広島大学学位請求論文

On Classification of Irreducible Quandle Modules over a Connected Quandle

(連結カンドル上の既約カンドル加群の分類について)

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目次

1. 主論文

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主論文

ON CLASSIFICATION OF IRREDUCIBLE QUANDLE MODULES OVER A CONNECTED QUANDLE

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Abstract. We study modules over quandles and classify irreducible quandle modules. The main result of this paper states that there is a correspondence between irreducible modules over a quandle Q and irreducible modules over certain groups: more specifically, irreducible modules over the fundamental group of Q and nontrivial irreducible modules over the associated group $\mathrm{As}(Q)$. As an application, we classify irreducible modules over generalized dihedral quandles, the quandles obtained from generalized dihedral groups, and connected quandles in $SL_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ denotes the finite field of $q = p^f$ elements.

Introduction

A quandle is an algebraic system given by an operation $\triangleright$ which generalizes the conjugation operation of groups, and quandles play an important role in knot theory. The notion of a quandle was first introduced by Joyce and Matveev in 1980s ([Joy], [Mat], see Definition 1.1). Just as in the cases of other algebraic objects such as groups and rings, it is expected that the quandle modules are important in studying quandles. The notion of a general quandle module was given by Andruskiewitsch and Graña [AG] and Jackson [Jac] (Definition 1.12). As an example of application of modules, homology of quandle modules is defined and some important homological invariants of quandle modules are found.

As suggested above, every group can be regarded as quandles by the conjugation operation. For a group G and $g, h \in G$, the operation $g \triangleright h = ghg^{-1}$ defines a quandle denoted by $\mathrm{Conj}(G)$, which is called the conjugation quandle of G. In the converse direction, a quandle Q naturally induces a group $\mathrm{As}(Q)$ called the associated group (Definition 1.4). These assignments give rise to functors $\mathrm{Conj} : \mathbf{Grp} \to \mathbf{Qd}$ and $\mathrm{As} : \mathbf{Qd} \to \mathbf{Grp}$ where $\mathbf{Grp}$ and $\mathbf{Qd}$ denote the categories of groups and quandles respectively, and these functors are adjoint to each other. A module over $\mathrm{As}(Q)$ naturally defines a module over Q. Such a module will be called a module induced from an $\mathrm{As}(Q)$-module. However, there also exist quandle modules which are not induced from $\mathrm{As}(Q)$-modules. This makes the classification of quandle modules more interesting.

In this paper, we study the problem of classifying irreducible modules over connected quandles. For a quandle Q, there is another group $\mathrm{Inn}(Q)$ called the inner automorphism group which is generated by left multiplication actions on Q. A quandle Q is said to be connected if the action of $\mathrm{Inn}(Q)$ on Q is transitive. Given a quandle module $\mathcal{M}$, we first look at the inner automorphism group $\mathrm{Inn}(\mathcal{M})$ of $\mathcal{M}$ regarded as a quandle. Then we can construct another quandle module $\mathcal{I}(\mathcal{M})$ over Q from $\mathrm{Inn}(\mathcal{M})$, which is induced from an $\mathrm{As}(Q)$-module, and a homomorphism $i_{\mathcal{M}} : \mathcal{M} \to \mathcal{I}(\mathcal{M})$ of quandle modules over Q. In particular, if $\mathcal{M}$ is an irreducible quandle module, $i_{\mathcal{M}}$ is either injective or zero. The main result of this paper is the followings:

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- An irreducible module $\mathcal{M}$ such that $i_{\mathcal{M}}$ is zero corresponds to an irreducible module over a group $\pi_1(Q, q)$ which is called the fundamental group of Q at $q \in Q$. (Theorem 3.4)
- Otherwise, $\mathcal{M}$ corresponds to an irreducible $\mathrm{As}(Q)$-module in a certain way. (Theorem 3.5)

As applications of the theorems, we classify irreducible modules over two series of finite quandles. The first one is the generalized dihedral quandle, the quandle of reflections in a generalized dihedral group. It can be also regarded as an Alexander quandle on an Abelian group. We classify the irreducible modules over dihedral quandles with coefficients in fields of characteristic 0. The second one is the connected quandle Q in the special linear group $SL_2(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ of $q = p^f$ elements. We classify the irreducible modules over Q with coefficients in certain fields of characteristic $\mathrm{char}(\mathbb{F}_q) = p$, applying Brauer theory on modular representations of finite groups.

This paper is organized as follows. In the first section, we recall the definitions of quandles, associated groups of quandles, and quandle modules and state some related results. The second section states some facts on quandle modules and their inner automorphism groups. We see that the inner automorphism group of a quandle module $\mathcal{M}$ has an Abelian normal subgroup $T(\mathcal{M})$ with an action of $\mathrm{As}(Q)$ in Proposition 2.5. We also define the quandle module $\mathcal{I}(\mathcal{M})$ and the homomorphism $i_{\mathcal{M}}$ for a quandle module $\mathcal{M}$ in Proposition 2.9.

In the third section, we prove the main theorems on irreducible quandle modules over a connected quandle Q. Note that Theorems 3.4 and 3.5 correpond to the cases $i_{\mathcal{M}} = 0$ and $i_{\mathcal{M}} \neq 0$ respectively.

In the fourth section, we explain how to list up irreducible modules over $\mathrm{As}(Q)$ for connected quandles Q. When Q is finite, $\mathrm{As}(Q)$ is written in the form of the semidirect product $N \rtimes \mathbb{Z}$ for some finite group N (Proposition 3.1, Corollary 1.11). We see how an irreducible module over $\mathrm{As}(Q)$ is obtained from an irreducible module over N in Proposition 4.4.

The last section gives explicit descriptions of irreducible modules over generalized dihedral quandles and connected quandles in $SL_2(\mathbb{F}_q)$.

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1. Preliminaries

1.1. **Quandles.** In this section, we explain the definitions and some basic facts on quandles and quandle modules. For recent development in related subjects, see [Nos1] and [Aki].

Definition 1.1. Let Q be a set and $\triangleright : Q \times Q \to Q$ be a binary operator. Then the pair $(Q, \triangleright)$ is called a *quandle* (of left action) if the following properties are satisfied:

(1) (Idempotency) For any $q \in Q$, $q \triangleright q = q$.
(2) (Left invertibility) For any $p \in Q$, the map $s_p : Q \to Q; q \mapsto p \triangleright q$ is a bijection. Denote by $p \triangleright^{-1} q$ the element $s_p^{-1}(q)$.
(3) (Left self-distributivity) For any $p, q, r \in Q$, $p \triangleright (q \triangleright r) = (p \triangleright q) \triangleright (p \triangleright r)$.

Let $(Q, \triangleright)$ and $(Q', \triangleright')$ be quandles. Then a map $f : Q \to Q'$ is called a *homomorphism* of quandles if $f(p \triangleright q) = f(p) \triangleright' f(q)$. We denote the category of quandles by **Qd**.

By (2) and (3) of the definition, the map s_p for $p \in Q$ is an automorphism of the quandle Q.

Definition 1.2. Let Q be a quandle.

(1) The group $\mathrm{Inn}(Q)$ generated by s_p for $p \in Q$ is called the *inner automorphism group* of Q.

(2) An orbit of $q \in Q$ under the action of $\mathrm{Inn}(Q)$ is called a *connected component* of Q. We denote the set of connected components by $\mathcal{C}(Q)$.
(3) A quandle Q is said to be *connected* (or *transitive*) if the action of $\mathrm{Inn}(Q)$ on Q is transitive. It is equivalent to saying that $\sharp\mathcal{C}(Q) = 1$.

Example 1.3. Let G be a group. Defining $g \triangleright h = ghg^{-1}$, G has a quandle structure. This quandle is called the *conjugation quandle* of G and is denoted by $\mathrm{Conj}(G)$. A group homomorphism is also a quandle homomorphism under this operation, hence Conj is a functor from the category **Grp** of the groups to **Qd**. In this case $\mathrm{Inn}(\mathrm{Conj}(G))$ is isomorphic to the inner automorphism group $\mathrm{Inn}(G) = G/Z(G)$ of the group G where $Z(G)$ denotes the center of the group G.

A union Q of some conjugacy classes in G forms a subquandle of $\mathrm{Conj}(G)$. Let H be the subgroup of G generated by elements in Q. Then an inner automorphism of Q as a quandle is regarded as an inner action of some $h \in H$ as a group. Since Q generates H, h acts trivially on Q if and only if $h \in Z(H)$. Therefore $\mathrm{Inn}(Q) \cong H/Z(H) = \mathrm{Inn}(H)$.

Definition 1.4. Let Q be a quandle. Then the group given by the group presentation

$$\mathrm{As}(Q) = \langle g_q \ (q \in Q) \mid g_{p \triangleright q} = g_p g_q g_p^{-1} \ (p, q \in Q) \rangle$$

is called the *associated group* of Q. A quandle homomorphism $f : Q \to Q'$ induces a group homomorphism $\mathrm{As}(f) : \mathrm{As}(Q) \to \mathrm{As}(Q'); g_q \mapsto g_{f(q)}$. Therefore As is a functor from **Qd** to **Grp**.

Proposition 1.5 ([FR, Proposition 2.1])**.** As : **Qd** $\to$ **Grp** is a left adjoint of Conj : **Grp** $\to$ **Qd**.

From the definitions above, we have the following elementary facts.

Proposition & Definition 1.6. Let Q be a quandle.

(1) The map $\pi_Q : \mathrm{As}(Q) \to \mathrm{Inn}(Q); g_q \mapsto s_q$ gives a well-defined surjective group homomorphism and defines an action of $\mathrm{As}(Q)$ on Q. For $x \in \mathrm{As}(Q)$ and $q \in Q$, denote by $x.q$ the action defined above (i.e. $x.q = \pi_Q(x)(q)$).
(2) For $x \in \mathrm{As}(Q)$ and $q \in Q$, $x g_q x^{-1} = g_{x.q}$.
(3) Let $Z(Q)$ be the kernel of π_Q. Then $Z(Q)$ is a central subgroup of $\mathrm{As}(Q)$.
(4) The map $\deg : \mathrm{As}(Q) \to \bigoplus_{c \in \mathcal{C}(Q)} \mathbb{Z}e_c; g_q \mapsto e_{[q]}$ where $[q]$ is the connected component containing q defines a well-defined surjective group homomorphism. Denote by $\mathrm{As}_0(Q)$ the commutator subgroup of $\mathrm{As}(Q)$. Then $\mathrm{As}_0(Q) = \ker(\deg)$. In particular the Abelianization $\mathrm{As}(Q)^{\mathrm{ab}}$ of $\mathrm{As}(Q)$ is isomorphic to $\mathbb{Z}^{\oplus \sharp\mathcal{C}(Q)}$.
(5) Q is connected if and only if $\mathrm{As}_0(Q)$ acts transitively on Q.

Proof. (1)-(4) are proved in [Nos2]. For (5), see [Eis, Remark 2.25]. □

Definition 1.7. Denote by $\mathrm{Inn}_0(Q)$ the image of $\mathrm{As}_0(Q)$ by π_Q and by $Z_0(Q)$ the kernel of $\pi_Q|_{\mathrm{As}_0(Q)} : \mathrm{As}_0(Q) \to \mathrm{Inn}_0(Q)$.

It is clear that $Z_0(Q) = \mathrm{As}_0(Q) \cap Z(Q)$.

To summarize, we have the following short exact sequences of groups:

$$1 \to Z(Q) \to \mathrm{As}(Q) \to \mathrm{Inn}(Q) \to 1, \qquad (*1)$$

$$1 \to Z_0(Q) \to \mathrm{As}_0(Q) \to \mathrm{Inn}_0(Q) \to 1, \qquad (*2)$$

$$1 \to \mathrm{As}_0(Q) \to \mathrm{As}(Q) \to \mathrm{As}(Q)^{\mathrm{ab}} \to 1. \qquad (*3)$$

To calculate $Z_0(Q)$, the following formula for group homologies is useful.

Theorem 1.8 (Five term exact sequence of group homology)**.** Let $1 \to N \to G \to H \to 1$ be an exact sequence of groups and A a G-module. Then there exists a natural exact sequence

$$H_2(G,A) \to H_2(H,A_N) \to H_1(N,A)_H \to H_1(G,A) \to H_1(H,A_N) \to 0.$$

Here

$$A_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A \cong A/\langle (1-g).a \mid a \in A,\ g \in G \rangle$$

where $\mathbb{Z}$ is regarded as a G-module by the trivial action.

Definition 1.9. For a group G and the trivial G-module $\mathbb{Z}$, the group $H_2(G,\mathbb{Z})$ is called the *Schur multiplier* of G and is denoted by $M(G)$.

In [Kar], the definition of $M(G)$ is given by the second cohomology group $H^2(G,\mathbb{C}^\times)$ where G acts on $\mathbb{C}^\times$ trivially. By [Kar, Theorem 2.7.3], if G is finite, $H_2(G,\mathbb{Z})$ is isomorphic to $H^2(G,\mathbb{C}^\times)$.

From five term exact sequence, we have the following result:

Proposition 1.10 ([Nos1, Lemma 3.9])**.** Let Q be a quandle. Then there is a natural surjective homomorphism

$$M(\mathrm{Inn}(Q)) = H_2(\mathrm{Inn}(Q),\mathbb{Z}) \to Z_0(Q).$$

In particular if Q is a finite quandle, $Z_0(Q)$ is a finite Abelian group.

From the exact sequence $(*2)$, we also have the existing result as follows. See also [Eis, Remark 1.13].

Corollary 1.11. If Q is finite, $\mathrm{As}_0(Q)$ is also a finite group.

1.2. **Quandle modules.** Now we recall the notion of quandle modules. We adopt the definition given by Jackson in [Jac]. Later we also refer to the definition given by Andruskiewitsch and Graña in [AG] before Jackson.

Definition 1.12. Let Q be a quandle. For each $p \in Q$, let an Abelian group A_p be given. Let $\mathcal{A} = \coprod_{p\in Q} A_p$ (the disjoint union as a set). For $p, q \in Q$, let $\eta_{p,q} : A_q \to A_{p\rhd q}$ be an isomorphism of groups and $\tau_{p,q} : A_p \to A_{p\rhd q}$ a homomorphism of groups. Then $\mathcal{A}$ together with $\{\eta_{p,q}\}$, $\{\tau_{p,q}\}$ is called a *quandle module* over Q (or simply a Q*-module*) if the following properties hold:

(1) $\eta_{p,q\rhd r}\eta_{q,r} = \eta_{p\rhd q,p\rhd r}\eta_{p,r}$.
(2) $\eta_{p,q\rhd r}\tau_{q,r} = \tau_{p\rhd q,p\rhd r}\eta_{p,q}$.
(3) $\tau_{p,q\rhd r} = \eta_{p\rhd q,p\rhd r}\tau_{p,r} + \tau_{p\rhd q,p\rhd r}\tau_{p,q}$.
(4) $\eta_{q,q} + \tau_{q,q} = \mathrm{id}_{A_q}$.

An element $a \in A_p \subset \mathcal{A}$ will be denoted by (a,p). The group A_p is called the *fiber* of $\mathcal{A}$ at p.

Note that if Q is connected, the fibers are isomorphic to each other.

Any quandle module $\mathcal{A}$ has a quandle structure.

Proposition 1.13 ([Jac, Proposition 2.1])**.** For $(a,p),(b,q) \in \mathcal{A}$, define $(a,p) \rhd (b,q) = (\eta_{p,q}b + \tau_{p,q}a, p \rhd q)$. Then $\mathcal{A}$ is a quandle.

Remark 1.14. There are some other definitions of quandle modules.

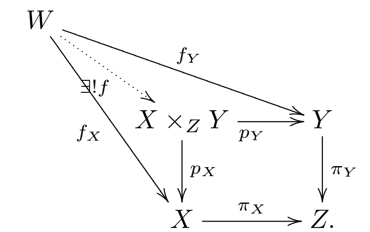
One of the definitions is as "Abelian group objects" in the category of quandles over a given quandle.

Let Q be a quandle and $\pi : \mathcal{A} \to Q, \pi' : \mathcal{A}' \to Q$ be quandle homomorphisms. Then the *fiber product* of $\mathcal{A}$ and $\mathcal{A}'$ over Q is the quandle

$$\mathcal{A} \times_Q \mathcal{A}' = \{(a,a') \in \mathcal{A} \times \mathcal{A}' \text{ (as a set)} \mid \pi(a) = \pi'(a')\}$$

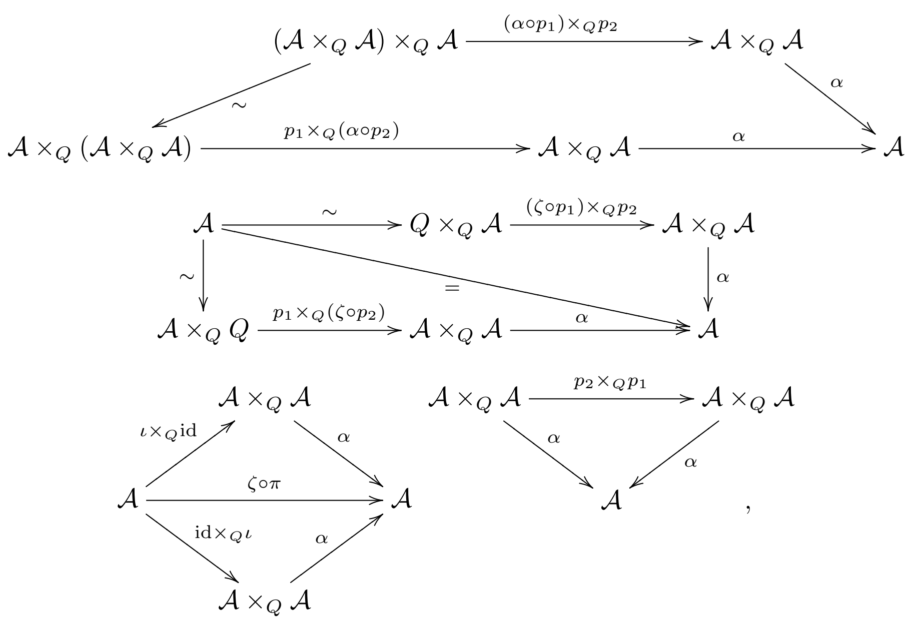
with binary operator $(a, a') \triangleright (b, b') = (a \triangleright b, a' \triangleright b')$.

It is easy to see that the fiber product of quandles is the category-theoretical fiber product in **Qd**. Recall that for a given $f_X : W \to X$ and $f_Y : W \to Y$ with $\pi_X \circ f_X = \pi_Y \circ f_Y$, there exists uniquely $f : W \to X \times_Z Y$ such that the following diagram is commutative:



We denote the map f by $f_X \times_Q f_Y$.

Let Q be a quandle and $\pi : \mathcal{A} \to Q$ a quandle homomorphism. Then $\mathcal{A}$ is called a quandle module over Q if it is endowed with quandle homomorphisms $\alpha : \mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$, $\zeta : Q \to \mathcal{A}$, and $\iota : \mathcal{A} \to \mathcal{A}$ such that the following diagrams are commutative:



where the map $p_i : \mathcal{A} \times_Q \mathcal{A} \to \mathcal{A}$ is the i-th projection. These four diagrams correspond respectively to associativity, existence of identity, existence of inverse, and commutativity. This definition turns out to be equivalent to Definition 1.12 [Jac, Theorem 2.6].

Another definition is as "modules over the algebra associated with the quandle" defined by Andruskiewitsch and Graña [AG]. Let F be the unital associative $\mathbb{Z}$-algebra generated by $\eta_{p,q}, \eta_{p,q}^{-1}, \tau_{p,q}$ for $p, q \in Q$ and I the two-sided ideal generated by the following elements:

(1) $\eta_{p,q \triangleright r} \eta_{q,r} - \eta_{p \triangleright q, p \triangleright r} \eta_{p,r}$,
(2) $\eta_{p,q \triangleright r} \tau_{q,r} - \tau_{p \triangleright q, p \triangleright r} \eta_{p,q}$,
(3) $\tau_{p,q \triangleright r} - \eta_{p \triangleright q, p \triangleright r} \tau_{p,r} - \tau_{p \triangleright q, p \triangleright r} \tau_{p,q}$,
(4) $\eta_{q,q} + \tau_{q,q} - 1$,

(5) $\eta_{p,q}\eta_{p,q}^{-1} - 1$, $\eta_{p,q}^{-1}\eta_{p,q} - 1$.

We define an algebra $\mathbb{Z}(Q) = F/I$. Then a module over Q is defined as a module A over $\mathbb{Z}(Q)$ in [AG]. If Q is connected, a quandle module can be identified with a module over $\mathbb{Z}(Q)$.

In this paper we will mainly be concerned with irreducible modules.

Definition 1.15. A nonzero quandle module $\mathcal{M}$ is said to be *irreducible* (or *simple*) if there is no non-trivial quandle submodule of $\mathcal{M}$.

Definition 1.16. Let $(\mathcal{A}, \eta_{*,*}, \tau_{*,*})$ and $(\mathcal{A}', \eta'_{*,*}, \tau'_{*,*})$ be Q-modules. Then a family of group homomorphisms $\{\phi_q : A_q \to A'_q\}_{q\in Q}$ is called a *homomorphism* of Q-modules if the following diagrams are commutative:

$$\begin{array}{ccc} A_q & \xrightarrow{\phi_q} & A'_q \\ \downarrow{\scriptstyle \eta_{p,q}} & & \downarrow{\scriptstyle \eta'_{p,q}} \\ A_{p\triangleright q} & \xrightarrow{\phi_{p\triangleright q}} & A'_{p\triangleright q}, \end{array} \qquad \begin{array}{ccc} A_p & \xrightarrow{\phi_p} & A'_p \\ \downarrow{\scriptstyle \tau_{p,q}} & & \downarrow{\scriptstyle \tau'_{p,q}} \\ A_{p\triangleright q} & \xrightarrow{\phi_{p\triangleright q}} & A'_{p\triangleright q}. \end{array}$$

The Q-modules $(\mathcal{A}, \eta_{*,*}, \tau_{*,*})$ and $(\mathcal{A}', \eta'_{*,*}, \tau'_{*,*})$ are said to be *isomorphic* if there exists a homomorphism $\{\phi_q : A_q \to A'_q\}_{q\in Q}$ of Q-module such that each ϕ_q is an isomorphism.

Notation 1.17. Let G be a group. Then denote by $\mathbb{Z}[G]$ the group algebra of G. For a commutative ring R, denote by $R[G]$ the group algebra over R. We sometimes denote them simply by $\mathbb{Z}G$, RG.

The following proposition states that an $\mathrm{As}(Q)$-module induces a Q-module.

Proposition 1.18. Let M be a $\mathbb{Z}[\mathrm{As}(Q)]$-module. Then the disjoint union $\mathcal{M} = \coprod_{q\in Q} M$ is a Q-module by $\eta_{p,q} = g_q, \tau_{p,q} = 1 - g_{p\triangleright q}$. Denote this Q-module by $\mathcal{M}_Q(M)$.

In terms of modules over rings, this corresponds to the pullback by the ring homomorphism $\mathbb{Z}(Q) \to \mathbb{Z}[\mathrm{As}(Q)]; \eta_{p,q} \mapsto g_q, \tau_{p,q} \mapsto 1 - g_{p\triangleright q}$. A homomorphism $f : M \to N$ of $\mathbb{Z}(Q)$-modules naturally induces a homomorphism $\mathcal{M}_Q(M) \to \mathcal{M}_Q(N); M_q \ni m \mapsto f(m) \in N_q$ where M_q, N_q are the fibers at q.

Definition 1.19. A Q-module $\mathcal{M}$ is said to be *induced from an* $\mathrm{As}(Q)$*-module* if $\mathcal{M}$ is isomorphic to $\mathcal{M}_Q(M)$ for some $\mathbb{Z}[\mathrm{As}(Q)]$-module M.

Now we extend the definitions above to modules over a commutative ring R. If the ring R is obvious from the context, we omit writing R.

Definition 1.20. Let R be a commutative ring.

(1) A Q-module $\mathcal{M}$ is called an RQ-module if M_q is an R-module for $q \in Q$ and parameters $\eta_{*,*}$ and $\tau_{*,*}$ are R-homomorphisms.
(2) An RQ-homomorphism of RQ-modules is a family of R-homomorphisms $\{\phi_q\}$ such that the diagrams in Definition 1.16 are commutative. Two RQ-modules are said to be R*-isomorphic* if they are isomorphic through R-isomorphisms $\{\phi_q\}_{q\in Q}$ and $\mathcal{M}$ is said to be R*-simple* if $\mathcal{M}$ is simple as an RQ-module.
(3) An RQ-module $\mathcal{M}$ is said to be *induced from an* $\mathrm{As}(Q)$*-module* if $\mathcal{M}$ is R-isomorphic to $\mathcal{M}_Q(M)$ for some $R[\mathrm{As}(Q)]$-module M. If R is a field and Q is connected, the dimension $\dim_R M_q$ for $q \in Q$ is called the *dimension* of the module $\mathcal{M}$. (Note that the dimension is well-defined since fibers are isomorphic.)

2. As(Q)-modules associated to a quandle module

Definition 2.1. For a Q-module $\mathcal{M} = \coprod_{q\in Q} M_q$, let $M(\mathcal{M}) = \bigoplus_{q\in Q} M_q$.

Now we show that $M(\mathcal{M})$ has a structure of As(Q)-module.

Proposition 2.2. Let $\mathcal{M} = \coprod_{q\in Q} M_q$ be a Q-module.

(1) For $p \in Q$, define a homomorphism $f_p : M(\mathcal{M}) \to M(\mathcal{M})$ as follows:
$$f_p(m_q) = \eta_{p,q} m_q \in M_{p\rhd q} \ (m_q \in M_q).$$
Then $\rho : \mathrm{As}(Q) \to \mathrm{Aut}(M(\mathcal{M})); g_p \mapsto f_p$ gives a structure of As(Q)-module on $M(\mathcal{M})$.

(2) If $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic, $M(\mathcal{M}) \cong M(\mathcal{M}')$ as As(Q)-modules.

Proof. The first statement is immediately due to Definition 1.12 (1). If $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic through $\{\phi_q : M_q \to M'_q\}_{q\in Q}$, $M(\mathcal{M}) \cong M(\mathcal{M}')$ through $M(\mathcal{M}) \supset M_q \ni m_q \mapsto \phi_q(m_q) \in M'_q \subset M(\mathcal{M}')$. □

Definition 2.3. The group $\mathrm{Tl}(\mathcal{M}) = \{\phi : \mathcal{M} \to \mathcal{M}; m_q \mapsto m_q + b_q \text{ for some } b_q \in M_q\}$ with composition of maps is called the *group of translations*.

Clearly $\mathrm{Tl}(\mathcal{M})$ is Abelian and is isomorphic to $\prod_{q\in Q} M_q$. It is also an As(Q)-module by
$$g_p.\{b_q\}_q = \{\eta_{p,p\rhd^{-1}q} b_{p\rhd^{-1}q}\}_q.$$
We again denote by $\mathrm{Tl}(\mathcal{M})$ the As(Q)-module defined as above. Note that this extends the As(Q)-action on $M(\mathcal{M})$. In particular if Q is finite, we can identify $\mathrm{Tl}(\mathcal{M})$ with $M(\mathcal{M})$.

Now we look at the structure of the group $\mathrm{Inn}(\mathcal{M})$.

Definition 2.4. For $(a,q) \in \mathcal{M}$, let $t_{a,q} = s_{(a,q)} s_{(0,q)}^{-1}$. Let $T(\mathcal{M})$ be the subgroup of $\mathrm{Inn}(\mathcal{M})$ generated by $t_{a,q}$ for $(a,q) \in \mathcal{M}$. Let $K(\mathcal{M})$ be the subgroup generated by $s_{(0,q)}$ for $q \in Q$.

Proposition 2.5. The followings hold:

(1) $T(\mathcal{M})$ is a subgroup of $\mathrm{Tl}(\mathcal{M})$. The element $t_{a,p}$ is represented by $\{\tau_{p,p\rhd^{-1}q} a\}_{q\in Q}$ as an element in $\mathrm{Tl}(\mathcal{M})$.

(2) $T(\mathcal{M})$ is normal in $\mathrm{Inn}(\mathcal{M})$. The inner action of $K(\mathcal{M})$ on $T(\mathcal{M})$ is compatible with the action of As(Q) on $\mathrm{Tl}(\mathcal{M})$ through $\mathrm{As}(Q) \to K(\mathcal{M}); g_q \mapsto s_{(0,q)}$.

(3) The map $M(\mathcal{M}) \to T(\mathcal{M}); M_q \ni a \mapsto t_{a,q}$ is a homomorphism of As(Q)-modules.

(4) $\mathrm{Inn}(\mathcal{M})$ is isomorphic to the semidirect product $T(\mathcal{M}) \rtimes K(\mathcal{M})$.

Proof. (1) Noting that $(a,p) \rhd^{-1} (b,q) = (\eta^{-1}_{p,p\rhd^{-1}q}(b - \tau_{p,p\rhd^{-1}q} a), p \rhd^{-1} q)$, we have
$$\begin{aligned} t_{a,p}(b,q) &= s_{(a,p)}(\eta^{-1}_{p,p\rhd^{-1}q} b, p \rhd^{-1} q) \\ &= (\eta_{p,p\rhd^{-1}q}\eta^{-1}_{p,p\rhd^{-1}q} b + \tau_{p,p\rhd^{-1}q} a, q) \\ &= (b + \tau_{p,p\rhd^{-1}q} a, q). \end{aligned}$$
Therefore $t_{a,p} = \{\tau_{p,p\rhd^{-1}q} a\}_{q\in Q}$ as an element in $\mathrm{Tl}(\mathcal{M})$.

(2),(3) For $(a,p),(b,q) \in \mathcal{M}$,
$$\begin{aligned} s_{(a,p)} t_{b,q} s_{(a,p)}^{-1} &= s_{(a,p)\rhd(b,q)} s^{-1}_{(a,p)\rhd(0,q)} \\ &= s_{(\eta_{p,q} b + \tau_{p,q} a, p\rhd q)} s^{-1}_{(\tau_{p,q} a, p\rhd q)}. \end{aligned}$$

We write $X = \eta_{p,q}b + \tau_{p,q}a$ and $Y = \tau_{p,q}a$. Now for $(c,r) \in \mathcal{M}$,

$$\begin{aligned} s_{(X,p\rhd q)}s^{-1}_{(Y,p\rhd q)}(c,r) &= s_{(X,p\rhd q)}(\eta^{-1}_{p\rhd q,(p\rhd q)\rhd^{-1}r}(c-\tau_{p\rhd q,(p\rhd q)\rhd^{-1}r}Y),(p\rhd q)\rhd^{-1}r) \\ &= (\eta_{p\rhd q,(p\rhd q)\rhd^{-1}r}\eta^{-1}_{p\rhd q,(p\rhd q)\rhd^{-1}r}(c-\tau_{p\rhd q,(p\rhd q)\rhd^{-1}r}Y)+\tau_{p\rhd q,(p\rhd q)\rhd^{-1}r}X,r) \\ &= (c+\tau_{p\rhd q,(p\rhd q)\rhd^{-1}r}(X-Y),r) \\ &= (c+\tau_{p\rhd q,(p\rhd q)\rhd^{-1}r}\eta_{p,q}(b),r) \\ &= t_{\eta_{p,q}b,p\rhd q}(c,r). \end{aligned}$$

Therefore

$$s_{(a,p)}t_{b,q}s^{-1}_{(a,p)} = t_{\eta_{p,q}b,p\rhd q} \in T(\mathcal{M}). \tag{E}$$

This shows the normality of $T(\mathcal{M})$. The compatibility in (2) holds since for $p' \in Q$ and $t_{a,p} \in T(\mathcal{M}) \subset \mathrm{Tl}(\mathcal{M})$,

$$\begin{aligned} g_{p'}.t_{a,p} &= \{\eta_{p',p'\rhd^{-1}q}\tau_{p,p\rhd^{-1}(p'\rhd^{-1}q)}a\}_{q\in Q} \\ &= \{\tau_{p'\rhd p,(p'\rhd p)\rhd^{-1}q}\eta_{p',p}a\}_{q\in Q} \\ &= t_{\eta_{p',p}a,p'\rhd p}. \end{aligned}$$

Note that the second equality holds from (2) of Definition 1.12, with p, q, r replaced by p', p, $p\rhd^{-1}(p'\rhd^{-1}q)$ respectively. Since we also have $t_{\eta_{p',p}a,p'\rhd p} = s_{(0,p')}t_{a,p}s^{-1}_{(0,p')}$ from (E), (3) holds.

(4) We have to show that $\mathrm{Inn}(\mathcal{M})/T(\mathcal{M})$ is represented by elements in $K(\mathcal{M})$ and that $T(\mathcal{M})\cap K(\mathcal{M}) = 1$. Since $s_{(a,p)} = t_{a,p}s_{(0,p)}$ which generates $\mathrm{Inn}(\mathcal{M})$, $\mathrm{Inn}(\mathcal{M})/T(\mathcal{M})$ is represented by elements in $K(\mathcal{M})$. Let $f \in T(\mathcal{M})\cap K(\mathcal{M})$. The condition $f \in T(\mathcal{M})$ implies that $f(M_q) = M_q$ for all q. Moreover $f \in K(\mathcal{M})$ implies $f((b,q)) = (\eta^{\varepsilon_1}_{p_1,q_1}\cdots\eta^{\varepsilon_r}_{p_r,q_r}b,q)$ for some $p_i, q_i \in Q$ and $\varepsilon_i \in \{\pm 1\}$, which is an additive action on each M_q. Since an additive map is also a translation if and only if it is the identity map, f must be the identity. □

By (1) of the proposition, the composition of maps in $T(\mathcal{M})$ is commutative and corresponds to the pointwize addition in $\mathrm{Tl}(\mathcal{M})$. Thus we write the group operation on $T(\mathcal{M})$ by $+$.

Notation 2.6. For $t \in T(\mathcal{M})$ and $k \in K(\mathcal{M})$, we denote by $(t;k)$ the element tk in $\mathrm{Inn}(\mathcal{M})$. Then the product is given by $(t_1;k_1)(t_2;k_2) = (t_1 + k_1.t_2; k_1k_2)$ where $k.t = ktk^{-1}$ is the conjugation of t by k. Similarly the inverse is given by $(t;k)^{-1} = (-k^{-1}.t; k^{-1})$. We write $k_q = s_{(0,q)} \in K(\mathcal{M})$.

Proposition 2.7. Let $\mathcal{M}$ be a Q-module and $I'_k = \{(t;k) \mid t \in T(\mathcal{M})\}$ for $k \in K(\mathcal{M})$. Then $\mathcal{I}'(\mathcal{M}) := \mathrm{Conj}(\mathrm{Inn}(\mathcal{M})) = \coprod_{k\in K(\mathcal{M})} I'_k$ is a $\mathrm{Conj}(K(\mathcal{M}))$-module. Moreover $\mathcal{I}'(\mathcal{M})$ is a module induced from an $\mathrm{As}(\mathrm{Conj}(K(\mathcal{M})))$-module.

Note that $\mathcal{I}'(\mathcal{M})$ is a module with fiber $T(\mathcal{M})$ through $T(\mathcal{M}) \ni t \mapsto (t;k) \in I'_k$, where each fiber is regarded as an additive group by $(t;k) + (t';k) = (t+t';k)$.

Proof. For $k_1, k_2 \in K(\mathcal{M})$ and $t_1, t_2 \in T(\mathcal{M})$,

$$\begin{aligned} (t_1;k_1)\rhd(t_2;k_2) &= (t_1;k_1)(t_2;k_2)(t_1;k_1)^{-1} \\ &= (t_1;k_1)(t_2;k_2)(-k_1^{-1}.t_1;k_1^{-1}) \\ &= (t_1 + k_1.t_2 - k_1k_2k_1^{-1}.t_1; k_1k_2k_1^{-1}) \\ &= (k_1.t_2 + (1-k_1k_2k_1^{-1}).t_1; k_1\rhd k_2). \end{aligned}$$

Therefore the parameters are given by $\eta_{k_1,k_2} = k_1, \tau_{k_1,k_2} = 1 - k_1k_2k_1^{-1}$. This means that $\mathcal{I}'(\mathcal{M})$ is a $\mathrm{Conj}(K(\mathcal{M}))$-module induced from the $\mathrm{As}(\mathrm{Conj}(K(\mathcal{M})))$-module $T(\mathcal{M})$. □

In general, for a group G and a G-module M, the semidirect product $M \rtimes G$ as a conjugation quandle is a quandle module over $\mathrm{Conj}(G)$ induced from the G-module M through the map $\mathrm{As}(\mathrm{Conj}(G)) \to G$ derived from the adjunction of identity map.

Definition 2.8. Let $f : Q \to Q'$ be a quandle homomorphism and $(\mathcal{M}' = \coprod_{q' \in Q'} M'_{q'}, \eta'_{*,*}, \tau'_{**})$ be a Q'-module. Then $f^*\mathcal{M}' = \coprod_{q \in Q} M'_{f(q)}$ is a Q-module by setting $\eta_{p,q} = \eta'_{f(p),f(q)}$ and $\tau_{p,q} = \tau'_{f(p),f(q)}$. This module is called the *pullback* of $\mathcal{M}'$ by f.

Note that in terms of ring modules, this corresponds to the pullback of modules by the natural ring homomorphism $R(Q) \to R(Q')$.

Proposition 2.9. (1) Let $\mathcal{I}(\mathcal{M})$ be the pullback of $\mathcal{I}'(\mathcal{M})$ by $Q \to \mathrm{Conj}(K(\mathcal{M})); q \mapsto k_q = s_{(0,q)}$. Then $\mathcal{I}(\mathcal{M}) = \coprod_{q \in Q} I_q$, where $I_q = I'_{k_q}$, is isomorphic to $\mathcal{M}_Q(T(\mathcal{M}))$. Denote by (t,q) the element $(t; k_q) \in I'_{k_q} = I_q$. Then $(t,q) \rhd (t',q') = (g_q.t' + (1 - g_{q \rhd q'}).t, q \rhd q')$. In particular, $\mathcal{I}(\mathcal{M})$ is a Q-module induced from the $\mathrm{As}(Q)$-module $T(\mathcal{M})$.

(2) The quandle homomorphism

$$\mathcal{M} \to \mathrm{Conj}(\mathrm{Inn}(\mathcal{M})); M_q \ni (a,q) \mapsto s_{(a,q)} \in I'_{k_q}$$

induces a module homomorphism $i_{\mathcal{M}} : \mathcal{M} \to \mathcal{I}(\mathcal{M})$ over Q. This implies that every quandle module has a homomorphism to a quandle module induced from the $\mathrm{As}(Q)$-module $T(\mathcal{M})$. Note that

$$s_{(a,q)} = t_{a,q} s_{(0,q)} = (t_{a,q}; k_q)$$

and hence the map $i_{\mathcal{M}}$ is defined by $(a,q) \mapsto (t_{a,q}, q)$.

Proof. The first statement follows from the construction. To see (2), let $\eta^I_{*,*}$ and $\tau^I_{*,*}$ denote the parameters of $\mathcal{I}(\mathcal{M})$. Then for $p, q \in Q$ and $a \in M_q$,

$$\begin{aligned} i_{\mathcal{M}}(\eta_{p,q}a, p \rhd q) &= (t_{\eta_{p,q}a, p \rhd q}, p \rhd q) \\ &= (g_p.t_{a,q}, p \rhd q) \\ &= (0,p) \rhd i_{\mathcal{M}}(a,q) \\ &= \eta^I_{p,q} i_{\mathcal{M}}(a,q) \end{aligned}$$

and, noting that the addition on $T(\mathcal{M})$ is the composition in $\mathrm{Aut}(\mathcal{M})$,

$$\begin{aligned} \tau^I_{q,p} i_{\mathcal{M}}(a,q) &= (t_{a,q}, q) \rhd (0,p) \\ &= ((1 - g_{q \rhd p}).t_{a,q}, q \rhd p) \\ &= (t_{a,q} s_{(0,q \rhd p)} t^{-1}_{a,q} s^{-1}_{(0,q \rhd p)}, q \rhd p) \\ &= (s_{(a,q)} s^{-1}_{(0,q)} s_{(0,q \rhd p)} s_{(0,q)} s^{-1}_{(a,q)} s^{-1}_{(0,q \rhd p)}, q \rhd p) \\ &= (s_{(a,q)} s_{(0,p)} s^{-1}_{(a,q)} s^{-1}_{(0,q \rhd p)}, q \rhd p) \\ &= (s_{(\tau_{q,p}a, q \rhd p)} s^{-1}_{(0,q \rhd p)}, q \rhd p) \\ &= (t_{\tau_{q,p}a, q \rhd p}, q \rhd p) \\ &= i_{\mathcal{M}}(\tau_{q,p}a, q \rhd p). \end{aligned}$$

Therefore (2) holds. □

The homomorphism $i_{\mathcal{M}}$ is not necessarily injective. For the extreme case, we make the following definition:

Definition 2.10. A Q-module $\mathcal{M}$ is called a *covering module* if $\tau_{p,q} = 0$ for any $p, q \in Q$.

This is equivalent to saying that $s_{(m,p)} = s_{(0,p)}$ in $\mathrm{Inn}(\mathcal{M})$ for all $m \in M_p$ and that $\mathcal{I}(\mathcal{M})$ is the zero module.

3. Irreducible Modules over Connected Quandles

Throughout this section, let Q be a connected quandle. We reduce the classification of irreducible Q-modules to that of irreducible modules over certain groups. We fix a commutative ring R.

Proposition 3.1. For a connected quandle Q, $\mathrm{As}(Q) \cong \mathrm{As}_0(Q) \rtimes \mathbb{Z}$.

Proof. This is obvious from (4) of Proposition 1.6 (note that an extension of $\mathbb{Z}$ by a group is a semidirect product). $\square$

Let G be a group and H be a subgroup of G. For an RG-module A and an RH-module B, denote by $B \uparrow_H^G = RG \otimes_{RH} B$ the induced module and by $A \downarrow_H^G$ the restricted module. Recall that the induction functor is the left adjoint of the restriction functor.

Definition 3.2. ([Eis, Definition 1.7]) For $q \in Q$, let $\mathrm{As}_q(Q) = \{x \in \mathrm{As}(Q) \mid x.q = q\}$. The group $\pi_1(Q, q) = \mathrm{As}_q(Q) \cap \mathrm{As}_0(Q)$ is called the *fundamental group* of Q at q.

Proposition 3.3. Let Q be a connected quandle and $q \in Q$.

(1) For a Q-module $\mathcal{M} = \coprod_{p \in Q} M_p$, $M(\mathcal{M})$ is isomorphic to the induced $R[\mathrm{As}(Q)]$-module of the $R[\mathrm{As}_q(Q)]$-module M_q through

$$M_q \uparrow_{\mathrm{As}_q(Q)}^{\mathrm{As}(Q)} \ni g \otimes m \mapsto g.m \in M(\mathcal{M}).$$

(Note that this map is the adjunction of the $\mathrm{As}_q(Q)$-homomorphism $M_q \hookrightarrow M(\mathcal{M})$.)

(2) $\mathrm{As}_q(Q) \cong \pi_1(Q, q) \times \langle g_q \rangle$.

Proof. For $p \in Q$, fix $x_p \in \mathrm{As}(Q)$ such that $x_p.q = p$. Then $\{x_p\}_{p \in Q}$ is a complete system of representatives for the set $\mathrm{As}(Q)/\mathrm{As}_q(Q)$ of the left cosets and $x_p.M_q = M_p$ in $M(\mathcal{M})$. This implies that $R[\mathrm{As}(Q)] \otimes_{R[\mathrm{As}_q(Q)]} M_q \cong M(\mathcal{M})$. The second statement holds since $\pi_1(Q, q)$ is normal in $\mathrm{As}_q(Q)$ and g_q centralizes $\pi_1(Q, q)$ by Proposition 1.6 (2). $\square$

Theorem 3.4. Let Q be a connected quandle. Fix $q \in Q$ and $x_p \in \mathrm{As}(Q)$ such that $x_p.q = p$ and let $X = \{x_p\}_{p \in Q}$. Then $x_{p \triangleright p'}^{-1} g_p x_{p'} \in \mathrm{As}_q(Q)$. For an $R[\pi_1(Q, q)]$-module M, we regard M as a module over $\mathrm{As}_q(Q) = \pi_1(Q, q) \times \langle g_q \rangle$ with g_q acting trivially. We write $\mathcal{MC}_{q,X}(M) = \coprod_{p \in Q} M$ and define an operation $\triangleright$ by

$$(m, p) \triangleright (m', p') = ((x_{p \triangleright p'}^{-1} g_p x_{p'}).m', p \triangleright p').$$

Then the followings hold:

(1) $\mathcal{MC}_{q,X}(M)$ is a covering RQ-module. For another representative $Y = \{y_p\}_{p \in Q}$ such that $y_p.q = p$, $\mathcal{MC}_{q,X}(M)$ and $\mathcal{MC}_{q,Y}(M)$ are naturally isomorphic.

(2) The assignments $\mathrm{res}_q; \mathcal{M} \mapsto M_q$ for a covering RQ-module $\mathcal{M}$ and $\mathcal{MC}_{q,X} : M \mapsto \mathcal{MC}_{q,X}(M)$ give a one-to-one correspondence between isomorphism classes of covering RQ-modules and $R[\pi_1(Q, q)]$-modules.

(3) A covering RQ-module $\mathcal{M}$ is irreducible if and only if M_q is irreducible as an $R[\pi_1(Q, q)]$-module.

Proof. First,

$$\begin{aligned}(x_{p\triangleright p'}^{-1}g_px_{p'}).q &= (x_{p\triangleright p'}^{-1}g_p).p' \\ &= x_{p\triangleright p'}^{-1}.(p\triangleright p') \\ &= q\end{aligned}$$

shows that $x_{p\triangleright p'}^{-1}g_px_{p'}\in \mathrm{As}_q(Q)$.

(1) Let M be an $R[\pi_1(Q,q)]$-module and regard it as an $R[\mathrm{As}_q(Q)]$ module. Let $\rho:\mathrm{As}_q(Q)\to \mathrm{Aut}(M)$ be the group homomorphism of the action on M. The above definition of $\triangleright$ corresponds to setting $\eta_{p,p'}=\rho(x_{p\triangleright p'}^{-1}g_px_{p'})$ and $\tau_{p,p'}=0$. Clearly the conditions (2) and (3) of Definition 1.12 hold since $\tau_{*,*}=0$. Since $\eta_{p,p}=\rho(x_p^{-1}g_px_p)=\rho(g_q)$ and g_q acts trivially on M, (4) also holds. For $p,p',p''\in Q$,

$$\begin{aligned}\eta_{p\triangleright p',p\triangleright p''}\eta_{p,p''} &= \rho((x_{p\triangleright(p'\triangleright p'')}^{-1}g_{p\triangleright p'}x_{p\triangleright p''})(x_{p\triangleright p''}^{-1}g_px_{p''})) \\ &= \rho(x_{p\triangleright(p'\triangleright p'')}^{-1}g_{p\triangleright p'}g_px_{p''}) \\ &= \rho(x_{p\triangleright(p'\triangleright p'')}^{-1}g_pg_{p'}x_{p''}) \\ &= \rho(x_{p\triangleright(p'\triangleright p'')}^{-1}g_px_{p'\triangleright p''}x_{p'\triangleright p''}^{-1}g_{p'}x_{p''}) \\ &= \eta_{p,p'\triangleright p''}\eta_{p',p''}.\end{aligned}$$

Therefore (1) holds and we see that $\mathcal{MC}_{q,X}(M)$ is a covering module. Let $Y=\{y_p\}$ be another representative. For $p\in Q$, let $a_p=x_p^{-1}y_p$. It is clear that $a_p\in \mathrm{As}_q(Q)$. Then

$$\begin{aligned}y_{p\triangleright p'}^{-1}g_py_{p'} &= y_{p\triangleright p'}^{-1}x_{p\triangleright p'}x_{p\triangleright p'}^{-1}g_px_{p'}x_{p'}^{-1}y_{p'} \\ &= a_{p\triangleright p'}^{-1}x_{p\triangleright p'}^{-1}g_px_{p'}a_p.\end{aligned}$$

Therefore $\mathcal{MC}_{q,X}(M)$ and $\mathcal{MC}_{q,Y}(M)$ are isomorphic through $\{\rho(a_p)\}_{p\in Q}$.

(2) Let $\mathcal{M}=\coprod_{p\in Q}M_p$ be a covering Q-module. Then let $\varphi:\mathcal{MC}_{q,X}(M_q)\to\mathcal{M};(m,p)\mapsto x_p.(m,q)\in M_p\subset\mathcal{M}$. It is clear that φ is bijective and additive. For $(m,p),(m',p')\in\mathcal{MC}_{q,X}(M_q)$,

$$\begin{aligned}\varphi((m,p))\triangleright\varphi((m',p')) &= (x_p.(m,q))\triangleright(x_{p'}.(m',q)) \\ &= (0,p)\triangleright((x_{p'}).(m',q)) \\ &= (g_px_{p'}).(m',q) \\ &= (x_{p\triangleright p'}x_{p\triangleright p'}^{-1}g_px_{p'}).(m',q) \\ &= x_{p\triangleright p'}.((x_{p\triangleright p'}^{-1}g_px_{p'}).m',q) \\ &= \varphi((m,p)\triangleright(m',p')).\end{aligned}$$

Note that the second equality holds since $\mathcal{M}$ is a covering module. Therefore φ is an isomorphism of Q-modules.

Conversely, let M be an $R[\pi_1(Q,q)]$-module. Regarding M as an $\mathrm{As}_q(Q)$-module, we have $M(\mathcal{MC}_{q,X}(M))\cong M\uparrow_{\mathrm{As}_q(Q)}^{\mathrm{As}(Q)}$. Therefore the fiber $M'=(\mathcal{MC}_{q,X}(M))_q$ is isomorphic to M.

(3) Since these mappings are equivalence of categories between the category of covering RQ-modules and the category of $R[\pi_1(Q,q)]$-module, the assertion holds. □

The word "covering" comes from quandle coverings in [Eis]. A quandle Q' is called a *quandle covering* of Q if there exists a surjective quandle homomorphism $\pi:Q'\to Q$ such that $\pi(p)=\pi(q)$

implies $s_p = s_q \in \mathrm{Inn}(Q')$ for $p, q \in Q'$. Eisermann showed that there is a one-to-one correspondence between the set of connected quandle coverings of Q and the set of subgroups of $\pi_1(Q, q)$.

Next we classify irreducible modules which are not coverings.

Theorem 3.5. Let Q be a connected quandle.

(1) Let $\mathcal{M}$ be an irreducible RQ-module which is not a covering. Then $T(\mathcal{M})$ is an irreducible $R[\mathrm{As}(Q)]$-module with nontrivial action. In particular, $\mathcal{M}$ is a submodule of a module induced from an irreducible $\mathrm{As}(Q)$-module.
(2) Let M be an irreducible $R[\mathrm{As}(Q)]$-module with nontrivial action. For $q \in Q$, let $M_q = (1 - g_q)M$. Then $\mathcal{MQ}(M) = \coprod_{q \in Q} M_q$ is an irreducible RQ-module which is not a covering.
(3) Let $\mathrm{Irr}_{\mathrm{nc}}(RQ)$ denote the set of isomorphism classes of irreducible RQ-modules which are not a covering module and $\mathrm{Irr}_{\mathrm{nt}}(R[\mathrm{As}(Q)])$ be the set of isomorphism classes of non-trivial irreducible $R[\mathrm{As}(Q)]$-modules. Then the assignments $\mathcal{M} \mapsto T(\mathcal{M})$ for an irreducible RQ-module $\mathcal{M}$ which is not a covering, and $M \mapsto \mathcal{MQ}(M)$ for a nontrivial irreducible $R[\mathrm{As}(Q)]$-module, give a one-to-one correspondence between $\mathrm{Irr}_{\mathrm{nc}}(RQ)$ and $\mathrm{Irr}_{\mathrm{nt}}(R[\mathrm{As}(Q)])$ which are inverse to each other.

Note that the dimension of the module is not neccesarily preserved under the correspondence. For an irreducible module $\mathcal{M}$, the dimension is preserved if and only if $\mathcal{M}$ is induced from $\mathrm{As}(Q)$-module.

To show the theorem, we give some lemmas.

Lemma 3.6. Let M be an $R[\mathrm{As}(Q)]$-module and $\{M_q\}_{q \in Q}$ a family of R-submodules of M. Then $\mathcal{M} = \coprod_{q \in Q} M_q$ forms an RQ-submodule of $\mathcal{M}_Q(M)$ if and only if $g_p.M_q = M_{p \triangleright q}$ and $(1 - g_q).M_p \subset M_q$ for any $p, q \in Q$.

Proof. Straightforward from the conditions $\eta_{p,q}(M_q) = M_{p \triangleright q}$ and $\tau_{p,q}(M_p) \subset M_{p \triangleright q}$. The second one is applied with q replaced by $p \triangleright^{-1} q$. □

Lemma 3.7. Let $\mathcal{M} = \coprod_{p \in Q} M_p$ be an RQ-module and T' an $R[\mathrm{As}(Q)]$-submodule of $T(\mathcal{M})$. Then $\mathcal{M}' = \coprod_{q \in Q} \left(M_q \cap i_{\mathcal{M}}^{-1}(T')\right)$ is a submodule of $\mathcal{M}$. (Recall that $\mathcal{I}(\mathcal{M}) = \mathcal{M}_Q(T(\mathcal{M}))$ by Proposition 2.9. Therefore $\mathcal{M}'$ is regarded as the inverse image of $\mathcal{M}_Q(T') \subset \mathcal{M}_Q(T(\mathcal{M}))$ by $i_{\mathcal{M}}$.)

Proof. This is easy to see since the inverse image of a submodule by a module homomorphism is a submodule. □

Definition 3.8. Let $\mathcal{M}$ be an RQ-module. Then a quandle automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ is called a *central translation* if $\varphi \in \mathrm{Tl}(\mathcal{M})$ and φ centralizes $\mathrm{Inn}(\mathcal{M})$ (i.e. for all $\psi \in \mathrm{Inn}(\mathcal{M}), \varphi \circ \psi = \psi \circ \varphi$).

Lemma 3.9. If $\mathcal{M}$ has a non-trivial central translation, $\mathcal{M}$ has a nonzero submodule which is a covering module.

Proof. Let $\varphi((a, q)) = (a + b_q, q)$ be a central translation. Then $\varphi(s_{(0,p)}((a, q))) = s_{(0,p)}(\varphi((a, q)))$. The left hand side equals to $\varphi(\eta_{p,q}a, p \triangleright q) = (\eta_{p,q}a + b_{p \triangleright q}, p \triangleright q)$ and the right hand side equals to $(0, p) \triangleright (a + b_q, q) = (\eta_{p,q}(a + b_q), p \triangleright q) = (\eta_{p,q}a + \eta_{p,q}b_q, p \triangleright q)$. This implies that $b_{p \triangleright q} = \eta_{p,q}b_q$ for all $p, q \in Q$ and $\mathcal{N} = \coprod_{q \in Q} Rb_q$ is closed under $\eta_{p,q}$. On the other hand, $(b_q, q) = \varphi((0, q))$ implies that $s_{(0,q)} = \varphi s_{(0,q)} \varphi^{-1} = s_{(b_q,q)}$, i.e. $\tau_{p,q} \equiv 0$ for all $p, q \in Q$. Therefore $\mathcal{N}$ is a submodule of $\mathcal{M}$ which is a covering module. □

Now we prove Theorem 3.5.

Proof. (1) First note that $T(\mathcal{M}) \neq 0$ since $T(\mathcal{M}) = 0$ implies that $\mathcal{M}$ is a covering module. Given $\mathcal{M}$, we have the module homomorphism $i_{\mathcal{M}} : \mathcal{M} \to \mathcal{I}(\mathcal{M})$. Since $\mathcal{M}$ is irreducible, $i_{\mathcal{M}}$ is injective or zero. Since $i_{\mathcal{M}} = 0$ implies that $\mathcal{M}$ is a covering module, $i_{\mathcal{M}}$ is injective. For $q \in Q$, let T_q be the image of M_q by $i_{\mathcal{M}}$. Then $T_q \cong M_q$ and when regarded as subgroups of $T(\mathcal{M})$, $\sum_{q\in Q} T_q = T(\mathcal{M})$ by definition (see Proposition 2.9 (2)).

Let T' be a proper $R[\mathrm{As}(Q)]$-submodule of $T(\mathcal{M})$. Then $\mathcal{M}' = \coprod_{q\in Q} M_q \cap i_{\mathcal{M}}^{-1}(T')$ is a submodule of $\mathcal{M}$ by Lemma 3.7. Since $\mathcal{M}$ is irreducible, $M_q \cap i_{\mathcal{M}}^{-1}(T') \cong T_q \cap T'$ is either M_q for all $q \in Q$ or zero for all $q \in Q$. Moreover, since $M_q \cong T_q$ generates $T(\mathcal{M})$ as an $R[\mathrm{As}(Q)]$-module, we must have $T_q \cap T' = 0$ for all q.

Now for $q \in Q$, by Lemma 3.6, $(1-g_q).T' \subset T' \cap T_q = 0$, which means that $(1-g_q).t' = 0$ for any $q \in Q$. Therefore T' is an $R[\mathrm{As}(Q)]$-submodule of $T(\mathcal{M})$ with trivial action. Recall that $\mathrm{As}(Q)$ acts on $T(\mathcal{M})$ via conjugation by $K(\mathcal{M})$. This means that an element t' in T' is a central translation. By Lemma 3.9, if $T' \neq 0$, $\mathcal{M}$ has a covering submodule, which contradicts the assumption that $\mathcal{M}$ is a non-covering irreducible RQ-module. Therefore $T' = 0$.

(2) By Lemma 3.6, it is straightforward to show that $\mathcal{MQ}(M)$ is an RQ-submodule of $\mathcal{M}_Q(M)$. Let $\mathcal{M}' = \coprod_{q\in Q} M'_q$ be an RQ-submodule of $\mathcal{MQ}(M)$. Then $M' = \sum_{q\in Q} M'_q$ is an $R[\mathrm{As}(Q)]$-submodule of M. Since M is irreducible, M' is either M or 0. If $M' = M$, by Lemma 3.6, $M'_q \supset \sum_{p\in Q}(1-g_q)M'_p = (1-g_q).M' = (1-g_q).M = M_q$. Therefore $M'_q = M_q$ for all q, and hence $\mathcal{M}' = \mathcal{MQ}(M)$.

Next we show that $\mathcal{MQ}(M)$ is not a covering module. For $(m,q),(n,p) \in \mathcal{MQ}(M)$, $t_{m,q}(n,p) = (n + \tau_{q,q\triangleright^{-1}p}(m), p) = (n + (1-g_p)m, p)$. Therefore $T(\mathcal{MQ}(M))$ is the set of translations $\{(1-g_p)m\}_{p\in Q}$ for $m \in M$. Since M is irreducible and nontrivial, the mapping $m \mapsto \{(1-g_p)m\}_{p\in Q}$ is injective. Moreover, for $q \in Q$,

$$\begin{aligned} \{(1-g_p).(g_q.m)\}_p &= \{(g_q(1-g_{q\triangleright^{-1}p})).m\}_p \\ &= g_q.\{(1-g_p).m\}_p \end{aligned}$$

implies that this mapping is a homomorphism of $\mathrm{As}(Q)$-modules. Therefore $T(\mathcal{MQ}(M)) \cong M \neq 0$. Therefore $\mathcal{MQ}(M)$ is not a covering module.

(3) Let $\mathcal{M}$ be an irreducible RQ-module which is not a covering. Then in the notation of (1) as we saw above, $(1-g_q)T(\mathcal{M}) \subset T_q$ for all $q \in Q$. Therefore $\mathcal{MQ}(T(\mathcal{M}))$ is a submodule of $\coprod_{q\in Q} T_q$. However since $\mathcal{M} \cong \coprod_{q\in Q} T_q$ which is irreducible, they must be equal. This implies that $\mathcal{M} \cong \mathcal{MQ}(T(\mathcal{M}))$.

The converse direction is also true since we have already shown that $T(\mathcal{MQ}(M)) \cong M$ for a nontrivial irreducible $R[\mathrm{As}(Q)]$-module M in proof of (2). □

4. As(Q)-modules

Throughout this section, let Q be a connected quandle. Let F be a field and $\bar{F}$ its algebraic closure. For a group G, denote by $\mathrm{Rep}(G)$ the set of isomorphism classes of finite dimensional $\bar{F}$-representations and by $\mathrm{Irr}(G)$ the set of isomorphism classes of finite dimensional irreducible $\bar{F}$-representations. An extension E of F is called a *decomposition field* of G if any irreducible representation of G is realized over E. Then the arguments in this section are valid over any decomposition field E.

Recall that $\mathrm{As}(Q) \cong \mathrm{As}_0(Q) \rtimes \mathbb{Z}$ by Proposition 3.1.

Definition 4.1. Let N be a group and $\varphi \in \mathrm{Aut}(N)$.

(1) For $(V, \rho : N \to \mathrm{Aut}(V)) \in \mathrm{Rep}(N)$, the representation $(V, \varphi^*\rho)$ defined by $\varphi^*\rho(n) = \rho(\varphi(n))$ is called the *pullback* of ρ. We sometimes denote $(V, \varphi^*\rho)$ by $\varphi^* V$. We write $\varphi^{*n}\rho = (\varphi^*)^n\rho$ and $\varphi^{*(-n)}\rho = ((\varphi^{-1})^*)^n\rho$ for a positive integer n.

(2) Let $(V, \rho),\ (U, \sigma) \in \mathrm{Rep}(N)$. Then a linear map $f : V \to U$ is called a φ-morphism if the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\rho(n)} & V \\ \downarrow f & & \downarrow f \\ U & \xrightarrow{\sigma(\varphi(n))} & U. \end{array}$$

This is equivalent to saying that f is an N-homomorphism from V to $\varphi^* U$.

(3) $(V, \rho),\ (U, \sigma) \in \mathrm{Rep}(N)$ are said to be *φ-equivalent* if there exists an integer n such that $(V, \varphi^{*n}\rho) \cong (U, \sigma)$.

(4) Let $(V, \rho) \in \mathrm{Rep}(N)$. Denote by $\mathrm{ord}_\varphi(\rho)$ (or $\mathrm{ord}_\varphi(V)$) the minimum positive integer n such that (V, ρ) and $(V, \varphi^n\rho)$ are isomorphic if such an n exists, otherwise ∞. This is called the *order* of ρ with respect to φ. Note that if φ is of order n in $\mathrm{Aut}(N)$, $\mathrm{ord}_\varphi(\rho)$ divides n.

Clearly the following properties hold:

Proposition 4.2. Let N be a group and $(V, \rho) \in \mathrm{Rep}(N)$. Then the followings hold:

(1) $H_V = \{\varphi \in \mathrm{Aut}(N) \mid V \cong \varphi^* V\}$ is a subgroup of $\mathrm{Aut}(N)$.

(2) For $\varphi \in \mathrm{Aut}(N)$, let $M_\varphi = \{f : V \to V \mid f \text{ is a } \varphi\text{-morphism}\}$. Then M_φ is a subspace of $\mathrm{End}_{\bar{F}}(V)$ and $M_\varphi M_\psi \subset M_{\varphi\psi}$ for $\varphi, \psi \in \mathrm{Aut}(N)$. The set M_{id} is the endomorphism ring of (ρ, V).

(3) For $\varphi \in H_V$, let $M_\varphi^\times \subset M_\varphi$ be the set of invertible (i.e. bijective) φ-morphisms (note that $M_\varphi^\times \neq \emptyset$ by the definition of H_V). Then for a fixed $f \in M_\varphi^\times$, there is a bijection between respectively M_{id} and M_φ, $M_{\mathrm{id}}^\times$ and $M_\varphi^\times$ through $a \leftrightarrow fa$ for $a \in M_{\mathrm{id}}$.

(4) If V is irreducible, every nonzero φ-morphism for $\varphi \in H_V$ is a linear automorphism. In particular, for $\varphi \in H_V$, $M_\varphi = f\bar{F}$ for some φ-isomorphism f.

Definition 4.3. Let N be a group and $\varphi \in \mathrm{Aut}(N)$. Let $G = N \rtimes \langle f \rangle$ where $\langle f \rangle \cong \mathbb{Z}$ and $fnf^{-1} = \varphi(n)$ for $n \in N$. For $(V, \rho) \in \mathrm{Rep}(N)$ such that $\mathrm{ord}_\varphi(\rho)$ is finite and $\alpha \in M^\times_{\varphi^{\mathrm{ord}_\varphi(\rho)}}$, let

$$V \uparrow^{\varphi,\alpha} = \bigoplus_{i=0}^{\mathrm{ord}_\varphi(V)-1} V_i$$

where $V_i = \varphi^{*i} V$ for each i and

$$\rho \uparrow^{\varphi,\alpha} (n) : V \uparrow^{\varphi,\alpha} \to V \uparrow^{\varphi,\alpha}; V_i \ni v_i \mapsto \varphi^{*i}\rho(n)(v_i) \in V_i,$$

$$\rho \uparrow^{\varphi,\alpha} (f) : V \uparrow^{\varphi,\alpha} \to V \uparrow^{\varphi,\alpha}; V_i \ni v_i \mapsto \left\{ \begin{array}{cc} v_i \in V_{i-1} & (i \neq 0) \\ \alpha(v_0) \in V_{\mathrm{ord}_\varphi(\rho)-1} & (i = 0) \end{array} \right. .$$

Then $\rho \uparrow^{\varphi,\alpha} (f) \rho \uparrow^{\varphi,\alpha} (n) \rho \uparrow^{\varphi,\alpha} (f^{-1}) = \rho \uparrow^{\varphi,\alpha} (\varphi(n)) = \rho \uparrow^{\varphi,\alpha} (fnf^{-1})$ and this implies that $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha}) \in \mathrm{Rep}(G)$. The representation $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha})$ is called the *φ-induced representation* of (V, ρ) by α.

Proposition 4.4. Let N be a finite group, $\varphi \in \mathrm{Aut}(N)$, and $G = N \rtimes_\varphi \mathbb{Z}$. Let $(V, \rho) \in \mathrm{Irr}(N)$ and $\alpha \in M^\times_{\varphi^{\mathrm{ord}_\varphi(\rho)}}$. Then $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha})$ is an irreducible G-representation. Conversely, a finite dimensional irreducible G-representation is obtained in this way.

Lemma 4.5. Let G be a group and $(W, \sigma) \in \mathrm{Irr}(G)$. Then for a positive integer n, $\mathrm{End}(\bigoplus_{i=1}^{n} W)$ is isomorphic to the matrix algebra $M_n(\bar{F})$.

Proof. Straightforward from the Schur's lemma. □

Now we prove Proposition 4.4.

Proof. As an N-representation, $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha}) \downarrow_N^G$ is the direct sum of mutually non-isomorphic irreducible representations. Hence any irreducible N-subrepresentation of $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha}) \downarrow_N^G$ is equal to one of the summands V_i in the definition above. By the definition of the action of f, V_i generates $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha})$ as a G-representaion. Therefore $(V \uparrow^{\varphi,\alpha}, \rho \uparrow^{\varphi,\alpha})$ is irreducible.

On the other hand, let $(W, \sigma) \in \mathrm{Irr}(G)$. Then there exists an irreducible N-subrepresentation U of $(W, \sigma) \downarrow_N^G$. Let $W_0 = \sum_{U \cong W' \subset W} W'$ (W' runs through all N-submodules isomorphic to U). Since W is finite dimensional, W_0 is isomorphic to U^r for some integer r as an N-representation. Moreover, for an irreducible submodule $W' \subset W_0$, $\sigma(f^{\mathrm{ord}_\varphi(U)})(W')$ is isomorphic to U, thus $\sigma(f^{\mathrm{ord}_\varphi(U)})(W') \subset W_0$. Therefore $\sigma(f^{\mathrm{ord}_\varphi(U)})$ is an automorphism of the N-representation W_0. By Lemma 4.5, this is represented by some matrix $A \in M_r(\bar{F})$. Note that a subspace of $M_r(\bar{F})$-module $\bar{F}^r$ corresponds to a subrepresentation of W_0. Let w be an eigenvector of A and let W' be the subrepresentation of W_0 corresponding to the eigenspace $\bar{F}w$. Then W' is irreducible and isomorphic to U, $\sigma(f^{\mathrm{ord}_\varphi(U)})(W') = W'$ and $\bigoplus_{i=0}^{\mathrm{ord}_\varphi(W')-1} \varphi^{*i} W'$ is a nonzero submodule of W. Since W is irreducible, $r = 1$ and the statement holds. □

5. Examples

In this section, we will describe irreducible modules over some connected quandles. Later C_n denotes the cyclic group of order n.

5.1. **Dihedral quandle.**

Definition 5.1. Let A be an Abelian group. Then the *generalized dihedral group* of A is the group $D_A = A \rtimes C_2$ where C_2 is generated by τ, with action $\tau a \tau^{-1} = a^{-1}$ for $a \in A$ (we write the group operation on A by multiplication as the subgroup of D_A).

Proposition 5.2. Let A be an Abelian group of odd order. Then $\{\tau a \mid a \in A\}$ forms a connected subquandle of $\mathrm{Conj}(D_A)$.

Proof. For $a, b \in A$, $(\tau a) \rhd (\tau b) = (\tau a)(\tau b)(\tau a)^{-1} = \tau a \tau b a^{-1} \tau^{-1} = \tau a b^{-1} a = \tau b^{-1} a^2$. Since A is of odd order, the map $a \mapsto a^2$ is an automorphism of A. □

Definition 5.3. Let A be an Abelian group of odd order. Denote by Q_A the quandle obtained as in Proposition 5.2.

More generally, for an Abelian group A with group operation $+$, the quandle A with the operator $a \rhd b = a + t(b - a)$ for some $t \in \mathrm{Aut}(A)$ is called an *Alexander quandle* on A. The dihedral quandles are special cases of Alexander quandles (the map t is the inversion map $a \mapsto -a$).

Since Q_A generates D_A and the center of D_A is trivial, the inner automorphism group $\mathrm{Inn}(Q_A)$ is isomorphic to D_A by Example 1.3 and it is easy to see that $\mathrm{Inn}_0(Q_A) \cong A$. As particular examples, we look at the case $A = C_n$ and $C_n \times C_n$ for an odd number n. To find irreducible modules, we have to study the structure of $\mathrm{As}(Q)$. By Proposition 1.10, there is a natural surjective homomorphism $M(\mathrm{Inn}(Q_A)) = H_2(\mathrm{Inn}(Q_A), \mathbb{Z}) \to Z_0(Q_A)$. Recall the exact sequence $(*2) : 1 \to Z_0(Q_A) \to \mathrm{As}_0(Q_A) \to \mathrm{Inn}_0(Q_A) \to 1$.

Now we study the structure of $\mathrm{As}(Q_A)$. The structure of the associated group of an Alexander quandle is given in [Cla].

Theorem 5.4 ([Cla, Theorem 1])**.** Let M be an Abelian group and T be an automorphism of M. Denote by $A(M,T)$ the Alexander quandle on M with the automorphism T. Suppose that the quandle $A(M,T)$ is connected. Let $\tau : M\otimes M \to M\otimes M; x\otimes y \mapsto (Ty)\otimes x$ and $S(M,T)$ the cokernel of $1-\tau$. Then the associated group $\mathrm{As}(A(M,T))$ is isomorphic to the group $F(M,T) = \mathbb{Z}\times M\times S(M,T)$ (as a set) with the operation

$$(k,x,\alpha)(m,y,\beta) = (k+m, T^m x + y, \alpha+\beta+[T^m x\otimes y])$$

where $[\gamma]$ in the third component denotes the element $\gamma \bmod (1-\tau)(M\otimes M)$. Moreover, the isomorphism is given by $\mathrm{As}(A(M,T)) \to F(M,T); g_x \mapsto (1,x,0)$.

In particular, if T is the inversion map of M, the subgroup $S(M,T)$ is the exterior square $M\wedge M$. Clauwens also gives a description of the fundamental group of Alexander quandles.

Remark 5.5 ([Cla])**.** Let M and T be as above and let the quandle $A(M,T)$ be connected. Then $\pi_1(A(M,T),0)$ is isomorphic to $S(M,T)$ through the isomorphism $\mathrm{As}(A(M,T)) \cong F(M,T)$ given in Proposition 5.4.

Definition 5.6. Let n be a positive integer. Then the group

$$\mathrm{He}_n = \langle S,\ T,\ U \mid S^n,\ T^n,\ U^n,\ [T,S]U^{-1},\ [U,S],\ [U,T]\rangle$$

is called the *Heisenberg group* of order n^3.

In general, for a commutative ring R, the subgroup of $GL_3(R)$ of the form

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c\in R \right\}$$

is called the *Heisenberg group* over R. The definition above is the case where $R=\mathbb{Z}/n\mathbb{Z}$.

Applying the results above, we have the following facts:

Proposition 5.7. Let n be an odd number.

(1) Let $A = C_n$ and $\varphi \in \mathrm{Aut}(A)$ be the inversion. Then $S(A,\varphi)=0$ and $\mathrm{As}(Q_A)\cong A\rtimes\mathbb{Z}$ with $1\in\mathbb{Z}$ acting by inversion. The fundamental group $\pi_1(Q_A,q)$ is trivial for any $q\in Q$.

(2) Let $A = C_n\times C_n$ and $\varphi\in\mathrm{Aut}(A)$ be the inversion. Then $S(A,\varphi)\cong C_n$ and $\mathrm{As}(Q_A)\cong \mathrm{He}_n\rtimes\mathbb{Z}$ where $1\in\mathbb{Z}$ acts on He_n by $S\mapsto S^{-1}, T\mapsto T^{-1}, U\mapsto U$ in the representation above. The fundamental group $\pi_1(Q_A,q)$ is $S(A,\varphi)\cong C_n$ for $q\in Q$

Proof. (1) As stated above, $S(A,\varphi) = C_n\wedge C_n = 0$ since n is odd. Moreover, let $u=(1,0,0)\in F(A,\varphi)$. Then $u^{-1}=(-1,0,0)$ and

$$u(0,1,0)u^{-1} = (1,0,0)(0,1,0)(-1,0,0) = (1,1,0)(-1,0,0) = (0,-1,0)$$

Therefore $\mathrm{As}(Q_A)\cong A\rtimes\mathbb{Z}$ with $1\in\mathbb{Z}$ acting by inversion. By Remark 5.5, $\pi_1(Q_A,q)$ is trivial.

(2) Let $A = C_n\times C_n = \langle x_1\rangle\times\langle x_2\rangle$. Then $S(A,\varphi) = (C_n\times C_n)\wedge(C_n\times C_n) = C_n$ generated by $[x_1\otimes x_2]$. Now let $s=(0,x_1,0), t=(0,x_2,0), u=(0,0,[-2x_1\otimes x_2])\in F(A,\varphi)$. Then for $i=1,2$,

$$\begin{aligned}(0,x_i,0)(0,-x_i,0) &= (0,0,[x_i\otimes(-x_i)]) \\ &= (0,0,0)\end{aligned}$$

since $x_i \otimes x_i \equiv 0 \bmod (1-\tau)(A \otimes A)$. Therefore $s^{-1} = (0, -x_1, 0)$, $t^{-1} = (0, -x_2, 0)$. It is clear that $su = us$, $tu = ut$. Moreover

$$\begin{aligned} tst^{-1}s^{-1} &= (0, x_2, 0)(0, x_1, 0)(0, -x_2, 0)(0, -x_1, 0) \\ &= (0, x_1 + x_2, [x_2 \otimes x_1])(0, -x_2, 0)(0, -x_1, 0) \\ &= (0, x_1, [x_2 \otimes x_1 - (x_1 + x_2) \otimes x_2])(0, -x_1, 0) \\ &= (0, 0, [-2x_1 \otimes x_2]) \\ &= u \end{aligned}$$

since $x_1 \otimes x_2 \equiv -x_2 \otimes x_1 \bmod (1-\tau)(A \otimes A)$. Therefore the group generated by s, t, u is isomorphic to He_n. Moreover, let $v = (1, 0, 0) \in F(A, \varphi)$. Then $v^{-1} = (-1, 0, 0)$ and

$$vsv^{-1} = (1, 0, 0)(0, x_1, 0)(-1, 0, 0) = (1, x_1, 0)(-1, 0, 0) = (0, -x_1, 0) = s^{-1},$$

$$vtv^{-1} = (1, 0, 0)(0, x_2, 0)(-1, 0, 0) = (1, x_2, 0)(-1, 0, 0) = (0, -x_2, 0) = t^{-1},$$

and

$$vu = uv.$$

Therefore $\mathrm{As}(Q_A) \cong \mathrm{He}_n \rtimes \mathbb{Z}$ where $1 \in \mathbb{Z}$ acts as in the statement. Remark 5.5 implies that $\pi_1(Q_A, q) \cong C_n = S(A, \varphi)$. □

Next we classify FQ_A-modules. Let F be an algebraically closed field of characteric 0 and consider irreducible $F[\mathrm{As}(Q_A)]$-modules. If $A = C_n$, $\mathrm{As}(Q_A) = C_n \rtimes \mathbb{Z}$ where $1 \in \mathbb{Z}$ acts on C_n through the inversion automorphism $\varphi \in \mathrm{Aut}(C_n)$. Let $\mathrm{As}_0(Q_A) = C_n = \langle \sigma \rangle$ and for $i \in \{0, \ldots, n-1\}$, let $\chi_i; \sigma \mapsto \zeta_n^i$ where $\zeta_n \in F$ is a fixed primitive n-th root of unity. Then $\mathrm{Irr}(C_n) = \{\chi_i \mid i \in \{0, \ldots, n-1\}\}$. The φ-equivalence classes are $\{\chi_0\}, \{\chi_1, \chi_{n-1}\}, \ldots, \{\chi_{\frac{n-1}{2}}, \chi_{\frac{n+1}{2}}\}$. By Proposition 4.4, $\mathrm{Irr}(\mathrm{As}(Q_A)) = \{\chi \uparrow^{\varphi,\alpha} \mid \chi \in \mathrm{Irr}(C_n),\ \alpha \in F^\times\}$.

Fix $\tau \in Q_A$. Since $\pi_1(Q_A, \tau) = 1$ by Proposition 5.7, there exists only one irreducible covering Q_A-module by Theorem 3.4. Let $\rho = \chi \uparrow^{\varphi,\alpha} \in \mathrm{Irr}(\mathrm{As}(Q_A))$ for $\chi \in \mathrm{Irr}(C_n)$, $\alpha \in F^\times$ and V be the representation space. If $\chi = \chi_0$, V is 1-dimensional. If $\alpha = 1$, ρ is the trivial $\mathrm{As}(Q)$-module. Otherwise, $\mathcal{MQ}(V)$ is isomorphic to $\mathcal{M}_Q(V)$ and is induced from the $\mathrm{As}(Q)$-module defined by $\mathrm{As}(Q) \to F^\times; g_\tau \mapsto \alpha$.

Let $\chi = \chi_i$ for $i \in \{1, \ldots, \frac{n-1}{2}\}$. Then $\rho(g_\tau)$ can be identified with the matrix $\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$. If $\alpha \neq 1$, the matrix has no eigenvector with respect to eigenvalue 1. Therefore $\mathcal{MQ}(V) = \mathcal{M}_Q(V)$ which is induced from an $\mathrm{As}(Q)$-modules. If $\alpha = 1$, the eigenvalues of the matrix are 1 and -1, each with multiplicity 1. Therefore $\mathcal{MQ}(V)$ is a 1-dimensional Q-module which is not induced from $\mathrm{As}(Q)$-modules.

Now we give a more concrete description of $\mathcal{MQ}(V)$ for $\rho = \chi_1 \uparrow^{\varphi,1}$. For an element $\tau\sigma^i \in Q_A \subset D_A$ where $A = \langle \sigma \rangle = C_n$, the representation ρ is given as follows:

$$\rho(g_{\tau\sigma^i}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^{-i} \end{pmatrix} = \begin{pmatrix} 0 & \zeta_n^{-i} \\ \zeta_n^i & 0 \end{pmatrix}.$$

Then $(1 - g_{\tau\sigma^i})V$ is a 1-dimensional subspace with basis $v_i = \begin{pmatrix} 1 \\ -\zeta_n^i \end{pmatrix}$. With this basis,

$$g_{\tau\sigma^i}.v_j = \begin{pmatrix} 0 & \zeta_n^{-i} \\ \zeta_n^i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\zeta_n^j \end{pmatrix} = \begin{pmatrix} -\zeta_n^{j-i} \\ \zeta_n^i \end{pmatrix} = -\zeta_n^{j-i} v_k,$$

$$(1-g_{\tau\sigma^{2i-j}}).v_i = \begin{pmatrix} 1 & -\zeta_n^{-(2i-j)} \\ -\zeta_n^{2i-j} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -\zeta_n^i \end{pmatrix} = \begin{pmatrix} 1+\zeta_n^{-(i-j)} \\ -\zeta_n^{2i-j}-\zeta_n^i \end{pmatrix} = (1+\zeta_n^{j-i})v_k$$

where $k \equiv 2i-j \bmod n, 0 \le k < n$. These imply that $\eta_{\tau\sigma^i,\tau\sigma^j} = -\zeta_n^{j-i}$, $\tau_{\tau\sigma^i,\tau\sigma^j} = 1+\zeta_n^{j-i}$.

Next let $A = C_p \times C_p$ where p is an odd prime. For the group He_p, the following holds:

Proposition 5.8. There are p^2+p-1 irreducible representations of He_p. Among them, p^2 are 1-dimensional and the others are p-dimensional.

Proof. Let $X_{i,j} = S^iT^j$ for $i,j \in \{0,\ \dots,\ p-1\}$. Recall that $TST^{-1} = SU$ and $STS^{-1} = TU^{-1}$. Therefore $SX_{i,j}S^{-1} = X_{i,j}U^{-j}$ and $TX_{i,j}T^{-1} = X_{i,j}U^i$. Hence the conjugacy classes are represented by $X_{i,j}$ for $i,j \in \{0,\ \dots,\ p-1\}$ except for $i=j=0$, and U^k for $k \in \{0,\ \dots,\ p-1\}$ and there are p^2+p-1 conjugacy classes. Since the number of 1-dimensional representations is equal to the order of the abelianization, there are p^2. Moreover since the dimension of an irreducible representation divides the order of the group by [NT, Chapter 3, Theorem 2.4] and $p^4 > p^3$, the others must be p-dimensional. Then $1\cdot p^2 + p^2\cdot(p-1) = p^3$. □

Specifically, every 1-dimensional irreducible representation is constructed by lifting an irreducible representation of $C_p^2 = \mathrm{He}_p^{\mathrm{ab}}$. Denote by $\rho_{s,t}^{(1)}$ the irreducible representation defined by $S \mapsto \zeta_p^s$, $T \mapsto \zeta_p^t$. Every p-dimensional representation is of the following form for $s \in \{1,\ \dots,\ p-1\}$:

$$S \mapsto \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix},$$

$$T \mapsto \begin{pmatrix} 1 & & & & \\ & \zeta_p^s & & & \\ & & \zeta_p^{2s} & & \\ & & & \ddots & \\ & & & & \zeta_p^{-s} \end{pmatrix},$$

$$U \mapsto \begin{pmatrix} \zeta_p^s & & \\ & \ddots & \\ & & \zeta_p^s \end{pmatrix}.$$

We denote by $\rho_s^{(p)}$ the irreducible representation above. Then its character is given as follows:

$$\chi_{\rho_s^{(p)}}(X_{i,j}) = 0 \text{ if } i \ne 0 \text{ or } j \ne 0,\ \chi_{\rho_s^{(p)}}(U^k) = p\zeta_p^{sk}.$$

Since two representations of a finite group are isomorphic if and only if their characters coincide, the following holds:

Proposition 5.9. Let φ be the automorphism of He_p defined by $S \mapsto S^{-1},\ T \mapsto T^{-1},\ U \mapsto U$. Then a φ-equivalence class is one of the followings:

(1) $\{\rho_{0,0}^{(1)}\}$,
(2) $\{\rho_{s,t}^{(1)},\ \rho_{p-s,p-t}^{(1)}\}$ for $s,t \in \{0,\ \dots,\ p-1\}$ except for $s=t=0$,
(3) $\{\rho_s^{(p)}\}$ for $s \in \{1,\ \dots,\ p-1\}$.

Let $\sigma \in \mathrm{Irr}(\mathrm{He}_p)$, $\rho = \sigma \uparrow^{\varphi,\alpha}$ and V be the representation space of ρ. The case $\dim \sigma = 1$ is similar to the case $A = C_p$. For $\sigma = \rho^{(1)}_{0,0}$ and $\alpha \neq 1$, $\mathcal{MQ}(V)$ is a module induced from an $\mathrm{As}(Q)$-module and for $\sigma = \rho^{(1)}_{s,t}$ and $\alpha = 1$, $\mathcal{MQ}(V)$ is a module which is not induced from $\mathrm{As}(Q)$-modules. If σ is p-dimensional, $\mathrm{ord}_\varphi(\sigma) = 1$ and a φ-automorphism α is of the form

$$P_a = a \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & \iddots & \\ & 1 & & \end{pmatrix}$$

for $a \in F^\times$. The matrix P_a has eigenvalues a of multiplicity $\frac{p+1}{2}$ and $-a$ of multiplicity $\frac{p-1}{2}$. Therefore if $a = 1$, $\mathcal{MQ}(V)$ is $\frac{p-1}{2}$-dimensional and if $a = -1$, $\mathcal{MQ}(V)$ is $\frac{p+1}{2}$-dimensional. Otherwise, $\mathcal{MQ}(V)$ is a module induced from an $\mathrm{As}(Q)$-module. In particular, if $p = 3$, there is a 1-dimensional module which is not induced from $\mathrm{As}(Q)$-modules.

On the other hand, by Proposition 5.7 $\pi_1(Q, q) = \langle U \rangle = C_p$, which implies that there are p irreducible covering modules.

5.2. **Connected quandles in** $\mathrm{Conj}(SL_2(\mathbb{F}_q))$**.** We consider the special linear group $SL_2(\mathbb{F}_q)$ where $\mathbb{F}_q$ denotes the field of $q = p^f$ elements for a prime p. First we state some basic facts on $SL_2(\mathbb{F}_q)$. A proof for odd p is in [Bon, Chapter 1] and the case $p = 2$ is similar.

Proposition 5.10. (1) The order of $SL_2(\mathbb{F}_q)$ is $q(q^2 - 1)$.

(2) If $q \geq 4$, $SL_2(\mathbb{F}_q)$ is a perfect group, i.e. its commutator subgroup is the whole group.

(3) Let $PSL_2(\mathbb{F}_q) = SL_2(\mathbb{F}_q)/Z(SL_2(\mathbb{F}_q))$ denote the projective special linear group where $Z(G)$ denotes the center of the group G. If $p \neq 2$, $Z(SL_2(\mathbb{F}_q)) = \{\pm I_2\}$ where I_2 is the identity matrix. If $p = 2$, $Z(SL_2(\mathbb{F}_q))$ is trivial.

(4) $PSL_2(\mathbb{F}_q)$ is a simple group if $q \geq 4$. $PSL_2(\mathbb{F}_3)$ is the alternating group $\mathfrak{A}_4$ and $SL_2(\mathbb{F}_2) = PSL_2(\mathbb{F}_2)$ is the symmetric group $\mathfrak{S}_3$ which are solvable.

It is also known that $SL_2(\mathbb{F}_4) \cong PSL_2(\mathbb{F}_5) \cong \mathfrak{A}_5$ and $PSL_2(\mathbb{F}_9) \cong \mathfrak{A}_6$.

Remark 5.11. Note that the group $PSL_2(\mathbb{F}_q)$ acts faithfully on the projective space $\mathbb{P}^1(\mathbb{F}_q)$. The isomorphism in (4) of Proposition 5.10 is obtained from this action. The commutator subgroup of $\mathfrak{S}_3 \cong PSL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2)$ is C_3 (generated by 3-cycles). The commutator subgroup of $\mathfrak{A}_4 \cong PSL_2(\mathbb{F}_3)$ is $C_2 \times C_2$ (generated by $(2,2)$-cycles). Through the surjective group homomorphism $\pi : SL_2(\mathbb{F}_3) \to PSL_2(\mathbb{F}_3)$, we have that the commutator subgroup of $SL_2(\mathbb{F}_3)$ is of order 8 (recall that the commutator subgroup of $SL_2(\mathbb{F}_3)$ is mapped onto the commutator subgroup of $PSL_2(\mathbb{F}_3)$ by π). In fact, the commutator subgroup is isomorphic to the quaternion group Q_8

On Schur multipliers of special linear groups, the following holds [Kar, Chapter 7]:

Theorem 5.12. $M(SL_2(\mathbb{F}_q)) = \begin{cases} 0 & (q \neq 4, 9), \\ \mathbb{Z}/2\mathbb{Z} & (q = 4), \\ \mathbb{Z}/3\mathbb{Z} & (q = 9). \end{cases}$

Theorem 5.13. $M(PSL_2(\mathbb{F}_q)) = \begin{cases} 0 & (q \text{ is even and } q \neq 4), \\ \mathbb{Z}/2\mathbb{Z} & (q = 4, \text{ or } q \text{ is odd and } q \neq 9), \\ \mathbb{Z}/6\mathbb{Z} & (q = 9). \end{cases}$

These theorems were originally proven by Steinberg in 1960s.

Next we look at conjugacy classes of $SL_2(\mathbb{F}_q)$. For a finite field F, denote by F^{qd} the quadratic extension of F. Fix a generator z of the multiplicative group $\mathbb{F}_q^\times$ and a primitive $(q+1)$-st root γ of unity. Then $\gamma \in \mathbb{F}_q^{\mathrm{qd}} = \mathbb{F}_{q^2}$.

Next we describe the conjugacy classes of $SL_2(\mathbb{F}_q)$. Let $D_{z^r} = \begin{pmatrix} z^r & 0 \\ 0 & z^{-r} \end{pmatrix}$ and $T_{\gamma^r} = \begin{pmatrix} 0 & -1 \\ 1 & \mathrm{Tr}(\gamma^r) \end{pmatrix}$. If q is odd, let $n \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$ and $N_{\pm,+} = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $N_{\pm,-} = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then the following facts are known.

Proposition 5.14. For $A \in SL_2(\mathbb{F}_q)$, let $n(A)$ be the size of the conjugacy class containing A.

(1) $n(D_{z^r}) = q(q+1)$ for $1 \le r \le \frac{q-3}{2}$ if p is odd, $1 \le r \le \frac{q}{2} - 1$ if $p = 2$.
(2) $n(T_{\gamma^r}) = q(q-1)$ for $1 \le r \le \frac{q-1}{2}$ if p is odd, $1 \le r \le \frac{q}{2}$ if $p = 2$.
(3) $n(N_{*,*})$ is $\frac{q^2-1}{2}$ if $p \neq 2$ and $q^2 - 1$ if $p = 2$.

Proof. See [Bon, Proposition 1.3.1] for odd q. Similar proof works for even q. □

Proposition 5.15. $A \in SL_2(\mathbb{F}_q)$ is conjugate to exactly one of the following matrices:

(1) I_2,
(2) $-I_2$ if $p \neq 2$,
(3) D_{z^r} for $1 \le r \le \frac{q-3}{2}$ if p is odd, $1 \le r \le \frac{q}{2} - 1$ if $p = 2$ (This occurs if $q \ge 4$.),
(4) T_{γ^r} for $1 \le r \le \frac{q-1}{2}$ if p is odd, $1 \le r \le \frac{q}{2}$ if $p = 2$,
(5) $N_{+,+}$,
(6) $N_{-,+}$ if $p \neq 2$,
(7) $N_{+,-}$ and $N_{-,-}$ where $n \in \mathbb{F}_q^\times$ is a non-square element, if $p \neq 2$.

Proof. See [Bon, Theorem 1.3.3] for odd q. Similar proof workss for even q. □

Proposition 5.16. If $q \ge 4$, any conjugacy class of $SL_2(\mathbb{F}_q)$ except for $\pm I_2$ generates the whole group. If $q < 4$, a conjugacy class C generates $SL_2(\mathbb{F}_q)$ if and only if C is the conjugacy class of $N_{*,*}$.

Proof. If $q \ge 4$, the group $SL_2(\mathbb{F}_q)$ is simple (if $p = 2$) or a non-split central extension of $PSL_2(\mathbb{F}_q)$ which is simple (if $p \neq 2$, for the extension is non-split since $SL_2(\mathbb{F}_q)$ is perfect by Proposition 5.10). Thus any proper normal subgroup of $SL_2(\mathbb{F}_q)$ is a subgroup of the center. Since every subgroup which is generated by some conjugacy class is normal in the whole group, the first statement holds.

If $q \le 3$, we saw in Remark 5.11 that $SL_2(\mathbb{F}_q)'$ is the 2- (resp. 3-)Sylow subgroup of $SL_2(\mathbb{F}_q)$ if $q = 3$ (resp. $q = 2$). Now the order of T_γ is 4 (resp. 3) if $q = 3$ (resp. $q = 2$.) which is a power of 2 (resp. 3). Therefore T_γ is contained in $SL_2(\mathbb{F}_q)'$ and T_γ does not normally generate the group $SL_2(\mathbb{F}_q)$. On the other hand, If $q = 3$, the order of $N_{+,*}$ (resp. $N_{-,*}$) is 3 (resp. 6). By Proposition 5.10, $PSL_2(\mathbb{F}_3) \cong \mathfrak{A}_4$ and $N_{*,*}$ is mapped to a 3-cycle. Since a 3-cycle normally generates the group $\mathfrak{A}_4$ and $SL_2(\mathbb{F}_3)$ is a non-split central extension (since the order of the Abelianization of $SL_2(\mathbb{F}_3)$ is 3 which is coprime to 2) of $PSL_2(\mathbb{F}_3)$, $N_{-,*}$ normally generates the group $SL_2(\mathbb{F}_3)$. If $q = 2$, the order of $N_{+,+}$ is 2. Since $SL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$ and a 2-cycle normally generates $\mathfrak{S}_3$, $N_{+,+}$ normally generates $SL_2(\mathbb{F}_2)$. □

Corollary 5.17. If $q \ge 4$, every non-central conjugacy class forms a connected subquandle of $\mathrm{Conj}(SL_2(\mathbb{F}_q))$. If $q < 4$, only the conjugacy classes of $N_{*,*}$ are connected subquandles of $\mathrm{Conj}(SL_2(\mathbb{F}_q))$.

If $q = 2$, the conjugation quandle Q generated by $N_{+,+}$ is isomorphic to the dihedral quandle of $A = C_3$. If $q = 3$, the quandle generated by $N_{*,*}$ is isomorphic to a subquandle of $\mathrm{Conj}(\mathfrak{A}_4)$ generated by $(1, 2, 3)$.

Next we give definitions and basic facts on modular representations, i.e. representations of groups over fields of positive characteristics. For details, see [NT, Chapter 3, §6].

Definition 5.18. Let p be a prime. For a natural number n, denote by n' the number satisfying $n = p^a n'$ and $\gcd(n', p) = 1$.

Definition 5.19. Let R be a complete discrete valuation ring of characteristic 0 with uniformizer π. Let F be the residue field $R/\pi R$, p the characteristic of F and K the field of fractions of R. Let G be a finite group.

(1) The triplet (K, R, F) is called a *p-modular system* for G if R contains every $\exp(G)$-th root of unity where $\exp(G)$ denotes the *exponent* of the group G, i.e. the least common multiple of $\mathrm{ord}(g)$ for $g \in G$.
(2) Let (K, R, F) be a p-modular system for G and F' be the subfield of F generated by roots of unity. Then the *Teichmüller character* t_R is defined on $(F')^\times$ as follows:
$$t_R : (F')^\times \to R; a \mapsto \zeta_a,$$
where ζ_a is the root of unity in R with $\zeta_a \equiv a \bmod \pi$. Such an element exists uniquely by Hensel's lemma.
(3) An element $g \in G$ is said to be *p-regular* if $\mathrm{ord}(g)$ is coprime to p. Otherwise, g is said to be *p-singular*. Denote by G'_p the set of p-regular elements in G.

Definition 5.20. Let G be a finite group and (K, R, F) be a p-modular system for G. Let (V, ρ) be a representation of G over F. For $g \in G'_p$, let $\alpha_1, \ldots, \alpha_r$ be the eigenvalues of $\rho(g)$, where $r = \dim \rho$ (note that eigenvalues are in $F^\times$ since F contains enough roots of unity). Then the map $\varphi_\rho : G'_p \to R; g \mapsto \sum_{i=1}^r t_R(\alpha_i)$ is called the *Brauer character* for (ρ, V) over a characteristic p. Denote by $\mathrm{IBr}_p(G)$ the set of *irreducible Brauer characters* over characteristic p, i.e. Brauer characters induced from irreducible F-representations. This is independent of the choice of the p-modular system up to identification.

One of the fundamental results in modular representations is stated as follows [NT, Chapter 3, Theorem 6.5]:

Theorem 5.21. $\sharp\mathrm{IBr}_p(G)$ is equal to the number of p-regular conjugacy classes.

Now we consider representations of $SL_2(\mathbb{F}_q)$ over fields of characteristic p. Fix a p-modular system (K, R, F) for $SL_2(\mathbb{F}_q)$. Since the exponent of $SL_2(\mathbb{F}_q)$ is
$$\begin{cases} p\frac{q^2-1}{2} & (p \neq 2), \\ 2(q^2-1) & (p = 2), \end{cases}$$
F contains $(\mathbb{F}_q)^{\mathrm{qd}} = \mathbb{F}_{q^2}$. An element $A \in SL_2(\mathbb{F}_q)$ is p-singular if and only if A is conjugate to $N_{*,*}$. Therefore thze number of p-regular conjugacy classes is
$$\begin{cases} 2 + \frac{q-3}{2} + \frac{q-1}{2} = q & (p \neq 2), \\ 1 + \frac{q}{2} - 1 + \frac{q}{2} = q & (p = 2). \end{cases}$$
Therefore the following holds:

Proposition 5.22. $\sharp(\mathrm{IBr}_p(SL_2(\mathbb{F}_q))) = q$.

Next we find out q irreducible representations. Recall that $q = p^f$. Let $\sigma : x \mapsto x^p \in \mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ (note that σ generates $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$). Now for $i = 0, \ldots, f-1$, let $\chi_i : SL_2(\mathbb{F}_q) \to GL_2(F); A \mapsto \sigma^i(A)$ (σ acts on each entry in A). Then χ_i is a 2-dimensional irreducible representation. Let $V_i \cong F^2$ be the representation space of χ_i. Then the action of $SL_2(\mathbb{F}_q)$ on V_i extends to an action on the symmetric algebra $S(V_i)$. Recall that for an n-dimensional vector space V, $S(V)$ is isomorphic to the polynomial ring in n variables. In this identification, for a polynomial h and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the action of $SL_2(\mathbb{F}_q)$ on $S(V_i)$ is written as follows:

$$A.h(X_i, Y_i) = h(\sigma^i(a)X_i + \sigma^i(c)Y_i, \sigma^i(b)X_i + \sigma^i(d)Y_i)$$

where X_i, Y_i denotes the standard basis for V_i. Then the subspace $V_{i,k+1} = S(V_i)_k$ of homogeneous polynomials of degree k for $k \geq 0$ is a $(k+1)$-dimensional subrepresentation of $S(V_i)$. We denote this representation by $\chi_{i,k+1}$. Clearly $\chi_{i,1}$ is the trivial representation and $\chi_{i,2} = \chi_i$.

With these notations, the following holds [Hum, Chapter 2.7 and 2.11]:

Proposition 5.23 (Steinberg Tensor Product Theorem for $SL_2(\mathbb{F}_q)$)**.** Let $r_i \in \{1, \ldots, p\}$ for $i = 0, \ldots, f-1$. Then

$$\rho_{r_0,\ldots,r_{f-1}} = \chi_{0,r_0} \otimes \cdots \otimes \chi_{f-1,r_{f-1}}$$

give distinct irreducible representations of $SL_2(\mathbb{F}_q)$.

Steinberg tensor product theorem gives distinct irreducible representations of groups of Lie type. The proposition is the special case for $SL_2(\mathbb{F}_q)$ which is of type A_1.

Next we give a concrete description of the group $\mathrm{As}(Q)$.

Proposition 5.24. Let $q \neq 2, 3, 4, 9$ and Q be a conjugacy class in $SL_2(\mathbb{F}_q)$ which generates the whole group (i.e. Q is a non-central conjugacy class of $SL_2(\mathbb{F}_q)$). Let $h : \mathrm{As}(Q) \to SL_2(\mathbb{F}_q)$ be the group homomorphism induced by adjunction from the inclusion map $Q \to \mathrm{Conj}(SL_2(\mathbb{F}_q))$. Then $\mathrm{As}(Q) \cong SL_2(\mathbb{F}_q) \times \mathbb{Z}$ by the map $g \mapsto (h(g), \deg(g))$.

Proof. Note that $\mathrm{Inn}(Q) = PSL_2(\mathbb{F}_q)$ by Example 1.3 and $\mathrm{Inn}(Q)$ is simple by Proposition 5.10. Recall that $Z_0(Q)$ is a quotient group of $M(PSL_2(\mathbb{F}_q))$ by Proposition 1.10 and that $\mathrm{Inn}_0(Q) \cong \mathrm{As}_0(Q)/Z_0(Q)$. By assumption and Theorem 5.13, $M(PSL_2(\mathbb{F}_q))$ is $\mathbb{Z}/2\mathbb{Z}$ if q is odd, otherwise 0. Therefore the order of $\mathrm{As}_0(Q)$ is at most $2(\sharp PSL_2(\mathbb{F}_q))$ if q is odd, $\sharp PSL_2(\mathbb{F}_q)$ otherwise, i.e. at most $\sharp(SL_2(\mathbb{F}_q))$ for both cases. Take $P \in Q$. Since $\mathrm{Inn}_0(Q) = \mathrm{Inn}(Q)$, there exists $x \in \mathrm{As}_0(Q)$ such that $x.A = g_P.A$ for any $A \in Q$. Then $x^{-1}g_P$ is in the center of $\mathrm{As}(Q)$ by Proposition 1.6. This shows that $\mathrm{As}(Q) \cong \mathrm{As}_0(Q) \times \langle x^{-1}g_P \rangle$. Since h is surjective and $SL_2(\mathbb{F}_q)$ is perfect by Proposition 5.10, $\mathrm{As}_0(Q)$ is mapped onto $SL_2(\mathbb{F}_q)$ by h. (Again note that the commutator subgroup is mapped onto the commutator subgroup by a surjective group homomorphism). By comparing the orders, we see that $\mathrm{As}_0(Q) \cong SL_2(\mathbb{F}_q)$. □

Let Q be a conjugacy class in $SL_2(\mathbb{F}_q)$ which generates the whole group where $q \neq 2, 3, 4, 9$. By Proposition 5.16, this is equivalent to saying that $q \neq 2, 3, 4, 9$ and Q is a non-central conjugacy class. Then $\pi_1(Q, P)$ for $P \in Q$ is the stabilizer of P in $\mathrm{As}_0(Q)$, hence is isomorphic to the centralizer of P in $SL_2(\mathbb{F}_q)$. By proof of Proposition 5.14, it is Abelian. Therefore there are $(\sharp\pi_1(Q, P))'$ irreducible covering modules (recall that n' denotes the prime-to-p part of n). Since $\mathrm{As}(Q) \cong SL_2(\mathbb{F}_q) \times \mathbb{Z}$, every irreducible representation of $\mathrm{As}(Q)$ is of the form $\chi \uparrow^{\mathrm{id},\alpha}$ for $\alpha \in F^\times$ and $\chi \in \mathrm{IBr}_p(SL_2(\mathbb{F}_q))$ by Proposition 4.4. We write the same symbol χ for the corresponding modular representation.

Let $t_R(z) = \zeta_z$ and $t_R(\gamma) = \zeta_\gamma$. Then ζ_z (resp. ζ_γ) is a $(q-1)$-st (resp. $(q+1)$-st) root of unity. Now the eigenvalues of $\chi_{i,r_i}(D_{z^r})$ (resp. $\chi_{i,r_i}(T_{\gamma^r})$) are

$$\zeta_z^{p^i r(r_i-1)},\ \zeta_z^{p^i r(r_i-3)},\ \ldots,\ \zeta_z^{p^i r(-r_i+3)},\ \zeta_z^{p^i r(-r_i+1)}$$

$$(\text{resp. } \zeta_\gamma^{p^i r(r_i-1)},\ \zeta_\gamma^{p^i r(r_i-3)},\ \ldots,\ \zeta_\gamma^{p^i r(-r_i+3)},\ \zeta_\gamma^{p^i r(-r_i+1)}).$$

Now a module which is not induced from $\mathrm{As}(Q)$-modules is obtained as $\mathcal{MQ}(\chi \uparrow^{\mathrm{id},\alpha^{-1}})$ for $\chi \in \mathrm{IBr}_p(SL_2(\mathbb{F}_q))$ and an eigenvalue α of $\chi(P)$ for some $P \in Q$ (note that α is an eigenvalue of $\chi(P)$ for any $P \in Q$ since Q is a conjugacy class of $SL_2(\mathbb{F}_q)$).

As an example, we classify all 1-dimensional Q-modules.

Proposition 5.25. Let Q be a non-central conjugacy class in $SL_2(\mathbb{F}_q)$ for $q \neq 2, 3, 4, 9$. Then any 1-dimensional Q-module is isomorphic to one of the followings:

(1) irreducible covering modules,
(2) $\mathcal{MQ}(\rho_{1,\cdots,1} \uparrow^{\mathrm{id},\alpha})$ for $\alpha \neq 1$,
(3) $\mathcal{MQ}(\chi_{i,2} \uparrow^{\mathrm{id},\alpha^{-1}})$ where $\alpha \in F^\times$ is an eigenvalue of $\chi_{i,2}(P)$ for $P \in Q$ for a p-regular conjugacy class Q (As stated above, α is an eigenvalue of $\chi_{i,2}(P)$ for any P.),
(4) $\mathcal{MQ}(\chi_{i,2} \uparrow^{\mathrm{id},\pm 1})$ where Q is a p-singular conjugacy class (i.e. Q is a class of $N_{*,*}$) and ± 1 is the eigenvalue of $P \in Q$,
(5) $\mathcal{MQ}(\chi_{i,3} \uparrow^{\mathrm{id},-1})$ where Q is a conjugacy class of order 4. (Note that this occurs if $p \neq 2$. Then since either $q-1$ or $q+1$ is divisible by 4, there exists an element of order 4.)

For the cases (4), (5), the module is defined over $\mathbb{F}_q$. For the case (3), the module is defined over $\mathbb{F}_q$ if and only if Q is the class of D_*. If Q is the class of T_*, the module is defined over $\mathbb{F}_q^{\mathrm{qd}} = \mathbb{F}_{q^2}$.

Proof. First note that if Q is a p-regular conjugacy class and χ is a representation of $\mathrm{As}_0(Q) \cong SL_2(\mathbb{F}_q)$, then $\chi(P)$ is diagonalizable for any $P \in Q$ and $\mathcal{MQ}(\chi \uparrow^{\mathrm{id},\alpha})$ is 1-dimensional for some α if and only if:

$$\chi(P) \text{ has 2 distinct eigenvalues with multiplicities } \dim\chi - 1 \text{ and 1 respectively.} \qquad (**)$$

Note that α is taken to be the inverse of the former eigenvalue.

As above, every irreducible covering module is 1-dimensional.

On the other hand, a non-covering 1-dimensional module is irreducible, and hence by Theorem 3.5, it is isomorphic to $\mathcal{MQ}(M)$ for some nontrivial irreducible $\mathrm{As}(Q)$-module M. Let $\chi \in \mathrm{IBr}_p(SL_2(\mathbb{F}_q))$ be its restiction to $\mathrm{As}_0(Q) = SL_2(\mathbb{F}_q)$. If $\dim\chi = 1$, $\chi = \rho_{0,\ldots,0}$ and this case corresponds to the case (2).

If $\dim\chi = 2$, $\chi = \chi_{i,2}$ for some i. If Q is a p-regular conjugacy class, every element $P \in Q$ has a common pair of eigenvalues in F. Therefore $\mathcal{MQ}(\chi_{i,2} \uparrow^{\mathrm{id},\alpha^{-1}})$ is a 1-dimensional Q-module if α is an eigenvalue of $\chi_{i,2}(P)$. If Q is a p-singular conjugacy class, every element $P \in Q$ has an eigenvalue ± 1 with multiplicity 2. Since $\chi_{i,2}(P) \neq I_2$, $\mathcal{MQ}(\chi_{i,2} \uparrow^{\mathrm{id},\pm 1})$ is 1-dimensional.

For the case $\dim\chi \geq 3$, by the following two lemmas we see that $\mathcal{MQ}(\chi)$ is 1-dimensional if and only if Q and χ are as in (5). □

Lemma 5.26. Let Q be a p-regular class. Let R be the set of f-tuples of integers from 1 to p indexed by $0, \ldots, f-1$. For $r = (r_i) \in R$, denote $\rho_r = \rho_{r_0,\ldots,r_{f-1}}$.

(1) Let $r \in R$ satisfy that just one of r_i is ≥ 3 and the others are 1. Then for $P \in Q$, the condition $(**)$ is satisfied for $\chi = \rho_r$ if and only if P is of order 4 and $r_i = 3$.

(2) Let $s = (s_i) \in R$ and $a \in \{1, \dots, p\}$. Choose an index i_0 such that $s_{i_0} = 1$ and let $r = (r_i) \in R$ where $r_i = s_i$ if $i \neq i_0$ and $r_{i_0} = a$. If ρ_s does not satisfy the condition $(**)$ for any $P \in Q$, neither does ρ_r.

(3) Let $r \in R$ satisfy that at least 2 of r_i's are ≥ 2. Then ρ_r does not satisfy the condition $(**)$ for any $P \in Q$.

Proof. (1) We may assume that the index i is 0. If Q is a p-regular class, as stated before the proposition, for an eigenvalue $\zeta \in F$ of P, the eigenvalues of $\rho_r(P)$ are $\zeta^{r_0-1}, \zeta^{r_0-3}, \dots, \zeta^{-r_0+3}, \zeta^{-r_0+1}$. Since $\zeta^2 \neq 1$, two neighboring eigenvalues cannot be equal. Thus if $(**)$ is satisfied, then we have $r = 3$ and $\zeta^2 = \zeta^{-2}$. Therefore we see that the condition $(**)$ is satisfied if and only if $r_0 = 3$ and ζ is a fourth root of unity.

(2) First note that ρ_s does not satisfy the condition $(**)$ for any $P \in Q$ if and only if at least one of the following conditions is satisfied:

- $\rho_s(P)$ has at least 3 distinct eigenvalues,
- the multiplicity of every eigenvalue of $\rho_s(P)$ is at least 2.

Note that ρ_r corresponds to $\rho_s \otimes \chi_{i_0,a}$. Since every eigenvalue of $\rho_s \otimes \chi_{i_0,a}(P)$ is of the form $\lambda\mu$ where λ (resp. μ) is an eigenvalue of $\rho_s(P)$ (resp. $\chi_{i_0,a}(P)$), if ρ_s satisfies one of the conditions above, the same condition is satisfied for ρ_r.

(3) By (1),(2), it is enough to show that the assertion for the case just 2 of r_i are 2 or 3 and the others are 1. Then $\rho_r = \chi_{i,a} \otimes \chi_{j,b}$ where $i, j \in \{0, \dots, f-1\}$ and $a, b \in \{2, 3\}$. We proceed by dividing into 3 cases:

(i) If $a = b = 2$, let λ, λ^{-1} (resp. μ, μ^{-1}) be the eigenvalues of $\chi_{i,2}(P)$ (resp. $\chi_{j,2}(P)$). Then $\rho_r(P)$ has the eigenvalues $\lambda\mu, \lambda\mu^{-1}, \lambda^{-1}\mu, \lambda^{-1}\mu^{-1}$. To prevent $\rho_r(P)$ from having 3 distinct eigenvalues, 2 of the 4 values must be equal. Since the order of P is not 2, $\lambda \neq \lambda^{-1}$ and $\mu \neq \mu^{-1}$. Therefore both $\lambda\mu = \lambda^{-1}\mu^{-1}$ and $\lambda\mu^{-1} = \lambda^{-1}\mu$ must hold. Then $\lambda\mu, \lambda\mu^{-1} \in \{\pm 1\}$ since these are equal to their inverses. Since $\mu \neq \mu^{-1}$, these are distinct. Therefore $\rho_r(P)$ has 2 eigenvalues with multiplicities 2,2 respectively.

(ii) If $a = 2$ and $b = 3$, note that Q must be a conjugacy class of order 4 and the eigenvalue of $\chi_{j,3}$ are $-1, -1, 1$. Since $P \in Q$ is of order 4, the eigenvalues of $\chi_{i,2}$ are 4th roots λ, λ^{-1} of unity. Therefore $\lambda^{-1} = -\lambda$. Therefore $\rho_r(P)$ has eigenvalues $\lambda, -\lambda$ with multiplicities $3, 3$ respectively.

(iii) If $a = b = 3$, $\rho_r(P)$ has eigenvalues $-1, 1$ with multiplicities $4, 5$ respectively. □

Lemma 5.27. Let Q be a p-singular class with eigenvalue ε and R the set defined in the previous lemma. Then for $P \in Q$ and $r = (r_i) \in R$, $\rho_r(P)$ has the unique eigenvalue $\varepsilon^{\sum_i (r_i - 1)}$. The codimension of the eigenspace is 1 if and only if $\dim(\rho_r) = 2$.

Proof. The first statement is clear. Let $P = \varepsilon \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in Q$. Then

$$\chi_{i,n}(P) = \varepsilon^{n-1} \begin{pmatrix} 1 & \sigma^i(a) & \sigma^i(a)^2 & \cdots & \sigma^i(a)^{n-1} \\ 0 & 1 & 2\sigma^i(a) & \cdots & (n-1)\sigma^i(a)^{n-2} \\ 0 & 0 & 1 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & (n-1)\sigma^i(a) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

for the basis $X_i^{n-1}, X_i^{n-2}Y_i, \dots, X_iY_i^{n-2}, Y_i^{n-1}$. Since $n \leq p$ and $a \neq 0$, the codimension of the eigenspace of $\chi_{i,n}(P)$, which is equal to the rank of $\chi_{i,n}(P) - \varepsilon^{n-1}I$, is $n - 1$. Therefore it is equal to 1 if and only if $n = 2$.

To complete the proof, let $r = (r_i) \in R$. Then $\rho_r = \chi_{0,r_0} \otimes \cdots \otimes \chi_{f-1,r_{f-1}}$ is a space spanned by $X_0^{s_0} Y_0^{(r_0-1)-s_0} \otimes \cdots \otimes X_{f-1}^{s_{f-1}} Y_{f-1}^{(r_{f-1}-1)-s_{f-1}}$, where $0 \leq s_i \leq r_i - 1$. Assume that r_i and r_j are ≥ 2 for distinct i, j. Then we write $Z_i = X_0^{r_0-1} \otimes \cdots \otimes Y_i^{r_i-1} \otimes \cdots \otimes X_{f-1}^{r_{f-1}-1}$ and $Z_j = X_0^{r_0-1} \otimes \cdots \otimes Y_j^{r_j-1} \otimes \cdots \otimes X_{f-1}^{r_{f-1}-1}$. Then $\rho_r(P)(Z_i) = \varepsilon^{\sum_k (r_k-1)} X_0^{r_0-1} \otimes \cdots \otimes (Y_i + \sigma^i(a) X_i)^{r_i-1} \otimes \cdots \otimes X_{f-1}^{r_{f-1}-1}$ and $\rho_r(P)(Z_j) = \varepsilon^{\sum_k (r_k-1)} X_0^{r_0-1} \otimes \cdots \otimes (Y_j + \sigma^j(a) X_j)^{r_j-1} \otimes \cdots \otimes X_{f-1}^{r_{f-1}-1}$. Since $2 \leq r_i, r_j \leq p$, it is clear that $\rho_r(P)(Z_i) - \varepsilon^{\sum_k (r_k-1)} Z_i$ and $\rho_r(P)(Z_j) - \varepsilon^{\sum_k (r_k-1)} Z_j$ are linearly independent. This shows that the rank of $\rho_r(P) - \varepsilon^{\sum_k (r_k-1)} I$ is at least 2. Therefore the number of indices i such that $r_i \geq 2$ is at most 1. □

For the case (5) of Proposition 5.25, a similar case occurs over characteristic 0. Let $G = SL_2(\mathbb{C})$ and Q be the subquandle of order 4 matrices in $\mathrm{Conj}(G)$. Note that Q is the set of matrices with eigenvalues $i, -i \in \mathbb{C}^\times$. Similarly to the positive characteristic case, extending the representation $G = SL_2(\mathbb{C}) \hookrightarrow GL(\mathbb{C}^2)$ to the representation on $S(\mathbb{C}^2)$, we have a 3-dimensional representation

$$\chi_3 : SL_2(\mathbb{C}) \to GL(\mathbb{C}^3); A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Then for $P \in Q$, $\chi_3(P)$ has eigenvalues $-1, -1, 1$. Now the map $P \mapsto -\chi_3(P)$ is a quandle homomorphism and this gives a group homomorphism $\varphi : \mathrm{As}(Q) \to GL(\mathbb{C}^3)$. Then $\mathcal{MQ}(\varphi) = \coprod_{P \in Q} (1 - g_P)\mathbb{C}^3$ is a 1-dimensional quandle module.

References

[AG] N. Andruskiewitsch, M. Graña, *From Racks to Pointed Hopf Algebras*, Adv. Math. **178** (2003) no. 2, 177–243.

[Aki] T. Akita, *The Adjoint Group of a Coxeter Quandle*, Kyoto J. Math. **60** (2020) no. 4, 1245–1260.

[Bon] C. Bonnafe, *Representations of $SL_2(\mathbb{F}_q)$*, Springer-Verlag London, London, 2011.

[Cla] F.J.-B.J. Clauwens, *The Adjoint Group of an Alexander Quandle*, preprint: arXiv:1011.1587.

[Eis] M. Eisermann, *Quandle Coverings and Their Galois Correspondence*, Fund. Math. **225** (2014) no. 1, 103–168.

[FR] R. Fenn, C. Rourke, *Racks and Links in Codimension Two*, J. Knot Theory Ramifications **1** (1992) no.4, 343-406.

[Hum] J. E. Humphreys, *Modular Representations of Finite Groups of Lie Type*, Cambridge University Press, Cambridge, 2006.

[Jac] N. Jackson, *Extensions of Racks and Quandles*, Homology Homotopy Appl. **7** (2005) no. 1, 151–167.

[Joy] D. Joyce, *A Classifying Invariant of Knots, the Knot Quandle*, J. Pure Appl. Algebra **23** (1982) no. 1, 37–65.

[Kar] G. Karpilovsky, *The Schur Multiplier*, The Clarendon Press, Oxford University Press, New York, 1987.

[Mat] S. V. Matveev, *Distributive Groupoids in Knot Theory*, Mat. Sb. (N.S.) **119** (1982) no. 1, 78–88.

[Nos1] T. Nosaka, *On Homotopy groups of Quandle Spaces and the Quandle Homotopy Invariant of Links*, Topology Appl. **158** (2011) no. 8, 996–1011.

[Nos2] T. Nosaka, *Central Extensions of Groups and Adjoint Groups of Quandles*, RIMS Kokyuroku Bessatsu, B66, Res. Inst. Math. Sci. (RIMS), Kyoto, 2017.

[NT] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, Boston, MA, 1989.

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