD}}(M, L; \Sigma) := d_{\mathcal{C}(\Sigma_L)}(\mathcal{PD}(V_L^+), \mathcal{PD}(V_L^-))$. It is easily checked that the following inequality holds:

$$d_{\mathcal{PD}}(M, L; \Sigma) \le d(M, L; \Sigma) \le d_{\mathcal{PD}}(M, L; \Sigma) + 2. \tag{1}$$

Furthermore, for a $(0, n)$-decomposition the following holds.

PROPOSITION 2.2 (Jang [**23**, Proposition 1.2]). *Suppose that $M = S^3$ and the genus of Σ is zero. Then,*

- *$d_{\mathcal{PD}}(M, L; \Sigma) = d(M, L; \Sigma)$ if $d_{\mathcal{PD}}(M, L; \Sigma) \ge 1$, and*
- *$d(M, L; \Sigma) = 0$ or 1 if $d_{\mathcal{PD}}(M, L; \Sigma) = 0$.*

We next define $d_{BS}(M, L; \Sigma)$, which was introduced by Bachman-Schleimer [**2**]. For trivial n-tangles $(V^\pm, V^\pm \cap L)$, we denote by $\mathcal{B}(V^\pm, V^\pm \cap L)$ the set of all vertices α of $\mathcal{AC}(\Sigma_L)$ such that

- $\alpha \in \mathcal{PD}(V_L^\pm)$, or
- α is an open arc in Σ_L such that $\partial\, \mathrm{Cl}(\alpha; \Sigma) \subset \Sigma \cap L$ and $\mathrm{Cl}(\alpha; \Sigma)$ cobounds a disk in $V^\pm$ with an arc of $V^\pm \cap L$.

We define $d_{BS}(M, L; \Sigma)$ by the distance between two sets $\mathcal{B}(V^+, V^+ \cap L)$ and $\mathcal{B}(V^-, V^- \cap L)$ in the arc and curve graph $\mathcal{AC}(\Sigma_L)$. By the argument of the proof of (1) in p.480 in Korkmaz-Papadopoulos [**32**], we have

$$\frac{1}{2} d(M, L; \Sigma) \le d_{\mathrm{BS}}(M, L; \Sigma) \le d(M, L; \Sigma). \tag{2}$$

See also [**3**]. We summarize a few facts needed in Sections 4 and 5. The following lemma is an extension of Haken's lemma [**17**].

LEMMA 2.3 ([**2**, Lemma 4.1]). *Let $(M, L; \Sigma)$ be a bridge decomposition of a link L in a closed orientable 3-manifold M. If M_L contains an essential 2-sphere, or if there exists a 2-sphere in M that intersects L transversely at a single point, then $d_{\mathcal{PD}}(M, L; \Sigma) = 0$.*

REMARK 2.1. Lemma 4.1 of [**2**] is stated only for knots, but their arguments hold for links.

Corollary 6.2 of Bachman-Schleimer [**2**] says that if $d_{\mathrm{BS}}(M, L; \Sigma) \geq 3$, then the complement of L admits a complete hyperbolic structure of finite volume. (Again, [**2**, Corollary 6.2] is stated for knots, but their arguments are valid even for links.) Combining this fact and the inequality (2), we have the following.

THEOREM 2.4. *Let $(M, L; \Sigma)$ be a bridge decomposition of a link L in a closed orientable 3-manifold M. If $d(M, L; \Sigma) \geq 5$, then M_L admits a complete hyperbolic structure of finite volume.*

REMARK 2.2. Here is a subtle remark on the various notions of distances introduced above. In [**23**], two notions of distance of bridge decompositions are discussed. One is d, which is denoted by d_T in [**23**], and the other is $d_{\mathcal{PD}}$, which is denoted by d_{BS} in the same paper. The important thing to note is that the definition of d_{BS} in [**23**] is different from the original one by Bachman-Schleimer [**2**]. Then, in [**21**, Theorem 5.1], it is claimed that if $(S^3, L; \Sigma)$ is a $(0, n)$-decomposition of a link L in S^3 and $d(S^3, L; \Sigma) \geq 3$, where $n \geq 3$, then the complement of L admits a complete hyperbolic structure of finite volume. The proof in that paper bases on two results. One is [**2**, Corollary 6.2]. The other is, however, not a relationship between d and d_{BS} but Proposition 2.2 above (literally this is described as a relationship between d_T and d_{BS} in [**23**]). Thus, we do not have a reasonable explanation of [**21**, Theorem 5.1]. If [**21**, Theorem 5.1] is still valid, then we can improve the distance estimation of Theorem 2.1 for $(0, n)$-decompositions of links in S^3.

2.4. Goeritz groups. Let M be an orientable manifold, and $Y_1, Y_2 \ldots, Y_k$ (possibly empty) subsets of M. In this chapter, we will work in the smooth categry for technical reasons. We use similar notations as in the previous chapter: Let $\mathrm{Diff}_+(M, Y_1, Y_2, \ldots, Y_k)$ denote the group of orientation-preserving self-diffeomorphisms of M that send Y_i to itself for $1 \leq i \leq k$. Let $\mathrm{MCG}_+(M, Y_1, \ldots, Y_k)$ denote the mapping class group of the $(k+1)$-tuple $(M, Y_1, Y_2 \ldots, Y_k)$, that is, the group of path-components of $\mathrm{Diff}_+(M, Y_1, Y_2 \ldots, Y_k)$.

DEFINITION. For a bridge decomposition $(M, L) = (V^+, V^+ \cap L) \cup_\Sigma (V^-, V^- \cap L)$, the mapping class group $\mathrm{MCG}_+(M, V^+, L)$ is called the *Goeritz group*, and it is denoted by $\mathcal{G}(M, L; \Sigma)$.

Let $(M, L; \Sigma)$ be a bridge decomposition of a link L in a closed orientable 3-manifold M. Since the natural map $\mathcal{G}(M, L; \Sigma) \to \mathrm{MCG}_+(\Sigma, \Sigma \cap L)$ obtained by restricting the maps of concern to Σ is injective, the Goeritz group can be thought of a subgroup of $\mathrm{MCG}_+(\Sigma, \Sigma \cap L)$. Thus, we can write

$$\mathcal{G}(M, L; \Sigma) = \mathrm{MCG}_+(V^+, V^+ \cap L) \cap \mathrm{MCG}_+(V^-, V^- \cap L) \subset \mathrm{MCG}_+(\Sigma, \Sigma \cap L).$$

We remark that the Goeritz group can also be defined as the group of path-components of $\mathrm{Aut}_+(M, V^+, L)$ as in Chapter 1: Indeed, it is well known that $\pi_0(\mathrm{Aut}_+(\Sigma, \Sigma \cap L))$ and $\pi_0(\mathrm{Diff}_+(\Sigma, \Sigma \cap L))$ $(= \mathrm{MCG}_+(\Sigma, \Sigma \cap L))$ are isomorphic, and hence $\pi_0(\mathrm{Aut}_+(M, V^+, L))$ and $\pi_0(\mathrm{Diff}_+(M, V^+, L))$ $(= \mathcal{G}(M, L; \Sigma))$ correspond to the same subgroup of $\mathrm{MCG}_+(\Sigma, \Sigma \cap L)$. (See for example [**4**].) So the definition of the Goeritz group in this chapter is compatible with that in Chapter 1.

3. Sweep-outs

We review the basic theory of sweep-outs. The main references of this section are Kobayashi-Saeki [**31**] and Johnson [**25**]. In the following, let M be a closed orientable 3-manifold, L a link in M, and $(M, L; \Sigma)$ a $(\text{genus}(\Sigma), n)$-decomposition of L throughout.

Definition. A function $f : M \to [-1, 1]$ is said to be a *sweep-out of* (M, L) *associated with the decomposition* $(M, L; \Sigma)$ if

- for all $s \in (-1, 1)$, $f^{-1}(s)$ is a bridge surface of L and the bridge decomposition $(M, L; f^{-1}(s))$ is equivalent to $(M, L; \Sigma)$; and
- $f^{-1}(1)$ and $f^{-1}(-1)$ are finite graphs, which are called *spines*, embedded in M.

We note that any bridge decomposition admits a sweep-out. For simplicity, we shall always assume further that the spines $f^{-1}(\pm 1)$ are uni-trivalent graphs, and the intersection of the spines and L is exactly the set of vertices whose valency is one. See Figure 1.

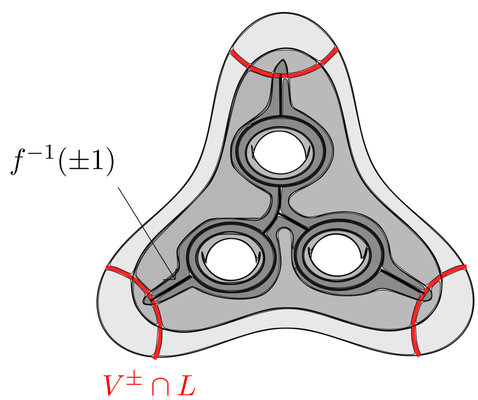


Figure 1. A spine in $V^{\pm}$.

A smooth map F from a 3-manifold N into $\mathbb{R}^2$ is said to be *stable* if there exists a neighborhood $U(F)$ of F in the space of smooth maps $C^{\infty}(N, \mathbb{R}^2)$ with the following property: for any $G \in U(F)$, there exist diffeomorphisms $\varphi : N \to N$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $G \circ \varphi = \psi \circ F$. The image of the set of singular points of the stable map is called the *discriminant set*.

Let f and g be sweep-outs of (M, L). Due to Kobayashi-Saeki [**31**], the map $f \times g : M \to [-1, 1] \times [-1, 1]$ can be perturbed so that $f \times g$ is stable in the complement of spines of f and g. In the following, whenever we consider the product of sweep-outs, we slightly perturb it to be stable. Let Γ be the closure in $[-1, 1] \times [-1, 1]$ of the union of the discriminant set of $f \times g$ and the image of L under the map $f \times g$. Then Γ is naturally equipped with a structure of a finite graph of valency at most four. Such a finite graph is called the *(Rubinstein-Scharlemann) graphic defined by* $f \times g$.

Each point (s,t) in the interior of the square $[-1,1]\times[-1,1]$ corresponds to the intersection of two level surfaces $\Sigma_s := f^{-1}(s)$ and $\Sigma'_t := g^{-1}(t)$. We note that the surfaces Σ_s and Σ'_t always intersect L transversely by definition. If the point (s,t) lies in the complementary region of the graphic Γ, then the surfaces Σ_s and Σ'_t intersect transversely and $\Sigma_s \cap \Sigma_t \cap L = \emptyset$. If (s,t) lies in the interior of an edge of Γ, then one of the following holds:

- Σ_s and Σ'_t share a single tangent point, and that point is a non-degenerate critical point of both $f|_{\Sigma'_t}$ and $g|_{\Sigma_s}$, see Figure 2 (i) and (ii); or
- Σ_s and Σ'_t intersect transversely, and $\Sigma_s \cap \Sigma'_t \cap L$ is a single point, see Figure 2 (iii).

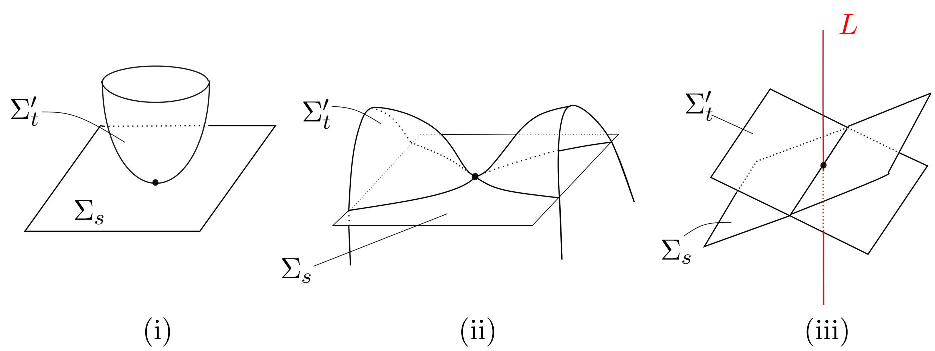


FIGURE 2. The surfaces Σ_s and Σ'_t when (s,t) lies in the interior of an edge of the graphic.

If (s,t) is at a 4-valent vertex of Γ, then either

- Σ_s and Σ'_t share exactly two tangent points, and those points are non-degenerate critical points of both $f|_{\Sigma'_t}$ and $g|_{\Sigma_s}$;
- Σ_s and Σ'_t share a single tangent point, and that point is a non-degenerate critical point of both $f|_{\Sigma'_t}$ and $g|_{\Sigma_s}$. Further, L intersects $\Sigma_s \cap \Sigma'_t$ at a point where Σ_s and Σ'_t intersect transversely; or
- Σ_s and Σ'_t intersect transversely, and $\Sigma_s \cap \Sigma'_t \cap L$ consists of exactly two points.

If (s,t) is at a 2-valent vertex of Γ, then Σ_s and Σ'_t share a single tangent point, and that point is a degenerate critical point of both $f|_{\Sigma'_t}$ and $g|_{\Sigma_s}$. See Figure 3. Each 1-

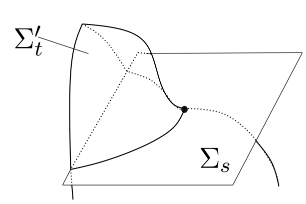


FIGURE 3. The surfaces Σ_s and Σ'_t when (s,t) is at a 2-valent vertex of the graphic.

or 3-valent vertex is in the boundary of the square, and it corresponds to the point where the level surface of one of the two sweep-outs is tangent to the spine of the other sweep-out.

Definition. The graphic defined by $f \times g$ is said to be *generic* if $f \times g$ is stable in the complement of the spines, and any vertical or horizontal arc in $[-1,1]\times[-1,1]$ contains at most one vertex of the graphic.

The following is Lemma 34 of Johnson [**25**].

Lemma 2.5. *Let f and g be sweep-outs associated to the bridge decomposition $(M,L;\Sigma)$. Let $\{\Phi_r : M \to M\}_{r\in[0,1]}$ be an ambient isotopy such that $\Phi_0 = \mathrm{id}_M$ and $\Phi_r(L) = L$ for all $r \in [0,1]$. Set $g_r := g \circ \Phi_r^{-1}$ for $r \in [0,1]$. Then we can perturb $\{\Phi_r\}_{r\in[0,1]}$ slightly, if necessary, so that the graphic defined by $f \times g_r$ is generic for all but finitely many $r \in [0,1]$. At each non-generic $r \in [0,1]$, the graphic fails to be generic due to one of the following two reasons:*

- *there exists a single vertical or horizontal arc in $[-1,1]\times[-1,1]$ containing two vertices of the graphic, or*
- *the map $f \times g_r$ is not stable in the complement of their spines.* (*This case corresponds to the six types of local moves shown in Figure* 5 *of* [**26**].)

Let f and g be sweep-outs associated to $(M,L;\Sigma)$. Let $s,t \in (-1,1)$. Set $\Sigma_s := f^{-1}(s)$, $\Sigma'_t := g^{-1}(t)$, $V_s{}^- := f^{-1}([-1,s])$, $V_s{}^+ := f^{-1}([s,1])$, $V_t'^- := g^{-1}([-1,t])$ and $V_t'^+ := g^{-1}([t,1])$.

Definition. We say that Σ_s is *mostly above* (*mostly below*, respectively) Σ'_t if each component of $\Sigma_s \cap V_t'^-$ ($\Sigma_s \cap V_t'^+$, respectively) is contained in a disk with at most one puncture in $\Sigma_s - L$.

Let $\mathscr{R}_a$ ($\mathscr{R}_b$, respectively) denote the set of all points $(s,t) \in [-1,1]\times[-1,1]$ such that Σ_s is mostly above (mostly below, respectively) Σ'_t. The regions $\mathscr{R}_a$ and $\mathscr{R}_b$ are bounded by the edges of the graphic. Note that a point (s,t) near $[-1,1]\times\{-1\}$ is labeled by $\mathscr{R}_a$ because $V_t'^-$ lies within a small neighborhood the spine of g, and all the intersections of $V_t'^-$ and Σ_s must consist of disks. Similarly, a point (s,t) near $[-1,1]\times\{1\}$ is labeled by $\mathscr{R}_b$. Also, by definition, both regions $\mathrm{Cl}(\mathscr{R}_a)$ and $\mathrm{Cl}(\mathscr{R}_b)$ are *vertically convex*, that is, if a point (s,t) is in $\mathrm{Cl}(\mathscr{R}_a)$ ($\mathrm{Cl}(\mathscr{R}_b)$, respectively), then so is (s,t') for any $t' \le t$ ($t' \ge t$, respectively).

Lemma 2.6. *Suppose that* $(\mathrm{genus}(\Sigma), n) \ne (0,1),(0,2),(1,1)$. *Then the closure of the regions $\mathscr{R}_a$ and $\mathscr{R}_b$ are disjoint.*

Proof. We first suppose that $\mathscr{R}_a \cap \mathscr{R}_b \ne \emptyset$. Let $(s,t) \in \mathscr{R}_a \cap \mathscr{R}_b$. Then there exists a component l of $\Sigma_s \cap \Sigma'_t$ such that l bounds once-punctured disks in Σ_s in both sides of l. Thus Σ_s is a twice-punctured sphere and $(\mathrm{genus}(\Sigma), n) = (0,1)$.

Next, suppose that $\mathrm{Cl}(\mathscr{R}_a)$ and $\mathrm{Cl}(\mathscr{R}_b)$ share an edge of the graphic. Let (s,t) be a point in the (interior of the) common edge of $\mathrm{Cl}(\mathscr{R}_a)$ and $\mathrm{Cl}(\mathscr{R}_b)$. A small neighborhood P in Σ_s of the component of $g|_{\Sigma_s}^{-1}(t)$ containing a critical point of $g|_{\Sigma_s}$ or a point of L is either a pair of pants or a once-punctured annulus, see Figure 4. By the assumption, each component of ∂P is inessential in $\Sigma_s - L$. Therefore Σ_s must be a twice-punctured sphere, and thus, we have $(\mathrm{genus}(\Sigma), n) = (0,1)$.

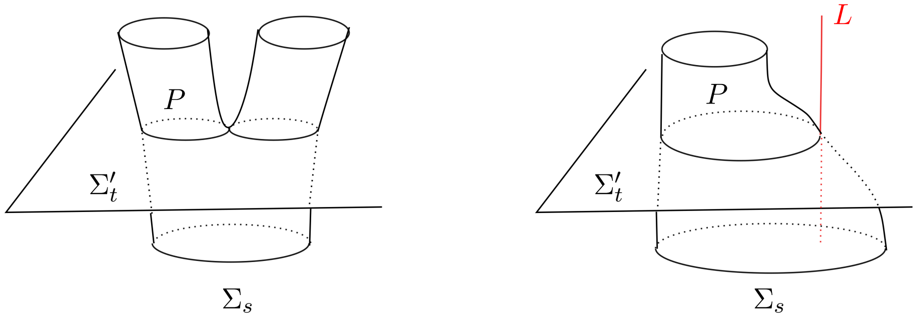


FIGURE 4. A small neighborhood P in Σ_s of the component of $g|_{\Sigma_s}^{-1}(t)$ containing a critical point of $g|_{\Sigma_s}$ or a point of L.

Finally, suppose that $\mathrm{Cl}(\mathscr{R}_a)$ and $\mathrm{Cl}(\mathscr{R}_b)$ do not share any edge, but they share a vertex of the graphic. Let $(s, t_\pm)$ be points near the vertex shown in Figure 5. There are exactly two critical points of $g|_{\Sigma_s}$ between $g|_{\Sigma_s}^{-1}(t_-)$ and $g|_{\Sigma_s}^{-1}(t_+)$: one is

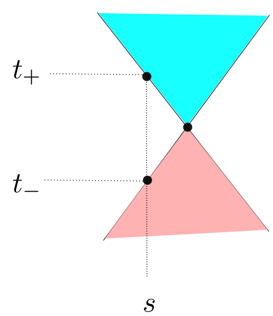


FIGURE 5. A neighborhood of the vertex

on $g|_{\Sigma_s}^{-1}(t_-)$ and the other is on $g|_{\Sigma_s}^{-1}(t_+)$. As in the above case, a small neighborhood $Q_\pm$ in Σ_s of the component of each $g|_{\Sigma_s}^{-1}(t_\pm)$ of concern is a pair of pants or a once-punctured annulus. In the surface $\Sigma_s - L$, each component of $\partial Q_\pm$ either bounds a once-punctured disk or cobounds an annulus with another component of $\partial Q_\pm$. Thus, we can check that Σ_s is either a four-times punctured sphere or a twice-punctured torus, which implies $(\mathrm{genus}(\Sigma), n) = (0, 2), (1, 1)$. □

In what follows, we assume that $(\mathrm{genus}(\Sigma), n) \neq (0, 1), (0, 2), (1, 1)$. We say that g *spans* f if there exists $t \in [-1, 1]$ such that the horizontal arc $[-1, 1] \times \{t\}$ intersects both $\mathscr{R}_a$ and $\mathscr{R}_b$. Otherwise, we say that g *splits* f. See Figure 6.

We say that g *spans* f *positively* if there exist points $(a, t) \in \mathscr{R}_a$ and $(b, t) \in \mathscr{R}_b$ with $b < a$.

LEMMA 2.7 ([**25**, Lemma 14]). *Let f be a sweep-out of (M, L), and g the result of perturbing f slightly so that the graphic defined by $f \times g$ is generic. Then g spans f positively.*

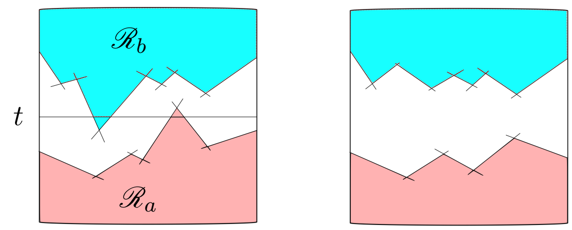


FIGURE 6. The function g spans f if some horizontal arc in the square intersects both $\mathscr{R}_a$ and $\mathscr{R}_b$ (left), and otherwise g splits f (right).

4. Upper bounds for the distance

Let $(M, L) = (V^+, V^+ \cap L) \cup_\Sigma (V^-, V^- \cap L)$ be a $(\text{genus}(\Sigma), n)$-decomposition of a link L in a closed orientable 3-manifold M, and suppose that $(\text{genus}(\Sigma), n) \neq (0, 1), (0, 2), (1, 1)$. Recall that $d_{\mathcal{PD}}(M, L; \Sigma) = d_{\mathcal{C}(\Sigma_L)}(\mathcal{PD}(V_L^+), \mathcal{PD}(V_L^-))$. The goal of this section is to show the following.

THEOREM 2.8. *If $d_{\mathcal{PD}}(M, L; \Sigma) \geq 4$, then the natural homomorphism $\eta : \mathcal{G}(M, L; \Sigma) \to \text{MCG}_+(M, L)$ is injective.*

We prove Theorem 2.8. We first note that, by Lemma 2.3, we may assume the following.

Assumption: Any meridional loop of L does not bound a disk in $M - L$.

LEMMA 2.9. *Let L and M be as above. Let Σ be a closed connected surface in M intersecting L transversely. Let D be a disk in M such that $D \cap \Sigma = \partial D$, $\partial D \cap L = \emptyset$, and D intersects L transversely in at most one point. Let Σ' be a component of a surface obtained by compressing Σ along D. Then we have $\chi(\Sigma' - L) \geq \chi(\Sigma - L)$, where $\chi(\cdot)$ denotes the Euler characteristic.*

REMARK 2.3. In the above lemma, we allow the case where D is not a compression disk for Σ, in other words, ∂D can be inessential in Σ.

PROOF. Suppose that $\chi(\Sigma' - L) < \chi(\Sigma - L)$. Then the only possibility is that $|D \cap L| = 1$, ∂D bounds a disk E in Σ with $E \cap L = \emptyset$, and $\Sigma' = (\Sigma - E) \cup D$. This contradicts our assumption stated right before the lemma. □

LEMMA 2.10. *Let Σ be a closed orientable surface, K the union of vertical arcs in $\Sigma \times [0, 1]$, and S a surface in $\Sigma \times [0, 1]$ that intersects K transversely. If S separates $\Sigma \times \{0\}$ from $\Sigma \times \{1\}$, then $\chi(S_K) \leq \chi(\Sigma_K)$. Furthermore, the equality holds if and only if S is isotopic to a horizontal surface keeping S transverse to K throughout the isotopy.*

PROOF. Let S' be the result of repeatedly compressing S_K so that S' is incompressible in $(\Sigma \times [0, 1]) - K$. The surface S' still separates $\Sigma \times \{0\}$ from $\Sigma \times \{1\}$, and it follows from Lemma 2.9 that $\chi(S_K) \leq \chi(S')$. Since any incompressible surface in $\Sigma_K \times [0, 1]$ is isotopic to a horizontal surface, we have $\chi(S_K) \leq \chi(\Sigma_K)$. □

Let $f : M \to [-1,1]$ be a sweep-out of (M, L) with $f^{-1}(0) = \Sigma$, and g the result of perturbing f slightly. Let $[\phi]$ be in the kernel of η. Then, there exists an ambient isotopy $\{\Phi_r : M \to M\}_{r\in[0,1]}$ such that $\Phi_0 = \mathrm{id}_M$, $\Phi_1 = \phi$, and $\Phi_r(L) = L$ for all $r \in [0,1]$. We can assume that $\{\Phi_r\}_{r\in[0,1]}$ satisfies the conditions described in Lemma 2.5, that is, only a finitely many element in the 1-parameter family $\{g_r := g \circ \Phi_r^{-1}\}_{r\in[0,1]}$ of sweep-outs of (M, L) is non-generic.

LEMMA 2.11. *If g_r spans f for all $r \in [0,1]$, then $\phi|_\Sigma$ is isotopic in Σ to the identity* $\mathrm{id}|_\Sigma$ *relative to the points $\Sigma \cap L$.*

PROOF. For each $r \in [-1,1]$, set $A_r := p_2(\mathrm{Cl}(\mathscr{R}_a))$ and $B_r := p_2(\mathrm{Cl}(\mathscr{R}_b))$, where $p_2 : [-1,1] \times [-1,1] \to [-1,1]$ denotes the projection onto the second coordinate. Since g_r spans f, A_r and B_r have nonempty intersection. Indeed, $A_r \cap B_r$ is a closed interval in $[-1,1]$ because $\mathrm{Cl}(\mathscr{R}_a)$ and $\mathrm{Cl}(\mathscr{R}_b)$ are vertically convex subsets of $[-1,1] \times [-1,1]$. Fix $r \in [-1,1]$. We define the map φ_r from the surface $g^{-1}(0)$ to $f^{-1}(0)$ that sends the points $g^{-1}(0) \cap L$ to $f^{-1}(0) \cap L$ as follows.

Let $t(r)$ be an interior point of the closed interval $A_r \cap B_r$. There are points $a(r)$ and $b(r)$ in $[-1,1]$ such that $(a(r), t(r)) \in \mathscr{R}_a$ and $(b(r), t(r)) \in \mathscr{R}_b$, respectively. Set $\Sigma_{a(r)} := f^{-1}(a(r))$, $\Sigma_{b(r)} := f^{-1}(b(r))$ and $\Sigma'_{t(r)} := g_r^{-1}(t(r))$. Since $\Sigma_{a(r)}$ is mostly above $\Sigma'_{t(r)} = g_r^{-1}(t(r))$ while $\Sigma_{b(r)}$ is mostly below $\Sigma'_{t(r)}$, we obtain a surface S_r lying within the product region between $\Sigma_{a(r)}$ and $\Sigma_{b(r)}$ by repeatedly compressing $\Sigma'_{t(r)}$ along the innermost disks intersecting L at most once in $\Sigma_{a(r)} \cup \Sigma_{b(r)}$ as long as possible. By the construction, the surface S_r separates $\Sigma_{a(r)}$ from $\Sigma_{b(r)}$. See Figure 7.

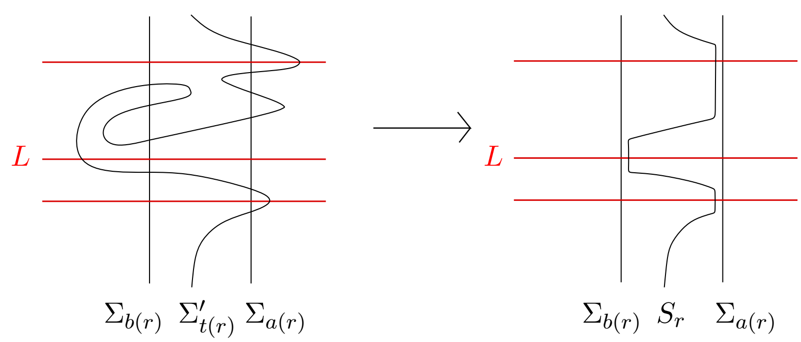


FIGURE 7. Compressing along innermost simple closed curves in $\Sigma_{a(r)} \cup \Sigma_{b(r)}$ iteratively to form the surface S_r.

We argue that S_r is canonically isotopic to a level surface of the sweep-out f keeping S_r transverse to the link L throughout the isotopy. By Lemma 2.9 we have $\chi(S_r - L) \geq \chi(\Sigma'_{t(r)} - L)$. On the other hand, since S_r separates $\Sigma_{a(r)}$ from $\Sigma_{b(r)}$, we have $\chi(S_r - L) \leq \chi(\Sigma'_{t(r)} - L)$ by Lemma 2.10. Thus, we have $\chi(S_r - L) = \chi(\Sigma'_{t(r)} - L)$. Again, by Lemma 2.10, S_r is isotopic to a level surface of the sweep-out f with keeping the surfaces transverse to L throughout.

By the argument above, it follows that S_r coincides with $\Sigma'_{t(r)}$ away from some disks, each of which intersects L at most once. Thus, there is a canonical identification of S_r with $\Sigma'_{t(r)}$. Therefore, we have the map:

$$g^{-1}(0) \to g_r^{-1}(t(r)) = \Sigma'_{t(r)} \to S_r \to f^{-1}(0).$$

Note that all maps are uniquely defined up to isotopy (with fixing the intersection points between the surface of concern and L). It is clear that the composition map can be chosen so that it sends $g^{-1}(0) \cap L$ to $f^{-1}(0) \cap L$. Define the map $\varphi_r : g^{-1}(0) \to f^{-1}(0)$ by such a composition map.

We shall now show that $\phi|_\Sigma$ is isotopic to the identity relative to $\Sigma \cap L$. There is the canonical identification of $f^{-1}(0) = \Sigma$ with $g^{-1}(0)$ because g is the result of perturbing f slightly. Under this identification, it holds that $\varphi_0 = \mathrm{id}_\Sigma$ and $\varphi_1 = \phi|_\Sigma$. It is clear that the values of $t(r)$ can be chosen so that it varies continuously. Although perhaps the points $a(r)$ and $b(r)$ do not vary continuously at some finitely many values of r, the deformation of φ_r remains to be continuous even around such values: it is easily seen that the choice of $a(r)$ or $b(r)$ does not affect the definition of the map $g_r^{-1}(t(r)) \to f^{-1}(0)$ in the above argument modulo isotopy. Thus, we conclude that for any $r, r' \in [0,1]$, φ_r and $\varphi_{r'}$ are isotopic fixing $\Sigma \cap L$, which shows the proof. □

Lemma 2.12. *If there exists $r \in [0,1]$ such that g_r splits f, then $d_{\mathcal{PD}}(M, L; \Sigma) \leq 3$.*

Proof. We denote by π_0 the natural projection from the preimage $f^{-1}((-1,1))$ of the open interval $(-1,1)$ to Σ that maps $f^{-1}((-1,1)) \cap L$ to $\Sigma \cap L$. By Lemma 2.7, g_0 spans f positively. Thus, there exists a time r_0 such that

- g_r spans f positively for all $r < r_0$, and
- $A_{r_0} \cap B_{r_0} = \{t\}$.

In the following, we consider the graphic defined by $f \times g_{r_0}$. By Lemma 2.5, the arc $[-1,1] \times \{t\}$ must intersect the region $\mathrm{Cl}(\mathscr{R}_a) \cup \mathrm{Cl}(\mathscr{R}_b)$ in exactly two vertices of the graphic. Let $(a,t) \in \mathrm{Cl}(\mathscr{R}_a)$ and $(b,t) \in \mathrm{Cl}(\mathscr{R}_b)$ be coordinates of such vertices. We note that $b < a$. Let us consider the points near these vertices shown in Figure 8: their coordinates are (a_-, t), (b_+, t), $(a_\pm, t_\pm)$ and $(b_\pm, t_\pm)$.

We set $\Sigma_s := f^{-1}(s)$, $\Sigma'_t := g_{r_0}^{-1}(t)$ as before. Set $f_0 := f|_{\Sigma'_t}$ and $f_\pm := f|_{\Sigma'_{t_\pm}}$. Note that the functions $f_\pm$ are Morse away from the preimages of ± 1.

We think about the function f_0. Let $\mathcal{L}_a$ be the set of simple closed curves of $f_0^{-1}(a_-)$ that are essential in $\Sigma_{a_-} - L$. Similarly, let $\mathcal{L}_b$ be the set of simple closed curves of $f_0^{-1}(b_+)$ that are essential in $\Sigma_{b_+} - L$. We note that $\mathcal{L}_a \neq \emptyset$ ($\mathcal{L}_b \neq \emptyset$, respectively) because (a_-, t) ((b_+, t), respectively) does not lie in $\mathscr{R}_a \cup \mathscr{R}_b$. Let l_a and l_b be arbitrary simple closed curves in $\mathcal{L}_a$ and $\mathcal{L}_b$, respectively.

By the choice of r_0, the same argument of the proof of Lemma 2.11 shows that we can find a natural map $\rho_0 : f_0^{-1}([b_+, a_-]) \to \Sigma_0 = \Sigma$ that extends to a homeomorphism $\hat{\rho}_0$ from the whole surface Σ'_t to Σ with $\hat{\rho}_0(\Sigma'_t \cap L) = \Sigma \cap L$. Since both l_a and l_b are level loops of $f_0 : \Sigma'_t \to (-1,1)$, they are disjoint. Therefore, the images $\rho_0(l_a)$ and $\rho_0(l_b)$ in $\Sigma_0 = \Sigma$ are also disjoint. In other words, we have

$$d_{\mathcal{C}(\Sigma_L)}(\rho_0(l_b), \rho_0(l_a)) \leq 1, \tag{3}$$

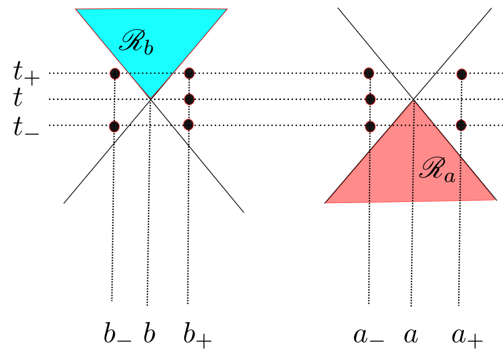


FIGURE 8. Small perturbed points of the vertices.

where we regard $\rho_0(l_a)$ and $\rho_0(l_b)$ as vertices of $\mathcal{C}(\Sigma_L)$. The projection π_0 also takes l_a and l_b to simple closed curves in Σ, which may have nonempty intersection. However, we see from the definition of ρ_0 that $\pi_0(l_a)$ and $\rho_0(l_a)$ ($\pi_0(l_a)$ and $\rho_0(l_a)$, respectively) are homotopic, hence, isotopic. Therefore, if we regard $\pi_0(l_a)$ and $\pi_0(l_b)$ as vertices of $\mathcal{C}(\Sigma_L)$, we can write

$$d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_b), \pi_0(l_a)) \leq 1. \tag{4}$$

We next show the inequality:

$$d_{\mathcal{C}(\Sigma_L)}(\pi_0(\mathcal{L}_a), \mathcal{PD}(V_L^+)) \leq 1. \tag{5}$$

We note that any level loop of $f_0^{-1}(a_-)$ can also be regard as loops of each of $f_\pm^{-1}(a_-)$ since the points (a_-, t) and $(a_-, t_\pm)$ are in the same component of the complementary region of the graphic. In the following, for simplicity, we shall not distinguish between a level loop of $f_0^{-1}(a_-)$ and the corresponding loops of $f_\pm^{-1}(a_-)$ in their notations. Let $l_a \in \mathcal{L}_a$. Let us first consider the function f_-. Since the point (a, t_-) lies within $\mathscr{R}_a$, as we pass from the level a_- to the level a, the simple closed curve l_a turns into one or two inessential simple closed curves in $\Sigma_a - L$. Therefore, in the surface $\Sigma_{a_-} - L$, either

- The simple closed curve l_a bounds a twice-punctured disk, see Figure 9 (i) and (ii); or
- The simple closed curve l_a cobounds with another essential simple closed curve l'_a an annulus that intersects L in at most one point, see Figure 9 (iii),

and the other simple closed curves of $f_-^{-1}(a_-)$ are inessential in $\Sigma_{a_-} - L$. As explained above, the natural map $\rho_0 : f_0^{-1}([b_+, a_-]) \to \Sigma_0 = \Sigma$ can be extended to the map $\hat{\rho}_0$ defined on the whole surface. The same thing still holds for the natural map from $f_-^{-1}([b_+, a_-])$ to Σ. Due to the existence of an extension of the natural map we see that, in the surface $\Sigma'_{t_-} - L$, either

- The simple closed curve l_a bounds a twice-punctured disk; or
- The simple closed curve l_a cobounds with another simple closed curve l'_a an annulus that intersects L in at most one point,

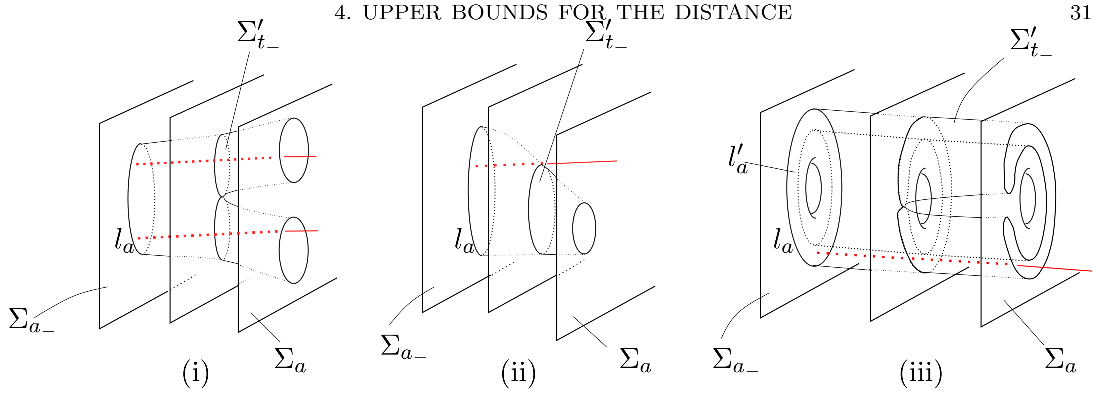


FIGURE 9. Potential configurations of l_a in Σ'_{t_-}.

according to which of the above two cases of the configuration of l_a in $\Sigma_{a_-} - L$ occurs. The other simple closed curves of $f_-^{-1}(a_-)$ are inessential even in $\Sigma'_{t_-} - L$.

Let us next consider the function f_+. Recall that we denote the simple closed curve in Σ'_{t_+} corresponding to $l_a \subset \Sigma'_t$ by the same symbol l_a.

Case A: The simple closed curve l_a bounds a twice-punctured disk in $\Sigma_{t_+} - L$ (and hence in $\Sigma_{a_-} - L$).

Let P be the twice-punctured disk in $\Sigma'_{t_+} - L$ bounded by l_a (note that such a subsurface is unique because $(\text{genus}(\Sigma), n) \neq (0, 2)$). We note that, in this case, $\mathcal{L}_a = \{l_a\}$. As we pass from the level a_- to the level a, there are four cases to consider.

Case A1: One or two new simple closed curves are created away from l_a (Figure 10).

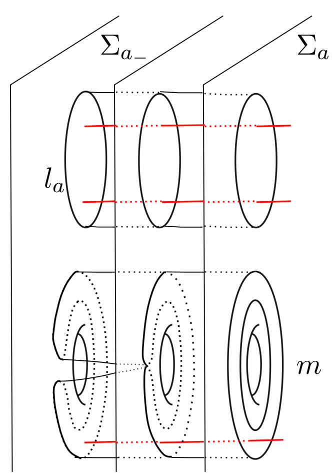


FIGURE 10. The simple closed curve $m \subset f_+^{-1}(a)$ in Σ'_{t_+} in Case A1.

We first see that at least one of the two new simple closed curves is essential in $\Sigma_a - L$. Suppose, contrary to our claim, that both of them are inessential in $\Sigma_a - L$. As we pass from the level a to the level a_+, the simple closed curve l_a turns into one or two inessential simple closed curves in $\Sigma_{a_+} - L$. Thus, all of the simple closed curves in $f_+^{-1}(a_+)$ are inessential in $\Sigma_{a_+} - L$. However, this contradicts the fact that the point (a_+, t_+) lies in the complement in $[-1, 1] \times [-1, 1]$ of $\mathscr{R}_a \cup \mathscr{R}_b$.

Let m be one of the new simple closed curves that is essential in $\Sigma_a - L$. Since each simple closed curve of $f_+^{-1}(a_-)$ is inessential in Σ'_{t_+} except for l_a, the curve m is contained in a disk with at most one puncture in $\Sigma'_t - L$. Thus, m is also inessential in $\Sigma'_{t_+} - L$. Let $D \subset \Sigma'_{t_+} - L$ be a disk with at most one puncture bounded by m. By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody $V_a^+ = f^{-1}([a, 1])$ such that $\partial D' = m$ and $|D' \cap L| \leq 1$. Thus, $\pi_0(m)$ is a vertex of $\mathcal{PD}(V_L^+)$. As shown in Figure 10, the inequality $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \pi_0(m)) \leq 1$ holds. In consequence, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq 1$.

Case A2: The simple closed curve l_a and another simple closed curve c in $f_+^{-1}(a_-)$ are pinched together to produce a new simple closed curve m (Figure 11).

Since the point (a, t_+) is in the complement in $[-1, 1] \times [-1, 1]$ of $\mathscr{R}_a \cup \mathscr{R}_b$, the simple closed curve m is essential in $\Sigma_a - L$. We also see that c bounds a once-punctured disk in $\Sigma_{a_-} - L$. Suppose, contrary to our claim, that c bounds a disk in $\Sigma_{a_-} - L$. Hence $\pi_0(m)$ is isotopic to $\pi_0(l_a)$ in $\Sigma - L$. As we pass from the level a to the level a_+, the simple closed curve m turns into one or two inessential simple closed curves in $\Sigma_{a_+} - L$, but this is impossible because the point (a_+, t_+) lies in the complement of $\mathscr{R}_a \cup \mathscr{R}_b$.

By the assumption, any meridional loop of L does not bound a disk in $M - L$. Thus, c bounds no disk in $\Sigma_{t_+} - L$. The possible configuration in P of l_a, m and c is shown in Figure 11. In particular, m bounds a once-punctured disk D in $P - L$. By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody V_a^+ such that $\partial D' = m$ and $|D' \cap L| = 1$. As the points (a_-, t_+) and (a, t_+) can be connected by a path that intersects the graphic once, $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \pi_0(m)) \leq 1$ holds. Therefore, it follows that $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq 1$.

Case A3: The simple closed curve l_a passes through a puncture and turns into a new simple closed curve m (Figure 12).

Since the point (a, t_+) is in the complement in $[-1, 1] \times [-1, 1]$ of $\mathscr{R}_a \cup \mathscr{R}_b$, m is essential in $\Sigma_a - L$. As shown in Figure 12, m bounds a once-punctured disk D in $P - L$. By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody V_a^+ such that $\partial D' = m$ and $|D' \cap L| = 1$. As $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a)), \pi_0(m)) \leq 1$, it follows that $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq 1$.

Case A4: The simple closed curve l_a is pinched to produce two simple closed curves m_1 and m_2 (Figure 13).

There are two possible configurations of l_a, m_1 and m_2 in P. See Figure 13.

First, suppose that both of the two simple closed curves m_1 and m_2 bound once-punctured disks, which are mutually disjoint, in $P - L$. Since the point (a, t_+) is

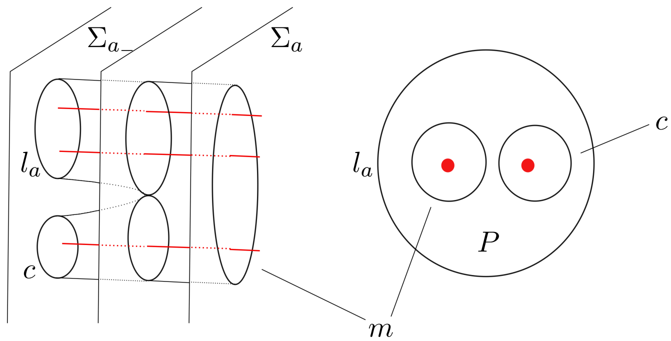


FIGURE 11. Case A2.

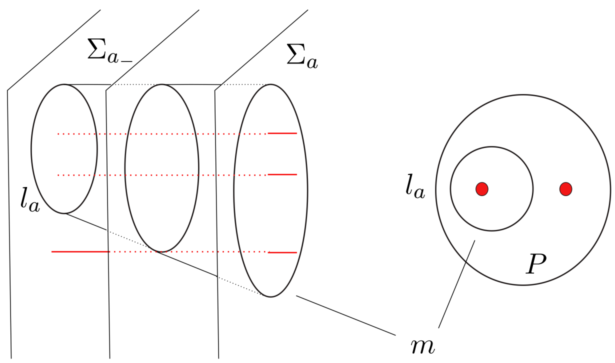


FIGURE 12. Case A3.

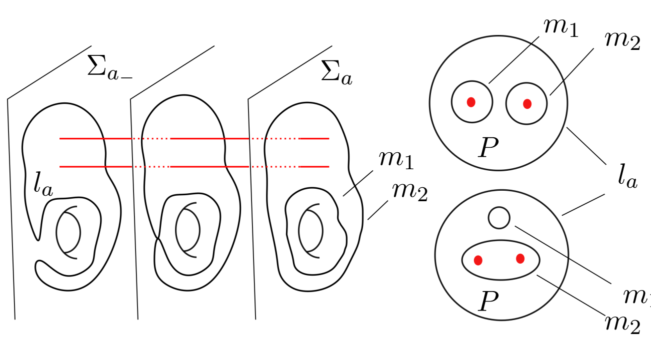


FIGURE 13. Case A4.

in the complement in $[-1,1] \times [-1,1]$ of $\mathscr{R}_a \cup \mathscr{R}_b$, one of m_1 and m_2 is essential in $\Sigma_a - L$. We may assume that m_1 is essential in $\Sigma_a - L$. Let $D \subset P$ be the disk such that $\partial D = m_1$ and $|D \cap L| = 1$. By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody V_a^+ such that $\partial D' = m_1$ and $|D' \cap L| = 1$. Thus, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \pi_0(m_1)) \leq 1$.

Next, suppose that m_1 bounds a disk in $P - L$. It suffices to show that m_1 must be essential in $\Sigma_a - L$. Indeed, if m_1 is essential in $\Sigma_a - L$, a similar argument as above shows that there exists a disk D in $V_a^+ - L$ such that $\partial D = m_1$. Thus, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq 1$.

Suppose, for the sake of contradiction, m_1 is inessential in $\Sigma_a - L$. By the assumption, any meridional loop of L does not bound a disk in $M - L$. Hence, m_1 must bound a disk in $\Sigma_a - L$. As we pass from the level a to the level a_+, the simple closed curve m_2 turns into one or two simple closed curves, which bound once-punctured disks in $P - L$. Note that $\pi_0(m_2)$ is isotopic to $\pi_0(l_a)$ in $\Sigma_0 - L$ because m_1 bounds a disk in $\Sigma_a - L$. It follows that as we pass from the level a to the level a_+, the simple closed curve m_2 turns into one or two simple closed curves that is inessential in $\Sigma_{a_+} - L$, and thus all of the simple closed curves of $f_+^{-1}(a_+)$ are inessential in $\Sigma_{a_+} - L$. This contradicts the fact that the point (a_+, t_+) does not lie in $\mathscr{R}_a \cup \mathscr{R}_b$. This completes the proof of the inequality (5) in Case A.

Case B: The simple closed curve l_a cobounds with another essential simple closed curve l'_a an annulus in Σ'_{t_+} (and hence in Σ_{a_-}) that intersects L in at most one point.

Let A be the annulus in Σ'_{t_+} bounded by l_a and l'_a (note that such an annulus is unique because $(\text{genus}(\Sigma), n) \neq (1, 1)$). We note that $\mathcal{L}_a = \{l_a, l'_a\}$. There are five cases to consider as we pass from the level a_- to the level a.

Case B1: A new simple closed curve m is created away from l_a and l'_a.

This case is same as Case A1.

Case B2: The simple closed curve l_a and another simple closed curve $c \neq l'_a$ of $f_+^{-1}(a_-)$ are pinched together to produce a single simple closed curve m (Figure 14).

We see that c bounds a once-punctured disk in $\Sigma_{a_-} - L$. Suppose, contrary to our claim, that c bounds a disk in $\Sigma_a - L$. Then, it follows that $\pi_0(m)$ is isotopic to $\pi_0(l_a)$ in $\Sigma_0 - L$. As we pass from the level a to the level a_+, the simple closed curves m and l'_a are pinched together to produce an inessential simple closed curve in $\Sigma_{a_+} - L$. This contradicts the fact that the point (a_+, t_+) does not lie in $\mathscr{R}_a \cup \mathscr{R}_b$.

By the assumption, any meridional loop of L does not bound a disk in $M - L$. Thus, it follows that $|A \cap L| = 1$ and c bounds a once-punctured disk in $A - L$. The possible configuration of l_a, l'_a, c and m in the annulus A is shown in Figure 14.

As we pass from the level a to the level a_+, the simple closed curves l'_a and m are pinched together to produce a new single curve m'. The simple closed curve m' is essential in $\Sigma_{a_+} - L$ because the point (a_+, t_+) lies in the complement of $\mathscr{R}_a \cup \mathscr{R}_b$. On the other hand, m' bounds a disk D in $A - L$. By repeatedly compressing D along the innermost disk in $\Sigma_{a_+} - L$, we obtain a disk D' in $V_{a_+}^+ - L$ such that $\partial D' = m'$. As $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \pi_0(m')) \leq 1$, it follows that $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \mathcal{PD}(V_L^+)) \leq 1$.

Case B3: The simple closed curve l_a passes through a puncture and turns into a new simple closed curve m (Figure 15).

In the annulus A, m cobounds with l'_a an annulus. See Figure 15. As we pass from the level a to the level a_+, the simple closed curves l'_a and m are pinched together to produce a new single curve m'. The simple closed curve m' is essential in $\Sigma_{a_+} - L$ because the point (a_+, t_+) lies in the complement of $\mathscr{R}_a \cup \mathscr{R}_b$. On the

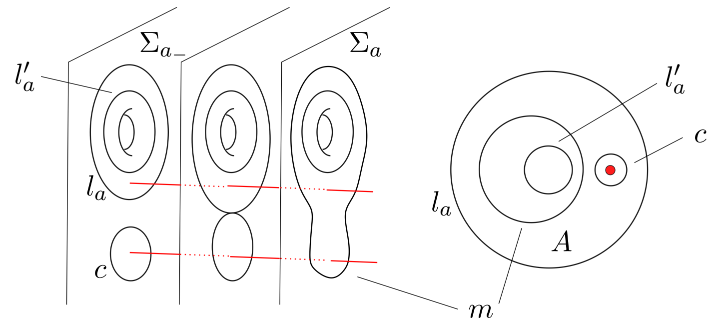


FIGURE 14. Case B2.

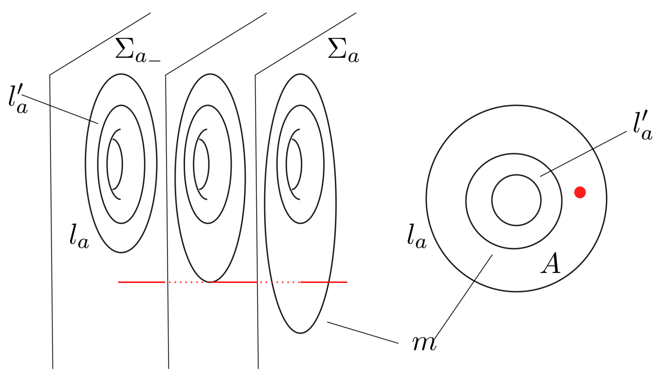


FIGURE 15. Case B3.

other hand, m' bounds a disk in $A - L$. By repeatedly compressing D along the innermost disk in $\Sigma_{a_+} - L$, we obtain a disk D' in $V^+_{a_+} - L$ such that $\partial D' = m'$. Therefore, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \mathcal{PD}(V^+_L)) \leq 1$.

Case B4: The simple closed curve l_a is pinched to produce two simple closed curves m_1 and m_2 (Figure 16).

There are two possible configurations of l_a, l'_a, m_1 and m_2 in the annulus A. See Figure 16.

First, suppose that m_1 bounds a disk D in $A - L$. By the assumption, any meridional loop of L does not bound a disk in its complement. Thus, the curve m_1 does not bound a once-punctured disk in $\Sigma_a - L$. We claim that m_1 does not bound a disk in $\Sigma_a - L$. Suppose, contrary to our claim, that m_1 bounds a disk in $\Sigma_a - L$. Then, $\pi_0(m_2)$ is isotopic to $\pi_0(l_a)$ in $\Sigma_0 - L$. As we pass from the level a to the level a_+, the simple closed curves m_2 and l'_a are pinched together to produce a single simple closed curve that is inessential in $\Sigma_{a_+} - L$. This contradicts the fact that the point (a_+, t_+) does not lie in $\mathscr{R}_a \cup \mathscr{R}_b$. Therefore, we conclude that m_1 must be essential in $\Sigma_a - L$.

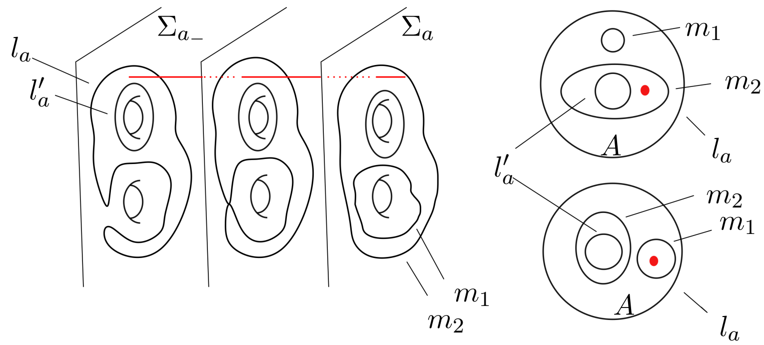


FIGURE 16. Case B4.

By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in $V_a^+ - L$ such that $\partial D' = m_1$. Since $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \pi_0(m_1)) \leq 1$, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \mathcal{PD}(V_L^+)) \leq 1$.

Next, suppose that m_1 bounds a once-punctured disk D in $A - L$. If m_1 is essential in $\Sigma_a - L$, by repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody V_a^+ such that $\partial D' = m_1$ and $|D' \cap L| = 1$. Since $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \pi_0(m_1)) \leq 1$, it follows that $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \mathcal{PD}(V_L^+)) \leq 1$. Thus, in the following, we shall assume that m_1 is inessential in $\Sigma_a - L$.

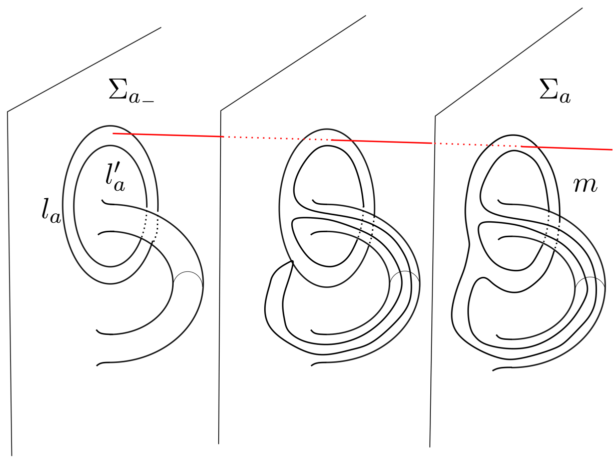
The simple closed curve m_2 cobounds an annulus with l'_a in $A - L$. As we pass from the level a to the level a_+, the simple closed curves l'_a and m_2 are pinched together to produce a new single simple closed curve m'. The curve m' is essential in $\Sigma_{a_+} - L$ because the point (a_+, t_+) lies in the complement of $\mathscr{R}_a \cup \mathscr{R}_b$. On the other hand, m' bounds a disk D in $A - L$. By repeatedly compressing D along the innermost disk in $\Sigma_{a_+} - L$ as long as possible, we finally obtain a disk D' in $V_{a_+}^+ - L$ such that $\partial D' = m'$. Since $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \pi_0(m')) \leq 1$, it follows that $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l'_a), \mathcal{PD}(V_L^+)) \leq 1$.

Case B5: The simple closed curves l_a and l'_a are pinched together to produce a single simple closed curve m (Figure 17).

Since the point (a, t_+) is in the complement in $[-1, 1] \times [-1, 1]$ of $\mathscr{R}_a \cup \mathscr{R}_b$, m is essential in $\Sigma_a - L$. In $A - L$, m bounds a disk D with at most one puncture. See Figure 17. By repeatedly compressing D along the innermost disk with at most one puncture in $\Sigma_a - L$ as long as possible, we finally obtain a disk D' in the handlebody V_a^+ such that $\partial D' = m$ and $|D' \cap L| \leq 1$. Therefore, we have $d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+)) \leq d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \pi_0(m)) \leq 1$, which completes the proof of the inequality (5) in Case B.

The symmetric argument of the proof of (5) shows the inequality

$$d_{\mathcal{C}(\Sigma_L)}(\pi_0(\mathcal{L}_b), \mathcal{PD}(V_L^-)) \leq 1. \tag{6}$$



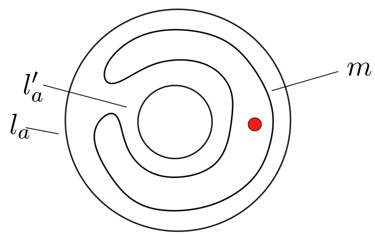


FIGURE 17. Case B5.

By the inequalities (4), (5) and (6), for some $l_a \in \mathcal{L}_a$ and $l_b \in \mathcal{L}_b$ we have

$$d_{\mathcal{PD}}(M, L; \Sigma) \leq d_{\mathcal{C}(\Sigma_L)}(\mathcal{PD}(V_L^-), \pi_0(l_b)) + d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_b), \pi_0(l_a)) + d_{\mathcal{C}(\Sigma_L)}(\pi_0(l_a), \mathcal{PD}(V_L^+))$$
$$\leq 1 + 1 + 1 \leq 3.$$

This completes the proof of Lemma 2.12. □

We now complete the proof of Theorem 2.8. By Lemma 2.12, g_r must span f for all $r \in [0, 1]$. Lemma 2.11 says that $\phi|_\Sigma$ is isotopic to $\mathrm{id}|_\Sigma$ relative to the points $\Sigma \cap L$. Therefore, ϕ represents the trivial element in $\mathcal{G}(M, L; \Sigma)$, which implies that the map η is injective.

5. Proof of Theorem 2.1

We are now in position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. Let $(M, L; \Sigma)$ be a bridge decomposition of L with the distance at least 6, where L is a link in a 3-manifold M. By Theorem 2.4, M_L admits a complete and finite volume hyperbolic structure. In this case it is well known that its mapping class group $\mathrm{MCG}_+(M_L)$ is a finite group, hence so is $\mathrm{MCG}_+(M, L)$. The first assertion now follows from Theorem 2.8 and (1). The second assertion can be shown by the same argument using Proposition 2.2 instead of (1). □

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