Behavior of Chameleon Mechanism on $F(R)$ Gravity

Yamato Matsuo [1*]

[1] *Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*

Abstract

We investigated $F(R)$ gravity, which is one of modified gravity theory. $F(R)$ gravity can describe slow roll inflation model and often describe Dark energy, Dark matter. There are many $F(R)$ model which explain unsolved cosmological problems. However observing of $F(R)$ gravity is difficult because of chameleon mechanism. In solar system the effect of $F(R)$ gravity is screened so we can't observe $F(R)$ gravity. Therefore we consider the case that we can observe $F(R)$ gravity. One of candidate is preheating era. We calculate behavior of chameleon mechanism at preheating era.

*ya-matsuo@hiroshima-u.ac.jp

Contents

1 Introduction

The equations of motion devised by Newton in the 17th century allowed the motion of objects to be treated mathematically. In Newtonian mechanics, force was understood as a remote action acting instantaneously on distant points of matter. In contrast, electromagnetic mechanics, invented by Maxwell in the 19th century, introduced a field called electromagnetic field to mediate the force. In this theory, an object such as a magnet generates a magnetic field, and this field acts on a distant substance to transmit force. This concept is called proximity action as opposed to distance action. With the birth of electromagnetism, it was discovered that light is a wave that propagates through an electromagnetic field. This electromagnetic field was thought to be transmitted through an unknown substance called the ether. According to this conventional idea, the speed of light changes depending on the relative speed of the inertial system and the ether. However, actual observation and experiments have shown that the speed of light does not depend on the inertial system. Einstein's special theory of relativity showed that there is no such thing as an absolutely stationary system in time and space, but rather that they are relative to each other and vary depending on the observer. This discovery led to a unified understanding of Newtonian mechanics and Maxwell's electrodynamics. In addition, Einstein developed the general theory of relativity to describe curved space. With this theory, he succeeded in calculating the movement of the perihelion of comets, which had been impossible to calculate with Newtonian mechanics.

Despite the success of general relativity, there are still unsolved problems. Three typical examples are Dark Energy (DE), inflation, and Dark Matter (DM). When the general theory of relativity was first completed, Einstein thought that the solution that the universe is expanding was unnatural. Therefore, he added a cosmological constant term to the Einstein equation to cancel out the expansion. However, Hubble's observation of the redshift of galaxies revealed that the distance between two galaxies increases the relative velocity of their separation from each other. This fact led to the discovery that the universe is expanding at an accelerating rate. To explain this accelerated expansion, we need an unknown energy (DE) whose density does not diminish with the expansion of the universe, and the cosmological constant added by Einstein corresponds to this DE. The cosmological constant written by Einstein is equivalent to this DE. There are various possible origins of this DE. For example, in quantum field theory, the vacuum has a constant energy. However, this energy has the Planck scale $M_{pl} \sim O(10^{38})GeV^2$, which is 123 orders of magnitude higher than the DE energy scale $\Lambda \sim O(10^{85})GeV^2$. Therefore, theories to explain this hierarchy and the effects that produce DE are currently being studied.

The second unsolved problem is the inflationary universe, which was the subject of the Big Bang theory proposed by G. Gamow in 1928, which states that the universe began as a ball of fire. In 1928, G. Gamow proposed the big bang theory that the universe began as a ball of fire. However, this theory raised the problems of flatness, in which the present universe is extremely flat, the horizon problem, in which there are correlations in regions of space that cannot be causally related, and the monopole problem, which has been predicted by the grand unified theory but has not yet been found [1, 2]. The inflationary universe was devised as a solution to these problems. In this theory, the universe expands rapidly at the beginning, and then the expansion energy is instantaneously dissipated into heat energy to produce particles. However, general relativity cannot induce the inflation that is said to have occurred at the beginning of the universe.

There are two ways to solve the problem of inflation: one is to add an inflaton, a particle that causes slow-roll inflation, and the other is to extend general relativity. The latter method is called the modified theory of gravity, in which the Einstein-Hilbert action R is replaced by a function $F(R)$ of an arbitrary Ricci scalar R, a scalar-tensor type one in which a scalar particle is added, and a tortion T instead of R [3,4,6,8,26,27]. In the present study, we will consider one of these methods, $F(R)$. $F(R)$ gravity is a theory proposed by H. A. Buchdahl in 1980, in which the action was rewritten as $\phi(R)$ instead of $F(R)$ [17]. As a result of this modification, the Einstein equation was rewritten in a modified form. In particular, the $F(R)$ gravity of R^2- proposed by Starobinksy [18], which adds a term of R^2 to the Einstein-Hilbert action, is now restricted by the observation of CMB fluctuations, e-folding number $N = 50 - 60$, curvature power spectrum $\ln(10^{10} As) = 3.043 \pm 0.014$, spectrum index $n_s = 0.9652 \pm 0.0042$ [28]. By adding logarithmic corrections to R^2, constant-roll inflation, which is a transition from the inflationary period to the present accelerating expansion universe, is now being studied [23,44].

On the other hand, galaxy rotation curves and gravitational lensing observations suggest the existence of invisible matter, DM. In the case of the galaxy rotation curve, a problem has arisen that galaxies are rotating faster than the expected rotation speed based on the observed total mass of galaxies. To explain this problem, invisible particles (DM) were needed in the galaxy clusters. It has also been observed that the light emitted from distant galaxies is bent by the gravitational lensing effect due to the invisible mass. This gravitational lensing effect is thought to be caused by the bending of space by DM. The gravitational lensing effect was also observed in the case of cluster collisions, where the DM did not interact with each other. This means that DM must be either non-interacting or very weakly interacting WIMPs (Weak Interaction Massive Particles), which also affect the structure formation of the universe. Simulations have shown that DM must have a non-relativistic (Cold) thermal velocity in order to reproduce the observed structure of the universe. Therefore, the DM must be non-interacting or WIMP in Cold. Candidates for the DM are the axion particle, whose existence is predicted by the strong CP problem of the SM, the SUSY particle predicted by supersymmetry theory, and black holes. Apart from these particles, an attempt has been made to solve the DM problem by using $F(R)$ gravity as a correction to the theory of gravity. In this paper, we try to solve the DM problem by $F(R)$ gravity based on the work of T. Katsuragawa and S. Matsuzaki [46,47]. In addition to the analysis of the $F(R)$ gravity treated here, models have been studied that focus on the contribution of the higher derivative of the curvature R [45], including coupling with scalar fields [48–50]

In section 2 we explain the inflation mechanism as slow roll inflation. In section 3 we explain preheating process, which is the next phase of universe from inflation era. In this phase in the early universe, the inflaton and scalar fields decay exponentially due to parametric resonance. We investigate this resonance by analyzing the matheu equation in a perturbative expansion. In section 4 we show $F(R)$ gravity as a modified gravity model. $F(R)$ gravity has typical feature as chameleon mechanism which is our main target of our work. We will focus on any $F(R)$ model. In particular, the logarithmic $F(R)$ model is a model that we have proposed and analyzed. In section 6 we numerically calculate inflaton time expansion at preheating era. Then we used symplectic numerical integral method, which conserve the value of Hamiltonian. In section 6 we discuss our results.

2 Slow roll inflation

In this section, we will first describe the slow roll inflation proposed by [39]as a theory to describe inflation. In this theory, the accelerated expansion of the universe that occurred in the early universe is caused by the vacuum energy of a scalar field called an inflaton. Fig.(1) schematically illustrates the behavior of inflaton in the early universe. First of all, the inflaton have an initial state that is displaced from the true vacuum. The inflaton slowly rolls down from this state to the true vacuum state. The inflaton then falls into the true vacuum, ending inflation, but the inflaton begins to oscillate around the vacuum. The cooled universe is then reheated as the vibrational energy of the inflaton decays into the interacting matter fields.

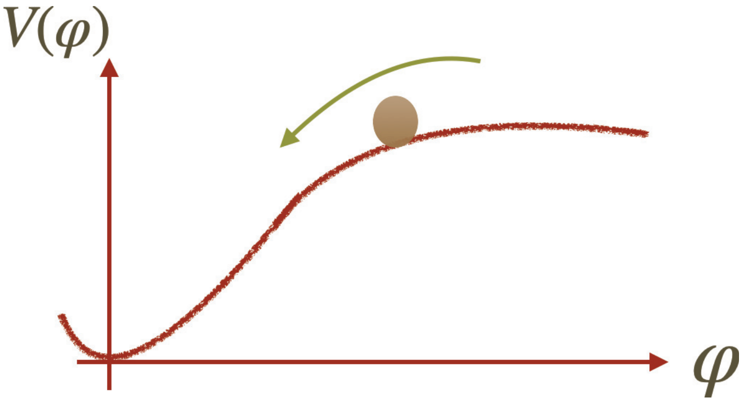

Figure 1: The potential of inflation and the behavior of inflaton in the early universe, where inflation slowly rolls down to a true vacuum.

2.1 Friedmann equation

First of all, I would like to review the accelerated expansion of the universe before looking at the behavior of the throw roll inflation in the early universe. In order to describe the expansion of the universe, we take the following Friedmann-Lemaitre-Robertson-Walker (FLRW) metric as the background space-time metric,

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \quad (1) \\ &= dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right). \quad (2) \end{aligned}$$

Here k is the curvature that determines the structure of the universe, and depending on the value of k, the universe can be classified as follows,

$$k = \begin{cases} > 0 & \text{closed space} \\ < 0 & \text{open space} \\ 0 & \text{flat space} \end{cases}. \quad (3)$$

The expansion of the universe is described using the Hubble parameter H. The H is defined by the scale factor $H = \dot{a}/a$. Here, $\dot{}$ represents the derivative with respect to time. The time expand of scale factor a is given by use H as

$$\frac{da}{a} = H(t)dt \tag{4}$$

$$a(t) = \exp\left(\int_{t_0}^{t} H(\tau)d\tau\right) \tag{5}$$

The power of exp of Eq(5) is represented N, e-folding number and the value should be $N \sim 60$.

First, we analyze the EInstein-Hilbert action, which describes general relativity. The action is defined as

$$S = \int d^4x \left(\sqrt{-g}\frac{1}{2\kappa^2}R - 2\Lambda\right), \tag{6}$$

where $\kappa = 1/M_{pi}$ and Λ describe dark energy term. The background space-time is described by metric $g^{\mu\nu}$. The equation of motion for the background spacetime, Friedmann equation, is obtained by varying the above equation to metric. Friedmann equation is given as

$$G_{00} = H^2 + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{\kappa^2\rho}{3} \tag{7}$$

$$G_{ij} = \delta_{ij}\left(2\dot{H} + 3H^2 + \frac{k}{a^2} - \Lambda\right) = -\delta_{ij}\kappa^2 P. \tag{8}$$

We will analyze the Hubble parameter H by solving this equation. At first we take derivative of Eq.(7) with respect to t, we obtain

$$2H\dot{H} - 2H\frac{k}{a^2} = \frac{\kappa^2\dot{\rho}}{3}. \tag{9}$$

We calculate -Eq(9) + H Eq(7),

$$3H\left(H^2 + \frac{k}{a^2} - \frac{\Lambda}{3}\right) = -\kappa^2\left(\frac{\dot{\rho}}{3} + HP\right) \tag{10}$$

$$\kappa^2 H\rho = -\kappa^2\left(\frac{\dot{\rho}}{3} + HP\right) \tag{11}$$

$$\dot{\rho} = -3H(\rho + P). \tag{12}$$

Eq.(12) corresponds to the law of the conservation of energy. We consider two case that matter effect is dominant and radiation effect is dominant. We place $k = \Lambda = 0$ because we don't focus the effect of curvature of universe and dark energy effect. First, we will focus on the Mattar dominant and Radiation dominant background fields to see how the expansion of the universe behaves.

2.1.1 Mattar dominant

First, let's look at the period when the material field prevails. In mattar dominant era, the term of P in Eq(12) is ignorable, then the equation becomes,

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} \tag{13}$$

$$\rho(t) = \rho_0 a(t)^{-3}. \tag{14}$$

The fact that matter density thins at the third order of spatial expansion is a result of intuition. The fact that the background field dilutes at the third order of spatial expansion determines the behavior of the expansion of the universe. Then we solve the Eq(7), the scale facter $a(t)$ expands as

$$H^2 = \frac{\kappa^2 \rho_0}{3} a^{-3} \tag{15}$$

$$a^{1/2} da = \sqrt{\frac{\kappa^2 \rho_0}{3}} dt \tag{16}$$

$$a(t) \propto t^{2/3}. \tag{17}$$

Thus, during the matter field dominance period, the universe decelerates and expands at the order of 2/3 of time.

2.1.2 Radiation dominant

When radiation effect is dominant, the EoM is described as relativistic and the relation of energy density and pressure is given as $\rho = 3P$. Then the law of the conservation of energy, Eq(12) is solved as

$$\dot{P} = -4HP \tag{18}$$

$$P(t) = P_0 a(t)^{-4}. \tag{19}$$

Then we solve the Eq(7), the scale factar $a(t)$ expands as

$$H^2 = \kappa^2 P_0 a^{-4} \tag{20}$$

$$a(t) \propto t^{1/2}. \tag{21}$$

Thus, during the matter field dominance period, the universe decelerates and expands at the order of 2/3 of time. While the matter field is diluted in the third order of space, radiation is diluted in the fourth order. Therefore, even if radiation dominant at first, it transitions to matter dominant as time goes by. Having reviewed the behavior of the expansion of the universe, we will now review inflation.

2.2 Slow roll inflation

Slow roll inflation is the one of inflation model, the scalar called as inflaiton expand universe. Inflation can be a particle that emerges from an extension of the Standard Model, or it can emerge from a modified theory of gravity. In slow roll inflation, the vacuum energy of the inflation causes the expansion of the universe. The action of slow roll inflation is given as,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \tag{22}$$

We consider scalar field is homogeneous, so we ignore the gradient term of scalar field. The EoM of scalar field ϕ is given as

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0. \tag{23}$$

In slow roll inflation, the inflaton must roll down the potential slowly. We define the slow roll condition for the inflaton to roll slowly down the potential. The slow roll conditions are defined as

$$\epsilon_V \equiv \frac{1}{2}\left(\frac{\partial_\phi V}{V}\right)^2 \tag{24}$$

$$\eta_V \equiv \frac{\partial_\phi^2 V}{V}, \tag{25}$$

and these parameters must satisfy $|\epsilon_V, \eta_V| \ll 1$ when universe was inflation era. Here, we calculate the magnitude of the e-folding number as the inflaton rolls from ϕ_1 to ϕ_2 as a measure of inflationary expansion. First, from the Friedmann equation, we obtain the following relationship between H and the scalar field ϕ,

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{26}$$

If ϕ satisfies the slow roll condition, the contribution of $\ddot{\phi}$ can be regarded as negligible compared to $\partial_\phi V$ and $\dot{\phi}$. In this case, the equation of motion of ϕ and the Friedmann equation can be approximated as follows

$$3H\dot{\phi} + \partial_\phi V = 0, \tag{27}$$

$$H^2 \simeq \frac{\kappa^2 V}{3}. \tag{28}$$

By using the above two equations, we can describe the *phi*-dependence of the e-folding number N. If we write N with t_1 as the starting state and t_2 as the ending state, we obtain

$$N = \int_{t_1}^{t_2} dtH = \int_{\phi_1}^{\phi_2} d\phi \frac{H}{\dot{\phi}} \simeq -\int_{\phi_1}^{\phi_2} d\phi \frac{\kappa^2 V(\phi)}{\partial_\phi V(\phi)}. \tag{29}$$

The e-folding number takes the value of $50 - 60$. The value of the final state ϕ_2 is determined by the value of ϕ, which breaks the slow roll condition, and the value of the starting state ϕ_1 is determined by calculating backwards from the value of the e-folding number.

Infratons fluctuate from their expected value during inflation. This fluctuation is the seed of the currently observed anisotropy of the CMB. Conversely, the observed CMB can be used to impose restrictions on the model of slow roll inflation.

$$A_s = \frac{\kappa^4}{4\pi^2}\frac{V(\phi)}{\epsilon_V}\bigg|_{\phi=\phi_0}, \quad n_s = (1 - 6\epsilon_V + 2\eta_V)|_{\phi=\phi_0}, \quad r = 16\epsilon_V|_{\phi=\phi_0}, \tag{30}$$

In slow roll inflation, the following relations are obtained for the power spectrum A_s, the spectrum index n_s, and the tensor-to-scalar ratio r [12, 14, 15]. In this study, we use these observations as a limit on slow roll inflation.

3 preheating process

When the inflation of universe finish, the universe enter a new phase. In this phase the energy of inflaton field transfer to elementary particle. At early universe, the universe must be reheat, because the universe became cold because of inflation. In the preheating phase, the exponential decay of the inflaton into a scalar field is observed due to the resonance effect. In this chapter, we will discuss how this resonance effect is caused.

3.1 Parametric resonance

Consider the case where the inflaton field has the simplest potential $1/2m_\phi^2\phi^2$ and interacts with the scalar field and $-1/2g^2\phi^2\chi^2$. In this case, the potential is given as follows.

$$V(\phi,\chi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{2}g^2\phi^2\chi^2 \tag{31}$$

In this case, assuming FLRW spacetime, the equations of motion for the inflaton ϕ and scalar field χ are

$$\left(\frac{d^2}{dt^2} + 3H\frac{d}{dt} + m_\phi^2 + g^2\chi^2\right)\phi = 0 \tag{32}$$

$$\left(\Box + m_\chi^2 + g^2\phi^2\right)\chi = 0, \tag{33}$$

where the effect of the gradient is dropped, assuming that ϕ is uniform, and the temperature of the preheating is cold, so it is assumed to be matter dominant. Therefore, the Habble scale H is given by $a(t) \propto t^{2/3}$, and H is $H(t) = \frac{2}{3t}$. Substituting this expression into Eq.(32), the equation of motion for the scalar field can be written as

$$\left(\frac{d^2}{dt^2} + \frac{2}{t}\frac{d}{dt} + m_\varphi^2\right)\phi = 0 \tag{34}$$

Here, the contribution of χ is assumed to be negligible with respect to the scale of ϕ. Now, to solve Eq.(34), we rewrite ϕ in terms of x as follows

$$\varphi = x(t)e^{-\int_{t0}^{t} d\tau \frac{1}{2\tau}} = \left(\frac{t}{t_0}\right)^{-\frac{1}{2}} x(t). \tag{35}$$

By using the function x given here, the derivative of ϕ with time can be written as follows.

$$\dot{\phi} = \left(\frac{\dot{x}}{x} - \frac{1}{2t}\right)\phi, \tag{36}$$

$$\ddot{\phi} = \left(\frac{\ddot{x}}{x} + \frac{1}{t}\frac{\dot{x}}{x} + \frac{3}{4t^2}\right)\phi \tag{37}$$

By substituting Eq.(36,37) into Eq.(34), we can rewrite the equation of motion of ϕ into a differential equation of x.

$$\frac{\ddot{x}}{x} + \frac{1}{t}\frac{\dot{x}}{x} + m^2 - \frac{1}{4t^2} = 0 \tag{38}$$

Here, by performing the transformation $t \to t/m$ for time t, Eq.(38) can be rewritten as follows.

$$\ddot{x}(t/m) - \frac{1}{t}\dot{x}(t/m) + \left(1 - \frac{1}{4t^2}\right)x(t/m) = 0 \tag{39}$$

The above equation corresponds to the differential equation of the bessel function, and by using the differential formula of bessel, the solution can be obtained as follows

$$x(t/m) = c_1 J_{1/2}(t/m) + c_2 Y_{1/2}(t/m) \tag{40}$$

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \qquad Y_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z), \tag{41}$$

where c_1, c_2 represent arbitrary indefinite coefficients, respectively. Using the solution of x obtained here, ϕ can be expressed as follows.

$$\phi_{back}(t) = \frac{A_1\cos(mt) + A_2\sin(mt)}{2mt}, \tag{42}$$

where A_1, A_2 are indefinite coefficients, respectively. Since the values of A_1 and A_2 can be chosen arbitrarily, we choose the following as the solution for ϕ.

$$\phi(t) \simeq \frac{\Phi\sin(m_\phi t)}{t} \tag{43}$$

Substituting the solution of ϕ obtained here into the equation of motion of χ and performing the Fourier transform, the following equation is obtained.

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + \frac{g^2\Phi^2\sin^2(m_\phi t)}{t^2}\right)\chi_k = 0, \tag{44}$$

where k corresponds to the wavenumber of the Fourier mode of χ. We also neglected the effect of the mass of χ. For Eq.(44), if the effect of the expansion of the universe is sufficiently negligible compared to the time evolution scale of ϕ, χ, the equation of motion of χ can be approximated as follows

$$\ddot{\chi}_k + \left(k^2 + g^2\tilde{\Phi}^2 + \frac{1}{2}(1 - \cos(2m_\phi t))\right)\chi_k = 0 \tag{45}$$

$$\ddot{\chi}_k + \left(\left(k^2 + \frac{1}{2}g^2\tilde{\Phi}^2\right) - \frac{1}{2}g^2\tilde{\Phi}^2\cos(2m_\phi t)\right) = 0. \tag{46}$$

The above equation can be approximated to an equation called the matheu equation.

3.2 Matheu equation

Eq.(46) can be rewritten as a differential equation called matheu equation as follows

$$\frac{d^2u}{d\tau^2} + (\delta - 2\epsilon\cos(2\tau))u = 0. \tag{47}$$

In order to treat this differential equation analytically, we perform a perturbation expansion under the condition $|\epsilon| \ll 1$. In this case, the parameters δ and $u(\tau)$ are expanded as follows

$$\delta = \delta_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + ..., \qquad x(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2(\tau) \tag{48}$$

Substituting this expansion equation into Eq.(47) and summarizing for each character of ϵ, the differential equation for δ, u is given for each order as

$$\left(\frac{d^2}{d\tau^2}+\delta_0\right)u_0(\tau)=0 \tag{49}$$

$$\left(\frac{d^2}{d\tau^2}+\delta_0\right)u_1(\tau)+(\delta_1-2\cos(2\tau))u_0(\tau)=0 \tag{50}$$

$$\left(\frac{d^2}{d\tau^2}+\delta_0\right)u_2(\tau)+(\delta_1-2\cos(2\tau))u_1(\tau)+\delta_2u_0(\tau)=0 \tag{51}$$

Here, $u(\tau)$ is assumed to behave stationary with respect to time evolution. Therefore, we exclude any solution for $u(\tau)$ that deviates from the oscillatory solution and proceed with the discussion. Assuming that $u(\tau)$ has 2π as its period, the value of δ_0 is $\delta_0 = 0, 1, 4, ...$ In the following, we will discuss the value of δ_0 in different cases.

3.2.1 $\delta_0 = 0$ case

In this case, u_0 is $u_0 = C_0+C_1t$ from the differential equation. Here, u_0 is a constant and $u_0 = C_0$ from the condition of stationary oscillatory solution for $u(\tau)$. The differential equation for $u_1(\tau)$ at this time is

$$\frac{d^2}{d\tau^2}u_1(\tau)+(\delta_1-2\cos(2\tau))C_0=0. \tag{52}$$

Here, from the steady-state oscillation condition for $u(\tau)$, we get $\delta_1 = 0$, and the solution for u_1 is

$$u_1(\tau)=-\frac{C_0}{2}\cos(2\tau). \tag{53}$$

Substituting the above table expression of $u_1(\tau)$ into the differential equation of $u_2(\tau)$, we obtain the following differential equation for $u_2(\tau)$.

$$\frac{d^2}{d\tau^2}u_2(\tau)+C_0\cos^2(2\tau)+\delta_2C_0=0. \tag{54}$$

If we choose $-1/2$ as the value of δ_2, the above equation can be rewritten as follows.

$$\frac{d^2}{d\tau^2}u_2(\tau)+\frac{C_0}{2}\cos(4\tau)=0. \tag{55}$$

In this case, the solution of $u_2(\tau)$ is as follows,

$$u_2(\tau)=\frac{C_0}{32}\cos(4\tau). \tag{56}$$

Thus, $\delta(epsilon)$ is then as follows,

$$\delta=-\frac{1}{2}\epsilon^2+O(\epsilon^3). \tag{57}$$

3.2.2 $\delta_0 = 1$ case

Next, we analyze the case where we choose 1 as the value of δ_0. In this case, the solution of $u_0(\tau)$ is given by the differential equation as follows,

$$u_0(\tau) = C_0 \cos(\tau). \tag{58}$$

Substituting the above solution of $u_0(\tau)$ into the differential equation of $u_1(\tau)$, Eq.(50), the differential equation of $u_1(\tau)$ is given by

$$\left(\frac{d^2}{d\tau^2} + 1\right) u_1(\tau) + (\delta_1 - 2\cos(2\tau))C_0 \cos(\tau) = 0. \tag{59}$$

This equation can be transformed using the trigonometric formula. If we transform the trigonometric function so that it becomes first order, the formula transforms as follows

$$\frac{d^2 u_1}{d\tau^2} + u_1(\tau) = -C_0 \left((\delta_1 - 1)\cos(\tau) - \cos(3\tau)\right). \tag{60}$$

At this time, if the right hand side contains a term proportional to $cos(\tau)$, a function whose period is 2π, the solution of $u_1(\tau)$ will be unstable because a term proportional to tau is included in the solution. Therefore, the condition for $u_1(\tau)$ to be stable is $\delta_1 = 1$. The solution to $u_1(\tau)$ is given by

$$u_1(\tau) = -\frac{C_0}{8}\left(2\cos(\tau) + \cos(3\tau)\right). \tag{61}$$

Substituting this result into Eq.(51), we obtain the following differential equation for $u_2(\tau)$,

$$\left(\frac{d^2}{d\tau^2} + 1\right) u_2(\tau) - (1 - 2\cos(2\tau))\frac{C_0}{8}\left(2\cos(3\tau) + \cos(3\tau)\right) + \delta_2 C_0 \cos(\tau) = 0. \tag{62}$$

If we transform the above equation to be the first order of the trigonometric function, the equation can be rewritten as

$$\frac{d^2 u_2}{d\tau^2} + u_2 = -C_0\left(\left(\delta_2 + \frac{1}{8}\right)\cos(\tau) + \frac{1}{8}\cos(3\tau) + \frac{1}{8}\cos(5\tau)\right). \tag{63}$$

Therefore, the condition to be stable for $u_2(\tau)$ is $\delta_2 = -1/8$. In this case, the ϵ dependence on δ is written as follows,

$$\delta(\epsilon) = 1 + \epsilon - \frac{1}{8}\epsilon^2 + O(\epsilon^3). \tag{64}$$

If δ satisfies this condition, $u(\tau)$ performs steady oscillations.

On the other hand, if we choose the sin function as the solution of $u_0(\tau)$, we get a different solution. In this case, we take the solution of $u_0(\tau)$ as follows,

$$u_0(\tau) = C_0 \sin(\tau) \tag{65}$$

Substituting the above solution of $u_0(\tau)$ into the differential equation of $u_1(\tau)$, Eq.(50), the differential equation of $u_1(\tau)$ is given by

$$\left(\frac{d^2}{d\tau^2}+1\right)u_1(\tau)+(\delta_1-2\cos(2\tau))C_0\sin(\tau)=0. \tag{66}$$

This equation can be transformed using the trigonometric formula. If we transform the trigonometric function so that it becomes first order, the formula transforms as follows

$$\frac{d^2u_1}{d\tau^2}+u_1(\tau)=-C_0\left((\delta_1+1)\sin(\tau)-\sin(3\tau)\right). \tag{67}$$

Therefore, the condition for $u_1(\tau)$ to be stable is $\delta_1=-1$. The solution to $u_1(\tau)$ is given by

$$u_1(\tau)=-\frac{C_0}{8}\left(2\sin(\tau)+\sin(3\tau)\right). \tag{68}$$

Substituting this result into Eq.(51), we obtain the following differential equation for $u_2(\tau)$,

$$\left(\frac{d^2}{d\tau^2}+1\right)u_2(\tau)-(1-2\cos(2\tau))\frac{C_0}{8}\left(2\sin(\tau)+\sin(3\tau)\right)+\delta_2C_0\sin(\tau)=0. \tag{69}$$

If we transform the above equation to be the first order of the trigonometric function, the equation can be rewritten as

$$\frac{d^2u_2}{d\tau^2}+u_2=-C_0\left(\left(\delta_2-\frac{3}{8}\right)\sin(\tau)+\frac{1}{8}\sin(3\tau)+\frac{1}{8}\sin(5\tau)\right). \tag{70}$$

Therefore, the condition to be stable for $u_2(\tau)$ is $\delta_2=3/8$. In this case, the ϵ dependence on δ is written as follows,

$$\delta(\epsilon)=1-\epsilon+\frac{3}{8}\epsilon^2+O(\epsilon^3). \tag{71}$$

3.2.3 $\delta_0 = 4$ case

Next, we analyze the case where we choose 1 as the value of δ_0. In this case, the solution of $u_0(\tau)$ is given by the differential equation as follows,

$$u_0(\tau)=C_0\cos(2\tau). \tag{72}$$

Substituting the above solution of $u_0(\tau)$ into the differential equation of $u_1(\tau)$, Eq.(50), the differential equation of $u_1(\tau)$ is given by

$$\left(\frac{d^2}{d\tau^2}+4\right)u_1(\tau)+(\delta_1-2\cos(2\tau))C_0\cos(2\tau)=0. \tag{73}$$

This equation can be transformed using the trigonometric formula. If we transform the trigonometric function so that it becomes first order, the formula transforms as follows

$$\frac{d^2u_1}{d\tau^2}+4u_1(\tau)=-C_0\left(\delta_1\cos(2\tau)-\cos(4\tau)-1\right). \tag{74}$$

At this time, if the right hand side contains a term proportional to $cos(\tau)$, a function whose period is $\pi/2$, the solution of $u_1(\tau)$ will be unstable because a term proportional to tau is included in the solution. Therefore, the condition for $u_1(\tau)$ to be stable is $\delta_1 = 0$. The solution to $u_1(\tau)$ is given by

$$u_1(\tau) = \frac{C_0}{12}(3 - \cos(4\tau)). \tag{75}$$

Substituting this result into Eq.(51), we obtain the following differential equation for $u_2(\tau)$,

$$\left(\frac{d^2}{d\tau^2} + 4\right) u_2(\tau) + (1 - 2\cos(2\tau))\frac{C_0}{12}(3 - \cos(4\tau)) + \delta_2 C_0 \cos(2\tau) = 0. \tag{76}$$

If we transform the above equation to be the first order of the trigonometric function, the equation can be rewritten as

$$\frac{d^2 u_2}{d\tau^2} + 4u_2 = -C_0 \left(\left(\delta_2 - \frac{5}{12} \right) \cos(\tau) + + \frac{1}{4} - \frac{1}{12}\cos(4\tau) + \frac{1}{12}\cos(6\tau) \right). \tag{77}$$

Therefore, the condition to be stable for $u_2(\tau)$ is $\delta_2 = 5/12$. In this case, the ϵ dependence on δ is written as follows,

$$\delta(\epsilon) = 4 + \frac{5}{12}\epsilon^2 + O(\epsilon^3). \tag{78}$$

If δ satisfies this condition, $u(\tau)$ performs steady oscillations.

On the other hand, if we choose the sin function as the solution of $u_0(\tau)$, we get a different solution. In this case, we take the solution of $u_0(\tau)$ as follows,

$$u_0(\tau) = C_0 \sin(2\tau) \tag{79}$$

Substituting the above solution of $u_0(\tau)$ into the differential equation of $u_1(\tau)$, Eq.(50), the differential equation of $u_1(\tau)$ is given by

$$\left(\frac{d^2}{d\tau^2} + 4\right) u_1(\tau) + (\delta_1 - 2\cos(2\tau))C_0 \sin(2\tau) = 0. \tag{80}$$

This equation can be transformed using the trigonometric formula. If we transform the trigonometric function so that it becomes first order, the formula transforms as follows

$$\frac{d^2 u_1}{d\tau^2} + 4u_1(\tau) = -C_0 (\delta_1 \sin(2\tau) - \sin(4\tau)). \tag{81}$$

Therefore, the condition for $u_1(\tau)$ to be stable is $\delta_1 = 0$. The solution to $u_1(\tau)$ is given by

$$u_1(\tau) = -\frac{C_0}{12}\sin(4\tau). \tag{82}$$

Substituting this result into Eq.(51), we obtain the following differential equation for $u_2(\tau)$,

$$\left(\frac{d^2}{d\tau^2} + 4\right) u_2(\tau) - (1 - 2\cos(2\tau))\frac{C_0}{12}\sin(4\tau) + \delta_2 C_0 \sin(2\tau) = 0. \tag{83}$$

If we transform the above equation to be the first order of the trigonometric function, the equation can be rewritten as

$$\frac{d^2u_2}{d\tau^2} + u_2 = -C_0\left(\left(\delta_2 + \frac{1}{12}\right)\sin(2\tau) - \frac{1}{12}\sin(4\tau) + \frac{1}{12}\sin(6\tau)\right). \tag{84}$$

Therefore, the condition to be stable for $u_2(\tau)$ is $\delta_2 = 3/8$. In this case, the ϵ dependence on δ is written as follows,

$$\delta(\epsilon) = 4 - \frac{1}{12}\epsilon^2 + O(\epsilon^3). \tag{85}$$

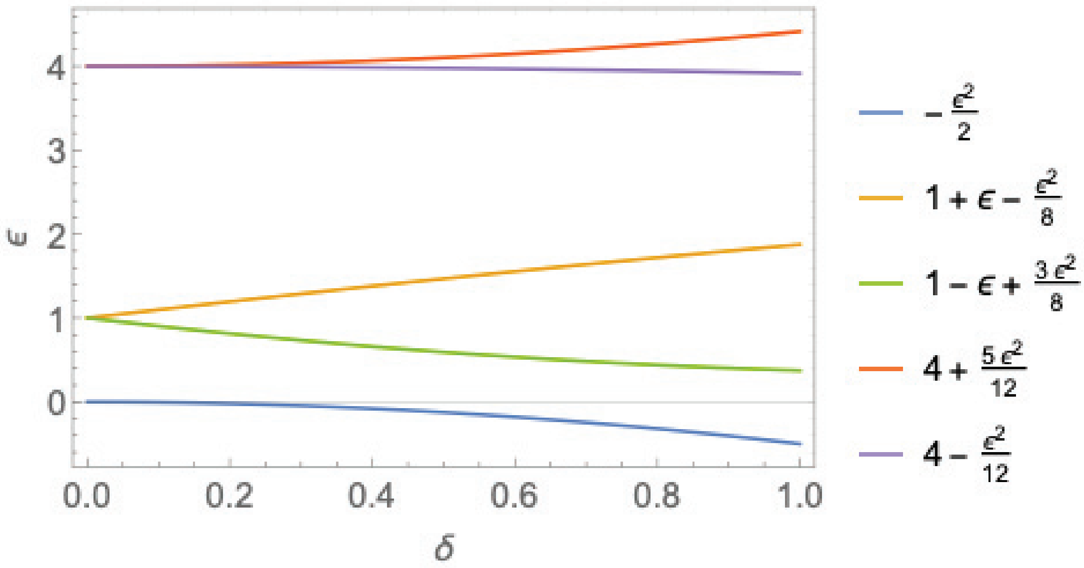

Figure 2: Graph of the relationship between δ and ϵ. When the parameters are on the line, $u(\tau)$ performs steady oscillation.

A plot of the relationship between δ and ϵ obtained above is shown in Fig.(2). The region of parameters below the yellow and red lines and below the green and blue lines corresponds to the solution where $u(\tau)$ behaves stably with respect to time evolution. Conversely, in the parameter region outside of this region, the solution of $u(\tau)$ behaves exponentially diverging with time evolution due to resonance with the friction term $\cos(2\tau)$. In the non-perturbative regime of the preheating phase in the early universe, it is speculated that this resonance effect causes an exponential decay from inflaton to scalar particles.

4 $F(R)$ gravity

The origin of the inflaton, a scalar particle introduced in chapter 2 to describe slow roll inflation, is not yet known. There is a way to introduce this inflaton from an extension of the Standard Model describing elementary particle theory. In this paper, we consider the modified gravity theory as the origin of this inflaton. One of the modified gravity theories, $F(R)$ gravity, has been well studied as a theory that can describe inflation [9–12]. On the other hand, it is worth mentioning that iflation can be written without introducing scalarons, but with the action of $F(R)$ [13].

We consider modified gravity theory as $F(R)$ gravity which is the model that the curvature R in Einstein-Hilbert action replace to arbitrary R's function,

$$S = \int d^4x\sqrt{-g}\frac{1}{2\kappa^2}F(R). \tag{86}$$

We introduce auxiliary field A, the action is rewritten as

$$S = \int d^4x\frac{1}{2\kappa^2}\left(F(A) + F'(A)(R - A)\right) \tag{87}$$

We take the derivative of A, we restore the action as Eq.(86). $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\kappa\varphi/\sqrt{6}}g_{\mu\nu}$. Then the curvature R is changed as

$$R = e^{2\kappa\varphi/\sqrt{6}}\left(\tilde{R} + \sqrt{6}\kappa\tilde{\Box}\varphi - \kappa^2\tilde{g}^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)\right) \tag{88}$$

We insert above equaiton into the action of $F(R)$ gravity, we obtain reformed action as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)\right), \tag{89}$$

where the potential $V(\varphi)$ is defined as

$$V(\varphi) \equiv \frac{1}{2\kappa^2}\frac{F'(A(\varphi))A(\varphi) - F(A(\varphi))}{F'^2(A(\varphi))}, \tag{90}$$

and we mach φ as $e^{2\kappa\varphi/\sqrt{6}} = F'(A(\varphi))$.

4.1 Chameleon mechanism

When matter fields exit, the part of modified gravity affects to the matter fields. The interaction of modified gravity and the matter fields can be described as the interaction between the scalar field and each fields. The action of $F(R)$ gravity and a matter field is given as,

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2\kappa^2}\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right) + S_{Matter}. \tag{91}$$

The metric $g^{\mu\nu}$ which is defined before Weyl transformation has 5 DoF and $\tilde{g}^{\mu\nu}$ has 4 DoF. The difference of DoF apere as the scalar field φ, so $\tilde{g}^{\mu\nu}$ don't depend on the scalar field, $\delta/\delta\varphi(x)\tilde{g}^{\mu\nu}(x') = 0$. We consider this point, we obtain the representation of the derivative with respect to φ as,

$$\begin{aligned}\frac{\delta}{\delta\varphi(x)} &= \frac{\partial}{\partial\varphi(x)} + \frac{\delta g^{\mu\nu}(x')}{\delta\varphi(x)}\frac{\delta}{\delta g^{\mu\nu}(x')} \\ &= \frac{\partial}{\partial\varphi(x)} + \frac{2\kappa}{\sqrt{6}}\delta^{(4)}(x - x')g^{\mu\nu}(x')\frac{\delta}{\delta g^{\mu\nu}(x')}.\end{aligned} \tag{92}$$

We consider Eq(), the EoM of scalaron field is given as,

$$
\begin{aligned}
\frac{\delta}{\delta\varphi(x)}S =& \int d^4x'\sqrt{-\tilde{g}(x')}\left(\delta^{(4)}(x-x')\tilde{\Box}\varphi(x') - \delta^{(4)}(x-x')V'(\varphi(x'))\right) \\
&+ \int d^4x'\sqrt{-g(x')}\frac{2\kappa}{\sqrt{6}}\delta^{(4)}(x-x')g^{\mu\nu}(x')\frac{\delta}{\delta g^{\mu\nu}(x')}\mathcal{L}_{Matter} \\
=& \sqrt{-g(x)}\left(\tilde{\Box}\varphi(x) - V'(\varphi(x)) - \frac{\kappa}{\sqrt{6}}e^{-4\kappa\varphi/\sqrt{6}}T^{\mu}{}_{\mu}(x)\right) = 0,
\end{aligned}
\tag{93}
$$

where the energy momentum tensor $T_{\mu\nu}$ is defined as

$$
T_{\mu\nu} \equiv \frac{\delta}{\delta g^{\mu\nu}}. \tag{94}
$$

We define the effective potential of scalaron field $V_{eff}(\varphi)$ as,

$$
V_{eff}(\varphi) \equiv V(\varphi) - \frac{1}{4}e^{-4\kappa\varphi/\sqrt{6}}T^{\mu}{}_{\mu}. \tag{95}
$$

When the scalaron stand on the vacuum state and scalaron field fluctuate around that point, we expansion the effective potential as,

$$
V_{eff}(\varphi) \simeq V_{eff}(\varphi_{min}) + \frac{1}{2}(\varphi - \varphi_{min})^2 V''_{eff}(\varphi_{min}) + ... \tag{96}
$$

We substitute this expansion form into Eq() and we ignore higher order of φ, we obtain Klein-Gordon equation for φ as,

$$
\left(\tilde{\Box} - V''_{eff}(\varphi_{min})\right)\varphi = 0, \tag{97}
$$

where we ignore constant factor of φ. Therefore we regard the second derivative of the effective potential as scalaron mass becouse of Eq.(97). Then the scalaron mass depend on $T^{\mu}{}_{\mu}$, so scalaron mass depend on the energy of back ground matter field. This mechanism called as chameleon mechanism,

$$
V''_{eff}(\varphi_{min}, T^{\mu}{}_{\mu}) = m_{\varphi}(T^{\mu}{}_{\mu}). \tag{98}
$$

4.2 Mattar field

ToEMT can obey the metric $g^{\mu\nu}$. Therefore ToEMT also depends on scalaron. Then the behavior of chameleon mechanism becomes bit difficult. We focus to interaction between each fields. We consider the wyle transformation on matter field, the action transform as following,

$$
S_{Matter} = \int d^4x\sqrt{-g}\mathcal{L}_{Matter}(g^{\mu\nu}, \Psi) = \int d^4x\sqrt{-\tilde{g}}e^{-4\kappa\varphi/\sqrt{6}}\mathcal{L}_{Matter}(e^{2\kappa\varphi/\sqrt{6}}\tilde{g}_{\mu\nu}, \Psi). \tag{99}
$$

We will investigate the interaction between scalaron particle and other particles.

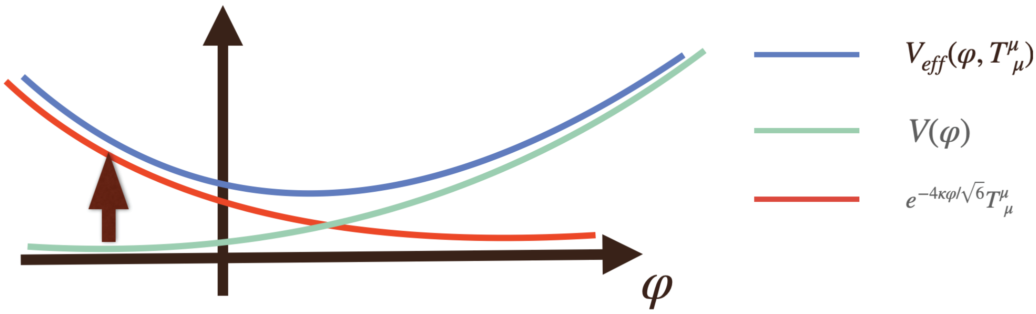

Figure 3: The constraints for α and R_C/Λ_{DE}. The colored area is permitted value for α and R_C/Λ_{DE}.

4.2.1 $m = 0$ vactor field

At first we consider massless vector field. The Lagrangian of massless vector field is given as

$$\mathcal{L}_V(g_{\mu\nu}, A_\mu) = -\frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}(A_\mu) F_{\alpha\beta}(A_\mu), \tag{100}$$

Then the tensor $F_{\mu\nu}$ is defiend as

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{101}$$

The definition of $F_{\mu\nu}$ is independent from metric $g^{\mu\nu}$, so $F_{\mu\nu}$ is invariant from wyle transformation. Therefore the Lagrangian of massless vector field transform with wyle transformaiton as following,

$$\mathcal{L}_V(g_{\mu\nu}, A_\mu) = -\frac{1}{4} e^{4\sqrt{1/6}\kappa\varphi} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu}(A_\mu) F_{\alpha\beta}(A_\mu). \tag{102}$$

We substitute above form into the action S_V, we obtain

$$\begin{aligned} S_V &= \int d^4x \sqrt{-\tilde{g}} [-\frac{1}{4} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta}] \\ &= \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_V(\tilde{g}_{\mu\nu}, A_\mu). \end{aligned} \tag{103}$$

Then the interaction between scalaron φ and massless vector field don't appear because the transformation of $\sqrt{-g}$ and $\mathcal{L}_V$ cancel. This result don't contradictory from the fact that the trace of energy momentum field becomes 0,

$$\begin{aligned} \sqrt{-g}\left(-\frac{1}{2} T_{V\mu\nu}\right) =& \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g} \left(-\frac{1}{4} g^{\alpha\beta} g^{\rho\sigma} F_{\alpha\rho} F_{\rho\sigma}\right) \\ =& -\frac{1}{4}\sqrt{-g}\left(\frac{1}{8} g_{\mu\nu}\left(g^{\alpha\beta} g^{\mu\sigma} F_{\alpha\rho} F_{\beta\sigma}\right) - \frac{1}{2} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}\right) \end{aligned} \tag{104}$$

$$T_V{}^{\mu}{}_{\mu} = 0. \tag{105}$$

Therefore we don't need to redefine the massless vector field.

4.2.2 $m = 0$ spinor field

Next we consider massless spinor field. The Lagrangian of massless spinor field is given as following,

$$\mathcal{L}_F(\gamma^\mu, \Psi) = i\bar{\Psi}\gamma^\mu \nabla_\mu \Psi. \tag{106}$$

Then the conformal derivative of Ψ is given as following,

$$\nabla_\mu \Psi = \partial_\mu \Psi + \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b]\Psi. \tag{107}$$

Then γ, ω are defined as

$$2g^{\mu\nu} = [\gamma^\mu, \gamma^\nu] = 2e^{2\kappa\varphi/\sqrt{6}}\tilde{g}^{\mu\nu} \tag{108}$$

$$\omega_{\mu ab} = e_{a\nu}(\partial_\mu e_b^\nu + \Gamma^\nu_{\mu\rho} e_b^\rho) = e_{a\nu}\nabla_\mu e_b^\nu, \tag{109}$$

where the bracket $[,]$ describe commutation relation and $\{,\}$ describe anti-commutation relation, Greek index describe Lorentz index and Latinum index describe spinor index. The relation between Lorentz and spinor index is given as

$$\gamma^\mu \equiv e_a^\mu \gamma^a \qquad\qquad g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b. \tag{110}$$

We use above relation, we obtain the wyle transformation of spinor is

$$e_\mu^a \to \tilde{e}_\mu^a = e^{\kappa\varphi(x)/\sqrt{6}} e_\mu^a \qquad\qquad \gamma_\mu \to \tilde{\gamma}_\mu = e^{\kappa\varphi(x)/\sqrt{6}}\gamma_\mu. \tag{111}$$

We use above formulas, the wyle transformation of ω is

$$\begin{aligned}\omega_{\mu ab} &= e^{-\sigma}\tilde{e}_{a\nu}[\partial_\mu(e^\sigma \tilde{e}_b^\nu) + (\tilde{\Gamma}^\nu_{\mu\rho} - \delta^\nu_\mu \sigma_{,\rho} - \delta^\nu_\rho \sigma_{,\mu} + \tilde{g}^{\nu\alpha}\tilde{g}_{\mu\rho}\sigma_{,\alpha})(e^\sigma \tilde{e}_b^\rho)] \\ &= \tilde{\omega}_{\mu ab} + \tilde{e}_{a\nu}[\tilde{e}_b^\nu \sigma_{,\mu} - \delta^\nu_\mu \tilde{e}_b^\rho \sigma_\rho - \tilde{e}_b^\nu \sigma_{,\mu} + \tilde{g}^{\nu\alpha}\tilde{e}_{b\mu}\sigma_{,\alpha}] \\ &= \tilde{\omega}_{\mu ab} - (\tilde{e}_{a\mu}\tilde{e}_b^\rho - \tilde{e}_{b\mu}\tilde{e}_a^\rho)\sigma_{,\rho}.\end{aligned} \tag{112}$$

We substitute Eq.(112) into Eq.(107), we obtain the wyle transformation of the conformal derivative of spinor field as,

$$\begin{aligned}\nabla_\mu \Psi &= \partial_\mu \Psi + \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b]\Psi \\ &= \partial_\mu \Psi + \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b]\Psi - \frac{1}{8}(\tilde{e}_{a\mu}\tilde{e}_b^\lambda - \tilde{e}_{b\mu}\tilde{e}_a^\lambda)[\gamma^a, \gamma^b]\Psi\sigma_{,\lambda} \\ &= \tilde{\nabla}_\mu \Psi - \frac{1}{8}\{[\tilde{\gamma}_\mu, \tilde{\gamma}^\lambda] - [\tilde{\gamma}^\lambda, \tilde{\gamma}_\mu]\}\sigma_{,\lambda}\Psi \\ &= \tilde{\nabla}_\mu \Psi - \frac{1}{4}[\tilde{\gamma}_\mu, \tilde{\gamma}^\lambda]\sigma_{,\lambda}\Psi.\end{aligned} \tag{113}$$

We use the commutation relation as $A[B,C] = \{AB, C\} - \{A, C\}B$, we obtain the formula of γ as,

$$\begin{aligned}\tilde{\gamma}^\mu[\tilde{\gamma}_\mu, \tilde{\gamma}^\lambda] &= \{\tilde{\gamma}^\mu \tilde{\gamma}_\mu, \tilde{\gamma}^\lambda\} - \{\tilde{\gamma}^\lambda, \tilde{\gamma}^\mu\}\tilde{\gamma}_\mu \\ &= 8\tilde{\gamma}^\lambda - 2\tilde{g}^{\lambda\mu}\tilde{\gamma}_\mu \\ &= 6\tilde{\gamma}^\lambda.\end{aligned} \tag{114}$$

We use above formulas, we obtain the Lagrangian on wyle transformation as

$$\begin{aligned}\mathcal{L}_F(\gamma^\mu,\Psi) &= i\bar{\psi}\gamma^\mu\nabla_\mu\Psi \\ &= e^{\sqrt{1/6}\kappa\varphi}i\bar{\Psi}\tilde{\gamma}^\mu\tilde{\nabla}_\mu\Psi - \frac{3i}{2}\sqrt{\frac{1}{6}}\kappa e^{\sqrt{1/6}\kappa\varphi}(\partial_\mu\varphi)\bar{\Psi}\tilde{\gamma}^\mu\Psi,\end{aligned} \tag{115}$$

and the action transform as following,

$$S_F = \int d^4x\sqrt{-\tilde{g}}e^{3\sqrt{1/6}\kappa\varphi}i\bar{\Psi}\tilde{\gamma}^\mu\left[\tilde{\nabla}_\mu - \frac{3}{2}\sqrt{\frac{1}{6}}\kappa(\partial_\mu\varphi)\right]\Psi. \tag{116}$$

Then scalaron φ couple with spinor field, so the interaction between scalaron and spinor field appear. For cancel this interaction, we redefine the spinor field as

$$\xi = e^{\frac{3}{2}\sqrt{1/6}\kappa\varphi}\Psi, \qquad \bar{\xi} = e^{\frac{3}{2}\sqrt{1/6}\kappa\varphi}\bar{\Psi}. \tag{117}$$

The conformal derivative of ξ is defied as

$$\tilde{\nabla}_\mu\xi = e^{-\frac{3}{2}\sqrt{1/6}\kappa\varphi}\left(\tilde{\nabla}_\mu - \frac{3}{2}\sqrt{\frac{1}{6}}\kappa(\partial_\mu\varphi)\right)\Psi. \tag{118}$$

Then we can show that the action of spinor field is invariant on wyle transformation,

$$S_F = \int d^4x\sqrt{-\tilde{g}}i\bar{\xi}\tilde{\gamma}^\mu\tilde{\nabla}_\mu\xi = \int d^4x\sqrt{-\tilde{g}}\mathcal{L}_F(\tilde{\gamma}^\mu,\xi). \tag{119}$$

The interaction between massless spinor field and scalaron disappear by redefinition of spinor field ξ. We mention that the gauge transformation, $\tilde{\nabla}_\mu \to \tilde{\nabla}_\mu - igA_\mu$ don't produce the interaction between A_μ and φ.

4.2.3 $m = 0$ scalar field

The Lagrangian of massless scalar field is given as

$$\mathcal{L}_S(g_{\mu\nu},\chi) = g^{\mu\nu}(\partial_\mu\chi^*)(\partial_\nu\chi). \tag{120}$$

Then the wyle transformation of action is given simply,

$$S_S = \int d^4x\sqrt{-\tilde{g}}e^{-2\sqrt{1/6}\kappa\varphi}\tilde{g}^{\mu\nu}(\partial_\mu\chi^*)(\partial_\nu\chi). \tag{121}$$

Then χ and φ couple, so the interaction between χ and φ appear. We redefine the scalar field as $\Theta = e^{-\sqrt{1/6}\kappa\varphi}\chi$, similarly to presubsection. Then the action of massless scalar field transform on wyle transformation as following,

$$\begin{aligned}S_S &= \int d^4x\sqrt{-\tilde{g}}e^{-2\sqrt{1/6}\kappa\varphi}\tilde{g}^{\mu\nu}(\partial_\mu e^{\sqrt{1/6}\kappa\varphi}\Theta^*)(\partial_\nu e^{\sqrt{1/6}\kappa\varphi}\Theta) \\ &= \int d^4x\sqrt{-\tilde{g}}\{\tilde{g}^{\mu\nu}(\partial_\mu\Theta^*)(\partial_\nu\Theta) + \tilde{g}^{\mu\nu}\frac{\kappa}{\sqrt{6}}[(\partial_\mu\varphi)\Theta^*(\partial_\nu\Theta) + (\partial_\mu\Theta^*)(\partial_\nu\varphi)\Theta] + \frac{\kappa^2}{6}\tilde{g}^{\mu\nu}[(\partial_\mu\varphi)(\partial_\nu\varphi)\Theta^*\Theta]\}\end{aligned} \tag{122}$$

In this case, the interaction between scalaron and massless scalar field remain on the second and third term of Eq.(122). For cancel the interaction, we add new term into the Lagrangian. We choose this new term which satisfy following conditions, (i) Lorentz invariant, (ii) Gauge invariant, (iii) Renormalization invariant. We can add a new term as a term which satisfy the three conditions as $aR\chi^*\chi$, where a is an arbitrary constant value. Then the Lagrangian is redefined as following,

$$\mathcal{L}_S = g^{\mu\nu}(\partial_\mu\chi^*)(\partial_\nu\chi) + aR\chi^*\chi. \tag{123}$$

The second term of Eq.(123) transform on wyle transformation as,

$$\begin{aligned}\int d^4x\sqrt{-g}aR\chi^*\chi &= \int d^4x\sqrt{-\tilde{g}}a[\tilde{R} + 6\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu(\sqrt{1/6}\kappa\varphi) - 6\tilde{g}^{\mu\nu}(\partial_\mu\sqrt{1/6}\kappa\varphi)(\partial_\nu\sqrt{1/6}\kappa\varphi)]\Theta^*\Theta \\ &= \int d^4x\sqrt{-\tilde{g}}a[\tilde{R} + \frac{\kappa}{\sqrt{6}}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\varphi - \kappa^2\tilde{g}^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi)]\Theta^*\Theta.\end{aligned} \tag{124}$$

Using these results, we can show the action is invariant on wyle transformation when we put a as $a = \frac{1}{6}$,

$$\begin{aligned}S_S &= \int d^4x\sqrt{-\tilde{g}}\{\tilde{g}^{\mu\nu}(\partial_\mu\Theta^*)(\partial_\nu\Theta) + \frac{1}{6}\tilde{R}\Theta^*\Theta + \tilde{g}^{\mu\nu}\frac{\kappa}{\sqrt{6}}[(\partial_\mu\varphi)\Theta^*(\partial_\nu\Theta) + (\partial_\mu\Theta^*)(\partial_\nu\varphi)\Theta + (\tilde{\nabla}_\mu\partial_\nu\varphi)\Theta^*\Theta]\} \\ &= \int d^4x\sqrt{-\tilde{g}}\{\tilde{g}^{\mu\nu}(\partial_\mu\Theta^*)(\partial_\nu\Theta) + \frac{1}{6}\tilde{R}\Theta^*\Theta + \tilde{\nabla}_\mu[\frac{\kappa}{\sqrt{6}}\tilde{g}^{\mu\nu}(\partial_\nu\varphi)\Theta^*\Theta]\} \\ &= \int d^4x\sqrt{-\tilde{g}}[\tilde{g}^{\mu\nu}(\partial_\mu\Theta^*)(\partial_\nu\Theta) + \frac{1}{6}\tilde{R}\Theta^*\Theta] \\ &= \int d^4x\sqrt{-\tilde{g}}\mathcal{L}_S(\tilde{g}_{\mu\nu},\Theta).\end{aligned} \tag{125}$$

Then scalaron φ disappear on the action of massless scalar field, so the interaction between scalaron and massless scalar field doesn't appear. Next we calculate the trace of momentum tensor of massless scalar field. Before redefining of scalar field,$\chi \to \Theta$, we perform the derivative of the action with respect to metric $g^{\mu\nu}$,

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}}S_s =& \frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}\left[g^{\alpha\beta}(\partial_\alpha\chi^*)(\partial_\beta\chi) + \frac{1}{6}R\chi^*\chi\right] \\ =&\sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\mathcal{L}_S + (\partial_\mu\chi^*)(\partial_\nu\chi) + \frac{1}{6}R_{\mu\nu}\chi^*\chi\right] = \sqrt{-g}\left(-\frac{1}{2}T_{\mu\nu}\right).\end{aligned} \tag{126}$$

Then we obtain the value of ${T^\mu}_\mu$ as ${T^\mu}_\mu = 2\mathcal{L}_S$. Next we calculate the value of ${T^\mu}_\mu$ when we perform the derivative of the action which is described by Θ with respect to φ. Then we consider the derivatives are given as Θ は $\delta/\delta\varphi\Theta = \delta/\delta\varphi e^{-\kappa\varphi/\sqrt{6}}\chi = -\kappa/\sqrt{6}\Theta$. Then the derivative of S_S with respect to φ is given as,

$$\begin{aligned}\frac{\delta}{\delta\varphi}S_S =& \frac{\delta}{\delta\varphi}\int d^4x\sqrt{-\tilde{g}}\left[\tilde{g}^{\mu\nu}(\partial_\mu\Theta^*)(\partial_\nu\Theta) + \frac{1}{6}\tilde{R}\Theta^*\Theta\right] \\ =&\sqrt{-\tilde{g}}\left[-\frac{2\kappa}{\sqrt{6}}\tilde{g}^{\mu\nu}(\partial_\mu\Theta)(\partial_\nu\Theta) - \frac{\kappa}{3\sqrt{6}}\tilde{R}\Theta^*\Theta\right] = -\frac{2\kappa}{\sqrt{6}}\sqrt{-\tilde{g}}\tilde{\mathcal{L}}_S.\end{aligned} \tag{127}$$

We use the relation as $\tilde{T}^{\mu}_{\mu} = 2\tilde{\mathcal{L}}_S$, the relation between $T^{\mu}{}_{\mu}$ and $\tilde{T}^{\mu}_{\mu}$ is given as

$$e^{-4\kappa\varphi/\sqrt{6}} T^{\mu}{}_{\mu} = \tilde{T}^{\mu}{}_{\mu}. \tag{128}$$

The differential of $T^{\mu}{}_{\mu}$ and $\tilde{T}^{\mu}_{\mu}$ is the factor as $e^{-4\kappa\varphi/\sqrt{6}}$.

We can show the interaction between scalaron and massless field can disappear when we redefine new field. This results are consist with the fact that massless field don't interact to gravitational field. Next we consider massive fields case.

4.2.4 $m \neq 0$ vector field

The Lagrangian of massive vector field is given as

$$\mathcal{L}_{V-mass}(g_{\mu\nu}, A^{\mu}) = -\frac{1}{2} m_V^2 g^{\mu\nu} A_{\mu} A_{\nu}, \tag{129}$$

and the action transform on wyle transformation as,

$$\begin{aligned} S_{V-mass} &= \int d^4x \sqrt{-\tilde{g}} e^{-4\sqrt{1/6}\kappa\varphi} \mathcal{L}_{V-mass}(g_{\mu\nu}, A^{\mu}) \\ &= \int d^4x \sqrt{-\tilde{g}} e^{-2\sqrt{1/6}\kappa\varphi} \left[-\frac{1}{2} m_V^2 \tilde{g}^{\mu\nu} A_{\mu} A_{\nu} \right]. \end{aligned} \tag{130}$$

We calculate the relation between $T^{\mu}{}_{\mu}$ and $\tilde{T}^{\mu}_{\mu}$. At first we perform the derivative with respect to $g^{\mu\nu}$ for obtaining $T^{\mu}{}_{\mu}$. We obtain $T^{\mu}{}_{\mu}$ as,

$$\frac{\delta}{\delta g^{\mu\nu}} S_{V-mass} = \sqrt{-g} \left[-\frac{1}{2} g_{\mu\nu} \mathcal{L}_{V-mass} + \frac{1}{2} m_V^2 A_{\mu} A_{\nu} \right] = \sqrt{-g} \left(-\frac{1}{2} T_{\mu\nu} \right), \tag{131}$$

and the relation with $\mathcal{L}_{V-mass}$ is given as $T^{\mu}{}_{\mu} = 2\mathcal{L}_{V-mass}$. On the other hand, the derivative with respect to φ is given as

$$\frac{\delta}{\delta\varphi} S_{V-mass} = \sqrt{-\tilde{g}} \left(-\frac{2\kappa}{\sqrt{6}} e^{-2\kappa\varphi/\sqrt{6}} \tilde{\mathcal{L}}_{V-mass} \right). \tag{132}$$

When we define $\tilde{T}^{\mu}{}_{\mu} = 2\tilde{\mathcal{L}}_{V-mass}$, the relation between$T^{\mu}{}_{\mu}$ and $\tilde{T}^{\mu}{}_{\mu}$ is given as

$$e^{-4\kappa\varphi/\sqrt{6}} T^{\mu}{}_{\mu} = e^{-2\kappa\varphi/\sqrt{6}} \tilde{\mathcal{T}}^{\mu}{}_{\mu}. \tag{133}$$

We expand around $|\kappa\varphi| \ll 1$ on Eq.(130), the interaction between scalaron and massive vector field is approximated as,

$$\begin{aligned} S_{V-mass} &= \int d^4x \sqrt{-\tilde{g}} \left[-\frac{1}{2} m_V^2 \tilde{g}^{\mu\nu} A_{\mu} A_{\nu} + \frac{\kappa m_V^2}{\sqrt{6}} \varphi \tilde{g}^{\mu\nu} A_{\mu} A_{\nu} \right] + O(\kappa^2\varphi^2) \\ &= \int d^4x \sqrt{-\tilde{g}} \left[\mathcal{L}_{V-mass}(\tilde{g}_{\mu\nu}, A^{\mu}) + \frac{\kappa m_V^2}{\sqrt{6}} \varphi \tilde{g}^{\mu\nu} A_{\mu} A_{\nu} \right] + O(\kappa^2\varphi^2). \end{aligned} \tag{134}$$

The interaction is given as three point coupling and the strength proportinal to the squared of vector field mass,

$$\mathcal{L}_{V-\varphi} = \frac{\kappa m_V^2}{\sqrt{6}} \varphi \tilde{g}^{\mu\nu} A_{\mu} A_{\nu}. \tag{135}$$

The interaction term don't disappear compare to massless case.

4.2.5 $m \neq 0$ spinor field

The Lagrangian of massive spinor field is given as

$$\mathcal{L}_{F-mass}(\Psi) = -m_F \bar{\Psi}\Psi. \tag{136}$$

We redefined spinor field in massless case, $\xi = e^{\frac{3}{2}\sqrt{1/6}\kappa\varphi}\Psi$, $\bar{\xi} = e^{\frac{3}{2}\sqrt{1/6}\kappa\varphi}\bar{\Psi}$, so we rewrite the Lagrangian by redefined field ξ. Then the Lagrangian is represented as

$$\begin{aligned} S_{F-mass} &= \int d^4x\sqrt{-\tilde{g}}e^{-4\sqrt{1/6}\kappa\varphi}\mathcal{L}_{F-mass}(\Psi) \\ &= \int d^4x\sqrt{-\tilde{g}}e^{-\sqrt{1/6}\kappa\varphi}\left[-m_F\bar{\xi}\xi\right]. \end{aligned} \tag{137}$$

We expand scalaron field similarly to vector field case, the action is approximated as,

$$\begin{aligned} S_{F-mass} &= \int d^4x\sqrt{-\tilde{g}}\left[-m_F\bar{\xi}\xi + \frac{\kappa m_F}{\sqrt{6}}\varphi\bar{\xi}\xi\right] + O(\kappa^2\varphi^2) \\ &= \int d^4x\sqrt{-\tilde{g}}\left[\mathcal{L}_{F-mass}(\xi) + \frac{\kappa m_F}{\sqrt{6}}\varphi\bar{\xi}\xi\right] + O(\kappa^2\varphi^2). \end{aligned} \tag{138}$$

Then the second term of Eq.(138) correspond to the interaction term,

$$\mathcal{L}_{F-\varphi} = \frac{\kappa m_F}{\sqrt{6}}\varphi\bar{\xi}\xi. \tag{139}$$

Therefore the interaction between scalaron is given as three point coupling.

4.2.6 $m \neq 0$ scalar field

The Lagrangian of massive scalar field is given as following,

$$\mathcal{L}_{S-mass}(\chi) = -m_S^2\chi^*\chi. \tag{140}$$

We rewrite the action by redefined scalar field, $\Theta = e^{-\sqrt{1/6}\kappa\varphi}\chi$, the action is represented as,

$$\begin{aligned} S_{S-mass} &= \int d^4x\sqrt{-\tilde{g}}e^{-4\sqrt{1/6}\kappa\varphi}\mathcal{L}_{S-mass}(\chi) \\ &= \int d^4x\sqrt{-\tilde{g}}e^{-2\sqrt{1/6}\kappa\varphi}[-m_S^2\Theta^*\Theta]. \end{aligned} \tag{141}$$

We calculate the trace of energy momentum tensor. At first we perform the derivative with respect to $g^{\mu\nu}$,

$$\frac{\delta}{\delta g^{\mu\nu}}S = \sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\mathcal{L}_{S-mass}\right] = \sqrt{-g}\left(-\frac{1}{2}T_{\mu\nu}\right). \tag{142}$$

Then the value of $T^\mu{}_\mu$ is given as $T^\mu{}_\mu = 4\mathcal{L}_{S-mass}$. On the other hand, the derivative of the action which is described by redefined field with respect to φ is given,

$$\frac{\delta}{\delta\varphi}S = \sqrt{-\tilde{g}}e^{-2\kappa\varphi/\sqrt{6}}\left[-\frac{2\kappa}{\sqrt{6}}\tilde{\mathcal{L}}_{S-mass} - \frac{2\kappa}{\sqrt{6}}\tilde{\mathcal{L}}_{S-mass}\right] = \sqrt{-\tilde{g}}e^{-2\kappa\varphi/\sqrt{6}}\left[-\frac{4\kappa}{\sqrt{6}}\tilde{\mathcal{L}}_{S-mass}\right]. \tag{143}$$

vector	spinor	scalar
$\frac{\kappa m_V^2}{\sqrt{6}}\varphi\tilde{g}^{\mu\nu}A_\mu A_\nu$	$\frac{\kappa m_F}{\sqrt{6}}\varphi\bar{\xi}\xi$	$\frac{2\kappa m_S^2}{\sqrt{6}}\varphi\Theta^*\Theta$

Table 1: The interaction term between scalaron and each fields

When we define $4\tilde{\mathcal{L}}_{S-mass} = \tilde{T}^\mu{}_\mu$, the relation between $T^\mu{}_\mu$ and $\tilde{T}^\mu{}_\mu$ of massive scalar field is given as

$$e^{-4\kappa\varphi/\sqrt{6}}T^\mu{}_\mu = e^{-2\kappa\varphi/\sqrt{6}}\tilde{T}^\mu{}_\mu. \tag{144}$$

We expand $\kappa\varphi$, we obtain

$$\begin{aligned} S_{S-mass} &= \int d^4x\sqrt{-\tilde{g}}\left[-m_S^2\Theta^*\Theta + \frac{2\kappa m_S^2}{\sqrt{6}}\varphi\Theta^*\Theta\right] + O(\kappa^2\varphi^2) \\ &= \int d^4x\sqrt{-\tilde{g}}\left[\mathcal{L}_{S-mass}(\Theta) + \frac{2\kappa m_S^2}{\sqrt{6}}\varphi\Theta^*\Theta\right] + O(\kappa^2\varphi^2). \end{aligned} \tag{145}$$

Then the interaction term between scalaron and massive scalar field is given as

$$\mathcal{L}_{S-\varphi} = \frac{2\kappa m_S^2}{\sqrt{6}}\varphi\Theta^*\Theta. \tag{146}$$

The interaction is given as three point coupling.

Table 1 show the interaction term between scalaron and each fields. The strength of coupling proportional to each field's mass, so the interaction becomes stronger when the fields are heavier.

4.2.7 Higgs mechanism

We showed that massive field interact with scalaron field. The mass of SM fields are produced by Higgs field, so the analysis of Higgs field on wyle transformation is important to understand the interaction between scalaron and massive particle. The action of Higgs field and vector, spinor fields are given as

$$\begin{aligned} S = \int d^4\sqrt{-g}\Big(& g^{\mu\nu}(\partial_\mu\phi)^\dagger\partial_\nu\phi - m_S^2|\phi|^2 + \frac{1}{6}R\phi^2 \\ & - \frac{1}{4}g^{\mu\rho}g^{\nu\lambda}G^a_{\mu\nu}G^a_{\rho\lambda} \\ & + \bar{\psi}iD\cdot\gamma\psi - \lambda_\psi\bar{\psi}\Phi\psi \\ & g^{\mu\nu}(D_\mu\Phi)(D_\nu\Phi) - \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 + \frac{1}{6}R\Phi^2\Big) \end{aligned} \tag{147}$$

where we define Higgs field as Φ. The conformal derivative D_μ is written as following,

$$D_\mu = \partial_\mu - igA_\mu. \tag{148}$$

At first we consider the wyle transformation on the kinetic part of Higgs fields. This term is expanded as following,

$$
\begin{aligned}
g^{\mu\nu}(D_\mu\Phi)^\dagger(D_\nu\Phi) =& g^{\mu\nu}[(\partial_\mu - igA_\mu)]^\dagger\Phi(\partial_\nu - igA_\nu)\Phi + \frac{1}{6}R\Phi^2 \\
=& g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + g^{\mu\nu}g^2A^a_\mu A^a_\nu\Phi^2 + \frac{1}{6}R\Phi^2.
\end{aligned} \tag{149}
$$

A curvature R transform on wyle tranformation as,

$$
\begin{aligned}
R &= e^{2\sigma}\tilde{R} + 2(D-1)e^{2\sigma}\tilde{\Box}\sigma - (D-2)(D-1)\tilde{g}^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma) \\
&= e^{2\sigma}\tilde{R} + 6e^{2\sigma}\tilde{\Box}\sigma - 6\tilde{g}^{\mu\nu}(\partial_\mu\sigma)(\partial_\nu\sigma).
\end{aligned} \tag{150}
$$

We replace scalar field Φ to h where h is defined as $\Phi = e^\sigma h$, the kinetic term of Higgs field transform. At first we perform wyle transformation, $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\sigma}g_{\mu\nu}$, the kinetic part of Higgs field transform as following,

$$
\begin{aligned}
g^{\mu\nu}(D_\mu\Phi)^\dagger(D_\nu\Phi) &= e^{4\sigma}\tilde{g}^{\mu\nu}(\partial_\mu h)(\partial_\nu h) + e^{4\sigma}\tilde{g}^{\mu\nu}g^2h^2A^a_\mu A^a_\nu + \frac{1}{6}e^{4\sigma}\tilde{R}h^2 + e^{4\sigma}\tilde{\Box}(h^2\sigma) \\
&= e^{4\sigma}\tilde{g}^{\mu\nu}(D_\mu h)^\dagger(D_\nu h) + e^{4\sigma}\tilde{\Box}(h^2\sigma)
\end{aligned} \tag{151}
$$

Next we consider kinetic term of vector field. The tensor $G^a_{\mu\nu}$ is defined as following,

$$
G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - ig_v f^{abc}[A^b_\mu, A^c_\nu]. \tag{152}
$$

Then we don't have to change the definition of vector field.

$$
g^{\mu\rho}g^{\nu\lambda}G^a_{\mu\nu}G^a_{\rho\lambda} = e^{4\sigma}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\lambda}G^a_{\mu\nu}G^a_{\rho\lambda} \tag{153}
$$

We rewrite the action by new difined field, h, ξ, we obtain following action,

$$
S = \int d^4\sqrt{-\tilde{g}}(\tilde{g}^{\mu\nu}(\partial_\mu\phi')^\dagger\partial_\nu\phi' - e^{-2\sigma}m_S^2|\phi'|^2 + \frac{1}{6}\tilde{R}\phi'^2 \tag{154}
$$

$$
-\frac{1}{4}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\lambda}G^a_{\mu\nu}G^a_{\rho\lambda} \tag{155}
$$

$$
+\bar{\xi}iD\cdot\gamma\xi - \lambda_\psi\bar{\xi}h\xi \tag{156}
$$

$$
\tilde{g}^{\mu\nu}(D_\mu h)(D_\nu h) - e^{-2\sigma}\mu^2h^\dagger h + \lambda(h^\dagger h)^2 + \frac{1}{6}\tilde{R}h^2). \tag{157}
$$

Next we replace σ to $\sigma = \kappa\varphi/\sqrt{6}$, the action is represented as following,

$$
S = \tilde{S} + (1 - e^{-2\kappa\varphi/\sqrt{6}})m_S^2|\phi'|^2 + (1 - e^{-2\kappa\varphi/\sqrt{6}})\mu^2h^\dagger h. \tag{158}
$$

Then the equation of motion for scalaron field is given as

$$
\tilde{\Box}\varphi - \partial_\varphi V(\varphi) + \frac{2\kappa}{\sqrt{6}}e^{-2\kappa\varphi/\sqrt{6}}\left(m_S^2|\phi'|^2 + \mu^2h^\dagger h\right) = 0, \tag{159}
$$

and the effective potential is written as

$$
V_{eff}(\varphi) = V(\varphi) + e^{-2\kappa\varphi/\sqrt{6}}\left(m_S^2|\phi'|^2 + \mu^2h^\dagger h\right) \tag{160}
$$

. Then only scalar field affect the effective potential of scalafon field. The observation of the decay of scalarons into photons is discussed [34].

4.3 Starobinsky model

The one of most famous model of $F(R)$ gravity is Starobinsky model which is the model that R^2 is joint to Einstein-Hilbert action. The action is given as

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \gamma R^2 - 2\Lambda\right). \tag{161}$$

Models that extend this R^2 term to arbitrary orders and so on are also well studied [41, 42]. When we perform Wyle transformation, the action is rewritten as

$$S = \int d^4x\sqrt{-\tilde{g}}\left(\frac{1}{2\kappa^2}\tilde{R} + V(\varphi)\right), \tag{162}$$

$$V(\varphi) = \frac{1}{2\kappa^2}\frac{F'(A(\varphi))A(\varphi - F(A(\varphi))}{F'^2(A(\varphi)} \tag{163}$$

At first we treat inflation with the Starobinsky model. We calculate the potential with the Starobinsky model, we insert R^2 into the potential. The first derivative of $F(R)$ with respect to R and the relation between an auxiliary field $A(\varphi)$ and scalaron φ is given as following,

$$\partial_A F(A) = 1 + 4\kappa^2\gamma A(\varphi) = e^{2\kappa\varphi/\sqrt{6}}, \tag{164}$$

$$A(\varphi) = \frac{1}{4\kappa^2\gamma}\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right). \tag{165}$$

We substitute above equation into Eq.(163), we obtain the effective potential of scalaron field on Starobinsky model is given as,

$$V_{eff}(\varphi) = \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left[2\kappa^2\gamma A(\varphi)^2 + 4\kappa^2\Lambda + \frac{\kappa^2 T^\mu_\mu}{2}\right] \tag{166}$$

$$= \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left[\frac{1}{8\kappa^2\gamma}(e^{2\kappa\varphi/\sqrt{6}} - 1)^2 + 4\kappa^2\Lambda + \frac{\kappa^2 T^\mu_\mu}{2}\right]. \tag{167}$$

We will focus on two era, early universe and late time universe.

4.3.1 Inflation era

At first we discuss about srow-roll inflation with Starobinsky model. We presume that there was only scalaron field as early universe, so we ignore $T_{\mu\nu}$ term. We show the effective potential as a function of scalaron. As we can see from Fig.4, the potential is flat as large φ region. Therefore the scalaron field roll to minimum slowly, and this field should introduce srow-roll inflation. We can fit theorical parameters by observed values. At first we define the end time point when the slow-roll inflation finish. The time srow-roll inflation finish is defined the condition that srow-roll parameters becomes order 1, $|\epsilon, \eta| \simeq 1$. For obtain srow-roll parameter ϵ, η, we calculate the first and second derivative of the potential with respect to φ,

$$V'(\varphi) = \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa\sqrt{6}}\left[\frac{1}{2\kappa^2\gamma}\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right) - 16\kappa^2\Lambda\right], \tag{168}$$

$$V''(\varphi) = \frac{e^{-4\kappa\varphi/\sqrt{6}}}{6}\left[\frac{1}{2\kappa^2\gamma}\left(2 - e^{2\kappa\varphi/\sqrt{6}}\right) + 32\kappa^2\Lambda\right]. \tag{169}$$

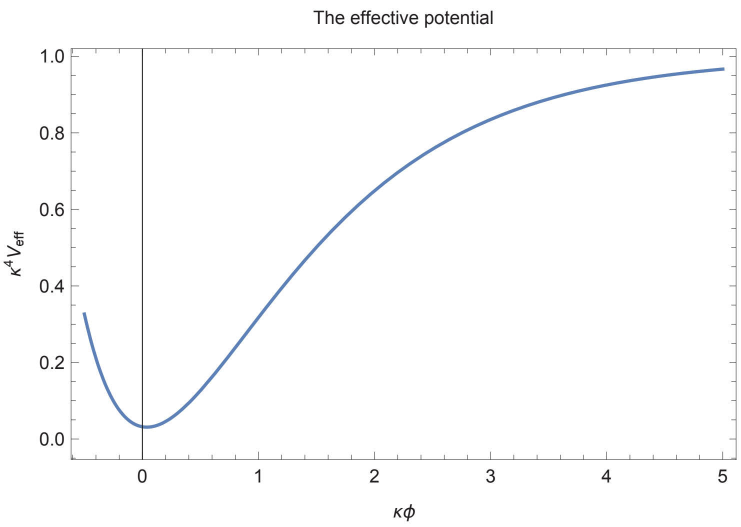

Figure 4: The effective potential as a function with respect to φ. The parameters are fitted, $\gamma = 1, \kappa^4\Lambda = 10^{-3}$.

We substitute Eq(168,169) into Eq(24,25), we obtain the representation of ϵ and η as,

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2 = \frac{4}{3}\left(\frac{e^{2\kappa\varphi/\sqrt{6}} - 1 - 32\kappa^4\gamma\Lambda}{\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right)^2 + 32\kappa^4\gamma\Lambda}\right)^2 \tag{170}$$

$$\eta = \frac{V''}{\kappa^2 V} = -\frac{4}{3}\left(\frac{e^{2\kappa\varphi/\sqrt{6}} - 2 - 64\kappa^4\gamma\Lambda}{\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right)^2 + 32\kappa^4\gamma\Lambda}\right). \tag{171}$$

As we can see from above equation, when $\kappa\varphi \gg 1$, ϵ and η becomes $|\epsilon, \eta| \ll 1$ because the index of exponential in denominator is large. On the other hand, when $\kappa\varphi \ll 1$, ϵ and η approach to 1. We solve the value of ϵ and η becomes 1, the value of φ_{end} is given as following,

$\epsilon = 1$;

$$e^{2\kappa\varphi_{end}/\sqrt{6}} = 1 + \frac{1}{\sqrt{3}} - \frac{\sqrt{1 - 32(2\sqrt{3}+3)\kappa^4\gamma\Lambda}}{\sqrt{3}}, \quad 1 + \frac{1}{\sqrt{3}} + \frac{\sqrt{1 - 32(2\sqrt{3}+3)\kappa^4\gamma\Lambda}}{\sqrt{3}}, \tag{172}$$

$$1 - \frac{1}{\sqrt{3}} - \frac{\sqrt{1 + 32(2\sqrt{3}-3)\kappa^4\gamma\Lambda}}{\sqrt{3}}, \quad 1 - \frac{1}{\sqrt{3}} + \frac{\sqrt{1 + 32(2\sqrt{3}-3)\kappa^4\gamma\Lambda}}{\sqrt{3}}, \tag{173}$$

$\eta = 1$;

$$e^{2\kappa\varphi_{end}/\sqrt{6}} = \frac{1}{3}\left(5 - 2\sqrt{-2 - 264\gamma\Lambda}\right), \quad \frac{1}{3}\left(5 + 2\sqrt{-2 - 264\gamma\Lambda}\right). \tag{174}$$

The solution of Eq(174) are complex, so we exclude these values. The value of φ_{end} should be the biggest one because scalaron field roll dawn to small φ from large φ. When the value of $\kappa^4\Lambda$ is so small, φ_{end} should be as

$$e^{2\kappa\varphi_{end}/\sqrt{6}} = 1 + \frac{1}{\sqrt{3}} + \frac{\sqrt{1 - 32(2\sqrt{3}+3)\gamma\Lambda}}{\sqrt{3}}. \tag{175}$$

We can define the initial point of φ because we defined the end point of φ. For defining the initial point of φ, we consider e-folding number. We define φ_N as a value of φ when the e-folding number bocomes N, e-folding numberis written as following,

$$N = \int_{\varphi_{end}}^{\varphi_N} d\varphi \frac{\kappa^2 V}{V'} = \int_{\varphi_{end}}^{\varphi_N} d\varphi \frac{\sqrt{6}\kappa}{4} \frac{\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right)^2 + 32\kappa^4\gamma\Lambda}{e^{2\kappa\varphi/\sqrt{6}} - 1 - 32\kappa^4\gamma\Lambda}$$

$$= \left[\frac{1}{4}\left(3e^{2\kappa\varphi/\sqrt{6}} - \sqrt{6}\kappa\varphi + 96\kappa^4\gamma\Lambda \ln\left[1 - e^{2\kappa\varphi/\sqrt{6}} + 32\kappa^4\gamma\Lambda\right]\right)\right]_{\varphi_{end}}^{\varphi_N} . \tag{176}$$

We can ignore third term of Eq(176) because Λ describe the dark energy and κ is the inverse of Planck mass. When the value of e-folding number is 60, the value of φ_0 becomes,

$$60 \simeq \frac{1}{4}\left(3e^{2\kappa\varphi_0/\sqrt{6}} - 3\ln\left(e^{2\kappa\varphi_0/\sqrt{6}}\right)\right) - \left(3e^{2\kappa\varphi_{end}/\sqrt{6}} - 3\ln\left(e^{2\kappa\varphi_{end}/\sqrt{6}}\right)\right) \tag{177}$$

$$\simeq \frac{1}{4}\left(3e^{2\kappa\varphi_0/\sqrt{6}} - 3\ln\left(e^{2\kappa\varphi_0/\sqrt{6}}\right)\right) - 3\left(1 + \frac{2}{\sqrt{3}}\right) + 3\ln\left(1 + \frac{2}{\sqrt{3}}\right), \tag{178}$$

$$e^{2\kappa\varphi_0/\sqrt{6}} \simeq 90. \tag{179}$$

curvature power spectrum A_s is given as

$$A_s = \frac{\kappa^4}{24\pi^2}\frac{V}{\epsilon}\bigg|_{\varphi=\varphi_N} = \frac{\kappa^2 e^{-4\kappa\varphi/\sqrt{6}}\left(\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right)^2 + 8\gamma\Lambda\right)^3}{256\pi^2\gamma\left(e^{2\kappa\varphi/\sqrt{6}} - 1 - 8\gamma\Lambda\right)^2}\Bigg|_{\varphi=\varphi_N} \xrightarrow{\kappa\varphi_N\to\infty} \frac{\kappa^2 e^{4\kappa\varphi/\sqrt{6}}}{256\pi^2\gamma}\bigg|_{\varphi=\varphi_N} . \tag{180}$$

When we take $\kappa\varphi_0 = 5.4$, we obtain $A_s = 2.5 \times \kappa^2/\gamma$. Therefore the order of $\kappa^2\gamma$ is given as 10^{-9}. Then the value of r and n_s is represented as

$$r \simeq \frac{12}{N^2}, \quad 1 - n_s \simeq \frac{2}{N}. \tag{181}$$

4.3.2 DE dominant era

Before subsubsection, we discussed the slow roll inflation with Starobinsky model. In this section, we will discuss about a scalaron in late time universe. In this era, we focus the effect of the chameleon mechanism. Compare to inflation era, the effect of $T_{\mu\nu}$ becomes important. The scalaron should be in vacuum state, so we will gain the minimum point of scalaron field.

$$V'_{eff}(\varphi) = \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left[-\frac{4\kappa}{\sqrt{6}}\left(\frac{1}{8\kappa^2\gamma}(e^{2\kappa\varphi/\sqrt{6}} - 1)^2 + 2\kappa^2\Lambda + \frac{\kappa^2 T^\mu_\mu}{2}\right) + \frac{1}{2\kappa\sqrt{6}\gamma}e^{2\kappa\varphi/\sqrt{6}}(e^{2\kappa\varphi/\sqrt{6}} - 1)\right] \tag{182}$$

$$= \frac{\kappa}{\sqrt{6}}\frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left[-\frac{1}{2\kappa^2\gamma}(e^{2\kappa\varphi/\sqrt{6}} - 1)^2 - 8\kappa^2\Lambda - 2\kappa^2 T^\mu_\mu + \frac{1}{2\kappa^2\gamma}(e^{4\kappa\varphi/\sqrt{6}} - e^{2\kappa\varphi/\sqrt{6}})\right] \tag{183}$$

$$= \frac{\kappa}{\sqrt{6}}\frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left[\frac{1}{2\kappa^2\gamma}\left(e^{2\kappa\varphi/\sqrt{6}} - 1\right) - 8\kappa^2\Lambda - 2\kappa^2 T^\mu_\mu\right]. \tag{184}$$

Therefore the minimum point of φ is given as following,

$$\kappa\varphi/\sqrt{6} = \frac{1}{2}\ln\left[1 + 16\kappa^4\gamma\Lambda + 4\gamma\kappa^4 T^{\mu}{}_{\mu}\right]. \tag{185}$$

We evaluate the effective mass of scalaron field at minimum point, we obtain

$$\begin{aligned} m_{\varphi}^2 &= V"_{eff}(\varphi)|_{\varphi=\varphi_{min}} \\ &= -\frac{4\kappa}{\sqrt{6}}V_{eff}(\varphi_{min}) + \left.\frac{e^{-2\kappa\varphi/\sqrt{6}}}{6\gamma}\right|_{\varphi=\varphi_{min}} \\ &= \frac{1}{12\kappa^2\gamma\left(1 + 16\kappa^4\gamma\Lambda + 4\gamma\kappa^4 T^{\mu}{}_{\mu}\right)}. \end{aligned} \tag{186}$$

Then scalaron mass has gap at

$$-\kappa^2 T^{\mu}{}_{\mu} = \frac{1}{2\gamma} + 4\Lambda. \tag{187}$$

We estimate the value of scalaron mass. For inflation condition, the value of γ is given as $\kappa^2/\gamma \sim 10^{-9}$, so the value of scalaron mass is given as

$$m_{\varphi}^2 = \frac{1}{\kappa^2}\frac{\kappa^2}{6\gamma} \sim 10^{28}\ \text{GeV}^2, \tag{188}$$

where we ignore the effect of Λ and $T^{\mu}{}_{\mu}$. Therefore the value of scalarom mass becomes Plack scale order. This value is much heavier than the constrain of Dark Matter candidate, $O(1)$ GeV. Starobinsky model can describe inflation in early universe, but one cannot become the dark matter candidate.

4.4 Starobinsky dark energy model

The Starobinsky model can describe inflation. Next time we only focus into dark energy dominant era. One of $F(R)$ gravity which can describe current universe expansion is following,

$$F(R) = R - \beta R_c\left(1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right), \tag{189}$$

where the value of βR_c is same as cosmological constant, $\beta R_c = 2\Lambda$. This model can describe current accelerating expansion of universe. When the value of curvature is large, which corresponds to the curvature of current universe, $R \gg R_c$, Eq(189) approximated to

$$F(R) \simeq R - \beta R_c. \tag{190}$$

The second term of Eq(190) corresponds to cosmological constant, so this approximation can describe current accelerating expansion. For describe the picture of scalaron field, we approximate at $R \gg R_c$ condition. Then the function $F(R)$ is approximated as

$$F(R) \simeq R - \beta R_c\left(1 - \left(\frac{R^2}{R_c^2}\right)^{-n}\right). \tag{191}$$

By this approximation, we perform the wyle transformation and the scalaron description. The relation between scalaron field φ and curvature is given as following,

$$F'(A) = 1 - 2n\frac{\beta A}{R_c}\left(\frac{A^2}{R_c^2}\right)^{-(n+1)}, \tag{192}$$

$$A(\varphi) = R_c\left(\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right)^{-\frac{1}{2n+1}}. \tag{193}$$

Next we substitute above equations into Eq(163), we obtain the effective potential of scalaron as following,

$$\begin{aligned}
V_{eff}(\varphi) =& \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left[1 - (2n+1)\left(\frac{A(\varphi)^2}{R_c^2}\right)^{-n}\right] - \frac{1}{4}e^{-4\kappa\varphi/\sqrt{6}}T^{\mu}{}_{\mu} \\
=& \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left[1 - (2n+1)\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}}\right] - \frac{1}{4}e^{-4\kappa\varphi/\sqrt{6}}T^{\mu}{}_{\mu}.
\end{aligned} \tag{194}$$

For calculate the minimum point of scalaron field, we calculate the first derivative of Eq(194) with respect to φ is,

$$\begin{aligned}
V'_{eff}(\varphi) =& \frac{\beta R_c}{2\kappa^2}\partial_\varphi\left(e^{-4\kappa\varphi/\sqrt{6}}\left[1 - (2n+1)\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}} - \frac{\kappa^2}{2\beta R_c}T^{\mu}{}_{\mu}\right]\right) \\
=& \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left(-\frac{4\kappa}{\sqrt{6}}\left[1 - (2n+1)\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}} - \frac{\kappa^2}{2\beta R_c}T^{\mu}{}_{\mu}\right]\right) \\
&+ \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left(\frac{2\kappa}{\beta\sqrt{6}}e^{2\kappa\varphi/\sqrt{6}}\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{-1}{2n+1}}\right) \\
=& \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left(-\frac{4\kappa}{\sqrt{6}}\left[1 - (2n+1)\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}} - \frac{\kappa^2}{2\beta R_c}T^{\mu}{}_{\mu}\right]\right) \\
&+ \frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left(-\frac{4\kappa}{\sqrt{6}}\left[-\frac{1}{2\beta}\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{-1}{2n+1}} + n\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}}\right]\right) \\
=& -\frac{4\kappa}{\sqrt{6}}\frac{\beta R_c}{2\kappa^2}e^{-4\kappa\varphi/\sqrt{6}}\left[-(n+1)\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}}\right. \\
&\left. - \frac{1}{2\beta}\left\{\frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi/\sqrt{6}}\right)\right\}^{\frac{-1}{2n+1}} - \frac{\kappa^2 T^{\mu}{}_{\mu}}{2\beta R_c}\right].
\end{aligned} \tag{195}$$

This formula is so messy and it is difficult to solve minimum point. Therefore we consider $|\kappa\varphi| \ll 1$ condition for simplify and we expansion exp part of Eq(195). We remark that this condition is not contradiction to small curvature condition because of Eq(193). Then we obtain approximated form as,

$$-(n+1)\left(-\frac{\kappa\varphi}{n\beta\sqrt{6}}\right)^{\frac{2n}{2n+1}} - \frac{1}{2\beta}\left(-\frac{\kappa\varphi}{n\beta\sqrt{6}}\right)^{\frac{-1}{2n+1}} - \frac{\kappa^2 T^{\mu}{}_{\mu}}{2\beta R_c} = 0. \tag{196}$$

When we ignore first term of Eq(196) because of the order of $\kappa\varphi$'s order, we can solve with respect to φ_{min},

$$\kappa\varphi_{min} = -n\beta\sqrt{6}\left(-\frac{\kappa^2 T^{\mu}{}_{\mu}}{R_c}\right)^{-(2n+1)} . \tag{197}$$

The mass of scalaron field is given as the second derivative of the effective potential at minimum point Eq.(197). The mass of scalaron is given as

$$\begin{aligned}
m_\varphi^2 &= V''_{eff}(\varphi)\big|_{\varphi=\varphi_{min}} \\
&= -\frac{4\kappa}{\sqrt{6}} V'_{eff}(\varphi)\big|_{\varphi=\varphi_{min}} - \frac{4\kappa}{\sqrt{6}}\frac{\beta R_c}{2\kappa^2} e^{-4\kappa\varphi/\sqrt{6}} \left[-\frac{2\kappa}{\sqrt{6}} \cdot \frac{2n(n+1)}{2n+1} \left\{ \frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi_{min}/\sqrt{6}}\right)\right\}^{\frac{2n}{2n+1}} \right. \\
&\quad \left. + \frac{\kappa(2n+3)}{\sqrt{6}\beta(2n+1)} \left\{ \frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi_{min}/\sqrt{6}}\right)\right\}^{\frac{-1}{2n+1}} - \frac{\kappa}{\sqrt{6}\beta^2 n(2n+1)} \left\{ \frac{1}{2n\beta}\left(1 - e^{2\kappa\varphi_{min}/\sqrt{6}}\right)\right\}^{-\frac{2(n+1)}{2n+1}} \right] \\
&\simeq -\frac{4\kappa}{\sqrt{6}}\frac{\beta R_c}{2\kappa^2} \left[-\frac{2\kappa}{\sqrt{6}} \cdot \frac{2n(n+1)}{2n+1} \left(-\frac{\kappa\varphi_{min}}{\sqrt{6}\beta n}\right)^{\frac{2n}{2n+1}} \right. \\
&\quad \left. + \frac{\kappa(2n+3)}{\sqrt{6}\beta(2n+1)} \left(-\frac{\kappa\varphi_{min}}{\sqrt{6}\beta n}\right)^{\frac{-1}{2n+1}} - \frac{\kappa}{\sqrt{6}\beta^2 n(2n+1)} \left(-\frac{\kappa\varphi_{min}}{\sqrt{6}\beta n}\right)^{-\frac{2(n+1)}{2n+1}} \right] \\
&\simeq \frac{R_c}{3\beta n(2n+1)} \left(-\frac{\kappa^2 T^{\mu}{}_{\mu}}{R_c}\right)^{2(n+1)} .
\end{aligned} \tag{198}$$

Then the mass of scalaron depend on the power of $T^{\mu}{}_{\mu}$ to the $2(n+1)$. When we assume the upper bound of scalaron mass, $m_\varphi \leq O(10)$ GeV, in solar system and $\beta R_c = \Lambda_{DE}$, the constant parameter β must satisfy $\beta < 10^{-30+\frac{13}{n}}$. Then R_C must satisfy $R_C > 10^{-55-\frac{13}{n}}$ GeV^2 and this order corresponds to solar system scale, $\kappa^2\rho_\odot \sim 10^{-55}$ GeV^2.

4.5 Logarithmic model

The scalaron with the Starobinsky dark energy model becomes so heavy, so we will consider more smaller scalaron mass model. We define the action as logarithmic form [35],

$$F(R) = R - \Lambda_{DE}\left(1 - \alpha\frac{R}{R_c}\ln\left(\frac{R}{R_c}\right)\right) + \kappa^2\gamma_0\left(1 + \gamma_1 \ln\left(\frac{R}{R_0}\right)\right) R^2, \tag{199}$$

where α, R_C andγ_0, γ_1, R_0 are free constant parameters. The corrections to log are the expected quantum corrections, and the corrections to log for R^2 are investigated in [43]. Fig.(5) schematically shows the approach to the Dark Energy, Dark Matter problem by the chameleon mechanism. scalaron potential has a flat potential in vacuum and a high curvature in the matter field. The potential of scalarons is flat in vacuum and has high curvature in the matter field, so that scalarons behave like Dark Energy as a constant potential in vacuum and become heavy in the vicinity of galaxies, which makes them candidates for Dark Matter. This logarithmic form is motivated by quantum correction. This model is stand from the starobinsky inflation model,

$$F(R) = R + f_{DE}(R) + \kappa^2\gamma R^2. \tag{200}$$

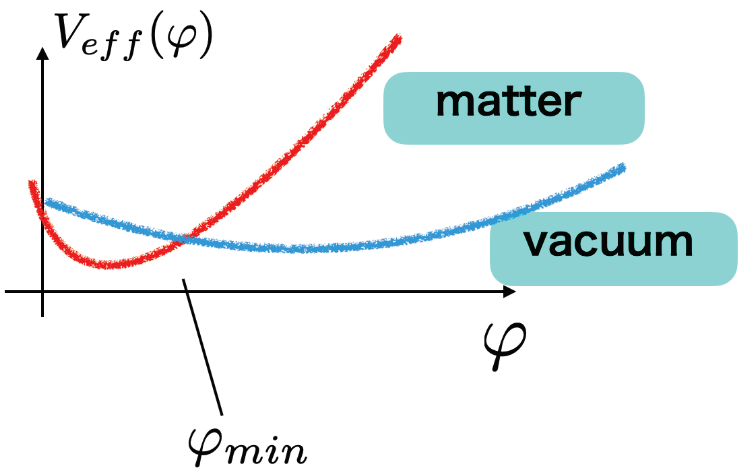

Figure 5: A chameleon mechanism approach to DE and EM problems. The red and blue lines suggest the behavior of the scalaron potential in a matter field and in a vacuum, respectively.

If we consider one-loop corrections to coupling constants in multiplicatively renormalizable higher-derivative quantum gravity [44], the coefficient γ in front of R^2 should become logalithmic form as eq.(199). We consider R_C is DE scale and R_0 is early time inflation scale. For explaining DE era and early time inflation era, this model should be approximated as following ;

In early time inflation era, the DE term can be neglected and this condition is represented as,

$$R_0, \quad \kappa^2\gamma_0[1+\gamma_1 \ln(O(1))]R_0^2 \gg \Lambda_{DE}\left[1-\alpha\frac{R_0}{R_C}\ln\left(\frac{R_0}{R_C}\right)\right]. \tag{201}$$

From above an inequality, we obtain the conditions for parameters as

$$\frac{R_C}{\Lambda_{DE}\ln(R_0/R_C)} \gg \alpha, \quad \gamma_0\gamma_1\frac{\kappa^2 R_0 R_C}{\Lambda_{DE}\ln(R_0/R_C)} \gg \alpha, \quad R_0 \gg \sqrt{\frac{\Lambda_{DE}}{\kappa^2\gamma_0\gamma_1}}. \tag{202}$$

We can constraint to some parameters γ_0, γ_1, R_0 by slow roll approximation. When DE term is dominant, the condition that inflation term not becomes effective is represented

$$R_C, \quad \Lambda_{DE}[1-\alpha\ln(O(1))] \gg \kappa^2\gamma_0\left[1+\gamma_1\ln\left(\frac{R_C}{R_0}\right)\right]R_C^2.$$

From above an inequality, we obtain the conditons for parameters as

$$\frac{1}{\kappa^2\gamma_0} \gg R_C, \quad \sqrt{\frac{\Lambda_{DE}}{\kappa^2\gamma_0}} \gg R_C, \quad \sqrt{\frac{\alpha\Lambda_{DE}}{\kappa^2\gamma_0}} \gg R_C. \tag{203}$$

It is known that this logalithmic $F(R)$ produce a inflation scenario in the Jordan frame [44].

4.5.1 Slow Roll Inflation

The initial value of ϕ is determined from the value of e-folding number, which is obtained from the effective potential of scalaron. We define the e-folding as

$$N = \int_{\varphi_{end}}^{\varphi_N} d\varphi \frac{V_{eff}(\varphi)}{V'_{eff}(\varphi)}, \tag{204}$$

where the value of e-folding is constrainted as $N = 50 \sim 60$. We define the constraint that srow-roll inflation finish as, the parameter ε is defined as

$$\varepsilon|_{\varphi_{end}} \equiv \frac{1}{2\kappa^2}\left(\frac{V'_{eff}(\varphi)}{V_{eff}(\varphi)}\right)^2\Bigg|_{\varphi_{end}}, \tag{205}$$

the parameter η is defined as

$$\eta|_{\varphi_{end}} \equiv \frac{1}{\kappa^2}\frac{V''_{eff}(\varphi)}{V_{eff}(\varphi)}\Bigg|_{\varphi_{end}}. \tag{206}$$

The power spectrum and the specrtum index, tensor-to-scalar ratio are defined and constraints as

$$P_s \equiv \frac{1}{24\pi^2}\frac{V_{eff}(\varphi)}{\varepsilon}\Bigg|_{\varphi_N} \tag{207}$$

$$n_s \equiv (1 - 6\varepsilon + 2\eta)|_{\phi_N} = 0.9652 \pm 0.0042 \tag{208}$$

$$r \equiv \frac{P_t}{P_s} = 16\varepsilon|_{\phi_N} < 0.106. \tag{209}$$

$\ln(10^{10}A_s) = 3.043 \pm 0.014$ [28]. The tensor-to-scalar ratio is obtained as If we tune the parameters as

$$\gamma_0 = (0.88 \sim 1.2) \times 10^9 \tag{210}$$

$$\gamma_1 = (1.0 \sim 1.4) \times 10^{-6} \tag{211}$$

$$R_0/\Lambda_{DE} = 1.8, \tag{212}$$

constraints (eq(208∼209)) are satisfied and the value of tensor-scalar ratio is given as

$$r = (2.94 \sim 4.10) \times 10^{-3}. \tag{213}$$

4.5.2 The Description of Scalaron

We have confirmed that the R^2 term can reproduce the slow roll inflation to match the current observation. Now we will analyze the current accelerated expansion. We consider DE dominant case,

$$F(R) \simeq R - \Lambda_{DE}\left[1 - \alpha\frac{R}{R_C}\ln\left(\frac{R}{R_C}\right)\right]. \tag{214}$$

The first we calculate the relation between R and φ,

$$R/R_C = \exp\left[\frac{R_C}{\alpha\Lambda_{DE}}(e^{2\kappa\varphi/\sqrt{6}} - 1) - 1\right]. \tag{215}$$

The small curvature corresponds DE era and the large curvature corresponds a inflation era, so the φ becomes large, the inflation effect appears and the shape of the effective potential transform becouse of the inflation effect. If α becomes large of R_C/Λ_{DE} becomes small, the slope of R graph becomes gentle. The regeon of φ that DE effect is dominant is depend of the slope of R graph, so on α and R_C/Λ_{DE}. The potential is given as follow,

$$\begin{aligned} V_{eff}(\varphi) &= \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left(RF'(R) - F(R) - \frac{\kappa^2}{2}T^{\mu}{}_{\mu}\right) \\ &= \frac{\alpha\Lambda_{DE}e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2}\left(\exp\left[\frac{R_C}{\alpha\Lambda_{DE}}(e^{2\kappa\varphi/\sqrt{6}} - 1) - 1\right] + \frac{1}{\alpha} - \frac{\kappa^2 T^{\mu}{}_{\mu}}{2\alpha\Lambda_{DE}}\right). \end{aligned} \tag{216}$$

Next we check the effective potential has the stable ground state or nor. If $\kappa\varphi/\sqrt{6} \ll 1$, the effective potential can be approximated as,

$$\begin{aligned} V_{eff}(\varphi) &\simeq \frac{\alpha\Lambda_{DE}}{2\kappa^2}\left(1 - 4\frac{\kappa\varphi}{\sqrt{6}}\right)\left(\exp\left[\frac{R_C}{\alpha\Lambda_{DE}}\frac{2\kappa\varphi}{\sqrt{6}} - 1\right] + \frac{1}{\alpha} - \frac{\kappa^2 T^{\mu}_{\mu}}{2\alpha\Lambda_{DE}}\right) \\ &\simeq \frac{\alpha\Lambda_{DE}}{2\kappa^2}\left(\frac{1}{e} + \frac{1}{\alpha} - \frac{\kappa^2 T^{\mu}_{\mu}}{2\alpha\Lambda_{DE}} + \left(-\frac{2}{e} - \frac{2}{\alpha} + \frac{R_C}{e\alpha\Lambda_{DE}} + \frac{\kappa^2 T^{\mu}_{\mu}}{\alpha\Lambda_{DE}}\right)\frac{2\kappa\varphi}{\sqrt{6}}\right). \end{aligned} \tag{217}$$

If the trace of energy tensor is much large, $\frac{\kappa^2 T^{\mu}_{\mu}}{\alpha\Lambda_{DE}} \ll 1$, a slope on garaph of the effective potential arround small φ becomes negative. On the other hand, If $\kappa\varphi/\sqrt{6} \gg 1$, the exponentialy term in the effective potential becomes dominant, so one can approximate the effective potential as,

$$V_{eff}(\varphi) \simeq \frac{\alpha\Lambda_{DE}}{2\kappa^2}\exp\left[\frac{R_C}{\alpha\Lambda_{DE}}(e^{2\kappa\varphi/\sqrt{6}} - 1)\right]. \tag{218}$$

Arround small φ, the effective potential $V_{eff}(\varphi)$ drop down as a linear function of φ if the trace of energy-momeutum tensor T^{μ}_{μ} is much large. Arround large φ, the effective potential $V_{eff}(\varphi)$ must inclease becouse of a effect of exponential function, eq(218). Therefore we can understand that if the tarace of energy-momentum tensor, the scalaron field has a stable ground state.The second derrevative of potential becomes sharp if R_C/Λ_{DE} becomes large and , so we can assume that the scalaron mass becomes heavy if R_C/Λ_{DE} becomes large and small if α becomes large. Next we calculate the scalaron mass analyticaly. The first we obtaine the φ that a minimum of the effective potential $V_{eff}(\varphi)$, we can obtain the minimum point of φ exactly as,

$$\kappa\varphi_{min}/\sqrt{6} = \frac{1}{2}\ln\left[\frac{\alpha\Lambda_{DE}}{R_C}\left(2 + W\left(\frac{e^{-1+\frac{R_c}{\alpha\Lambda_{DE}}}\left(2 - \frac{\kappa^2 T^{\mu}_{\mu}}{\Lambda_{DE}}\right)}{\alpha}\right)\right)\right], \tag{219}$$

where W is a lambelt W-function that obey following equation,

$$z = W(z)e^{W(z)}. \tag{220}$$

A lambelt W-function is complecly if we treat this model analyticaly, so we approximate eq(219) by elementary function. When $\frac{\kappa^2 T^\mu}{\alpha\Lambda_{DE}} \gg 1$ and $\frac{R_C}{\Lambda_{DE}} \gg \alpha$, one can apploximate eq.(219) as

$$\kappa\varphi_{min}/\sqrt{6} \simeq \frac{1}{2}\ln\left[1+\frac{\alpha\Lambda_{DE}}{R_C}\left(1+\ln\left[\frac{2\Lambda_{DE}-\kappa^2 T^\mu_\mu}{R_C}\right]\right)\right], \tag{221}$$

where we used the apploximation for $W(x)$ as [32]

$$W(x) \simeq \ln[x] - \ln[\ln[x]] \quad for \quad x \gg 1. \tag{222}$$

We substitute eq(221) into the second derivetive of the effective potential $V_{eff}(\varphi)$, wje obtaine a scalaron mass as

$$m_\varphi^2 \simeq -\frac{\kappa^2 R_C T^\mu_\mu}{3\alpha\Lambda_{DE}}. \tag{223}$$

We drow the scalaron mass depends on the trace of energy-momentum tensor. The approximation we adopted is viable. It is known that a scalaron decay two photons and two gluons through massive fermion and gauge boson loops. If a scalaron's life time is so short, a scalaron decay soon and there is no scalaron in current universe. Therefore we assume a scalaron's life time is longer than the age of current uneverse,

$$\tau_\varphi > \tau_{\text{uni}} \sim 10^{17} s. \tag{224}$$

From this constraint, we find the upper bound for the scalaron mass [**?**],

$$m_\varphi < O(1)[\text{GeV}]. \tag{225}$$

Now we consider current relic DM, so we substitute T^μ_μ as solar system scale $\rho_\odot \simeq 10^{-17}\text{GeV}^4$. Substituting this value to eq.(225), we obtain the constraint for R_C and α,

$$\frac{R_C}{\Lambda_{DE}} < \frac{3O(1)^2}{\kappa^2\rho_\odot}\alpha \simeq 3\times 10^{55}\alpha. \tag{226}$$

4.6 Power low like model

Next we consider power low like model which is treated in [16, 40, 44] at $n = m$,

$$F(R) = R\left(1+\left(\frac{R}{R_0}\right)^n\right)^m. \tag{227}$$

This model can be approximated when $(R/R_0)^n \ll 1$. This model can be approximated as

$$F(R) \simeq R + m\left(\frac{R}{R_0}\right)^n. \tag{228}$$

In this approximation, the term is like the Einstein-Hilbert action plus R^n, and is expected to reproduce slow roll inflation when $n \sim 2$. On the other hand, when $(R/R_0)^n \gg 1$ is satisfied, it can be approximated as follows

$$F(R) \simeq \frac{R^{mn+1}}{R_0}. \tag{229}$$

In this case, action is expressed as a pure power of curvature R.

4.6.1 Acceleration in DE era

Perform an approximate calculation for the current universe, where the value of curvature R is sufficiently small. At this time, the approximation is divided into cases according to the magnitude of the value of n. The density of matter at that time is calculated as follows

$(H^2/R_0)^n \ll 1$

$$\Omega_m = 1 - m\left((2n^2+n)q + \left(2n^2+2n-1\right) - \frac{n(1+n)}{q-1}\frac{\dot{q}}{H}\right)(6(-q+1))^n\left(\frac{H^2}{R_0}\right)^n. \tag{230}$$

$(H^2/R_0)^n \gg 1$

$$\begin{aligned}\Omega_m = &\left(\left(\frac{H^2}{R_0}\right)^{-n}\right)^{-m} (6(-q+1))^{mn} \\ &\times\left(-m^2n^2q + 1 - mn - m^2n^2 + \frac{\dot{q}mn(1+mn)}{H(q-1)} + \right. \\ &\left. m\left(1 + (\dot{q}/H - q^2 - 1)\frac{n(-1+m+n)}{q-1}\right)\left(\frac{H^2}{R_0}\right)^{-n}(6(-q+1))^{-n}\right).\end{aligned} \tag{231}$$

At this point, we can discuss the expansion and contraction of the universe depending on the value of m, n.

4.6.2 Tensor scalar picture

By performing a Wyle transformation on the metric, we can rewrite it in Einstein-Hilbert and scalaron form. The effective potential of the scalaron is then given by

$$V_{eff}(\varphi) = \frac{e^{-4\kappa\varphi/\sqrt{6}}}{2\kappa^2} m \cdot nr_0 r(\varphi)^{n+1}(1+r(\varphi)^n)^{m-1} + e^{-4\kappa\varphi/\sqrt{6}}\rho, \tag{232}$$

where $r(\varphi)$ is dimension less value, $r(\varphi) = R(\varphi)/R_0$. In this section, we treat $r(\varphi)$ substitute to curvature R. φ is solved as

$$e^{2\kappa\varphi/\sqrt{6}} \equiv (1 + (1 + m\cdot n)r(\varphi)^n)(1 + r(\varphi)^n)^{m-1} \tag{233}$$

The potential of the scalaron can be determined analytically. The effective mass of the scalaron can be obtained as follows

$$m_\varphi^2(\rho) = \frac{r_0}{12mn(n+1)}\left(\frac{2\kappa^2\rho}{r_0}\right)^{1-n}. \tag{234}$$

Thus, when $n > 1$, scalarons behave in an inverse chameleonic manner, rather than being lightened in mass by the presence of matter fields.

5 Symplectic numerical integral

The $F(R)$ gravity is difficult to observe on the ground due to the chameleon mechanism. On the other hand, if the chameleon mechanism is demonstrated, it will be a proof of the existence of $F(R)$ gravity. In Fig.(6), we summarize the possible influence of the chameleon mechanism in each phase of the early universe. In the preheating phase, the background matter field may increase exponentially due to the resonant effect of the scalaron and scalar fields, as discussed in Chapter 3. This process may have a verifiable effect on the chameleon mechanism. In the following, we discuss the $F(R)$ gtavity in the preheating phase.

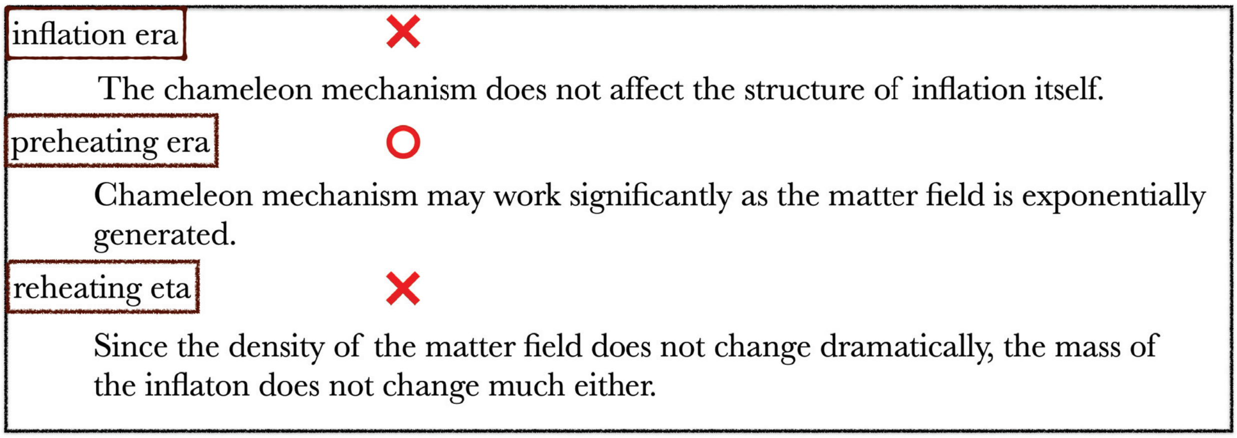

Figure 6: Possible influence of the chameleon mechanism for each phase in the early universe.

In this chapter, we will actually calculate the behavior of the scalaron in the preheating phase numerically. In the equation of motion of the scalaron in the preheating phase in $F(R)$ gravity, the behavior of its potential is an important factor. In addition, unlike the general field, the kinetic term of the matter field on $F(R)$ gravity has a factor of $e^{-4\kappa\varphi/\sqrt{6}}$, which means that the kinetic term does not have a canonical form. Therefore, it is not possible to perform an approximate calculation as we did in chapter 3. Therefore, we will use the full equation of motion of scalarons and calculate it numerically. However, when we try to solve the equation of motion numerically, there is a risk that the value of the Hamiltonian may deviate from the true value by general numerical methods because the kinetic term has a non-canonical form. In this study, we apply the method of [37] to solve the equation of motion symplectically for $F(R)$ gravity.

5.1 Second order

We will first look at the symplectic numerical integral method here. Canonical equation of motion conserve the value of Hamiltonian because, the time derivative is shown as the poisson bracket belong to Hamiltonian H, $\dot{f} = \{f, H\}$.

$$\dot{H} = \{H, H\} = 0 \tag{235}$$

Therefore the time expantion should be canonical transformation to conserve the form of equation.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = I_{ij} \frac{\partial H}{\partial r_i}, \tag{236}$$

where matrix I_{ij} and r_i is defined as

$$I_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad r_i = \begin{pmatrix} q \\ p \end{pmatrix}. \tag{237}$$

The time derivative of values $P(p,q), Q(p,q)$ is written as

$$\dot{R}_i = \frac{\partial R_i}{\partial r_j} \dot{r}_j = \frac{\partial R_i}{\partial r_j} I_{jk} \frac{\partial H}{\partial r_k} = \frac{\partial R_i}{\partial r_j} I_{jk} \frac{\partial R_l}{\partial r_k} \frac{\partial H}{\partial R_l}, \tag{238}$$

The condition that $p, q \to P(p,q), Q(p,q)$ becomes canonical transformation is,

$$\frac{\partial R_i}{\partial r_j} I_{jk} \frac{\partial R_l}{\partial r_k} = I_{il}. \tag{239}$$

More symply above equation is written as following,

$$\begin{pmatrix} \{Q, Q\}_{q,p} & \{Q, P\}_{q,p} \\ \{P, Q\}_{q,p} & \{P, P\}_{q,p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{240}$$

The exact solution of EoM is represented as

$$f(t) = e^{\hat{H}(t-t_0)} f(q_0, p_0), \tag{241}$$

where we define the operator $\hat{H} f = \{f, H\}_{p,q}$. So this representation should conserve symplectic symmetry, but the representation of Eq.(241) is difficult because of commutation relation of the kinetic term K and potential V.

$$e^{\hat{H}\tau + O(\tau^2)} \to e^{\hat{V}\tau} e^{\hat{K}\tau}, \tag{242}$$

then each exponential operator is represented as

$$e^{\hat{V}\tau} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p - \frac{\partial V}{\partial q}\tau \end{pmatrix}, \quad e^{\hat{K}\tau} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q + \frac{\partial K}{\partial p}\tau \\ p \end{pmatrix}. \tag{243}$$

This operator, when applied to any function $F(q,p)$ of p, q, performs a transformation that transitions the variables q, p, and

$$e^{\hat{V}\tau} F(q,p) = F\left(q, p - \frac{\partial V}{\partial q}\tau\right), \quad e^{\hat{K}\tau} F(q,p) = F\left(q + \frac{\partial K}{\partial p}\tau, p\right), \tag{244}$$

The transition operator defined above is used to describe the time evolution by the symplectic integral. The time step is defined by Eq.(242). The first order symplectic integral step is shown as,

$$\begin{pmatrix} q_{i+1} \\ p_{i+1} \end{pmatrix} = e^{\tau V} e^{\tau K} \begin{pmatrix} q^n \\ p^n \end{pmatrix} = e^{\tau V} \begin{pmatrix} q^n + \tau \frac{\partial K}{\partial p}(p^n) \\ p^n \end{pmatrix} = \begin{pmatrix} q^n + \tau \frac{\partial K}{\partial p}(p^{n+1}) \\ p^n - \tau \frac{\partial V}{\partial q}(q^n) \end{pmatrix}. \tag{245}$$

The time update defined above preserves the value of the Hamiltonian. This is guaranteed by the fact that the Eq.(245) transformation has symplectic symmetry. This transform conserve symplectic symmetry,

$$\begin{aligned}\{q_{i+1}, p_{i+1}\} &= \left\{q_i + \frac{\partial K}{\partial p}(p^{n+1})\tau,\ p_{i+1}\right\} \\ &= \{q_i, p_{i+1}\} = \left\{q_i,\ p_i - \tau\frac{\partial V}{\partial q}(q_i)\right\} = \{q_i, p_i\}. \end{aligned} \tag{246}$$

In fact, the symplectic integral defined here does not conserve the exact amount of Hamiltonian. Therefore, we will find the conserved conservative by this transformation. The transformation by the transformation Eq.(245) transforms τ in quadratic order as follows

$$e^{\tau V}e^{\tau K} = \exp\left(\tau(K+V) + \frac{\tau^2}{2}[V, K]\right). \tag{247}$$

So the value which conserve along symplectic integral is

$$\begin{aligned}H' &= K + V + \frac{\tau}{2}\{V, K\}_{q,p} \\ &= K + V + \frac{\tau}{2}\frac{\partial V}{\partial q}\frac{\partial K}{\partial p}. \end{aligned} \tag{248}$$

This quantity is the one that is preserved by the transformation by Eq.(245).

5.2 Leap flog

We will look at transformations that preserve the Hamiltonian at a higher order than the transformation defined by Eq.(245). Leap flog which is second order of symplectic integral is shown as

$$\begin{pmatrix} p^{n+1} \\ q^{n+1} \end{pmatrix} = e^{\tau K/2}e^{\tau V}e^{\tau K/2}\begin{pmatrix} p^n \\ q^n \end{pmatrix}. \tag{249}$$

For the transformation defined by the above, we obtain the specific sign of the time evolution step by letting the transition operator act on p^n, q^n. We perform the poisson bracket operator, we obtain

$$\begin{aligned}\begin{pmatrix} p^{n+1} \\ q^{n+1} \end{pmatrix} &= e^{\tau K/2}e^{\tau V}e^{\tau K/2}\begin{pmatrix} p^n \\ q^n \end{pmatrix} \\ &= e^{\tau K/2}e^{\tau V}\begin{pmatrix} p^n \\ q^n + \frac{\tau}{2}\frac{\partial K}{\partial p}(p^n) \end{pmatrix} \\ &= e^{\tau K/2}\begin{pmatrix} p^n - \tau\frac{\partial V}{\partial q}(q^n) \\ q^n + \frac{\tau}{2}\frac{\partial K}{\partial p}\left(p^n - \tau\frac{\partial V}{\partial q}(q^n)\right) \end{pmatrix} \\ &= \begin{pmatrix} p^n - \tau\frac{\partial V}{\partial q}\left(q^n + \frac{\tau}{2}\frac{\partial K}{\partial p}(p^n)\right) \\ q^n + \frac{\tau}{2}\frac{\partial K}{\partial p}(p^n) + \frac{\tau}{2}\frac{\partial K}{\partial p}\left(p^n - \tau\frac{\partial V}{\partial q}(q^n)\right) \end{pmatrix} \end{aligned} \tag{250}$$

The right hand side of the above transformation has a complicated form. Therefore, we can rewrite the update of the time step in a simpler form by defining half step. Above transformation is rewritten we introduce half step,

$$\begin{cases} q^{n+1/2} &= q^n + \tau\frac{\partial H}{\partial p}(p^n) \\ p^{n+1} &= p^n - \tau\frac{\partial H}{\partial q}(q^{n+1/2}) \\ q^{n+1} &= q^{n+1/2} + \tau\frac{1}{2}p^{n+1}, \end{cases} \tag{251}$$

Naturally, we can confirm that the leap frog method also has symplectic symmetry. This transformation conserve symplectic symmetry,

$$\begin{aligned} \{q_{i+1}, p_{i+1}\} &= \left\{q^{n+1/2} + \tau\frac{1}{2}p^{n+1}, p^{n+1}\right\} \\ &= \{q^{n+1/2}, p^{n+1}\} \\ &= \left\{q^{n+1/2}.p^n - \tau\frac{\partial H}{\partial q}\left(q^{n+1/2}\right)\right\} \\ &= \{q^{n+1/2}, p^n\} \\ &= \left\{q^n + \tau\frac{\partial H}{\partial p}\left(p^n\right), p^n\right\} \\ &= \{q^n, p^n\}. \end{aligned} \tag{252}$$

It was confirmed that the leap frog method also has symplectic symmetry, which means that there is a conserved quantity that is conserved by the time update by Eq.(252). To obtain the conserved quantity by updating Eq.(252), we use the following formula

$$\begin{aligned} e^A e^B e^C = \exp\Bigg(& A + B + C + \frac{1}{2}([A-C, B] + [A, C]) \\ & + \frac{1}{12}\left([A,[A,B+C]] + [B,[B,A+C]] + [C,[C,A+B]] + 3[[A,B],C] + [B,[A,C]]\right)\Bigg). \end{aligned} \tag{253}$$

Using the above formula, we can rewrite the transition operator of the leap frog method in the third order of time. The second order of symplectic integral is represented as

$$e^{\tau K/2} e^{\tau V} e^{\tau K/2} = \exp\left(\tau(K+V) + \frac{\tau^3}{12}\left[V + \frac{K}{2}, [V, K]\right]\right). \tag{254}$$

So the value which conserve along symplectic integral is

$$H' = K + V + \frac{\tau^2}{12}\left\{V + \frac{K}{2}, \{V, K\}_{q,p}\right\}_{q,p} \tag{255}$$

$$= K + V + \frac{\tau^2}{12}\left(\frac{\partial^2 K}{\partial p^2}\left(\frac{\partial V}{dq}\right)^2 - \frac{1}{2}\left(\frac{\partial K}{\partial p}\right)^2\frac{\partial^2 V}{\partial q^2}\right). \tag{256}$$

This quantity is the conserved quantity that is truly conserved by the leap frog method. A new quantity of the second order of time is added to the true Hamiltonian.

5.3 Advance to our model

In the previous chapter, we reviewed the basics of the symplectic integral method, which can only be used in theories that originally had symplectic symmetry. However, our model does not have symplectic symmetry because the matter field has a non-canonical kinetic term. Therefore, by applying the method of [37], we will apply the symplectic integral method to the model we study.

We will adopt the simplest R^2 model as the $F(R)$ gravity and the case where it couples non-minimally with the scalar field χ. The lagrangian is given as following,

$$\mathcal{L} = \sqrt{-g_J}\left[\frac{M_{pl}^2}{2}R_J + \gamma R_J^2 + \xi R\chi^2 - \frac{1}{2}g_J^{\mu\nu}(\partial_m u\chi)(\partial_\nu\chi) - \frac{1}{2}m_\chi^2\chi^2\right]. \tag{257}$$

The Lagrangian given here can also be rewritten in the Einstein-Hilbert part and the scalaron part by performing the Wyle transformation on the metric $g_J^{\mu\nu}$, as explained in Chapter 4. The Lagrangian rewritten in this way can be written as follows

$$\mathcal{L} = \sqrt{-g_E}\left[\frac{M_{pl}^2}{2}R_E - \frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2}e^{-2/(\sqrt{6}M_{pl})\varphi}g_E^{\mu\nu}(\partial_\mu\chi)(\partial_\nu\chi) - V(\varphi,\chi)\right], \tag{258}$$

where the effective potential of the scalaron φ is defined as follows

$$V(\varphi,\chi) = e^{-4\kappa\varphi/\sqrt{6}}\left(\frac{1}{16\kappa^4\gamma}\left(e^{2\kappa\varphi/\sqrt{6}} - 1 - 2\kappa^2\xi\chi^2\right) + \frac{1}{2}g^{\mu\nu}(\partial_\mu\chi)(\partial_\nu\chi) + \frac{1}{2}m_\chi^2\chi^2\right) \tag{259}$$

and the relation of scalaron and curvature R is given as following,

$$R(\varphi) = \frac{1}{4\kappa^2\gamma}\left(1 - e^{-2\kappa\varphi/\sqrt{6}} - 2\kappa^2\xi\chi^2\right). \tag{260}$$

5.3.1 Einstein-Hilbert term

At first we calculate the Einstein-Hilbert part. We consider metric as following,

$$ds^2 = dt^2 + g_{ij}dx^i dx^j, \tag{261}$$

$$g_{ij} = a(t)^2\left(\delta_{ij} + h_{ij}\right) \tag{262}$$

where h_{ij} is the perturbation part of Minkowski metric. In generally the basic parameter of gravity is the metric $g_{\mu\nu}$, but we introduce the other definition. For analytic treatment, we define the matrix β which is defined as the logarithmic of g as

$$\beta_{ij} = (\ln g)_{ij}. \tag{263}$$

Next we expansion around the Minkowski metric because of $|h| \ll 1$,

$$(\ln g)_{ij} = \left(\ln(a^2(\delta + h))\right)_{ij} = 2\delta_{ij}\ln a + h_{ij} + O(h^2). \tag{264}$$

As a parameter to describe the background space-time, we will use β instead of h. Then the perturbation parameter h is written by β,

$$h_{ij} \simeq \beta_{ij} - 2\delta_{ij}\ln a. \tag{265}$$

To rewrite the metric $G^{\mu\nu}$ in β, we also define the traceless part of β. We define the traceless part of β as,

$$\gamma_{ij} = \beta_{ij} - \frac{\beta}{3}\delta_{ij} = h_{ij} - \frac{h}{3}\delta_{ij} + O(h^2). \tag{266}$$

where β is the trace of β matrix. This implies that the matrix h and γ is same order, $O(\gamma^n) \leq O(h^n)$. Therefore, we can regard γ as a perturbation parameter.

$$(\ln g)_{ij} = \beta_{ij} = \frac{\beta}{3}\delta_{ij} + \gamma_{ij} \tag{267}$$

$$g_{ij} \simeq \left(e^{\beta/3\delta+\gamma}\right)_{ij}. \tag{268}$$

$g^{\mu\nu}$ is defined above. Now we need to make a definition for $g_{\mu\nu}$. The matrix g^{ij} is defined as the inverse of g_{ij}, so the elements of g^{ij} is given as,

$$g^{ij} \simeq e^{-\beta/3}\left(\delta - \gamma + \frac{1}{2}\gamma^2\right)^{ij}. \tag{269}$$

Using the above definition, we describe the trace, off diagonal part of $g^{\mu\nu}$. The relation of g^{ij} is given as following,

$$g^{11} \simeq e^{-\beta_{11}} + \frac{e^{-2\beta_{11}/3}}{2}\left(\beta_{12}^2 e^{-\beta_{22}/3} + \beta_{13}^2 e^{-\beta_{33}/3}\right) \tag{270}$$

$$g^{22} \simeq e^{-\beta_{22}} + \frac{e^{-2\beta_{22}/3}}{2}\left(\beta_{23}^2 e^{-\beta_{33}/3} + \beta_{21}^2 e^{-\beta_{11}/3}\right) \tag{271}$$

$$g^{33} \simeq e^{-\beta_{33}} + \frac{e^{-2\beta_{33}/3}}{2}\left(\beta_{31}^2 e^{-\beta_{11}/3} + \beta_{32}^2 e^{-\beta_{22}/3}\right) \tag{272}$$

$$g^{12} \simeq -\beta_{12}e^{-(\beta_{11}+\beta_{22})/2} + \frac{1}{2}\beta_{31}\beta_{23}e^{-\beta/3} \tag{273}$$

$$g^{23} \simeq -\beta_{23}e^{-(\beta_{22}+\beta_{33})/2} + \frac{1}{2}\beta_{12}\beta_{31}e^{-\beta/3} \tag{274}$$

$$g^{31} \simeq -\beta_{31}e^{-(\beta_{33}+\beta_{11})/2} + \frac{1}{2}\beta_{23}\beta_{12}e^{-\beta/3}. \tag{275}$$

The Einstein-Hilbert action R_E gives rise to the kinetic term β by this rewrite. The kinetic term of β is given as following,

$$K_\beta = \int d^3x \frac{e^{\beta/2}}{4\kappa^2}\left(\dot\beta_{12}^2 + \dot\beta_{23}^2 + \dot\beta_{31}^2 - \dot\beta_{11}\dot\beta_{22} - \dot\beta_{22}\dot\beta_{33} - \dot\beta_{33}\dot\beta_{11}\right). \tag{276}$$

In addition to the kinetic term, the effect of the gradient can be obtained as a potential for β. The potential corresponding to the gradient of β can be obtained as follows

$$\begin{aligned} H_{\nabla\beta} = \int d^3x \frac{1}{4\kappa^2}(&\beta_{xy,z}^2 + \beta_{yz,x}^2 + \beta_{zx,y}^2 \\ &- 2\beta_{xy,z}\beta_{yz,x} - 2\beta_{yz,x}\beta_{zx,y} - 2\beta_{zx,y}\beta_{xy,z} \\ &- 2\beta_{xx,z}\beta_{yy,z} - 2\beta_{yy,z}\beta_{zz,x} - 2\beta_{zz,y}\beta_{xx,y} \\ &+ 2\beta_{xy,x}\beta_{zz,y} - 2\beta_{yz,y}\beta_{xx,z} - 2\beta_{zx,z}\beta_{yy,x}). \end{aligned} \tag{277}$$

The behavior of the background space-time is described by the kinetic term and potential of β.

In order to proceed with the discussion in the Symplectic integral method, it is necessary to rewrite the picture in Hamiltonian picture instead of Lagrangian picture. We rewrite the Hamiltonian picture, the conjugate momentum Π_β is introduced and the Hamiltonian is given as,

$$\Pi_{\beta_{ii}} = \frac{\delta K_\beta}{\delta \beta_{ii}(x)} = \frac{e^{\beta/2}}{4\kappa^2}(\dot{\beta}_{ii} - \dot{\beta}), \qquad \Pi_{\beta_{ij}} = \frac{\delta K_\beta}{\delta_{\beta_{ij}(x)}} = \frac{e^{\beta/2}}{2\kappa^2}\dot{\beta}_{ij}. \tag{278}$$

In the above equation, the trace and off-diagonal parts of Pi_β are defined separately, but both can be written together. This formula can be simplified as

$$\Pi_{\beta_{ij}} = \frac{1}{4\kappa^2} e^{\beta/2}(2 - \delta_{ij})(\dot{\beta}_{ij} - \dot{\beta}\delta_{ij}). \tag{279}$$

By using Π_β defined here, we can rewrite the kinetic term of β in Hamiltonian picture. The kinetic part's Hamiltonian is defined as following,

$$H_{\beta_{ii}} = \int d^3x e^{-\beta/2}\kappa^2 \left(2\sum_{i=1,2,3} \Pi^2_{\beta_{ii}} - \left(\sum_{i=1,2,3} \Pi_{\beta_{ii}} \right)^2 \right), \tag{280}$$

$$H_{\beta_{ij}} = \int d^3x e^{-\beta/2}\kappa^2 \left(\Pi^2_{\beta_{12}} + \Pi^2_{\beta_{23}} + \Pi^2_{\beta_{31}} \right). \tag{281}$$

5.3.2 Apply to symplectic integration

In the previous section, we obtained the Hamiltonian picture of the Einstein-Hilbert part as a function of β. Now we need to rewrite the Hamiltonian picture for the scalaron and the scalar field χ. If we rewrite the Hamiltonian picture, the whole Hamiltonian is given as follows,

$$\begin{aligned} H =& K_\beta + H_{\nabla\beta} + \left[e^{-\beta/2}\frac{\Pi^2_\varphi}{2} \right] + \left[\frac{1}{2} e^{\beta/2} g^{ij} (\nabla\varphi)^2 \right] \\ &+ \left[e^{-\beta/2} e^{2/(\sqrt{6}M_{pl})\varphi} \frac{\Pi^2_\chi}{2} \right] + \left[\frac{1}{2} e^{\beta/2} e^{-2/(\sqrt{6}M_{pl})\varphi} g^{ij} (\nabla\chi)^2 \right] + e^{\beta/2} V(\varphi, \chi), \end{aligned} \tag{282}$$

Note that the whole thing is multiplied by $\sqrt{-g}$, which is e^β. Then the conjugate momentum of φ, χ are defined as following

$$\Pi_\varphi = e^{\beta/2}\dot{\varphi}, \tag{283}$$

$$\Pi_\chi = e^{\beta/2} e^{-2/\sqrt{6}M_{pl}\varphi}\dot{\chi}. \tag{284}$$

As can be seen from Eq.(282), the kinetic term of φ is subject to the function β and the kine term of χ is subject to the function β, φ, so χ, φ is non -canonical. For this reason, applying the symplectic integral method to this model requires ingenuity. In the general symplectic integral method, the Hamiltonian is decomposed into two parts: the kinetic term and the potential. Here, the Hamiltonian is decomposed into three parts. These three parts are defined as follows,

$$e^{\tau K_1}\begin{pmatrix}\phi\\ \chi\\ \beta_{ii}\\ \beta_{ij}\\ \Pi_\phi\\ \Pi_\chi\\ \Pi_{\beta ii}\\ \Pi_{\beta ij}\end{pmatrix}=\begin{pmatrix}\phi\\ \chi+e^{-\beta/2}e^{2/\sqrt{6}\phi}\Pi_\chi\tau\\ \beta_{ii}\\ \beta_{ij}+2\kappa^2e^{-\beta/2}\Pi_{\beta ij}\tau\\ \Pi_\phi-\frac{2\kappa}{\sqrt{6}}K_\chi\tau\\ \Pi_\chi\\ \Pi_{\beta ii}+\frac{K_1}{2}\tau\\ \Pi_{\beta ij}\end{pmatrix}, \tag{285}$$

$$e^{\tau K_2}\begin{pmatrix}\phi\\ \chi\\ \beta_{ii}\\ \beta_{ij}\\ \Pi_\phi\\ \Pi_\chi\\ \Pi_{\beta ii}\\ \Pi_{\beta ij}\end{pmatrix}=\begin{pmatrix}\phi+e^{-\beta/2}\Pi_\phi\tau\\ \chi\\ \beta_{ii}+\frac{2e^{-\beta/2}}{M_p^2}\left(\Pi_{\beta ii}-\Pi_{\beta jj}-\Pi_{\beta kk}\right)\\ \beta_{ij}\\ \Pi_\phi\\ \Pi_\chi\\ \Pi_{\beta ii}+\frac{1}{2}e^{-\beta/2}\left(\frac{1}{M_{pl}^2}\left(2\sum\Pi_{\beta ii}^2-\left(\sum\Pi_{\beta ii}\right)^2\right)+\frac{\Pi_\phi^2}{2}\right)\tau\\ \Pi_{\beta ij}\end{pmatrix}, \tag{286}$$

$$e^{\tau V}\begin{pmatrix}\phi\\ \beta_{ii}\\ \beta_{ij}\\ \Pi_\phi\\ \Pi_{\beta ii}\\ \Pi_{\beta ij}\end{pmatrix}=\begin{pmatrix}\phi\\ \beta_{ii}\\ \beta_{ij}\\ \Pi_\phi-\frac{\partial V}{\partial\phi}\tau\\ \Pi_{\beta ii}-\frac{V}{2}\tau\\ \Pi_{\beta ij}-\frac{\partial V}{\partial\beta ij}\tau\end{pmatrix}. \tag{287}$$

Indeed we can solve Eq(285) and Eq(287) exactly because they only transform $\chi, \beta_{ij}, \Pi_\phi, \Pi_{\beta_{ii}}$ and right side only depends $\beta_{ii}, \phi, \Pi_\chi, \Pi_{\beta_{ij}}$. Therefore, if the time evolution takes place only in K_1, V, the update is guaranteed to have symplectic symmetry. Therefore, we will pseudo-reproduce the symplectic integral method by replacing only the transition by K_2 with an integral method such as Runge-Kutta method. This method is the one used in [37]. Here, the operator of the time transition will adopt the following as the sixth order of time,

$$\begin{aligned}e^{(\mathbf{A}+\mathbf{B}+\mathbf{C})dt}=&e^{c_3\mathbf{A}dt/2}e^{c_3\mathbf{B}dt/2}e^{c_3\mathbf{C}dt}e^{c_3\mathbf{B}dt/2}e^{(c_3+c_2)\mathbf{A}dt/2}\\ &\times e^{c_2\mathbf{B}dt/2}e^{c_2\mathbf{C}dt}e^{c_2\mathbf{B}dt/2}e^{(c_2+c_1)\mathbf{A}dt/2}\\ &\times e^{c_1\mathbf{B}dt/2}e^{c_1\mathbf{C}dt}e^{c_1\mathbf{B}dt/2}e^{(c_1+c_0)\mathbf{A}dt/2}\\ &\times e^{c_0\mathbf{B}dt/2}e^{c_0\mathbf{C}dt}e^{c_0\mathbf{B}dt/2}e^{(c_0+c_1)\mathbf{A}dt/2}\\ &\times e^{c_1\mathbf{B}dt/2}e^{c_1\mathbf{C}dt}e^{c_1\mathbf{B}dt/2}e^{(c_1+c_2)\mathbf{A}dt/2}\\ &\times e^{c_2\mathbf{B}dt/2}e^{c_2\mathbf{C}dt}e^{c_2\mathbf{B}dt/2}e^{(c_2+c_3)\mathbf{A}dt/2}\\ &\times e^{c_3\mathbf{B}dt/2}e^{c_3\mathbf{C}dt}e^{c_3\mathbf{B}dt/2}e^{c_3\mathbf{A}dt/2}\\ &+O(dt^7),\end{aligned} \tag{288}$$

and we substitute $\mathbf{A}=K_1, \mathbf{B}=K_2, \mathbf{C}=V$. With the time step thus defined, we calculate the time evolution of the scalaron and χ in the preheating period.

First of all, for the sake of simplicity, we will limit our discussion to the case where the space is zero-dimensional and has no expanse. In this case, the effect of the gradient of χ does not appear in the equation of motion. Therefore, it is assumed that the parametric resonance like effect discussed in chapter 3 is not caused by the wavenumber vector k, but by the value of the mass m_χ of the scalar field χ. The time evolution of Φ, χ and the conjugate momenta Π_ϕ, Π_χ are plotted below, where the result in Fig. (7) corresponds to the case where $\xi = 0, m_\chi = 0.1$. In this case, the amplitude of the time evolution of scalaron and χ decreases with time due to the expansion of the universe. With this combination of parameters, parametric resonance is not observed. The calculation result in Fig.(8) corresponds to the case where $\xi = 0, m_\chi = 0.5$. The

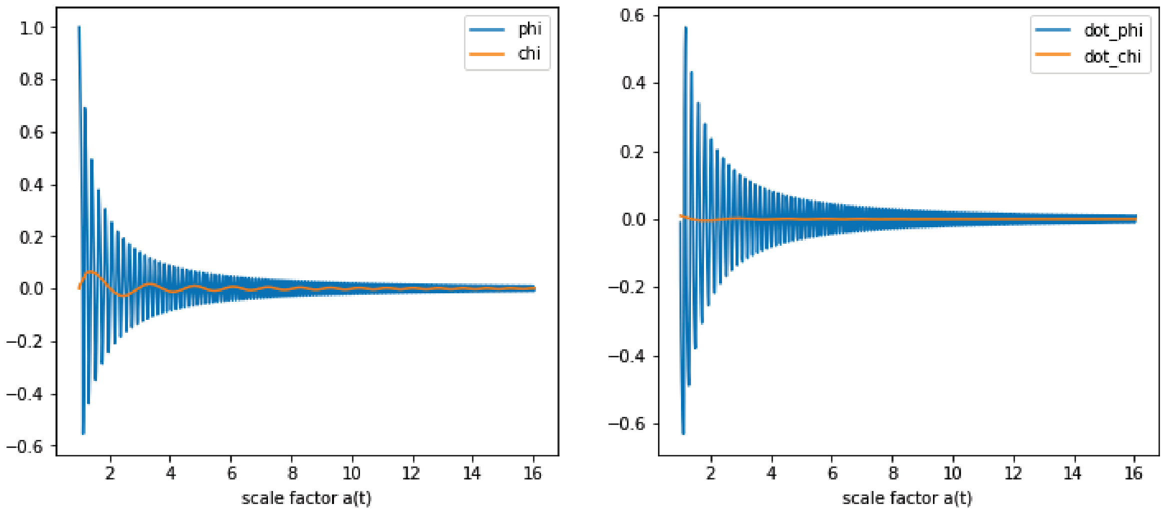

Figure 7: Graph of the time evolution of the scalar field for $\xi = 0, m_\chi = 0.1$. The horizontal axis is the scale factor. The left figure shows the time evolution of scalarons and scalar field χ. The right figure shows the momentum-conjugate quantities.

results of Fig. (8) correspond to the case where $\xi = 0, m_\chi = 0.5$, where the amplitude of both scalaron and χ increases due to parametric resonanace. Since the number density of particles in the background scalar field increases exponentially, the mass of scalarons is expected to change significantly through the chameleon mechanism.

Fig.(9) plots the time evolution of the effective masses of scalaron and χ for the case $\xi = 0, m_\chi = 0.5$. As can be seen from this result, the effective mass of scalaron converges to a certain value as time evolves. If the effect of the chameleon mechanism comes into play, the effective mass of the scalaron should increase with the increase in the number of χ particles. In the results of this numerical calculation, the effect of the chameleon mechanism is not seen because the scalaron mass converges to a certain value.

6 Summary

In this paper, we studied $F(R)$ gravity, which is one of the modified gravity theories. As mentioned in chapter 4, $F(R)$ gravity is a model that can explain the slow roll inflation. The $F(R)$

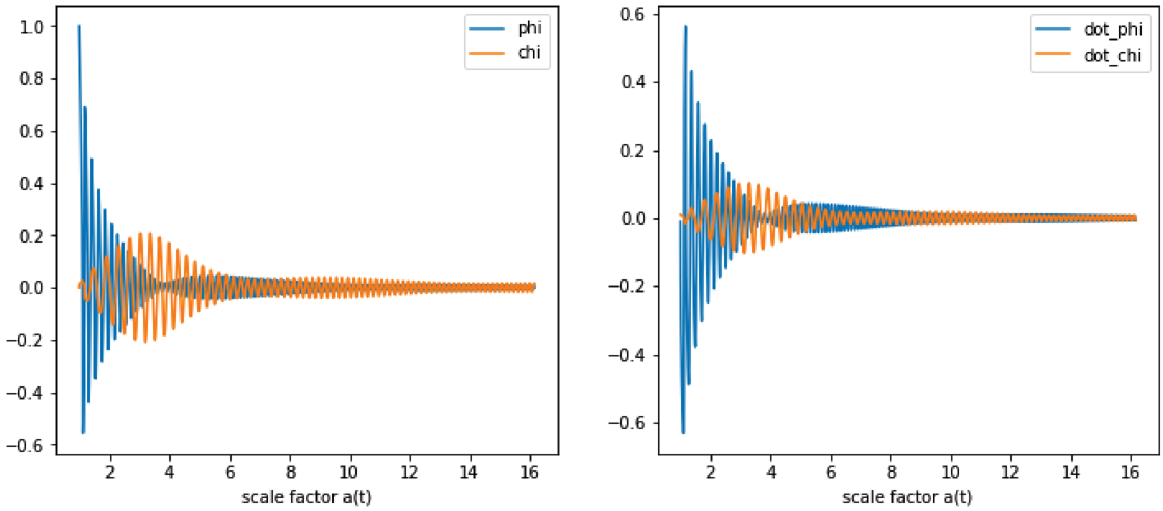

Figure 8: Graph of the time evolution of the scalar field for $\xi = 0, m_\chi = 0.5$. The horizontal axis is the scale factor. The left figure shows the time evolution of scalarons and scalar field χ. The right figure shows the momentum-conjugate quantities.

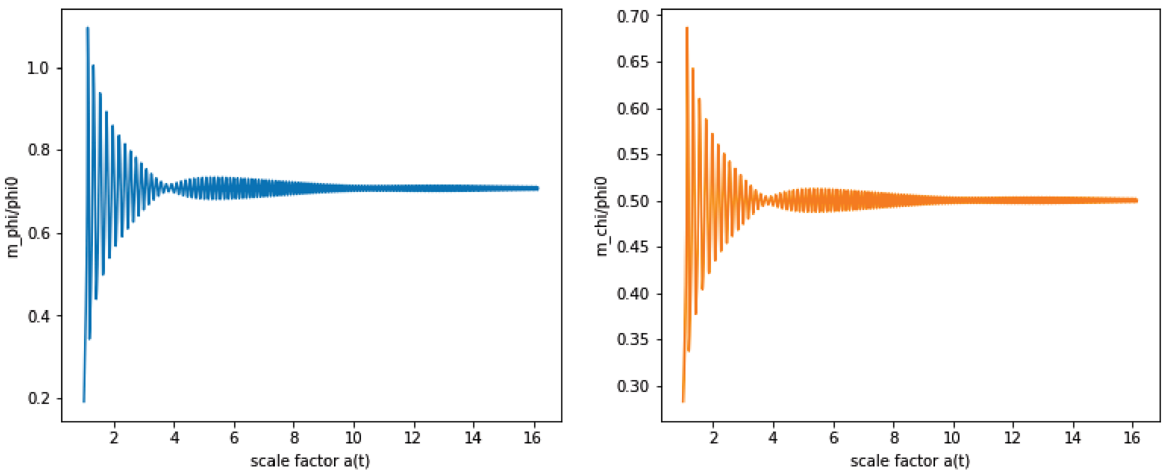

Figure 9: Plot of the time evolution of the effective masses of scalarons and scalar fields χ when the parameter values are taken to be $\xi = 0, m_\chi = 0.1$.

gravity can be rewritten in the form of Einstein-Hilbert action and scalar particles by performing Wyle transformation on the metric tensor $g^{\mu\nu}$. The scalar particle with the new degree of freedom introduced here is called scalaron. In the application of $F(R)$ gravity to inflation, slow-roll inflation can be explained by identifying this scalaron with inflation. In particular, as we saw in chapter 4, the R^2-Starobinsky inflation can explain the slow-roll inflation in a way that fits the current cosmological observations. In this paper, we investigate the application of $F(R)$ gravity to the accelerated expansion of the current universe and the dark matter problem, in addition to inflation.

The scalaron has a chameleon mechanism in which its mass changes depending on the background matter field. In chapter 4.4, we specifically look at the possibility that scalarons are candidates for dark matter. In chapter 4.4, we will discuss in particular whether scalarons are candidates for dark matter. In the Starobinsky model, which includes the dark energy region, the mass of scalarons in a matter field increases with the power of the energy of the background matter field. The condition for a dark matter candidate is that no two-photon decay from scalarons is observed. In this case, when the background is solar scale ($O(10^{-17})\text{GeV}^4$), the scalaron mass is $m_\varphi < O(1)\text{GeV}$. In chapter 4.5, we analyze the logarithmic $F(R)$ model that we devised. In chapter 4.5, we analyzed a logarithmic model of $F(R)$, which is characterized not only by the slow-roll inflation due to the R^2 term, but also by the dark matter behavior due to the lobarithmic Λ_{DE} term corresponding to the cosmological constant. In this chapter, we have shown that the logarithmic F(R) model can be a candidate for dark matter by choosing the model parameters.

In chapter 5, the behavior of scalarons in the preheating phase was calculated numerically by applying symplectic numerical integral. The verification of the $F(R)$ gravity on the ground is difficult due to the chameleon mechanism. Therefore, we focused on the early universe as a possibility to verify the $F(R)$ gravity. The early universe immediately after inflation is divided into a preheating period in which inflaton fluctuations behave non-perturbatively, and a reheating period in which they perturbatively decay into scalar particles. In the preheating phase, the inflaton fluctuation resonates with the motion of the scalar particle, and the inflaton fluctuation exponentially decays into the scalar particle. When we consider the verification of $F(R)$ gravity, the background matter field increases exponentially in the preheating phase, and the mass of the scalar changes significantly due to the chameleon mechanism, which is expected to have an effect on the final reheating temperature.

Since $F(R)$ gravity does not have symplectic symmetry, it is not possible to apply the symplectic integral method as it is. Therefore, the symplectic integral method cannot be applied as it is. In this study, the calculation was performed assuming a zero dimension with no spatial extent. Since the scalar field does not have a wavenumber vector, the mass of the scalar field m_χ and the non-minimal coupling ξ are the parameters that work for parametric resonance. We have confirmed that parametric resonance is induced when $\xi = 0, m_\chi = 0.5$. However, even in this case, no contribution of the chameleon mechanism was found. For a more realistic model, calculations should include the effect of the gradient of the scalar field for the three-dimensional case.

Acknowledgement The authors would like to thank H. Sakamoto and H. Shimoji for valuable discussions. I would also like to take this opportunity to express my gratitude to my supervisor, T. Inagaki, for all his help over the years.

References

[1] D. Langlois, Lect. Notes Phys. **800**, 1-57 (2010) doi:10.1007/978-3-642-10598-2_1 [arXiv:1001.5259 [astro-ph.CO]].

[2] A. Linde, doi:10.1093/acprof:oso/9780198728856.003.0006 [arXiv:1402.0526 [hep-th]].

[3] T. Clifton and J. D. Barrow, Phys. Rev. D **72**, no.10, 103005 (2005) [erratum: Phys. Rev. D **90**, no.2, 029902 (2014)] doi:10.1103/PhysRevD.72.103005 [arXiv:gr-qc/0509059 [gr-qc]].

[4] S. Nojiri and S. D. Odintsov, eConf **C0602061**, 06 (2006) doi:10.1142/S0219887807001928 [arXiv:hep-th/0601213 [hep-th]].

[5] S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59-144 (2011) doi:10.1016/j.physrep.2011.04.001 [arXiv:1011.0544 [gr-qc]].

[6] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. **513**, 1-189 (2012) doi:10.1016/j.physrep.2012.01.001 [arXiv:1106.2476 [astro-ph.CO]].

[7] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. **692**, 1-104 (2017) doi:10.1016/j.physrep.2017.06.001 [arXiv:1705.11098 [gr-qc]].

[8] S. Capozziello, R. D'Agostino and O. Luongo, Int. J. Mod. Phys. D **28**, no.10, 1930016 (2019) doi:10.1142/S0218271819300167 [arXiv:1904.01427 [gr-qc]].

[9] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. **95**, 71-78 (1996) doi:10.1143/PTP.95.71 [arXiv:astro-ph/9507001 [astro-ph]].

[10] C. M. Peterson and M. Tegmark, Phys. Rev. D **87**, no.10, 103507 (2013) doi:10.1103/PhysRevD.87.103507 [arXiv:1111.0927 [astro-ph.CO]].

[11] X. Gao, T. Li and P. Shukla, JCAP **10**, 008 (2014) doi:10.1088/1475-7516/2014/10/008 [arXiv:1403.0654 [hep-th]].

[12] J. Elliston, D. Seery and R. Tavakol, JCAP **11**, 060 (2012) doi:10.1088/1475-7516/2012/11/060 [arXiv:1208.6011 [astro-ph.CO]].

[13] D. J. Brooker, S. D. Odintsov and R. P. Woodard, Nucl. Phys. B **911**, 318-337 (2016) doi:10.1016/j.nuclphysb.2016.08.010 [arXiv:1606.05879 [gr-qc]].

[14] J. O. Gong and T. Tanaka, JCAP **03**, 015 (2011) [erratum: JCAP **02**, E01 (2012)] doi:10.1088/1475-7516/2012/02/E01 [arXiv:1101.4809 [astro-ph.CO]].

[15] J. O. Gong, JCAP **05**, 041 (2015) doi:10.1088/1475-7516/2015/05/041 [arXiv:1409.8151 [astro-ph.CO]].

[16] L. Sebastiani, G. Cognola, R. Myrzakulov, S. D. Odintsov and S. Zerbini, Phys. Rev. D **89**, no.2, 023518 (2014) doi:10.1103/PhysRevD.89.023518 [arXiv:1311.0744 [gr-qc]].

[17] H. A. Buchdahl, Mon. Not. Roy. Astron. Soc. **150**, 1 (1970).

[18] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980) [Phys. Lett. **91B**, 99 (1980)] [Adv. Ser. Astrophys. Cosmol. **3**, 130 (1987)]. doi:10.1016/0370-2693(80)90670-X

[19] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Class. Quant. Grav. **34**, no. 24, 245012 (2017) doi:10.1088/1361-6382/aa92a4 [arXiv:1704.05945 [gr-qc]].

[20] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **96**, no. 2, 024029 (2017) doi:10.1103/PhysRevD.96.024029 [arXiv:1704.02931 [gr-qc]].

[21] S. D. Odintsov, V. K. Oikonomou and L. Sebastiani, Nucl. Phys. B **923**, 608 (2017) doi:10.1016/j.nuclphysb.2017.08.018 [arXiv:1708.08346 [gr-qc]].

[22] S. D. Odintsov, D. Saez-Chillon Gomez and G. S. Sharov, Eur. Phys. J. C **77**, no. 12, 862 (2017) doi:10.1140/epjc/s10052-017-5419-z [arXiv:1709.06800 [gr-qc]].

[23] H. Motohashi and A. A. Starobinsky, Eur. Phys. J. C **77**, no. 8, 538 (2017) doi:10.1140/epjc/s10052-017-5109-x [arXiv:1704.08188 [astro-ph.CO]].

[24] T. Katsuragawa and S. Matsuzaki, Phys. Rev. D **95**, no. 4, 044040 (2017) doi:10.1103/PhysRevD.95.044040 [arXiv:1610.01016 [gr-qc]].

[25] T. Katsuragawa and S. Matsuzaki, Phys. Rev. D **97**, no. 6, 064037 (2018) Erratum: [Phys. Rev. D **97**, no. 12, 129902 (2018)] doi:10.1103/PhysRevD.97.129902, 10.1103/PhysRevD.97.064037 [arXiv:1708.08702 [gr-qc]].

[26] S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59 (2011) doi:10.1016/j.physrep.2011.04.001 [arXiv:1011.0544 [gr-qc]].

[27] S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. **692**, 1 (2017) doi:10.1016/j.physrep.2017.06.001 [arXiv:1705.11098 [gr-qc]].

[28] N. Aghanim *et al.* [Planck Collaboration], arXiv:1807.06209 [astro-ph.CO].

[29] J. Khoury and A. Weltman, Phys. Rev. D **69**, 044026 (2004) doi:10.1103/PhysRevD.69.044026 [astro-ph/0309411].

[30] P. Brax, C. van de Bruck, A. C. Davis and D. J. Shaw, Phys. Rev. D **78**, 104021 (2008) doi:10.1103/PhysRevD.78.104021 [arXiv:0806.3415 [astro-ph]].

[31] J. A. R. Cembranos, Phys. Rev. Lett. **102**, 141301 (2009) doi:10.1103/PhysRevLett.102.141301 [arXiv:0809.1653 [hep-ph]].

[32] Corless, R.M., Gonnet, G.H., Hare, D.E.G. et al. Adv Comput Math (1996) 5: 329. https://doi.org/10.1007/BF02124750

[33] A. A. Starobinsky, JETP Lett. **86**, 157 (2007) doi:10.1134/S0021364007150027 [arXiv:0706.2041 [astro-ph]].

[34] T. Katsuragawa, S. Matsuzaki and K. Homma, [arXiv:2107.00478 [gr-qc]].

[35] T. Inagaki, Y. Matsuo and H. Sakamoto, Int. J. Mod. Phys. D **28** (2019) no.12, 1950157 doi:10.1142/S0218271819501578 [arXiv:1905.05503 [gr-qc]].

[36] T. Inagaki, Y. Matsuo and H. Shimoji, Symmetry **11** (2019) no.4, 451 doi:10.3390/sym11040451 [arXiv:1903.04244 [hep-th]].

[37] Z. Huang, (2011), https://www.cita.utoronto.ca/ zqhuang/hlat/lattice.pdf

[38] C. Fu, P. Wu and H. Yu, Phys. Rev. D **99** (2019) no.12, 123526 doi:10.1103/PhysRevD.99.123526 [arXiv:1906.00557 [astro-ph.CO]].

[39] A. D. Linde, Phys. Lett. B **108** (1982), 389-393 doi:10.1016/0370-2693(82)91219-9

[40] T. Inagaki and H. Sakamoto, Int. J. Mod. Phys. D **29**, no.02, 2050012 (2020) doi:10.1142/S0218271820500121 [arXiv:1909.07638 [gr-qc]].

[41] H. Motohashi, Phys. Rev. D **91**, 064016 (2015) doi:10.1103/PhysRevD.91.064016 [arXiv:1411.2972 [astro-ph.CO]].

[42] L. H. Liu, doi:10.1007/s10773-018-3809-0 [arXiv:1807.00666 [gr-qc]].

[43] I. Ben-Dayan, S. Jing, M. Torabian, A. Westphal and L. Zarate, JCAP **09**, 005 (2014) doi:10.1088/1475-7516/2014/09/005 [arXiv:1404.7349 [hep-th]].

[44] S. D. Odintsov, V. K. Oikonomou and L. Sebastiani, Nucl. Phys. B **923**, 608-632 (2017) doi:10.1016/j.nuclphysb.2017.08.018 [arXiv:1708.08346 [gr-qc]].

[45] R. Myrzakulov, S. Odintsov and L. Sebastiani, Phys. Rev. D **91**, no.8, 083529 (2015) doi:10.1103/PhysRevD.91.083529 [arXiv:1412.1073 [gr-qc]].

[46] T. Katsuragawa and S. Matsuzaki, Phys. Rev. D **95**, no.4, 044040 (2017) doi:10.1103/PhysRevD.95.044040 [arXiv:1610.01016 [gr-qc]].

[47] T. Katsuragawa and S. Matsuzaki, Phys. Rev. D **97**, no.6, 064037 (2018) [erratum: Phys. Rev. D **97**, no.12, 129902 (2018)] doi:10.1103/PhysRevD.97.064037 [arXiv:1708.08702 [gr-qc]].

[48] J. Mathew, J. P. Johnson and S. Shankaranarayanan, Gen. Rel. Grav. **50**, no.7, 90 (2018) doi:10.1007/s10714-018-2410-4 [arXiv:1705.07945 [gr-qc]].

[49] J. P. Johnson, J. Mathew and S. Shankaranarayanan, Gen. Rel. Grav. **51**, no.3, 45 (2019) doi:10.1007/s10714-019-2531-4 [arXiv:1706.10150 [gr-qc]].

[50] D. D. Canko, I. D. Gialamas and G. P. Kodaxis, Eur. Phys. J. C **80**, no.5, 458 (2020) doi:10.1140/epjc/s10052-020-8025-4 [arXiv:1901.06296 [hep-th]].