A hierarchical structural interpretation of 1-dimensional 2-state number conserving cellular automata (1次元2状態保存的セルオートマトンの階層構造による解釈)

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Abstract

A cellular automaton(CA), introduced by Von Neumann as a self-reproducing model, is a discrete dynamics system that evolves in discrete space and discrete time. A CA consists of a grid (finite dimension) of cells of which states are finite numbers. Each cell evolves by a local function of which arguments are its neighborhood cells. CA is widely used as a modeling tool for a wide variety of fields, especially for physical modeling. CAs for conserving mass or any quantity have also been studied. One of them, the Number-Conserving CA(NCCA), can be interpreted as a model for particle interaction. The state number of each cell is regarded as the number of particles in the cell. The evolution of the NCCA is described by the particle movements between cells. In addition, a motion representation which expressed NCCA as the movement of particles was introduced. Unlike the rule table expressing CA, the motion representation more intuitively represents the movement of the particles by the NCCA.

In the first part of this thesis, we propose an hierarchical motion representation (HMR) that can be summarized and expressed more simply according to the complexity of each motion (pattern length, number of 1) in a motion representation. The relation between n-cell NCCA and $(n-1)$-cell NCCA, one of the main principles of HMR, shows that NCCAs of different sizes can be efficiently expressed through HMR. Through this, we propose an HMR tree that can express all NCCAs for one neighborhood size at once.

Any two-state NCCA with the state set, $\{0, 1\}$ keeps the number of 1s on the configuration constant. In other words, all the 1s on the configuration move without disappearing or appearing at any time step. When 1s on a configuration are moved by motions defined in a motion representation of a two-state NCCA, these motions are determined by the related argument patterns of its local function of which value is 1.

In the second part of this thesis, we define a bundle quad (length $n-2$) meaning 4 length n patterns and a bundle pair (length $n-1$) meaning 2 length n patterns. Using that structure, we showed that there are NCCAs that have perfectly identical evolution between neighborhood size n and $n-2$, and show some properties according to the structure.

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List of Publications

The following papers have been published and/or presented. The content of this thesis is based on the papers.

Referred Journals

- G. T. Kong, K. IMAI, and T. NAKANISHI, “The structure of Hierarchical Motion Representation of 2-state Number Conserving Cellular Automata," Journal of Cellular Automata vol.14, No.5-6, pp.397-416, 2019.
- G. T. Kong, K. IMAI, and T. NAKANISHI, “A new structure of 2-state number-conserving cellular automata," *IEICE Transactions on Information and Systems*, Vol. E104-D, No.05, 2021, in press.

Referred International Conferences

- G.T. Kong, K. IMAI, and T. NAKANISHI, “Hierarchical Motion Representation of 2-State Number Conserving Cellular Automata," Proc. 5th International Workshop on Applications and Fundamentals of Cellular Automata (CANDAR-AFCA), IEEE Xplore DOI 10.1109/CANDAR.2017.105, pp. 194-199. 2017..

Other International Conferences

- G.T. Kong, and K. IMAI, “On enumeration of number conserving cellular automata by value-1 patterns,” The 12th Annual Meeting of the Asian Association for Algorithms and Computation (AAAC 2019), April 19 – 21, 2019, Seoul.

Contents

Chapter 1

Introduction

1.1 Backgrounds

A cellular automaton (CA), which was introduced by Von Neumann as a biological self-reproducing model, is a discrete dynamical system that evolves in discrete space and time [26]. A CA consists of a grid (finite dimension) of cells of which states are finite numbers. Each cell evolves by a local function of which arguments are its neighborhood cells. One of the famous CA is Conway's "the game of life". In this CA, the state of each cell at time t is either dead or alive. The state of the cell at time $t + 1$ is determined by the states of its neighbors and a predetermined local function. CAs can be used in a variety of applications depending on cell shape, neighborhood, state, and local function [7, 23, 8]. Among many examples, it is often used as a physical model such as car traffic, forest fires, earthquakes, urban growth rates, and lattice gas [10, 12, 9]. CAs for conserving mass or any quantity have also been studied. One of them, the Number-Conserving CA(NCCA), can be interpreted as a model for particle interaction [20, 21]. The state number of each cell is regarded as the number of particles in the cell. The evolution of the NCCA is described by the particle movements between cells.

Number-conserving condition of CA was first discussed by Hattori et al. [11]. Boccara et al. studied NCCA on circular conditions in [2]. There have also been many studies of two-dimensional NCCA [22, 25, 14]. Especially, Durand et al. studied the relation between several boundary conditions and showed that the condition of number-conservation is equivalent for both finite and infinite configurations [6]. The NCCA is widely used to the research of lattice gas, traffic flow, etc. [28, 23] and it has been continuously studied until recently [27].

Boccara et al. also introduced motion representation [1, 2, 3]. Motion representation is another representation of an NCCA. The state number of each cell is regarded as the number of particles in the cell. The evolution of the NCCA is described by the particle movements between cells. Fukś and Pivato showed that one-dimensional NCCA always has motion representation, but various motion representations are possible for an NCCA [4, 5]. However, Moreira et al. have shown that all one-dimensional NCCAs can be uniquely represented as a canonical form of motion representation in [21]. And they used motion representation to effectively express the 1 movement of each two-state NCCA rule.

1.2 Our contributions

The evolution of the NCCA is described by the particle movements between cells. Alhazov and Imai constructed a universal finite NCCA with five particles [13]. We have shown the non-universality of NCCA with three or fewer particles, but four particles an open problem [15].

An NCCA and a motion representation are inherently different computing models. For an NCCA, its neighborhood size is essential in contrast to a motion representation. Even in the case of a two-state simple shift NCCA for a large neighborhood size n, you have to give a length-2^n table. But in the case of motion representation, just the information of a cell of state-1 is enough to identify the value-1 to be moved. Thus only the motion representation is enough to describe the simple shift CA for any neighborhood size.

However, a complicated NCCA also has a complicated motion representation. We introduced a hierarchical motion representation (HMR) that minimizes the motion expression according to the complexity of the motions [16]. In contrast to the normal motion representations, motions in an HMR are arranged in the order of the size of matching pattern and the number of 1's in the pattern [17].

Any two-state NCCA with the state set, $\{0, 1\}$, keeps the number of 1s on the configuration constant. In other words, all the 1s on the configuration move without disappearing or appearing at any time step. Among the two-state NCCAs, there are cases where the evolution is exactly the same, but the different neighborhood size. In this case, the number of value-1 patterns of the NCCAs is different, but they have the same HMR. The relation between n-cell NCCA and $(n-1)$-cell NCCA, one of the main principles of HMR, shows that NCCAs of different sizes can be efficiently expressed through HMR. Through this, we propose an HMR tree that can express all NCCAs of a neighborhood size at once. [17]

When 1s on a configuration are moved by motions defined in a motion representation of a two-state NCCA, these motions are determined by the related argument patterns of its local function of which value is 1.

We proposed a relation (bundle) between patterns of length n and $n-1$, and showed some properties of NCCAs with neighborhood size n and $n-1$ in [17].

Also in [18], we proposed a new structure of value-1 patterns using a relation between patterns of length n and $n-2$. Using that structure, we showed that there are NCCAs that have perfectly identical evolution between neighborhood size n and $n-2$, and show some properties according to the structure.

1.3 Contents of this thesis

The remaining of this thesis is organized as follows.

Chapter 2 describes NCCA and motion representation. We introduce some definitions that are often used in this thesis, and one of the applications of NCCA. In addition, we define some useful relations such as bundle which is defined between different-length patterns. And we also shows the universality of NCCA with less than 3 particles introduced as one of the applications of NCCA.

Chapter 3 proposes hierarchical motion representation (HMR), and shows an algorithm which can compute an HMR from an NCCA, and also proposes an HMR tree that expresses all NCCAs for a neighborhood size using HMR at once.

Chapter 4 proposes a new structure of NCCAs. This chapter introduces bundle pair and bundle quad, which are the relation between patterns of the same length appeared in an NCCA, and introduces some properties according to the structure of NCCAs.

Finally, Chapter 5 concludes this thesis together with some future works.

Chapter 2

Preliminaries

In this chapter, we describes NCCA and bundle that defines the relation between length n and $n-1$ patterns and simple definitions related to them. And we introduces definition of motion representation and some properties. And we shows the universality of NCCA with less than 3 particles introduced as one of the applications of NCCA.

2.1 Number Conserving Cellular Automata

Definition 1 (one-dimensional two-state Cellular Automata)**.** *[18] A one-dimensional two-state cellular automaton A is defined as $A = (n, f)$, where its neighborhood size n is a non-negative finite integer and $f : \{0,1\}^n \to \{0,1\}$ is a mapping called the local function of A. A configuration over $\{0,1\}$ is a mapping $c : \mathbf{Z} \to \{0,1\}$, where $\mathbf{Z}$ is the set of all integers. Then,* $\mathrm{Conf}(\{0,1\}) = \{c|c : \mathbf{Z} \to \{0,1\}\}$ *is the set of all configurations over $\{0,1\}$. The global function F of A is defined as $F : \mathrm{Conf}(\{0,1\}) \to \mathrm{Conf}(\{0,1\})$, i.e.,*

$$\forall c \in \mathrm{Conf}(\{0,1\}),\ \forall i \in \mathbf{Z} : F(c)(i) = f(c(i) \cdots c(i+n-1)).$$

Note that we use the Wolfram numbering [29] $W(f)$ to represent a local function f: $W(f) = \sum f(a_1 \cdots a_n) 2^{2^{n-1}a_1 + 2^{n-2}a_2 + \cdots + 2^0 a_n}$ where the sum is applied on $a_1 \cdots a_n \in \{0,1\}^n$. To represent a CA, we also use a pair of its neighborhood size and its Wolfram number instead of its local function. The local function f is also referred to as the *rule* of A.

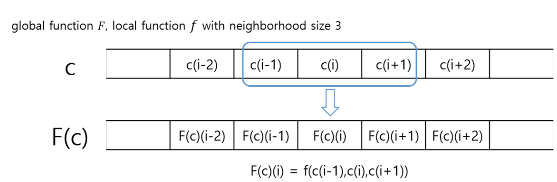

Figure 2.1: Configurations c and $F(c)$ with a 3-cell CA

In this paper, we simply call the CA with the neighborhood size n local function n-cell CA. Figure 2.1 shows two configurations c and $F(c)$ with a one-dimensional 3-cell CA. Figure 2.2 shows the rule table and its space-time diagram of rule 110 which is one of the famous CA rules.

Previous pattern	111	110	101	100	011	010	001	000	Rule
Next state	0	1	1	1	1	0	0	0	110

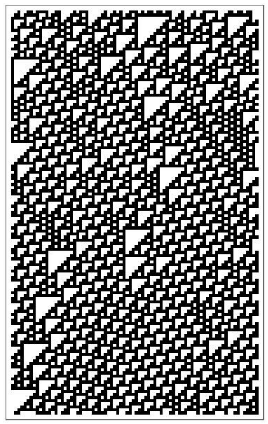

Figure 2.2: The rule table and a space-time diagram of rule 110

State-1 (state-0) cells are shown in black (white), respectively in the diagram.

Definition 2 (Number Conserving CA)**.** *[18] A cellular automaton $A = (n, f)$ is said to be number-conserving iff $F(\alpha_0) = \alpha_0$ and*

$$\lim_{n\to\infty} \frac{\mu_n(F(\alpha))}{\mu_n(\alpha)} = 1 \text{ for all } \alpha \in Conf(\{0,1\}) - \alpha_0$$

where F is the global function of A, α_0 is zero configuration, i.e., the value of every cell is 0, and $\mu_n(\alpha) = \sum_{i=-n}^{n} \alpha(i)$

Since Durand et al. [6] showed that finite-number-conserving is equivalent to the general infinite case, it is enough to show the number-conservation of a CA even for the case of infinite number of nonzero cells.

Fig. 2.3 is the rule table of all 1-dimensional 3-cell NCCAs. In the five 3-cell NCCA rules, 240, 204, and 170 are simple shifts as shown in Fig. 2.4. Especially, rules 184 and 226 are the most famous NCCA rules, the car traffic rules. As shown in Fig. 2.5, during the time evolution of rule 184, the sum of all 1s (black cells) is not changed. The NCCA is famous as a simple model of traffic flow which describes the property that each vehicle moves forward only if there is a space in front of it [23].

NCCA, which the sum of states is conserved, can also be viewed as a particle-based modeling of the mass conservation method. Changing of the number of states of a cell can be regarded as a movement of 'particles' in each time step. Hence Boccara et al. proposed a Motion Representation(MR) to describe the motions of states in [2].

111	110	101	100	011	010	001	000	Rule
1	1	1	1	0	0	0	0	240
1	1	0	0	1	1	0	0	204
1	0	1	0	1	0	1	0	170
1	0	1	1	1	0	0	0	184
1	1	1	0	0	0	1	0	226

Figure 2.3: 3-cell NCCA rules

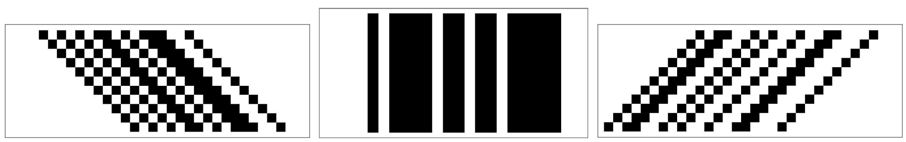

Figure 2.4: Space-time diagram of 240,204,170

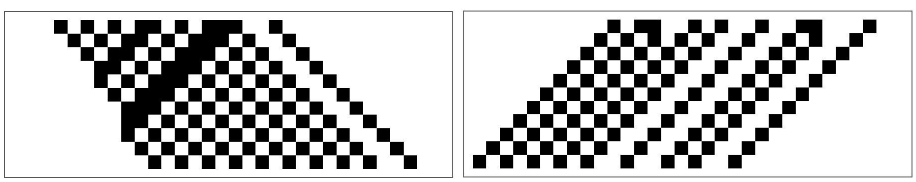

Figure 2.5: Space-time diagram of 184,226

2.2 Bundle

In this paper, we denote a configuration $c = \cdots, c(i), c(i+1), \cdots, c(i+n-1), \cdots$ by $\cdots c_i c_{i+1} \cdots c_{i+n-1} \cdots$ as an abbreviation. We regard a sub configuration of finite size as a pattern and use it as the argument list of a local function f, i.e., we also denote $f(c(i), \cdots, c(i+n-1))$ by $f(c_i \cdots c_{i+n-1})$.

We use the notation, $|\ \ |$, in several different ways. For a pattern set P, $|P|$ means the number of patterns of P. For a pattern p, $|p|$ means the length of p as a pattern string. But it is used in a slightly different way for configurations, i.e., for a configuration c, $|c|$ means the number of 1s.

Definition 3 (Pattern)**.** *[17] A pattern $p = a_1 a_2 \cdots a_n$ is a sequence of $a_i \in \{0, 1\}$ of a finite length n.*

We also use the notation of concatenation of two or more patterns to represent another pattern. For example, if $p = 010$, then $0p = 0010$ and $p1 = 0101$.
In addition, a pattern containing the wildcard character "_" which represents both 0 and 1 is called an extended pattern. For example, _010 means two patterns: 1010 and 0010.

Definition 4 (Bundle)**.** *[17] For a length $n(\geq 1)$ pattern r, if length $n+1$ patterns p and q satisfy the condition*

$$p = 0r,\ q = 1r \text{ (resp. } p = r0,\ q = r1),$$

then we call $p(q)$ l-bundle (resp. r-bundle) of r. When r, p and r, q satisfy both cases, we call r the bundle pattern of p and q.

As shown in Fig.2.6, pattern 011 can be l-bundle of 11 with 111, or an r-bundle of 01 with 010.

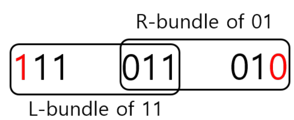

Figure 2.6: An example of bundle

Definition 5 (Value-1 pattern set)**.** *For a CA $A = (n, f)$, we call the pattern set $P_A = \{p | f(p) = 1\}$ the value-1 pattern set of A.*

Definition 6 (Bundle pattern set)**.** *Let P be a pattern set of length n patterns and $l \equiv |P|$ is even. For any length $l/2$ set of pair p_i, q_j $(i \neq j, p_i, p_j \in P)$, if there exists a set B s.t. each pairs in B has a bundle pattern $b_k, (k = 1, \cdots, l/2)\}$ then we call $\hat{P} \equiv \{b_1, \cdots, b_{l/2}\}$ a bundle pattern set of P.*

If there is no such set, there is no bundle pattern set of P.

For example, $\{01, 11\}$ can be a bundle pattern set of $\{001, 101, 110, 111\}$. Because 01 can be a bundle pattern of 001, 101 and 11 can be a bundle pattern of 110, 111. The following sets have no bundle pattern set:

$$\{001, 010, 110, 111\}, \{011, 110, 111\}, \{01, 11, 110, 111\}.$$

2.3 Motion representation

Because an evolution of NCCA can be regarded as the movement of particles, there is another representation of an NCCA rule, motion representation [1, 21].

Definition 7 (Motion Representation)**.** *Let $\tilde{p}$ be an extended pattern for invoking a motion μ. A motion μ is defined as $\mu = (\tilde{p}, s, e, v)$ where s and e is a start location and an end location, respectively. v is a finite nonzero integer which represents a moving value from s to e. Let $\mathcal{M}$ be a set of motions $\{\mu_1, \ldots, \mu_n\}$. For any configuration c of an NCCA $A = (n, f)$ with the global function F, for each position in c to which a translated p_i (of μ_i) matches, subtract v_i from the cell s_i and add v_i to the cell e_i simultaneously. If the resulting configuration is equal to $F(c)$ for any c, $\mathcal{M}$ is a motion representation of A.*

We also graphically represent a motion μ by an arrow over $\tilde{p}$ from s to e whose suffix is v. Fig. 2.7 is the graphical representation of

$$\mu = \{(\_01, 1, 3, 1), (11, 1, 2, 1)\}.$$

The suffix is omitted when v=1.

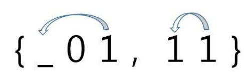

Figure 2.7: Graphical representation of motion representation

It is shown that the motion representation of any 2-state CA can be composed of motions with $v = 1$ [14]. We show the outline of the proof: suppose there is a motion $\mu_1 = (p_1, s_1, e_1, 2)$ in a motion representation. The start cell should be the destination of another motion, say $\mu_2 = (p_2, s_2, e_2, v_2)$. Then if $v_2 = 1$ these motions can be replaced by $\mu_1' = (p_1, s_1, e_1, 1)$ and $\mu_3 = (p_3, s_3, e_3, 1)$ where p_3 is a union of μ_1 and μ_2 and s_3 (e_3) is the related position to s_2 (e_1), respectively. If $v_2 = 2$ then $\mu_2' = (p_2, s_2, e_2, 1)$ is also remained. In turn, it is possible to replace all motions of 2 (or more particles).

2.4 On particle complexity of number conserving cellular automata

A two-state NCCA with only 0 or 1 states always keeps the number of 1s on the configuration. That is, all 1s in the configuration will move without disappearing or appearing. Motion representation was used to effectively represent the movement of 1 in each 2-state NCCA rule [1, 2]. But when motion representation is applied to a configuration, the state of each cell can be negative, and motion representation must be applied to all cells. In contrast, HMR applies motion only to state-1 cells on a configuration, and the state of cells should be 0 or positive. We have shown that all one-dimensional two-state NCCAs have HMR, and we also introduced the algorithm [17]. This two-state NCCA is suitable for representing particle movement, and Alhazov and Imai have shown that a universal finite NCCA with five particles can be constructed [13].

In this section, we show that NCCAs with two or three particles are not strong Turing-universal, using ultimate periodicity and partwise ultimate periodicity.

Definition 8 (Strong Turing universality)**.** *A cellular automaton A is said to be strongly Turing-universal if it can simulate any deterministic Turing machine from a finite configuration.*

Definition 9. *A configuration c of A is ultimately periodic if $\exists i, j (0 \leq i < j)$, $\forall x \in$ $\mathbf{Z}, F^i(c(x)) = F^j(c(x+\sigma))$ where F is the global function of A and σ is a constant.*

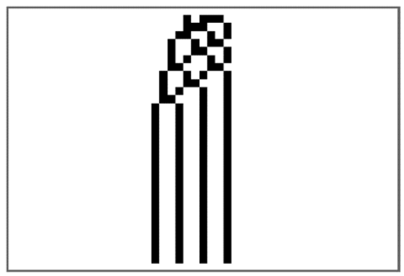

Figure 2.8: An example of ultimately periodic configuration

Also, if a configuration c can be divided into two ultimately periodic configurations, c_1 and c_2, for some $p \in \mathbf{Z}$ as follows:
$c_1(i) = \begin{cases} c(i) & i \leq p \\ 0 & i > p \end{cases}$ $c_2(i) = \begin{cases} 0 & i \leq p \\ c(i) & i > p \end{cases}$ We call *c partwise ultimately periodic*.

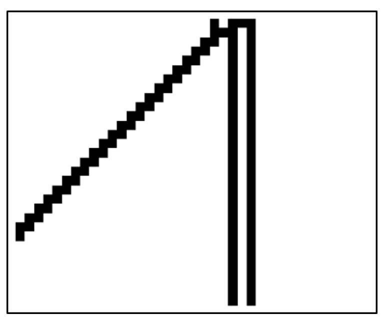

Figure 2.9: An example of partwise ultimately periodic configuration

Definition 10. *A cellular automaton A is ultimately periodic if c is ultimately periodic for any configuration c.*

Definition 11. *Two configurations c_1 and c_2 are independent if $\exists p, F^t(c_1(s)) = 0, s > p$ and $F^t(c_2(s)) = 0, s <= p, \forall t > \tau$ for some τ.*

2.4.1 On particle complexity of NCCA

In this section, we show that NCCAs with two or three particles are not strongly Turing-universal.

There is no universal finite NCCA with only one particle. Because the particle always follow the basic motion of A.

Theorem 1. *Finite NCCA with two particles is ultimately periodic.*

Proof: We prove finite NCCA with only two particles is not strongly Turing-universal by using that those NCCAs are ultimately periodic. Let A be an n-cell NCCA (n is finite) and l be a gap length(the number of cells between the particles) of two particles. There are only two cases of initial configurations with two particles: $l_0 > n$ and $l_0 \leq n$.

If $l_0 > n$ in the initial configuration, the two particles will each follow the basic motion of A, and $l_t = l_0$ for any $t(> 0)$. Then A is ultimately periodic.

If $l_0 \leq n$ in the initial configuration, there exists a time step τ and $l_\tau > n$ or $l\tau = l_t(0 \leq t < \tau)$, since n is finite. Thus the both cases are ultimately periodic. □

Theorem 2. *Finite NCCA with three particles is not strongly Turing-universal.*

Proof: We prove finite NCCA with only three particles are not strongly Turing-universal by using ultimately periodicity or partwise ultimately periodicity.

Let A be an n-cell NCCA (n is finite) and l_t, r_t be two gap lengths of three particles on a configuration $F^t(c)$ of time $t(\geq 0)$. There are three cases of the initial configuration c with three particles as follows.

(a) $l_0 > n-2,\ r_0 > n-2$

(b) $l_0 \leq n-2,\ r_0 > n-2$ or $l_0 > n-2,\ r_0 \leq n-2$

(c) $l_0 + r_0 \leq 2n-4$

If the initial configuration c satisfies (a), because l_0 and r_0 are longer than the size of the neighborhood n, the three particles follow the basic motion of A, then for any t, $l_t = l_{t+1}$ and $r_t = r_{t+1}$. Then A is ultimately periodic.

If c satisfies (b), the three particles are split into one and two, and t satisfies $\forall t, l_t + r_t > 2n-4$ or $\exists t,$ s.t. $l_t + r_t \leq 2n-4$.

If t satisfies $\forall t, l_t + r_t > 2n-4$, one particle follow the basic motion of A, and the two particles are ultimately periodic, as shown in Theorem 1. Since the three particles split never meet, the configuration $F^u(c)(u > t)$ also can be split into configuration c_1 containing one particle and configuration c_2 containing two particles. $\exists p, \forall t > u, F^t(c_1(s)) = 0, s > p$ and $F^t(c_2(s)) = 0, s <= p$ Then, A is partwise ultimately periodic with the two independent configurations c_1 and c_2.

If t satisfies $\exists t,$ s.t. $l_t + r_t \leq 2n-4$, it is the same as in the case (c) described below.

If c satisfies (c), t satisfies $\forall t, l_t + r_t \leq 2n-4$ or $\exists t,$ s.t. $l_t + r_t > 2n-4$. If t satisfies $\forall t, l_t + r_t \leq 2n-4$, since n is finite, $\exists t'$ s.t. $l_{t'} = l_t, r_{t'} = r_t$. Then A is ultimately periodic. If t satisfies $\exists t,$ s.t. $l_t + r_t > 2n-4$, the three particles are split and closed repeatedly. Then $\exists i, j(> t)$ s.t. $l_i = l_j, r_i = r_j$ because n is finite. Then A is ultimately periodic.

Finally, A is ultimately periodic or partwise ultimately periodic with two independent configurations. Then A is not strongly Turing-universal. □

2.4.2 Conclusion

In this section, we showed that NCCA with two particles was shown to be ultimately periodic, whereas NCCA with three particles was shown to be ultimately periodic or partwise ultimately periodic, showing that it is not strongly Turing-universal. In [13], Alhazov et al. showed that it is possible to construct a universal NCCA with five particles, so the case of four particles is still open.

Chapter 3

The structure of Hierarchical Motion Representation of 2-state Number Conserving Cellular Automata

An NCCA and a motion representation are inherently different computing models. For an NCCA, its neighborhood size is essential in contrast to a motion representation. Even in the case of a two-state simple shift NCCA for a large neighborhood size n, you have to give length 2^n table. But in the case of motion representation, just the information of a cell of state 1 is enough to identify the value 1 to be moved. Thus only the motion representation is enough to describe the simple shift CA for any neighborhood size. The simplest car traffic rule 184 [23], can be regarded as the combination of such a basic shift and a motion depending on a size-two pattern, even the evolution can be embedded into an NCCA of any neighborhood size which is larger than three. So any motion representation of a two-state NCCA seems to be constructed by the set of motions that are ordered by their pattern size.

In this chapter, we propose a hierarchical motion representation (HMR) and an algorithm to compute the HMR from an NCCA rule which is first appeared in our article [16]. And we introduce a binary tree structure called a bundle tree which represents a relation between a pattern set of the local function arguments of an n-cell NCCA and that of less than n. We show the proof of the underline properties and give the algorithm of computing HMR.

3.1 The Bundle tree

In this section, we show some properties of NCCA and the main principle of bundle tree for any NCCA, they give the theoretical basis for the algorithm1.

From now, we show the value-1 pattern set of each $(n-1)$-cell NCCA is a bundle pattern set of an n-cell NCCA as Theorem 3 which is the basis for an algorithm of computing hierarchical motion representation (Algorithm 1).

Lemma 1. *For a pattern p in a value-1 pattern set of an NCCA, there exists a pattern q in the set and a pattern r s.t. p and q are either l-bundle or r-bundle of r.*

Proof: Let F be the global function of an NCCA $A = (n, f)$. For a configuration $\cdots 0c_1c_2\cdots c_n0\cdots$, the following equation holds:

$$|F(\cdots 0c_1c_2\cdots c_n0\cdots)| = f(0\cdots 0c_1) + f(0\cdots 0c_1c_2)$$
$$+\cdots + f(c_1c_2\cdots c_n) + \cdots + f(c_{n-1}c_n0\cdots 0) + f(c_n0\cdots 0).$$

Considering the configuration:
$c = \cdots 0c_1c_2\cdots c_n\underbrace{0\cdots 0}_{k}\bar{c_1}c_2c_3\cdots c_n\underbrace{0\cdots 0}_{k}c_1c_2\cdots c_{n-1}\bar{c_n}0\cdots (k > n)$,
the following equation holds:

$$|c| = 2\sum_{k=1}^{n} c_k + (\bar{c_1} + c_2 + \cdots + c_{n-1} + \bar{c_n}),$$

where $\bar{c}$ is the negation of c, i.e., $c + \bar{c} = 1$. Since $f(0\cdots 0) = 0$ then

$$\begin{aligned}|F(c)| &= |F(\cdots 0c_1\cdots c_n0\cdots)| + |F(\cdots 0\bar{c_1}c_2\cdots c_n0\cdots)| + |F(\cdots 0c_1\cdots c_{n-1}\bar{c_n}0\cdots)|\\ &= 2|F(0\cdots 0c_1\cdots c_n0\cdots 0)| - f(c_1c_2\cdots c_n) + f(0\cdots c_1)\\ &+\cdots + f(\bar{c_1}c_2\cdots c_n) + f(c_1\cdots c_{n-1}\bar{c_n}) + f(c_2\cdots c_{n-1}\bar{c_n}0) + \cdots + f(\bar{c_n}0\cdots 0).\end{aligned}$$

Moreover the next formula holds because F is the global function of an NCCA,

$$2\sum_{k=1}^{n} c_k + (\bar{c_1} + c_2 + \cdots + c_{n-1} + \bar{c_n}) = |F(c)|. \tag{3.1}$$

Because $2\sum_{k=1}^{n} c_k = |F(0\cdots 0c_1\cdots c_n0\cdots 0)|$ and $\bar{c_1} + c_2 + \cdots + c_{n-1} + \bar{c_n} = |F(0\cdots 0\bar{c_1}c_2\cdots c_{n-1}\bar{c_n}0\cdots 0)|$, we can get the following formula from (3.1):

$$f(\bar{c_1}c_2\cdots c_{n-1}\bar{c_n}) + f(c_1c_2\cdots c_n)$$
$$= f(\bar{c_1}c_2\cdots c_n) + f(c_1\cdots c_{n-1}\bar{c_n}) \tag{3.2}$$

From (3.2), we can get the following result:
if $f(c_1c_2\cdots c_n) = 1$ then $f(\bar{c_1}c_2\cdots c_n) = 1$ or $f(c_1c_2\cdots c_{n-1}\bar{c_n}) = 1$. □

By Lemma 1, we can know that there are always bundle pattern for all patterns in value-1 pattern set of an NCCA. For example, When a pattern can be paired with two different elements in the value-1 pattern set P of an NCCA, for example, $\{101, 100, 001\} \subset P$, we show that 000 is also in P by the next Lemma 2.

Lemma 2. *Let P_A is a value-1 pattern set of an NCCA $A(n, f)$. If three patterns $a_1 \cdots a_n, a_1 \cdots a_{n-1}\bar{a}_n, \bar{a}_1 a_2 \cdots a_n$ are in P_A then pattern $\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n$ is also in P_A.*

Proof: Suppose $\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n \notin P_A$ and $a_1 \cdots a_n, a_1 \cdots a_{n-1}\bar{a}_n, \bar{a}_1 a_2 \cdots a_n \in P_A$, in other words, $f(a_1 \cdots a_n) = f(a_1 \cdots a_{n-1}\bar{a}_n) = f(\bar{a}_1 a_2 \cdots a_n) = 1$, and $f(\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n) = 0$. Because A is NCCA, we can get following formula:

$$\begin{aligned} a_1 &= f(a_1 \cdots a_n) + f(0a_1 \cdots a_{n-1}) + \cdots + f(0 \cdots 0a_1) - \{f(0a_2 \cdots a_n) + \cdots + f(0 \cdots 0a_2)\} \\ &= f(a_1 \cdots a_{n-1}\bar{a}_n) + f(0a_1 \cdots a_{n-1}) + \cdots + f(0 \cdots 0a_1) - \{f(0a_2 \cdots a_{n-1}\bar{a}_n) \\ &\quad + \cdots + f(0 \cdots 0a_2)\}. \end{aligned}$$

Then $0 = f(a_1 \cdots a_n) - f(0a_2 \cdots a_n) - \{f(a_1 \cdots a_{n-1}\bar{a}_n) - f(0a_2 \cdots a_{n-1}\bar{a}_n\}$ is necessary. Since $0 = f(a_1 \cdots a_n) = f(a_1 \cdots a_{n-1}\bar{a}_n) = 1$,

$$f(0a_2 \cdots a_n) = f(0a_2 \cdots a_{n-1}\bar{a}_n) \tag{3.3}$$

In the same way,

$$\begin{aligned} \bar{a}_1 &= f(\bar{a}_1 a_2 \cdots a_n) + f(0\bar{a}_1 a_2 \cdots a_{n-1}) + \cdots + f(0 \cdots 0\bar{a}_1) - \{f(0a_2 \cdots a_n) + \cdots + f(0 \cdots 0a_2)\} \\ &= f(\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n) + f(0\bar{a}_1 a_2 \cdots a_{n-1}) + \cdots + f(0 \cdots 0\bar{a}_1) - \{f(0a_2 \cdots a_{n-1}\bar{a}_n) \\ &\quad + \cdots + f(0 \cdots 0a_2)\}. \end{aligned}$$

Then $0 = f(\bar{a}_1 a_2 \cdots a_n) - f(0a_2 \cdots a_n) - \{f(\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n) - f(0a_2 \cdots a_{n-1}\bar{a}_n\}$.
Since $f(\bar{a}_1 a_2 \cdots a_n) = 1,\ f(\bar{a}_1 a_2 \cdots a_{n-1}\bar{a}_n) = 0$,

$$f(0a_2 \cdots a_n) = 1 + f(0a_2 \cdots a_{n-1}\bar{a}_n) \tag{3.4}$$

Formula (3.3) and (3.4) be a contradiction. □

In the making process of a bundle pattern set from the value-1 pattern set of an NCCA, a pattern might have two different candidates for pairing to form a bundle. For example, the value-1 pattern set of an NCCA rule 204 is $\{010, 011, 110, 111\}$. In this case, 010 can not only make a pair with 011 to 01 but also 110 to 10. But by Lemma 2, there exist another pattern 111 which can be paired with 011 and 110. Therefore we can make two distinct pairs like $\{010, 011\}, \{110, 111\}$ or $\{010, 110\}, \{011, 111\}$. Thus, all patterns in the value-1 pattern set of an n-cell NCCA can always be paired, and the bundle pattern set has exactly 2^{n-2} patterns.

Therefore for all NCCA A, there exist a bundle pattern set $\hat{P}_A$ where $|\hat{P}_A| = |P_A|/2$.

Theorem 3. *For an n-cell NCCA A $(n \geq 2)$ with $|P_A| = 2^{n-1}$, an $(n-1)$-cell CA B satisfying $P_B = \hat{P_A}$ is an $(n-1)$-cell NCCA when $|\hat{P}_A| = 2^{n-2}$.*

Proof: Let F be the global function of an NCCA $A = (n, f)$. Because A is an NCCA, $|c| = |F(c)|$ for any configuration $c(= \cdots c_{-1}c_0c_1 \cdots)$. Let G be the global function of a CA $B = (n-1, g)$. If $|G(c)| = |F(c)|$ then $|c| = |F(c)| = |G(c)|$. i.e., B can be an NCCA. Then we will show $|G(c)| = |F(c)|$ from now.
For each state-1 cell $F(c)(k) = 1$, $k \in \mathbb{Z}$ on $F(c)$,

$$\exists p = c_k \cdots c_{k+n-1} \in P_A \text{ s.t. } f(p) = 1.$$

Also for a pattern $q \in \hat{P}$, which is a bundle pattern of p, it can be satisfied $g(q) = 1$(i.e., $q \in P_B$). Thus if $F(c)(k) = 1$ then either $G(c)(k) = 1$ or $G(c)(k+1) = 1$ holds. When p is an l-bundle of q, $G(c)(k) = 1$ and when p is an r-bundle of q, $G(c)(k+1) = 1$ like Fig. 3.1.

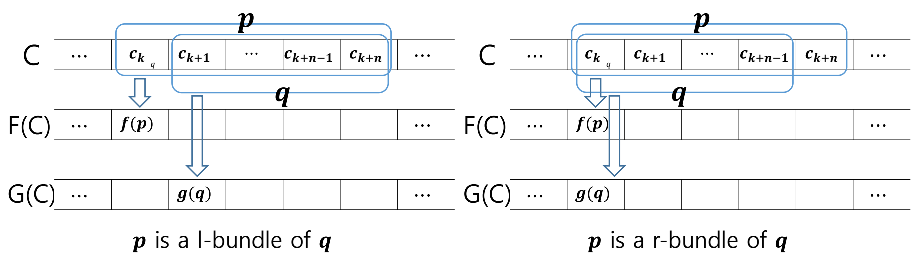

Figure 3.1: Two evolutions according to the relation between p and q

If $F(c)(k) = F(c)(k+1) = 1$ then
(1) by $F(c)(k) = 1$, either $G(c)(k) = 1$ or $G(c)(k+1) = 1$ holds,
(2) by $F(c)(k+1) = 1$, either $G(c)(k+1) = 1$ or $G(c)(k+2) = 1$ holds.
So if $G(c)(k+1) = 1$ occur simultaneously in (1) and (2), an overlap occurs. Thus $|F(c)| = |G(c)|$ is satisfied if there is no overlap.
Suppose that the above case has occurred. Then there are four patterns $p = c_k \cdots c_{k+n-1}$, $q = c_{k+1} \cdots c_{k+n-1}$ which is an l-bundle of p and $p' = c_{k+1} \cdots c_{k+n}, q' = c_{k+1} \cdots c_{k+n-1}$ which is an r-bundle of p. Then q is the same with q'. Because of two patterns q, q' are the same in P_B, $|P_B| = |P_A|/2 - 1$. This contradicts the assumption, $|P_B| = |P_A|/2$.
In the result, $G(c)$ is a 2-state configuration with $|G(c)| = |F(c)| = |c|$. Then B is an NCCA.

The value-1 pattern set of 1-cell NCCA is $\{1\}$, so the number of elements in P_A of n-cell NCCA ($n \geq 2$) is always 2^{n-1}. □

Theorem 3 shows that the value-1 pattern sets of an n-cell and an $(n-1)$-cell NCCA have a kind of hierarchical relation and we can extract the relation as a tree structure as follows:
Let P_n be the value-1 pattern set of an NCCA $A_n = (n, f_n)$. By Theorem 3, we can get a sequence of value-1 pattern sets $P_i (n \geq i \geq 1)$ of $A_i = (i, f_i)$ where $P_{i-1} = \hat{P}_i$ and $|P_i| = 2^{i-1}$. For each element r in P_i $(1 < i < n-1)$, there are two elements $p, q \in P_{i+1}$ where $p(q)$ is an l-(r-)bundle of r, respectively. We can construct a tree $T_{A_n} = (V, E)$ where V is the set of all elements in all sets P_i and E is the set of all edges (r, p) and (r, q) described above. Clearly, T_{A_n} is a complete binary tree and its root vertex is 1 and its height is $n-1$. The height-i vertices of T_{A_n} are the elements of the value-1 pattern set of A_i. We call T_{A_n} a *bundle tree* of A_n. Fig. 3.2 is a bundle tree of the CA $(4, 62600)$.

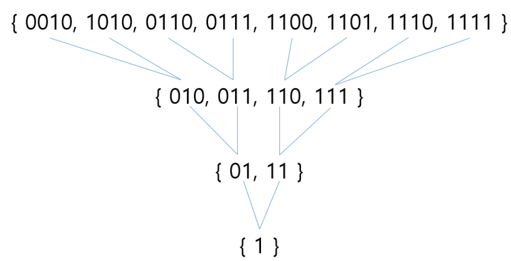

Figure 3.2: A bundle tree of (4,62600)

By Lemma 1, it is clear that a bundle tree exists for any NCCA rules. Moreover we can get Corollary 1 and Theorem 4 by Theorem 3.

Corollary 1. *For a bundle tree of A_n, $\forall i \in \mathbb{Z}$, A_i is an i-cell NCCA.*

Theorem 4. *Bundle tree of any NCCA is always a binary tree with root* $\{1\}$.

Proof: The number of elements of value-1 pattern set of an NCCA (n, f) is always 2^{n-1}. Then it is clear that the number of elements of pattern sets at i th level is 2^{i-1} by corollary2. Moreover the smallest cell NCCA is 1-cell NCCA 1. Then bundle tree of any NCCA be a binary tree with root(the 1st level) 1. □

In the next section, we introduce HMR and an algorithm to compute an HMR from an NCCA rule that was introduced in [16].

3.2 Hierarchical Motion Representation of NCCA

A Motion representation can illustrate an evolution of any NCCA by the movement of particles when the number of each cell state is thought to repesent the number of “particles” in the cell. In motion representation, where each motion consists of a start and an end locations and a pattern, the start and end locations are set to the position of the cell in the pattern, so it is difficult to specify the particle to be moved when the pattern contains two or more state-1 cells.

In Hierarchical Motion Representation (HMR), which we propose in this paper, motions are ordered by the complexity of their patterns, and only state-1 cells can be the starting location of motions. In contrast to MR, a motion is not always applied to any matching pattern in HMR. Although a simple algorithm is needed to choose the applicable motion, it is easy to find particles to be moved.

In motion representation, any cells can be the start location regardless of cell state (even the state 0), while in HMR it is only possible to move a particle in any non-zero state cell (only state-1 cells can be the start locations). For each state-1 cell in a configuration c, we have to determine its movement for the next configuration. An assignment of arguments for the local function which value is 1 is responsible for the movement. We only need to consider the patterns in a motion representation as its argument. The only problem is which 1 in the pattern is moved. We call the 1 $the\ focus$1. And we express the location of the focus 1 in the pattern (or extended pattern) by index as follows.

$$0101_4$$

It means that if the pattern 0101 is in a configuration then the pattern 010 is a neighbor of the focus 1 (located at the 4th number) which can get the movement (move to left 3 cells). Moreover the focus 1 is an essential element to get the hierarchical motion representation which will be proposed as follows.

Definition 12 (Hierarchical Motion Representation (2-state))**.** *Let $\tilde{p}$ be an extended pattern for invoking a motion μ. A motion μ is defined as $\mu = (\tilde{p}, l)$ where l is the location of the focus 1. Let HMR $\hat{\mathcal{M}}$ be a list of motions $\{\mu_1, \ldots, \mu_n\}$. For $i \in \{1, 2, \cdots\}$, the $\tilde{p}_i$ of μ_i has the shorter length pattern which is excluded "_" than $\tilde{p}_{i+1}$'s or $\tilde{p}_i$ has the fewer the number of 1s than $\tilde{p}_{i+1}$. Also we call μ_1 of an HMR the basic motion.*

Also, the basic motion means the movement of the 1 when there is only one 1 on the configuration as Fig. 3.3.

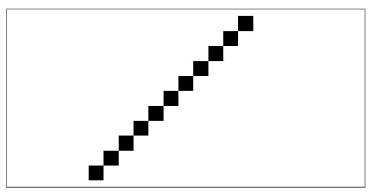

Figure 3.3: A space-time diagram of a basic motion _1

In addition to when applying the HMR on a configuration, the motion to the right applies first.

[Application scheme of $\hat{\mathcal{M}}$] The application of $\hat{\mathcal{M}}$ to a configuration c is defined as follows: for each $x \in L(c)$, check that $\tilde{p}_i$ of μ_i matches to c at the position x (i.e., check $\tilde{p}_i = c(x - s + 1) \cdots c(x + |\tilde{p}_i| - s)$ or not) for i from n to 1. If $\tilde{p}_k$ ($n \geq k \geq 1$) first matches to c then 1 at the cell x will move to the cell $x - s + 1$, i.e., the first position related to $\tilde{p}_k$.

Contrary to motion representation, there is an application order of motions from right to left in HMR. To place a motion with a shorter pattern to the left position, we can represent a hierarchy of applicable motions.

Fig. 3.4 is an example of the application of an HMR to a configuration c. For each cell of state-1 in c, check the pattern of each motion in $\hat{\mu}$ as its start location is overlapped to the cell. This check is performed for all motions in $\hat{\mu}$ from right to left. If the matching motion is found, then the end location to move the 1 is determined.

Let c_i be a state-1 cell. If $c_{i+1} = 0, c_{i+2} = 1$, then the 1 of c_i will move to c_{i-1}. If $c_{i+1} = 1$, then the 1 of c_i will move to c_{i-1}. In other cases, the 1 will stay at c_i. Through

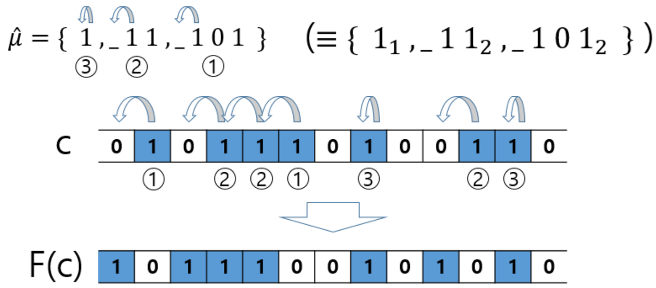

Figure 3.4: Evolution by the HMR of (4,58336)

this process, we can get $F(c)$ in Fig.3.4.

The location of the focus 1 should be inherited through from the bundle patterns. For example, we can think about the location of the focus 1 of 1101_2 be inherited from 110_2 or 101_1. If 1101 is an r-bundle of 110 (resp. l-bundle of 101), the location of the focus 1 will be not changed (resp. the location of the focus 1 will be moved to "+1"-th number). Thus the process of finding the location of the focus 1 of patterns called "tracking the focus 1", and the figure formed in the process called a "constructing the bundle tree".

In this case, the location of the focus 1 is changed from 1 to 2 because the location of the focus 1 is shifted to the second cell of 1101. If 1101_2 is an r-bundle of 110.

(neighborhood size, wolfram number)	Motion representation	HMR
(2, 12)	{ 1 }	{ 1 }
(3, 226)	{ _ 0 1, 1 1 }	{ _ _ 1, 1 1 }
(4, 48770)	{ _ 0 0 1 , 1 0 1, 1 1 0 , _ 1 1 1}	{ _ 0 0 1 , 1 1 , 1 0 1 , _ 1 1 1}

Figure 3.5: Motion representations and HMRs of rules

Fig. 3.5 shows motion representations and HMRs of several NCCA rules as examples. There are two differences between them. First, in (2,12) case, the stable motion needs to be explicitly described in HMR contrary to its motion representation. Second, motions in HMR has an ordering. Especially HMR always contains a motion with only one 1 what we called basic motion. And it is placed on leftmost in a HMR. There exist n basic motions in all HMR of n-cell NCCA then we can classify every HMR of the same cell NCCAs.

We can compute the HMR of the n-cell NCCA of Wolfram number w by Algorithm 1.

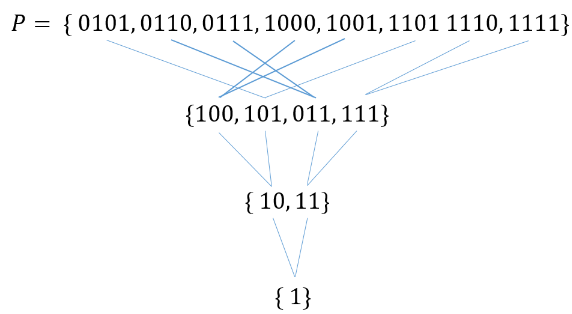

Figure 3.6: A bundle tree of $(4, 58336)$

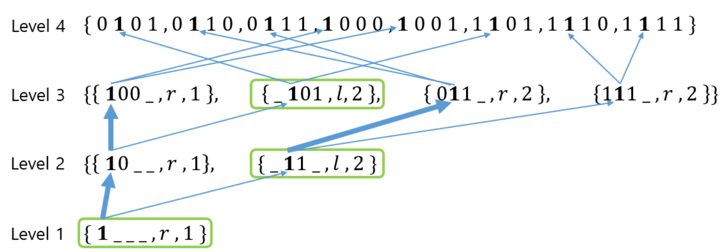

Figure 3.7: An HMR formation process by Algorithm 1

Fig.3.6 and Fig.3.8 show the bundle tree and the tracking process of the focus 1 by this algorithm about $(4, 58336)$. We briefly describe the algorithm as follows:
Starting from the value-1 pattern set of A, for a pattern p which has the smallest number of 1, find a bundle pattern w of p;
First, if p is an r-bundle of w ($w0 = p$) with another r-bundle $q(= w1)$ of w will be in P, then $\{w\_, r\}$ will be a node of the one-step lower level. Otherwise, (if p is an l-bundle of w ($0w = p$) with there is not another r-bundle of w) then $\{\_ w, l\}$ will be a node of the one-step lower level. By repeating the above process, the one-step lower level will be consist of nodes {extended pattern, direction}. For each w in the above, repeat the above process until the length of w will be 1. Then we can get a bundle tree of A like Fig.3.6.
According to the bundle tree, we can track the location of the focus 1 (bold 1 in Fig. 3.7) through the trees (represented by arrows in Fig. 3.7). Between three nodes connected by bundle, when the location of the focus 1 of a mother node is i th number, the location of the focus 1 of two nodes of the one-step upper level be according to the direction

Algorithm 1 HMR(n,w)

```
//INPUT n: neighborhood size, w: Wolfram number
//OUTPUT HMR
Constant l, r                    // direction
E : a two dimensional array of lists // each element has the structure
{extendedpattern, direction, [bundlepattern, [offset]]}
{Constructing the bundle tree:}
P := a value-1 pattern set of the rule (w, n) with ascending order as the list of binary
representations of integers.
for k = 1 to n − 1 do
  for each element e in E[k] do
    Append(Remove[e[1], "_"]) to P   // When k = 1, E[k] = {} thus this line
    is skipped.
  end for
  while Length[P] > 1 do
    p := the pattern which contains the smallest number of 1 in the patterns in P
    a :=Subseq(p, 1, 1)      // Subseq(p, i, j) returns p_i p_{i+1} ··· p_j
    b :=Subseq(p, n, n);  w :=Subseq(p, 2, n − 1)
    if a w b̄ ∈ P then
      add {aw_, r} to E[k]
      remove awb, awb̄ in P
      if k > 1 then
        i :=Position[P, awb];  j :=Position[P, awb̄]
        E[k − 1][i] :=Append{E[k − 1][i], aw}  // add bundle pattern
        E[k − 1][j] :=Append{E[k − 1][j], aw}
      end if
    else
      if āwb ∈ P then
        add {_wb, l} to E[k]
        remove awb, āwb in P
        if k > 1 then
          i :=Position[P, awb];  j :=Position[P, āwb]
          E[k − 1][i] :=Append{E[k − 1][i], wb}
          E[k − 1][j] :=Append{E[k − 1][j], wb}
        end if
      else
        return {}            // Non NCCA
      end if
    end if
  end while
end for
TrackingTheFocus1(n, E)  // Update E
return GeneratingHMR(n, E)
```

Algorithm 2 Tracking the focus $1(n, E)$

```
//INPUT n: neighborhood size, E : two dimensional array of lists
//OUTPUT null   //   but E will be modified.
e := E[n − 1][1]   //   E[n] only has a value at position 1.
if e[2] == l then
   e[3] := 2
else
   e[3] := 1
end if
for k = n − 2 down to 1 do
   P := {}
   for each element e in E[k + 1] do
      Append(Remove[e[1], "_"]) to P
   end for
   for each element e in E[k] do
      i :=Position[P, e[3]]
      if e[2] == l then
         replace 4th element of e to the 4th element of E[k + 1][i] + 1
      else
         replace 4th element of e to the 4th element of E[k + 1][i]
      end if
   end for
end for
```

Algorithm 3 Generating HMR

```
//INPUT n: neighborhood size, E : two dimensional array of lists
//OUTPUT HMR : list of motions {extendedpattern, offset}
H := {E[n − 1][1]}
for k = 1 to n − 2 do
   for each element e in E[k + 1] do
      Append(Remove[e[1], "_"]) to P
   end for
   for each element e in E[k] do
      i :=Position[P, e[3]]
      if e[4] ≠ 4th element of E[k + 1][i] then
         Append e to H
      end if
   end for
end for
for each element h in H do
   Collect(h[1], h[4]) to result   //   h[4] be an suffix of h[1]
end for
return result
```

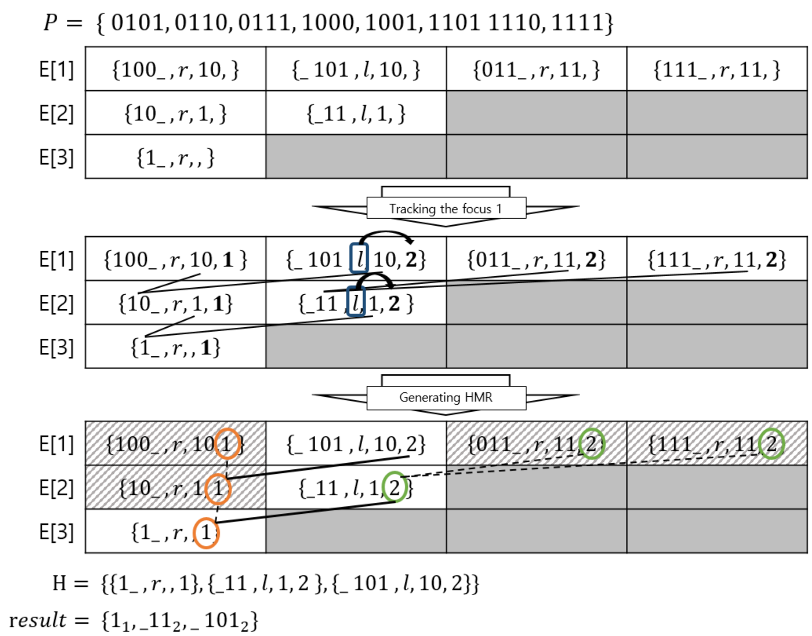

Figure 3.8: Process of Algorithm1 (4,58336)

of each node. If the direction of a node is r then the location of the focus 1 be i th number, otherwise, the location of the focus 1 be $i + 1$ th in the extended pattern. For example, in Fig. 3.6, because the location of the focus 1 of 1 _ _ _ is 1, the location of the focus 1 of 10 _ _ , _ 11 _ be 1 and 2 because of the direction is r and l respectively.

From the above step of the algorithm, the location of the focus 1 is added as the third element of each node. If the directions of two nodes from a mother node are the same, the motions can be replaced by that of their mother node. Thus the set of motions depicted in the blue boxes is remained to be the HMR of the rule. In the final step of the algorithm, blanks on the right-hand side of each pattern are removed and the locations of the focus 1 are shifted by the suffix of the pattern then we can get the HMR $\{1_1,\ \_11_2,\ \_101_2\}$.

3.3 Representation of NCCAs by an HMR tree

In this section, we show HMRs of all NCCAs of neighborhood size 2, 3, 4, and 5. We employ a tree form representation of HMRs (HMR tree). Each HMR of all NCCAs is computed by the algorithm 1 from the rule based enumerated results of NCCA rules in Fig. 3.10 and Fig. 3.12. Although number-conservation is a strong condition and the number of NCCAs is very small, the number of 5-cell NCCAs is 428 and it is quite difficult to understand the overall properties of them by just watching the list of all rules. Employing the HMR trees, we can express the structure of all NCCAs briefly. The HMR trees of a k-cell NCCAs are composed by k-trees. The root node of each tree is one of the k basic motions. Gathering motion(s) in each child node, we can get all HMRs categorized in the basic motion.

There are three types of motions in a node. Motions in $< >$ means that one of the elements in $< >$ is chosen. For example, $\left\{\_\,\_\,\_\,\_\;1_5,\ \_\;111_4,\ <^{\_\;101_4}_{1011_4, 0101_4}>\right\}$ means $\{\_\,\_\,\_\,\_\;1_5,\ \_\;111_4,\ \_\;101_4\}$ and $\{\_\,\_\,\_\,\_\;1_5,\ \_\;111_4, 1011_4, 0101_4\}$. The symbol [ ] means that each element in the power set of all elements in [ ] is taken as the element in the node[1]. For example, a branch with a node $\{1_1, [\_\;11_2]\}$ means that there is two branches, i.e., the branch with $\{1_1\}$ and the branch with $\{1_1,\ \_\;11_2\}$.

Fig. 3.9 and 3.11 are HMR trees of up to 4-cell NCCA rules.

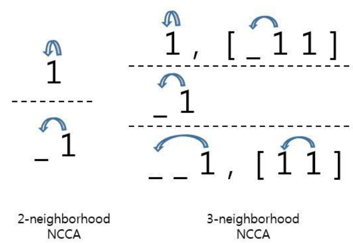

Figure 3.9: HMR trees of 2 and 3-cell NCCAs

1111	1110	1101	1100	1011	1010	1001	1000	111	0110	0101	0100	0011	0010	0001	0000	rule
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	43690
1	0	1	0	1	0	1	1	1	0	1	0	1	0	0	0	43944
1	0	1	1	1	0	0	0	1	0	1	1	1	0	0	0	47288
1	0	1	1	1	1	0	0	1	0	0	0	1	1	0	0	48268
1	0	1	1	1	1	1	0	1	0	0	0	0	0	1	0	48770
1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	49024
1	1	0	0	1	0	0	0	1	1	1	1	1	0	0	0	51448
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	52428
1	1	0	0	1	1	1	0	1	1	0	0	0	0	1	0	52930
1	1	0	0	1	1	1	1	1	1	0	0	0	0	0	0	53184
1	1	0	1	1	1	0	0	1	1	0	1	0	0	0	0	56528
1	1	1	0	0	0	0	0	1	1	1	0	1	1	0	0	57580
1	1	1	0	0	0	1	0	1	1	1	0	0	0	1	0	58082
1	1	1	0	0	0	1	1	1	1	1	0	0	0	0	0	58336
1	1	1	0	1	0	1	0	0	0	1	0	1	0	1	0	59946
1	1	1	0	1	0	1	1	0	0	1	0	1	0	0	0	60200
1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	61680
1	1	1	1	0	1	0	0	1	1	0	0	0	1	0	0	62600
1	1	1	1	1	0	0	0	0	0	1	1	1	0	0	0	63544
1	1	1	1	1	1	0	0	0	0	0	0	1	1	0	0	64524
1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	0	65026
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	65280

Figure 3.10: 4-cell NCCA rules

It is easily seen that HMR trees have two features in the structure of NCCA rules. First, the HMR trees of k-cell NCCAs consist of k-trees. Second, the half of rule are symmetric and you don't need to show.

[1] $\emptyset$ is also included.

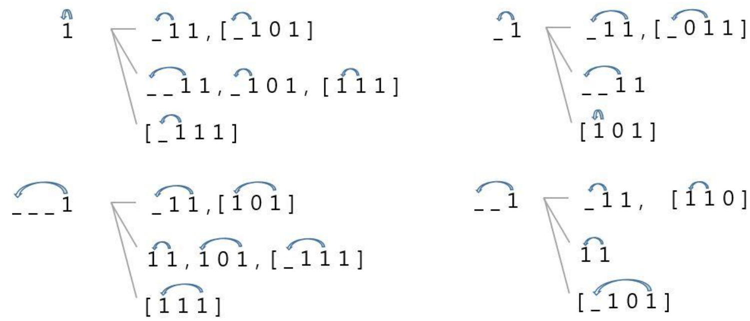

Figure 3.11: HMR trees of 4-cell NCCA rules

Definition 13. *[16] [Symmetric motions] Let μ, μ' are motions of some HMRs of n-cell NCCAs such that $\mu = \{\tilde{p}, s\}$, $\mu' = \{\tilde{p}', s'\}$ and $\tilde{p} = a_1 a_2 \cdots a_n$, $\tilde{p} = b_1 b_2 \cdots b_n$ are extended pattern. If $a_1 = b_n, a_2 = b_{n-1}, \cdots, a_n = b_1$ with $s = n + 1 - s'$, μ, μ' are symmetric motions. Moreover we call two HMR $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_2$ are symmetric when if $|\hat{\mathcal{M}}_1| = |\hat{\mathcal{M}}_2|$ and for each motion μ of $\hat{\mathcal{M}}_1$, there exists a symmetric motion of μ in $\hat{\mathcal{M}}_2$.*

In HMR trees of n-cell NCCAs, HMRs in k-th and $(n - k + 1)$-th HMR tree are mutually symmetric. If n is odd then the half of HMRs in $(n - 1)/2 + 1$-th HMR tree have their symmetric ones in the tree. Thus half of HMR trees is enough to represent the all structures of HMRs.

Fig. 3.13 shows one of the HMR trees of 5-cell NCCAs. We can graphically represent the structure of all NCCAs.

3.4 Conclusion

In this chapter, we showed some properties for the value-1 patterns of NCCAs, and show that all n-cell NCCAs can be derived from $(n-1)$-cell NCCAs. Moreover we introduced a binary tree structure called a bundle tree which represents a relation between a pattern set of the local function arguments of an n-cell NCCA and that of less than n. Using the relation, we also proposed hierarchical motion representation. Whereas normal motion representation focuses on the position of the cell, hierarchical motion representation focuses on actual moving particles. It is also convenient to describe the structure of all NCCAs of a neighborhood size by merging their shared motions.

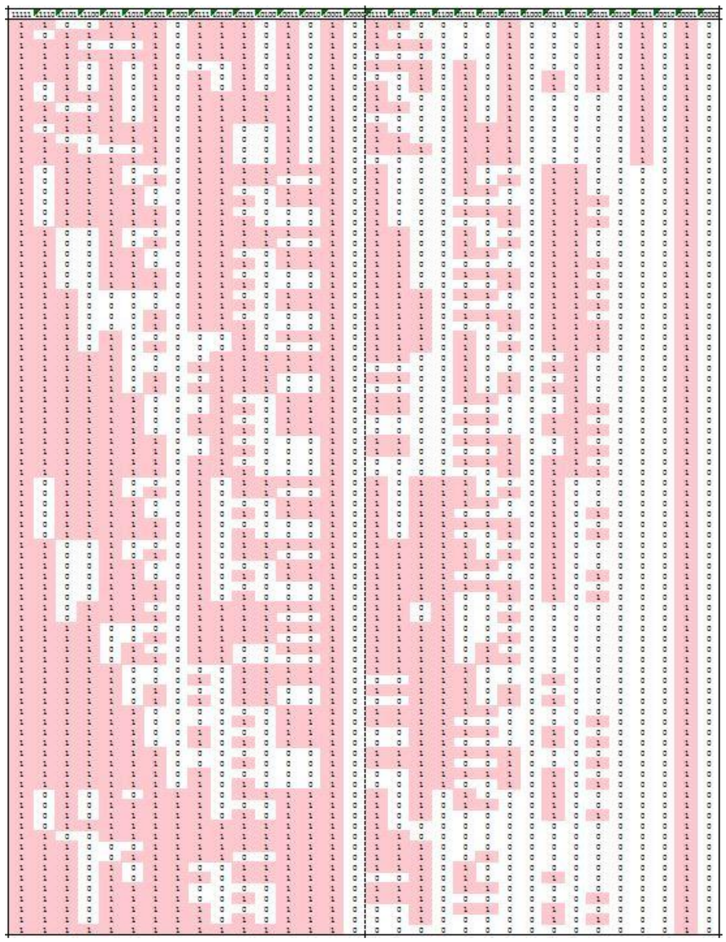

Figure 3.12: 5-cell NCCA rules with a basic motion $\{\_\ \_\ \_\ \_\ 1_5\}$

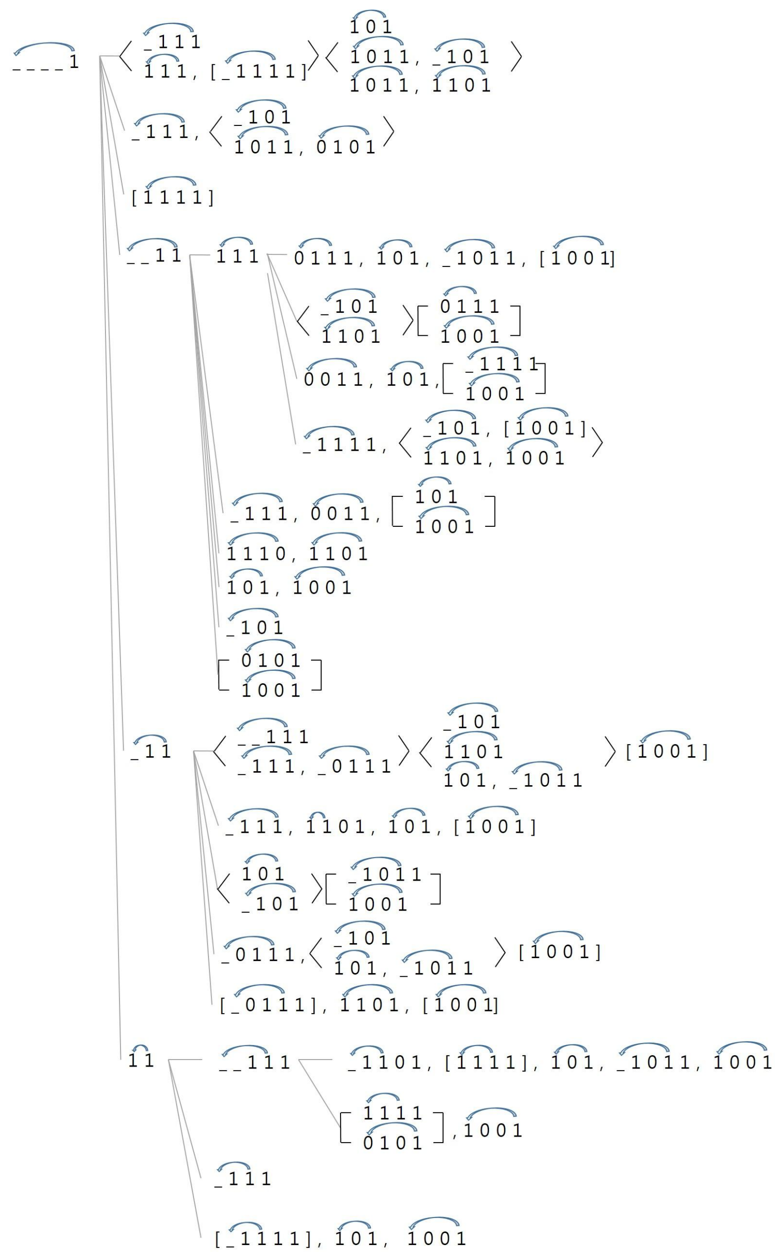

Figure 3.13: An HMR tree of 5-cell NCCA rules with $\{\_\,\_\,\_\,\_\,1_5\}$

Chapter 4

A new structure of two-state NCCA

Two-state NCCA with the state set, $\{0, 1\}$, keeps the number of 1s on the configuration constant. In other words, all the 1s on the configuration move without disappearing or appearing at any time step. When 1s on a configuration have motion by two-state NCCA, that motion is determined by the value-1 patterns of the NCCA rule. We introduced hierarchical motion representation that systematically and simply expresses the value-1 patterns, and thus introduced the HMR tree that NCCAs with similar motions can express together. In this process, we found that the value-1 patterns of n-cell and $(n-1)$-cell NCCA rules were related to each other, and the relation was shown using bundle in [17].

In this chapter we introduce bundle pair representing a relation between patterns of length n and $n-1$, and bundle quad representing a relation between patterns of length n and $n-2$. Value-1 pattern sets of all NCCA rules always have a structure consisting of bundle pairs and bundle quads, and have different characteristics depending on what structure they have. We introduce some properties of NCCA rules with only bundle quads and NCCA rules without bundle quad.

4.1 Bundle pair

In this section, we introduce bundle pairs of value-1 patterns, and some related properties.

Definition 14. *[18] [Bundle pair] Two patterns p and q are a bundle pair if p and q are both l-bundle or r-bundle of a certain pattern r.*

We use the following corollary of theorem3 in chapter 3, i.e., the value-1 pattern set of an NCCA is a set of bundles.

Corollary 2. *Let Q be the value-1 pattern set of an $(n+1)$-cell NCCA with 2^n elements. There exists a pattern set $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$, which is the value-1 pattern set of an*

n-cell NCCA, where for each element p_i, two patterns $\alpha, \beta \in Q$ are bundle of p_i, and α, β are not for any other $p_j (j \neq i)$.

In brief, corollary 2 is directly derived from theorem 3 and means that $(n-1)$-cell NCCA with the number of elements of the value-1 pattern set is 2^{n-2} can be obtained from n-cell NCCA with the number of elements of the value-1 pattern set of elements is 2^{n-1}.

Because theorem 3 is a characterization of the value-1 pattern set by its bundle patterns, the following corollary is also derived:

Corollary 3. *Let $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$ be the value-1 pattern set of an n-cell NCCA. There exists a pattern set $Q = \{q_j | 2 \leq j \leq 2^n + 1\}$, which is the value-1 pattern set of an $(n+1)$-cell NCCA, where q_{2i}, q_{2i+1} are a bundle of p_i for $1 \leq i \leq 2^{n-1}$.*

Corollary 3 means that n-cell NCCAs can be obtained from each $(n-1)$-cell NCCA, in contrast to corollary 2.

The bundle patterns from the value-1 pattern set of an NCCA forms a multiset, but its multiplicity is at most two by the following lemma:

Lemma 3. *Let $Q = \{q_j | 1 \leq j \leq 2^n\}$ be the value-1 pattern set of an $(n+1)$-cell NCCA. There always exists a multiset $M = \{m_i | 1 \leq i \leq 2^{n-1}\}$ s.t. m_i is a bundle pattern of two patterns q_α, q_β in Q, and q_α, q_β are not for any $m_j (j \neq i)$. Each element of M has a multiplicity at most two.*

Proof: Let Q be the value-1 pattern set of an $(n+1)$-cell NCCA. Then, each element in Q is one of a bundle pair by lemma 1. If there are three patterns p, q, r such that p and q are l-bundle of a certain pattern and q and r are r-bundle of a certain pattern. Then there must be a pattern s that can be r-bundle with p or l-bundle with r by lemma 2. Then, a multiset $M = \{m_i | 1 \leq i \leq 2^{n-1}\}$ always exists when m_i is a bundle pattern of two patterns p, q in Q. □

For example, there are sixteen 4-cell CAs of which bundle pattern set is the value-1 pattern set of Rule 184. Eight cases among them are NCCA. Fig. 4.1 shows one of them, 4-cell NCCA Rule 60200.

Until now, we show the relation between n-cell and $(n+1)$-cell NCCAs using bundle. From now, we introduce the relation between n-cell and $(n+2)$-cell NCCAs using either l-bundle or r-bundle. Lemma 4 is the description of the relation between n-cell and $(n+1)$-cell NCCAs using only l-bundle.

Lemma 4. *Let $P = \{p_i | 1 \leq i \leq 2^{n-1}\}$ be the value-1 pattern set of an n-cell NCCA. If $Q = \{q_j | 2 \leq j \leq 2^n + 1\}$ is a pattern set, where q_{2i}, q_{2i+1} are l-bundle of p_i for $1 \leq i \leq 2^{n-1}$, then Q is the value-1 pattern set of an $(n+1)$-cell NCCA.*

Proof: Each element of Q is a pattern in P prefixed with a number 0 or 1. Then, the number of elements of Q is 2^n without any duplicated patterns. Hence, Q is the value-1 pattern set of an $(n+1)$-cell NCCA by corollary 3 □.

When q_{2i}, q_{2i+1} are r-bundle of p_i in lemma 4, Q is also the value-1 pattern set of an $(n+1)$-cell NCCA.

Next, we think about the relation between n-cell and $(n+2)$-cell NCCAs using the combination of l- and r-bundle.

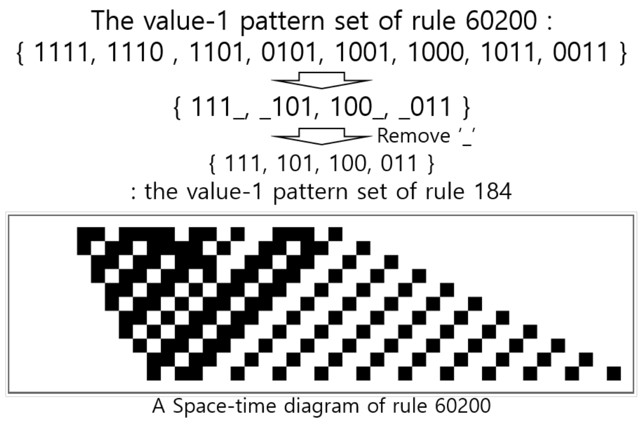

Figure 4.1: 4-cell NCCA rule 60200 whose bundle pattern set is the value-1 pattern set of rule 184.

Lemma 5. *Let $P = \{p_i | 1 \le i \le 2^{n-1}\}$ be the value-1 pattern set of an n-cell NCCA and $R = \{r_k | 4 \le k \le 2^{n+1} + 3\}$ be a pattern set satisfied that $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are l-bundle of r-bundle of p_i for $1 \le i \le 2^{n-1}$. Then, R is the value-1 pattern set of an $(n+2)$-cell NCCA.*

Proof: Each element in R is a pattern in P prefixed with a number, 0 or 1, and suffixed with a number, 0 or 1. Then, the number of elements of R is 2^{n+1} without any duplicated patterns. Hence, R is the value-1 pattern set of an $(n+2)$-cell NCCA by corollary 3 □.

When $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are r-bundle of l-bundle of p_i in lemma 5, R is also the value-1 pattern set of an $(n+2)$-cell NCCA.

By lemma 5, the value-1 pattern set can be extended from n-cell NCCA to $(n+2)$-cell NCCA without overlapping elements using the operations of generating l-bundle and r-bundle as shown in Fig. 4.2.

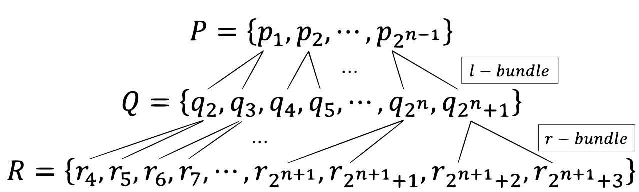

Figure 4.2: Extension of value-1 pattern sets from n-cell to $(n+2)$-cell

4.2 Bundle quad

Although NCCA rules should be recursively generated by extending bundle pairs, enumeration or indexing of NCCA rules is still difficult, because a similar value-1 pattern set could be generated by some different extending operations. To figure out the difficulty, we introduce another structure on the value-1 patterns, a bundle quad.

Definition 15. *[Bundle quad] Four patterns $p, q, r,$ and s are a bundle quad if p (resp.s) is a bundle pair with q and r simultaneously.*

For example, in Fig 4.3, the four patterns 0010, 1010, 0011, and 1011 are a bundle quad because 0010 (resp. 1011) and 1010 are l-bundle (resp. r-bundle) of 010 (resp. 101). Also, 0010 (resp. 1011) and 0011 are r-bundle (resp. l-bundle) of 001 (resp. 011). Further, those four patterns are l-bundle of r-bundle (or r-bundle of l-bundle) of 01.

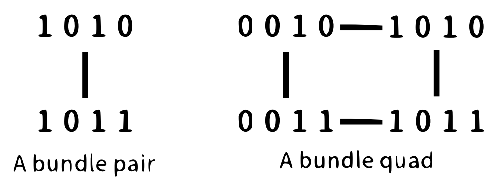

Figure 4.3: A bundle pair and a bundle quad

Let four patterns $p, q, r,$ and s of length $n + 2$ be a bundle quad and two patterns α and β of length $n + 1$ be the bundle patterns of p, q and r, s, respectively. Moreover, let a length n pattern γ be a bundle pattern of α, β. Then, p, q, r, s are l-bundle of r-bundle of γ. We call the pattern γ, the seed of the bundle quad, and we denote the bundle quad $\boxed{\gamma}$. For example, in Fig 4.3, the seed of 0010, 1010, 0011, 1011 is 01, and we denote them as $\boxed{01}$.

We showed that the elements of the value-1 pattern set of any NCCA A form bundle pairs. Moreover, some pairs of the bundle pairs may form bundle quads. Therefore, we can represent the value-1 pattern set of A by the set of bundle quads and bundle pairs. We call it the quad and pair set of A. The following theorem relates to the case where value-1 patterns only form bundle quads.

Theorem 5. *The number of n-cell NCCAs with 2^{n-3} bundle quads is equal to the number of $(n-2)$-cell NCCAs.*

Proof: Let X be the number of n-cell NCCA rules which has 2^{n-3} bundle quads and Y be the number of $(n-2)$-cell NCCA rules.
First, let $P = \{p_i | 1 \leq i \leq 2^{n-3}\}$ be the value-1 pattern set of an $(n-2)$-cell NCCA rule. By def. 15, there exists only one pattern set $\{r_k | 4 \leq k \leq 2^{n+1} + 3\}$ that satisfies the condition that $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are l-bundle of r-bundle of p_i ($\boxed{p_i}$) for $1 \leq i \leq 2^{n-3}$. Also, the pattern set is the value-1 pattern set of an n-cell NCCA by lemma 5. Then, $Y \leq X$.

Second, let $R = \{r_k | 4 \leq k \leq 2^{n+1} + 3\}$ be the value-1 pattern set of an n-cell NCCA rule with 2^{n-3} bundle quads. Then, there exists only one multiset $\{p_i | 1 \leq i \leq 2^{n-3}\}$ that satisfies the condition that $r_{4i}, r_{4i+1}, r_{4i+2}, r_{4i+3}$ are $\boxed{p_i}$ for $1 \leq i \leq 2^{n-3}$ by def. 15 and lemma 3. Also, the multiset is the value-1 pattern set of an $(n-2)$-cell NCCA rule by corollary 3. Then, $X \leq Y$.
Therefore, the number of n-cell NCCA rules which has 2^{n-3} bundle quads X is equal to the number of $(n-2)$-cell NCCA rules Y. □

For example, there are only two 2-cell NCCAs, 10 and 12. Also, there are only two (2^{4-3}) 4-cell NCCAs with two bundle quads, 52428 and 61680.
First, the value-1 pattern set of rule 10 is $\{11, 01\}$, and $\boxed{11}$ (resp. $\boxed{01}$) is $1111, 1110, 0111, 0110$ (resp. $1011, 1010,$
$0011, 0010$). A set of patterns $\{1111, 1110, 0111, 0110, 1011,$
$1010, 0011, 0010\}$ is the value-1 pattern set of the 4-cell NCCA rule, 52428. Next, the value-1 pattern set of rule 61680 is $\{1111, 1110, 0111, 0110, 1101, 1100, 0101, 0100\}$, and $1111, 1110, 0111, 0110$ (resp. $1101, 1100, 0101, 0100$) are $\boxed{11}$ (resp. $\boxed{10}$). The set $\{11, 10\}$is the value-1 pattern set of the 2-cell NCCA rule 12.

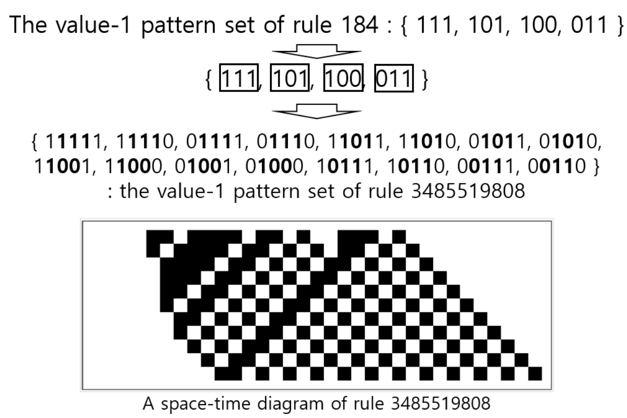

Figure 4.4: 5-cell NCCA rule 3485519808 having the same space-time diagram as rule 184

As shown in Fig 4.4, an n-cell NCCA with full quads have the same space-time diagram as any $(n-2)$-cell NCCA of which value-1 pattern set is the set of the seeds of the bundle quads of the n-cell NCCA rule.

Corollary 4. *The maximum number of bundle quads of the value-1 pattern set of an n-cell NCCA is* 2^{n-3}.

n-cell NCCA rules can be classified by the structure of their value-1 patterns as illustrated in Fig. 4.5. From the enumeration result of NCCAs using a computer[19, 21] we show the number of NCCA rules for each combination of bundle pairs and bundle

Neighborhood size	**Structure of value-1 patterns** (bundle quads & bundle pairs)	The number of rules
1-cell	x	**1**
2-cell	1 bundle pair	**2**
3-cell	1 bundle quad	**1**
	0 bundle quad & 2 bundle pairs	**4**
	Total	**5**
4-cell	2 bundle quads	**2**
	1 bundle quad & 2 bundle pairs	**12**
	0 bundle quad & 4 bundle pairs	**8**
	Total	**22**
5-cell	4 bundle quads	**5**
	3 bundle quads & 2 bundle pairs	**64**
	2 bundle quads & 4 bundle pairs	**185**
	1 bundle quad & 6 bundle pairs	**150**
	0 bundle quad & 8 bundle pairs	**24**
	Total	**428**
6-cell	8 bundle quads	**22**
	7 bundle quads & 2 bundle pairs	**576**
	6 bundle quads & 4 bundle pairs	**5482**
	5 bundle quads & 6 bundle pairs	**23416**
	4 bundle quads & 8 bundle pairs	**46256**
	3 bundle quads & 10 bundle pairs	**40956**
	2 bundle quads & 12 bundle pairs	**14632**
	1 bundle quads & 14 bundle pairs	**1780**
	0 bundle quads & 16 bundle pairs	**64**
	Total	**133184**

Figure 4.5: The number of NCCA rules by combination of bundle pairs and bundle quads

quads up to 6-cell in Fig.4.5. By Theorem5, it is easily verified that the number of n-cell NCCA rules with 2^{n-3} bundle quads, i.e., the number of 'full quad' rules is equal to the number of $(n-2)$-cell NCCA rules.

4.3 Full pair

In contrast to the number of the 'full quad' case, that of the 'full pair' case, i.e., no bundle quad and 2^{n-2} bundle pairs, is not shown in general. The following properties are found so far:

Proposition 1. *For any n-cell NCCA $A = (n, f_A)$ $(n \geq 2)$ of full pair with its quad and pair set U, the following properties hold:*

1. *Let U' be the set of all symmetric elements in U, i.e., for each extended pattern*

$a_1 a_2 \cdots a_n \in U$, $a_n \cdots a_2 a_1 \in U'$. *$U'$ is also a quad and pair set of full pair.*

2. *Either* $1 \cdots 1\_ \in U$ *or* $\_1 \cdots 1 \in U$ *holds. If* $1 \cdots 1\_ \in U$ *(resp.* $\_1 \cdots 1 \in U$*),* $\{\_1 \cdots 1\} \cup (U \backslash \{1 \cdots 1\_\})$ *(resp.* $\{1 \cdots 1\_\} \cup (U \backslash \{\_1 \cdots 1\})$*) is also a quad and pair set of an n-cell NCCA.*

The first property in Proposition 1 is trivial because the symmetric value-1 pattern set of an NCCA is also that of another NCCA and U' is also a full pair set because the operation just changes r- or l-bundle pattern to its opposite direction.

The outline of a proof of the second property is as follows: $f_A(1 \cdots 11) = f_A(1 \cdots 10) = 1$ and $f_A(01 \cdots 1) = 0$ holds by the assumption. For a sub-configuration $s_t = 1 \cdots 10$ (with $f_A(s_t) = 1$) of a configuration c_t, the evolved sub-configuration is $s_{t+1} = 1 \cdots 10$. In the case of $A' = (n, f_{A'})$, where $f_{A'}(1 \cdots 11) = f_{A'}(01 \cdots 1) = 1$ and $f_{A'}(1 \cdots 10) = 0$, thus the evolved sub-configuration is changed to $s'_{t+1} = 01 \cdots 1$. But there are only two cases of its left-extension either $0s_t$ or $1s_t$. If $0s_t$, the evolved configuration is $1s'_{t+1}$ and the application of $f_{A'}$ only switches these positions of 0 and 1. If $1s_t$, we repeatedly check the above extension whether $01 \cdots 1s_t$ or $11 \cdots 1s_t$.Because we start from s_t, the case $01 \cdots 1s_t$ should appear for any finite cyclic configuration including s_t. Because A is a full-pairs NCCA, A' is also a full-pairs NCCA.

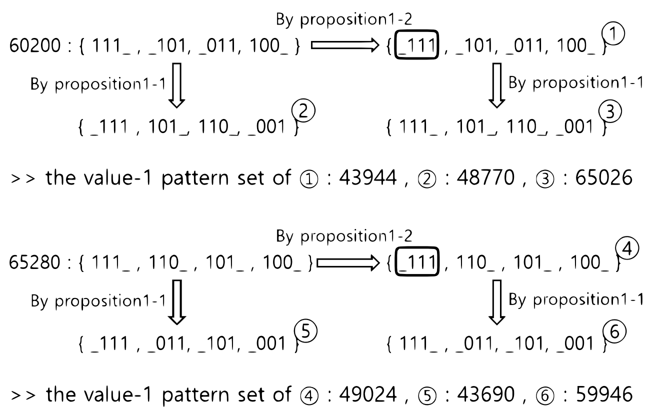

Figure 4.6: Six 4-cell rules derived from two NCCA rules 60200, 65280

Fig. 4.6 shows quad and pair sets of two 4-cell full pair rules 60200 and 65280. By Proposition 1, we get the other six rules from the above two rules in 4-cell full pair cases as in Fig.4.6.

Proposition 2. *For the quad and pair set U of any n-cell NCCA* $(2 \leq n \leq 4)$*, the derived set by deleting '_' from all elements of U is the value-1 pattern set of a certain* $(n-1)$*-cell NCCA rule. For* $n \geq 5$*, there exist n-cell NCCA rules which do not satisfy the property.*

Proposition 2 is easily verified for the case of $2 \leq n \leq 4$ by the enumeration result of NCCA rules.

As Fig. 4.6, deleting '_' form each element of their quad and pair sets, we get $\{111, 101, 011, 100\}, \{111, 101, 110, 001\}, \{111, 110, 101, 100\}$, and $\{111, 011, 101, 001\}$. These sets are the value-1 pattern sets of 3-cell full pair rules $184, 226, 240$, and 170. In the case of n-cell full pair ($n \geq 5$), the most of the rules, such as the rule 4294901760 in Fig. 4.7, satisfy Proposition 2 but there are some other rules which do not satisfy it, e.g., rule 4021231776 in Fig.4.7. There are several extended patterns which share a

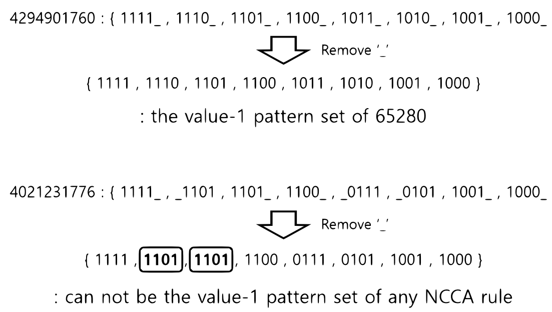

Figure 4.7: Quad and pair sets of 5-cell NCCA rules 4294901760, 4021231776

same pattern from second to $(n-1)$-th numbers but different extending directions in the quad and pair sets of the rules.

Nevertheless, the following Proposition holds for any full pair case in general.

Proposition 3. *For any n-cell NCCA ($n \geq 2$) of full pair, its quad and pair set U has the following form:*

$$U = \{u_i | u_i = \text{ either } a_i p_i\_ \text{ or } \_\, p_i a_i\}$$

where p_i is the zero-padded binary representation of i of the length $n-2$ and $a_i \in \{0, 1\}$ for $0 \leq i \leq 2^{n-2} - 1$.

Proof: Let U be the quad and pair set of any n-cell NCCA ($n \geq 2$) of full pair and let p_i and p_j be the length $n-2$ sub-patterns from second to $n-1$ positions of two different element u_i and u_j of U ($i \neq j$). Suppose $p_i = p_j$. There are only two cases according to the extending directions of u_i and u_j. First, if they are the same then $u_i = \_p_i a_i, u_j = \_p_i a_j$ (or $u_i = a_i p_i\_, u_j = a_j p_i\_$) with $a_i \neq a_j$. These two bundle pairs form a bundle quad, but U is a quad and pair set of full pair. Secondly, if they are different, then $u_i = \_p_i a_i, u_j = a_j p_i\_$ ($a_i, a_j \in \{0, 1\}$). u_i is the l-bundle of $0p_i a_i, 1p_i a_i$ and u_j is the r-bundle of $a_j p_i 0, a_j p_i 1$. Among the four patterns, at least two patterns are the same for any a_i and a_j. But all elements in the value-1 pattern set of an NCCA should be different and the number of them is 2^{n-2}. The case of $u_i = a_i p_i\_, u_j = \_p_i a_j$ is shown in the same way and thus $p_i \neq p_j$. □

For example, Fig.4.8 shows a set derived from the quad and pair sets of the 6-cell 16987224761694323240.

In contrast to the full quad case, the value-1 pattern sets of the full pair case differ each other only on the leftmost and rightmost number of each pattern. Although we did not clearly characterized the full pair case as full quad derived by $(n-2)$-cell NCCA

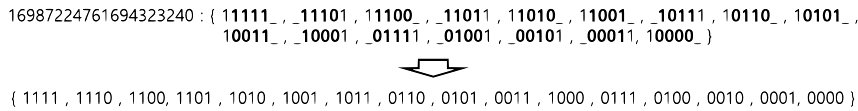

Figure 4.8: Quad and pair set of 6-cell 16987224761694323240

rules, but it considered to be derived from all patterns of $n-2$ cells. We also conjecture that U in proposition 3 satisfies the following condition:

$$U = \{u_i | u_i = \text{ either } 1p_i\_ \text{ or } \_\ p_i 1\}.$$

If the conjecture is true, only the assignment of extending directions accumulate the recursive information of $n-1$ or less NCCA rules.

4.4 Conclusion

In this chapter, we introduced a new structure of the value-1 patterns of each NCCA, a bundle pair and a bundle quad. Any NCCA rule can be represented by some combinations of bundle pairs and bundle quads. The structure reflects the relation between NCCAs of different neighborhood sizes and we show the relation that the number of $(n-2)$-cell NCCA rules is equal to the number of n-cell NCCA rules only composed of bundle quads. In addition, we show some propositions of full pair case and the full pair case can be derived from all kinds of $(n-2)$-cell patterns, unlike the full quad. If the rules of large-neighborhood NCCAs are represented by the quads and pairs of smaller-neighborhood NCCA, we can analyze large-neighborhood NCCAs more easily.

Chapter 5

Conclusion

In this thesis, we showed NCCAs with different size at once using HMR, and showed the relation between the NCCAs using the newly proposed structure.

First of all, we showed some properties about the value-1 patterns of NCCA rules, and as the result, it was shown that all n-cell NCCAs can be derived from $(n-1)$ cell NCCAs. In addition, we introduced HMR, which plays an important role for the characterization. Unlike MR, which focuses on the location of all state-1 cells, HMR only focuses on the particles actually to be moved. Also, since HMR has a tree structure according to the order of the 'complexity' of patterns which invoke the movements of 1s, it is easy to express NCCAs of the same neighborhood length as an HMR tree. Second, we proposed a new structure, bundle pair and bundle quad, and showed some properties of the value-1 patterns of NCCAs using the structure. We showed that all NCCAs can be represented as a combination of bundle pairs and bundle quads. We also showed that the number of n-cell NCCAs consisting of only bundle quads is the same as the number of all $(n-2)$ cell NCCAs. In addition, we show some propositions of full pair case and the full pair case can be derived from all kinds of $(n-2)$-cell patterns, unlike the full quad case.

We focused on the relations of NCCAs of different neighborhood sizes. Ultimately, we expect to find any method to find NCCAs that have their desired movements of 1s regardless of their neighborhood size employing the structure of HMR in future study.

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