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Relation	

Fundamental properties and asymptotic shapes of the singular and classical radial solutions for supercritical semilinear elliptic equations

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Abstract

We study singular radial solutions of the semilinear elliptic equation $\Delta u + f(u) = 0$ on finite balls in $\mathbf{R}^N$ with $N \geq 3$. We assume that f satisfies either $f(u) = u^p + o(u^p)$ with $p > (N+2)/(N-2)$ or $f(u) = e^u + o(e^u)$ as $u \to \infty$. We provide the existence and uniqueness of the singular radial solution, and show the convergence of regular radial solutions to the singular solution. Some applications to the bifurcation diagram of an elliptic Dirichlet problem are also given. Our results generalize and improve some known results in the literature.

MSC: 35J61, 35A05, 35A24
Keywords: Semilinear elliptic equation, Singular solution, Supercritical, Uniqueness, Existence

1 Introduction

We study singular radial solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in } \Omega \setminus \{0\},$$

where $\Omega \subset \mathbf{R}^N$, with $N \geq 3$, is a finite ball centered with the origin and $f \in C^1[0,\infty)$. In the study, we consider solutions to the ordinary differential equation

$$u'' + \frac{N-1}{r}u' + f(u) = 0 \quad \text{for } r > 0. \tag{1.1}$$

By a singular solution $u^*(r)$ of (1.1) we mean that $u^*(r)$ is a classical solution of the equation (1.1) for $0 < r \leq r_0$ with some $r_0 > 0$ and it satisfies $u^*(r) \to \infty$ as $r \to 0$. For $\alpha > 0$, we

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denote by $u(r,\alpha)$ a regular solution of (1.1) satisfying $u(0)=\alpha$ and $u'(0)=0$. In (1.1) we assume that $f(u)$ satisfies either

$$f(u)=u^p+o(u^p) \quad \text{or} \quad f(u)=e^u+o(e^u) \quad \text{as } u\to\infty,$$

where $p>p_S=(N+2)/(N-2)$. It is well known that the singular solution plays a key role in the study of the solution structure of the supercritical problem (1.1). See, e.g. [13, 17, 26, 27, 30]. In this paper, we prove the existence and uniqueness of the singular solution of (1.1), and also the convergence of the regular solution $u(r,\alpha)$ to the singular solution as $\alpha\to\infty$. These properties have been studied extensively in the literature, because of various potential applications for both elliptic and parabolic problems. When $f(u)=u^p$, (1.1) has the exact singular solution $u^*(r)=Ar^{-2/(p-1)}$ if $p>N/(N-2)$, where

$$A=\left\{\frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right)\right\}^{1/(p-1)}. \tag{1.2}$$

It was shown by Serrin–Zou [34, Proposition 3.1] that, if $p>p_S$, the singular solution of (1.1) is unique. On the other hand, it was shown in [34, 7] that, if $N/(N-2)<p<p_S$, (1.1) has a continuum of singular solutions. When $f(u)=u^p+u$ with $p>p_S$, it was shown by Merle–Peletier [24] that (1.1) has a singular solution and the regular solution $u(r,\alpha)$ converges to the singular solution as $\alpha\to\infty$. See also [4]. When $f(u)=u^p-u$, Chern *et al.* [8] investigated the entire structure of radial solutions according to their behaviors at the origin and infinity, and showed that (1.1) possesses a unique singular solution in the case $N>10$ and p is large. In [26] the first author of the present paper showed the existence of the singular solution in the case $f(u)=u^p+g(u)$ with $p>p_S$, where g satisfies

$$|g(u)|\le C_0u^{p-\delta} \quad \text{and} \quad |g'(u)|\le C_0u^{p-1-\delta} \quad \text{for } u\ge u_0$$

with some constants $C_0>0$, $\delta>0$ and $u_0\ge 0$. In this case, the existence of singular extremal solutions was studied in the previous paper [30].

In the case $f(u)=e^u$, it was shown by Mignot-Puel [25] that $u^*(r)=-2\log r+\log 2(N-2)$ is the unique singular solution of (1.1) when $N\ge 3$. On the other hand, (1.1) has no singular solution when $N=1$, and (1.1) has a continuum of singular solutions when $N=2$. (See Tello [35, Theorem 1.1].) In [27] the author studied the case $f(u)=e^u+g(u)$, where g satisfies

$$|g(u)|\le C_0e^{(1-\delta)u} \quad \text{and} \quad |g'(u)|\le C_0e^{(1-\delta)u}$$

with some constants $C_0>0$, $\delta>0$ and $u_0\ge 0$. In [27], the existence of the singular solution and the convergence property of regular solutions were obtained. On the other hand, the uniqueness of the singular solution was left open.

For general supercritical nonlinearity, the existence of the singular solution was obtained by Lin [21]. Recently, by [28], the existence and asymptotic behavior of the singular solution were shown for a certain class of supercritical nonlinearities. On the other hand, in [21, 28], the uniqueness of the singular solution was left open, and it was unclear whether regular solutions converge to the singular solution.

In this paper, we will show the qualitative properties of the singular solution to (1.1) with the following two types of nonlinearities (F1) and (F2) by a unified approach.

(F1) $f \in C^1[0,\infty)$ has the form $f(u) = u^p + g(u)$ for $u \geq 0$ with $p > p_S$, where $g(u)$ satisfies

$$g(u) = o(u^p) \quad \text{and} \quad g'(u) = o(u^{p-1}) \quad \text{as } u \to \infty. \tag{1.3}$$

(F2) $f \in C^1[0,\infty)$ has the form $f(u) = e^u + g(u)$ for $u \geq 0$, where $g(u)$ satisfies

$$g(u) = o(e^u) \quad \text{and} \quad g'(u) = o(e^u) \quad \text{as } u \to \infty. \tag{1.4}$$

Our main result is the following.

Theorem 1.1. *Assume that either* (F1) *or* (F2) *holds. Then there exists a unique singular solution $u^*(r)$ of* (1.1) *for $0 < r \leq r_0$ with some $r_0 > 0$, and the regular solution $u(r,\alpha)$ satisfies*

$$u(r,\alpha) \to u^*(r) \quad \text{in } C^2_{\mathrm{loc}}(0,r_0] \quad \text{as } \alpha \to \infty. \tag{1.5}$$

Furthermore, if (F1) *holds, the singular solution u^* satisfies*

$$u^*(r) = Ar^{-2/(p-1)}(1+o(1)) \quad \text{as } r \to 0, \tag{1.6}$$

where A is a constant defined by (1.2), *and if* (F2) *holds, u^* satisfies*

$$u^*(r) = -2\log r + \log 2(N-2) + o(1) \quad \text{as } r \to 0. \tag{1.7}$$

Remark 1.1. The Morse index of u^* is determined by the linearized operator $-\Delta - f'(u^*)$. See [3, 26, 28] for detail. From (1.6) and (1.7), we see that

$$f'(u^*(r)) = \frac{C}{r^2} + o(r^{-2}) \quad \text{as } r \to 0,$$

where $C = pA^{p-1}$ if (F1) holds and $C = 2(N-2)$ if (F2) holds.

In the proof of Theorem 1.1, we obtain the following characterization of the singular solution of (1.1).

Theorem 1.2. *Let u be a positive solution of* (1.1) *for $0 < r \leq r_0$ with some $r_0 > 0$.*

(i) *Assume that* (F1) *holds. Then u is a singular solution if and only if u satisfies*

$$\limsup_{r\to 0} r^{2/(p-1)} u(r) > 0. \tag{1.8}$$

(ii) *Assume that* (F2) *holds. Then* u *is a singular solution if and only if* u *satisfies*

$$\limsup_{r\to 0}(u(r)+2\log r) > -\infty. \tag{1.9}$$

Remark 1.2. It is interesting to point out that both of the conditions (1.8) and (1.9) can be reduced to a unified condition

$$\limsup_{r\to 0}\frac{r^2}{G(u(r))} > 0, \quad \text{where} \quad G(u)=\int_u^\infty \frac{1}{f(t)}dt.$$

The function G and its inverse function are effectively used in the recent paper [28].

As an application of Theorem 1.1, we consider the following nonlinear eigenvalue problem

$$\begin{cases} \Delta v+\lambda f(v)=0 & \text{in } B,\\ v>0 & \text{in } B,\\ v=0 & \text{on } \partial B, \end{cases} \tag{1.10}$$

where B is a unit ball in $\mathbf{R}^N$ with $N\geq 3$ and $\lambda>0$ is a parameter. By the symmetry result of Gidas-Ni-Nirenberg [16], every regular solution v of (1.10) must be radially symmetric at the origin. It was shown by [5, 33, 14] that, if a solution v of (1.10) has the singularity at $x=\{0\}$, then v must be radially symmetric about the origin.

Denote by $(\lambda(\alpha), v(r,\alpha))$ a solution of (1.10) with $v(0,\alpha)=\alpha>0$. It is well known that the solution set of (1.10) can be described as the curve $\{(\lambda(\alpha), v(r,\alpha)) : 0<\alpha<\infty\}$. (See, e.g., [20, 26, 27].) The bifurcation diagram of the problem (1.10) was completely characterized by Joseph-Lundgren [19] in the cases $f(u)=(u+1)^p$ and $f(u)=e^u$. In these cases, (1.10) can be transformed into the autonomous system by the special changes of variables, and the explicit singular solution can be obtained as a critical point of the system. For a general nonlinearity we cannot expect to find such a change of variables, but we obtain the following result as a consequence of Theorem 1.1.

Corollary 1.1. *Assume that either* (F1) *or* (F2) *holds. Assume, in addition, that* $f(u)>0$ *for* $u\geq 0$*. Then the following* (i) *and* (ii) *hold.*

(i) *The problem* (1.10) *has a unique singular radial solution* (λ^*, v^*)*, that is, there exists a unique* $\lambda^*>0$ *such that the problem* (1.10) *with* $\lambda=\lambda^*$ *has a singular radial solution* v^**, and the solution* v^* *is a unique singular radial solution of* (1.10) *with* $\lambda=\lambda^*$.

(ii) *As* $\alpha\to\infty$*, the solution* $(\lambda(\alpha), v(r,\alpha))$ *of* (1.10) *described above satisfies*

$$\lambda(\alpha)\to\lambda^* \quad \textit{and} \quad v(r,\alpha)\to v^*(r) \quad \textit{in } C^2_{loc}(0,1], \tag{1.11}$$

where (λ^*, v^*) *is the singular radial solution in* (i).

Remark 1.3. (i) In Corollary 1.1, the curve of the solution set emanates from $(\lambda, v) = (0, 0)$, since f satisfies $f(u) > 0$ for $u \geq 0$. The assertions (i) and (ii) also hold if f satisfies $f(u) > 0$ for $u > 0$, $f(0) = 0$ and $f'(0) \neq 0$. In this case, the solution curve emanates from the point $(\mu_1/f'(0), 0)$, where μ_1 is the first eigenvalue of $-\Delta$ in B with Dirichlet boundary data.

(ii) It is well known that the singular solution plays an important role in the study of the bifurcation diagram of the problem (1.10). Brezis-Vázquez [3] gave a necessary and sufficient condition for the non-existence of a turning point of the solution curve by using the singular solution. Guo-Wei [17] investigated the Morse index of the singular solution, and showed that the divergence of the Morse index indicates the existence of infinitely many turning points of the solution curve. See also [26, 27, 28].

(iii) As mentioned above, the singular solution is radial if the singularity set is the origin [5, 33, 14]. On the other hand, (1.10) may have a nonradial singular solution with the singularity at $x \neq \{0\}$. See, e.g., [23, 32, 7, 9].

(iv) For the study of singular radial solutions to the problem (1.10) with the critical or subcritical growth of $f(u)$, we refer to [2, 12, 15]

Next we consider the existence of radially symmetric solutions of the problem

$$\begin{cases} \Delta v + f(v) = 0 & \text{in } \mathbf{R}^N \setminus \{0\}, \\ v > 0 & \text{in } \mathbf{R}^N \setminus \{0\}, \end{cases} \tag{1.12}$$

where $N \geq 3$. As a consequence of Theorem 1.1, we obtain the following result.

Corollary 1.2. *Assume that either* (F1) *or* (F2) *holds. Then one of the following alternatives holds:*

(i) (1.10) *has a unique singular radial solution* (λ^*, v^*) *and* (1.12) *has no singular radial solution.*

(ii) (1.10) *has no singular radial solution and* (1.12) *has a unique singular radial solution.*

Remark 1.4. (i) It was shown by Ni-Serrin [31, Theorem 3.1] that, if there exists $\alpha \geq (N+2)/(N-2)$ such that $(\alpha+1)\int_0^u f(t)dt \leq uf(u)$ for all $u \geq 0$, then (1.10) has no singular radial solution. In this case, (ii) occurs and then (1.12) has a unique singular radial solution.

(ii) In [34], Serrin-Zou fully described the set of positive radial solutions to (1.12) in the case $f(u) = u^p + u^q$ with $N/(N-2) < q < p$. See also [1, 6, 10].

(iii) The existence of nonradial singular solution of (1.12) was obtained by Dancer-Guo-Wei [11] when $f(u) = u^p$ with $N \geq 4$ and some range of p, and by [29] when $f(u) = e^u$ with $4 \leq N \leq 10$.

(iv) In [18], Johnson-Pan-Yi showed the existence of singular positive solutions in the subcritical case $f(u) = -u + u^p$ with $1 < p < p_S$.

To prove Theorem 1.1, we derive some a priori estimates of positive regular and singular solutions near $r = 0$, and give the characterization of the singular solution. By using Pohozaev's identity and some comparison argument, we obtain the singular solution as a limit of a sequence of regular solutions even if $u(r, \alpha)$ is not necessarily increasing in α. In [24, 26, 27], the singular solution is constructed by the contraction mapping theorem, and in the proof, it was crucial to obtain the precise asymptotic expansions of the singular solutoin by using the presice form of $f(u)$, assuming the existence of the solution. However, it seems difficult to extend their results to more general class of $f(u)$ as (F1) or (F2).

The paper is organized as follows. In Section 2, we obtain apriori estimates of both the regular and singular solutions near $r = 0$, and in Section 3, we give the proof of Theorem 1.2. We show the uniqueness of the singular solution in Section 4, and give the proofs of Theorem 1.1 and Corollaries 1.1 and 1.2 in Section 5.

2 Preliminaries

2.1 A priori estimates of positive solutions near the origin in the case (F1)

In this subsection we assume that (F1) holds. Then there exist constants $u_0 > 0$, $C^* > 1 > C_* > 0$ such that

$$0 < C_* u^p \leq f(u) \leq C^* u^p \quad \text{for } u \geq u_0. \tag{2.1}$$

Since $f'(u) = pu^{p-1} + o(u^{p-1})$ as $u \to \infty$, we may assume that

$$f'(u) > 0 \quad \text{for } u \geq u_0. \tag{2.2}$$

We consider a solution of (1.1) satisfying

$$u(r) \geq u_0 \quad \text{for } 0 < r \leq r_0 \tag{2.3}$$

with some $r_0 > 0$. Note that u satisfies (2.3) if $u(r) \to \infty$ as $r \to 0$. First we show the following results.

Lemma 2.1. *Assume that* (F1) *holds, and that* u_0 *is the constant in* (2.1). *Let* u *be a positive solution of* (1.1) *satisfying* (2.3) *with some* $r_0 > 0$. *Then*

$$u(r) \leq C_1 r^{-2/(p-1)} \quad \text{and} \quad 0 \leq -u'(r) \leq C_2 r^{-(p+1)/(p-1)} \quad \text{for } 0 < r \leq r_0,$$

where constants $C_1 > 0$ *and* $C_2 > 0$ *are independent of* u. *Furthermore,* u *satisfies*

$$-r^{N-1}u'(r) = \int_0^r s^{N-1} f(u(s))ds \quad \text{for } 0 < r \leq r_0. \tag{2.4}$$

Proof. From (2.1) and (2.3), we have

$$0 < C_* u(r)^p \le f(u(r)) \le C^* u(r)^p \quad \text{for } 0 < r \le r_0. \tag{2.5}$$

We will show that

$$u'(r) \le 0 \quad \text{for } 0 < r \le r_0. \tag{2.6}$$

In fact, put $V(t) = u(r)$ with $r = t^{-1/(N-2)}$. Then $V(t)$ satisfies

$$V''(t) = -\frac{1}{(N-2)^2} t^{-2(N-1)/(N-2)} f(V(t)) < 0 \quad \text{for } t \ge t_0$$

with $t_0 = r_0^{2-N}$. Hence, $V'(t)$ is decreasing for $t \ge t_0$. We will show that $V'(t) \ge 0$ for $t \ge t_0$. Assume to the contrary that there exists $t_1 \ge t_0$ such that $V'(t_1) < 0$. Since $V'(t)$ is decreasing, we have $V'(t) \le V'(t_1) < 0$ for $t \ge t_1$. This implies that $V(t) \to -\infty$ as $t \to \infty$, which is a contradiction. Thus we obtain $V'(t) \ge 0$ for $t \ge t_0$, which implies that (2.6) holds.

Take $r_1 \in (0, r_0)$ arbitrarily. Integrating (1.1) on (r_1, r) with $r \le r_0$, we obtain

$$-r^{N-1}u'(r) = -r_1^{N-1}u'(r_1) + \int_{r_1}^{r} s^{N-1} f(u(s))ds \ge \int_{r_1}^{r} s^{N-1} f(u(s))ds.$$

Since $r_1 > 0$ is arbitrary, we obtain

$$-r^{N-1}u'(r) \ge \int_0^r s^{N-1} f(u(s))ds. \tag{2.7}$$

Then it follows that

$$-r^{N-1}u'(r) \ge C_* \int_0^r s^{N-1} u(s)^p ds \ge C_* u(r)^p \int_0^r s^{N-1} ds = \frac{C_*}{N} u(r)^p r^N.$$

This implies that

$$-\frac{u'(r)}{u(r)^p} \ge \frac{C_*}{N} r.$$

Integrating the above on $[\rho, r]$, and letting $\rho \to 0$, we obtain $u(r)^{1-p} \ge (p-1)C_* r^2/(2N)$, and hence we obtain

$$u(r) \le C_1 r^{-2/(p-1)} \tag{2.8}$$

with $C_1 = \{(p-1)C_*/(2N)\}^{1/(p-1)}$. Next we will show that

$$\liminf_{r \to 0} (-r^{N-1}u'(r)) = 0. \tag{2.9}$$

Assume to the contrary that $\liminf_{r\to 0}(-r^{N-1}u'(r)) = c > 0$. Then there exists $r_1 > 0$ such that

$$-r^{N-1}u'(r) \ge \frac{c}{2} \quad \text{for } 0 < r \le r_1.$$

Integrating above on (r, r_1), we obtain

$$u(r) \geq u(r_1) + \frac{c}{2(N-2)}(r^{2-N} - r_1^{2-N}).$$

This contradicts (2.8). Thus we obtain (2.9).

By (2.9) there exists $r_k \to 0$ such that $r_k^{N-1}u'(r_k) \to 0$ as $k \to \infty$. Integrating (1.1) on $[r_k, r]$, and letting $k \to \infty$, we obtain (2.4). It follows from (2.8) that

$$-r^{N-1}u'(r) \leq C^* \int_0^r s^{N-1}u(s)^p ds \leq C^* C_1^p \int_0^r s^{N-1-2p/(p-1)} ds = C_2 r^{N-2p/(p-1)}$$

with $C_2 = C^* C_1^p/(N - 2p/(p-1))$. Thus we obtain $-u'(r) \leq C_2 r^{-(p+1)/(p-1)}$ for $0 < r \leq r_0$. □

Let u be a positive solution of the equation (1.1) for $0 < r \leq r_0$. Define

$$w(t) = r^{2/(p-1)}u(r) \quad \text{with } t = -\log r. \tag{2.10}$$

Then w satisfies

$$w'' - aw' - A^{p-1}w + e^{-p\theta t} f(e^{\theta t} w(t)) = 0 \quad \text{for } t \geq t_0, \tag{2.11}$$

where $\theta = 2/(p-1)$, $a = N - 2 - 2\theta > 0$, and $t_0 = -\log r_0$. Since f has the form $f(u) = u^p + g(u)$, the equation (2.11) can be written as

$$w'' - aw' - A^{p-1}w + w^p + e^{-p\theta t} g(e^{\theta t} w(t)) = 0 \quad \text{for } t \geq t_0. \tag{2.12}$$

Lemma 2.2. *Assume that* (F1) *holds. Let u be a singular solution of* (1.1). *Then*

$$\limsup_{r \to 0} r^{2/(p-1)}u(r) > 0.$$

Proof. Assume to the contrary that

$$\lim_{r \to 0} r^{2/(p-1)}u(r) = 0. \tag{2.13}$$

Since $u(r) \to \infty$ as $r \to 0$, there exists $r_0 > 0$ such that (2.3) holds. First we will show that

$$(r^{2/(p-1)}u(r))' \geq 0 \quad \text{for } 0 < r < r_1 \tag{2.14}$$

with some $r_1 \in (0, r_0]$. Define $w(t)$ by (2.10). Then w satisfies (2.11). From (2.5) we have

$$w'' - aw' - A^{p-1}w + C^* w^p \geq 0 \quad \text{for } t \geq t_0.$$

From (2.13) we have $w(t) \to 0$ as $t \to \infty$. Then there exists $t_1 \geq t_0$ such that

$$(e^{-at}w(t)')' \geq e^{-at}(A^{p-1} - C^* w(t)^{p-1})w(t) > 0 \quad \text{for } t \geq t_1,$$

which implies that $e^{-at}w'(t)$ is increasing for $t \geq t_1$. We will show that

$$w'(t) \leq 0 \quad \text{for all } t \geq t_1. \tag{2.15}$$

Assume to the contrary that there exists $t_2 \geq t_1$ such that $w'(t_2) > 0$. Then $e^{-at}w'(t) \geq e^{-at_2}w'(t_2) > 0$ for $t \geq t_2$. This implies that $w'(t) > e^{-at_2}w'(t_2)e^{at}$ for $t > t_2$, and hence $w'(t) \to \infty$ as $t \to \infty$. This contradicts that $w(t) \to 0$ as $t \to \infty$. Thus we obtain (2.15), which implies that (2.14) holds.

From (2.5) and (2.13) we have

$$\frac{r^2 f(u(r))}{u(r)} \leq C^* r^2 u(r)^{p-1} = C^*(r^\theta u(r))^{p-1} \to 0 \quad \text{as } r \to 0.$$

Take $\varepsilon > 0$ so that $\varepsilon < 1/p$. Then there exists $r_1 \in (0, r_0]$ such that

$$r^2 f(u(r)) < (N-2-\theta)\varepsilon u(r) \quad \text{for } 0 < r < r_1.$$

Note here that $N - 2 - \theta > 0$. From (2.14) we have

$$\begin{aligned}\int_0^r s^{N-1} f(u(s))ds &\leq (N-2-\theta)\varepsilon \int_0^r s^{N-3}u(s)ds \\ &\leq (N-2-\theta)\varepsilon r^\theta u(r) \int_0^r s^{N-3-\theta}ds = \varepsilon r^{N-2}u(r)\end{aligned}$$

for $0 < r < r_1$. From (2.4) it follows that

$$-r^{N-1}u'(r) = \int_0^r s^{N-1} f(u(s))ds \leq \varepsilon r^{N-2}u(r) \quad \text{for } 0 < r < r_1.$$

This implies that $(r^\varepsilon u(r))' \geq 0$ for $0 < r \leq r_1$, and hence $r^\varepsilon u(r) \leq r_1^\varepsilon u(r_1)$ for $0 < r < r_1$. Then we obtain $u(r) = O(r^{-\varepsilon})$ as $r \to 0$. From (2.4) and (2.5), we obtain

$$-r^{N-1}u'(r) \leq C^* \int_0^r s^{N-1}u(s)^p ds \leq C \int_0^r s^{N-1-p\varepsilon}ds = C' r^{N-p\varepsilon}$$

with some constants $C, C' > 0$. Thus $u'(r) = O(r^{1-p\varepsilon})$ as $r \to 0$. Since $\varepsilon < 1/p$, we have $u'(r) \to 0$ as $r \to 0$, and hence $\lim_{r\to 0} u(r) < \infty$. This is a contradiction. Thus we obtain $\limsup_{r\to 0} r^{2/(p-1)}u(r) > 0$. □

2.2 A priori estimates of positive solutions near the origin in the case (F2)

In this subsection we assume that (F2) holds. Then there exist constants $u_0 > 0$, $C^* > 1 > C_* > 0$ such that

$$0 < C_* e^u \leq f(u) \leq C^* e^u \quad \text{for } u \geq u_0. \tag{2.16}$$

We may assume that (2.2) holds. We obtain the following results.

Lemma 2.3. *Assume that* (F2) *holds, and that* u_0 *is the constant in* (2.16). *Let* u *be a positive solution of* (1.1) *satisfying* (2.3) *with some* $r_0 > 0$. *Then*

$$u(r) \leq -2\log r + C_1 \quad \textit{and} \quad 0 \leq -u'(r) \leq \frac{C_2}{r} \quad \textit{for } 0 < r \leq r_0,$$

where constants $C_1 \in \mathbf{R}$ *and* $C_2 > 0$ *are independent of* u. *Furthermore,* u *satisfies* (2.4).

Proof. From (2.16) and (2.3), we have

$$0 < C_* e^{u(r)} \leq f(u(r)) \leq C^* e^{u(r)} \quad \text{for } 0 < r \leq r_0. \tag{2.17}$$

By a similar argument as in the proof of Lemma 2.1, we obtain (2.6) and (2.7). Then it follows that

$$-r^{N-1}u'(r) \geq C_* \int_0^r s^{N-1} e^{u(s)} ds \geq C_* e^{u(r)} \int_0^r s^{N-1} ds = \frac{C_*}{N} e^{u(r)} r^N.$$

This implies that

$$-\frac{u'(r)}{e^{u(r)}} \geq \frac{C_*}{N} r.$$

Integrating the above on $[\rho, r]$, and letting $\rho \to 0$, we obtain $e^{-u(r)} \geq C_* r^2/(2N)$, and hence

$$u(r) \leq -2\log r + C_1 \tag{2.18}$$

with $C_1 = \log(2N/C_*)$. By a similar argument as in the proof of Lemma 2.1, we obtain (2.4). From (2.17) and (2.18) we have

$$-r^{N-1}u'(r) \leq C^* \int_0^r s^{N-1} e^{u(s)} ds \leq C^* e^{C_1} \int_0^r s^{N-1-2} ds = C_2 r^{N-2}$$

with $C_2 = C^* e^{C_1}/(N-2)$. Thus we obtain $-u'(r) \leq C_2/r$ for $0 < r \leq r_0$. $\square$

Let u be a positive solution of the equation (1.1) for $0 < r \leq r_0$. Put $\kappa = \log 2(N-2)$. Define

$$w(t) = u(r) + 2\log r - \kappa \quad \text{with } t = -\log r. \tag{2.19}$$

Then w satisfies

$$w'' - (N-2)w' - 2(N-2) + e^{-2t} f(w + 2t + \kappa) = 0 \quad \text{for } t \geq t_0 \tag{2.20}$$

with $t_0 = -\log r_0$. Since f has the form $f(u) = e^u + g(u)$, the equation (2.20) can be written as

$$w'' - (N-2)w' + 2(N-2)(e^w - 1) + e^{-2t} g(w + 2t + \kappa) = 0 \quad \text{for } t \geq t_0. \tag{2.21}$$

Lemma 2.4. *Assume that* (F2) *holds. Let* u *be a singular solution of* (1.1). *Then*

$$\limsup_{r \to 0} (u(r) + 2\log r) > -\infty.$$

Proof. Assume to the contrary that

$$\lim_{r\to 0}(u(r) + 2\log r) = -\infty. \tag{2.22}$$

Since $u(r) \to \infty$ as $r \to 0$, there exists $r_0 > 0$ such that (2.3) holds. We will show that

$$ru'(r) + 2 \geq 0 \quad \text{for } 0 < r < r_1 \tag{2.23}$$

with some $r_1 \in (0, r_0]$. Define $w(t)$ by (2.19). Then w satisfies (2.20) and $w(t)+2t+\kappa = u(r)$. Since (2.17) holds, we have

$$f(w(t) + 2t + \kappa) \leq C^* e^{w(t)+2t+\kappa} = 2(N-2)C^* e^{2t} e^{w(t)} \quad \text{for } t \geq t_0$$

with some $t_0 \in \mathbf{R}$. Then w satisfies

$$w'' - (N-2)w' + 2(N-2)(C^* e^w - 1) \geq 0 \quad \text{for } t \geq t_0.$$

From (2.22) we have $w(t) \to -\infty$ as $t \to \infty$. Then there exists $t_1 \geq t_0$ such that

$$(e^{-(N-2)t} w')' \geq -2(N-2)e^{-(N-2)t}(C^* e^{w(t)} - 1) > 0 \quad \text{for } t \geq t_1.$$

This implies that $e^{-(N-2)t}w'(t)$ is increasing for $t \geq t_1$. We will show that

$$w'(t) \leq 0 \quad \text{for all } t \geq t_1. \tag{2.24}$$

Assume to the contrary that there exists $t_2 \geq t_1$ such that $w'(t_2) > 0$. Then $e^{-(N-2)t}w'(t) \geq e^{-(N-2)t_2}w'(t_2) > 0$ for $t \geq t_2$. This implies that

$$w'(t) > e^{-(N-2)t_2} w'(t_2) e^{(N-2)t} \quad \text{for } t > t_2,$$

and hence $w'(t) \to \infty$ as $t \to \infty$. This contradicts that $w(t) \to -\infty$ as $t \to \infty$. Thus we obtain (2.24), which implies (2.23).

From (2.17) and (2.22) we obtain

$$r^2 f(u(r)) \leq C^* r^2 e^{u(r)} = C^* e^{u(r)+2\log r} \to 0 \quad \text{as } r \to 0.$$

Take $\varepsilon \in (0, 1)$. Then there exists $r_1 \in (0, r_0]$ such that

$$r^2 f(u(r)) < (N-2)\varepsilon \quad \text{for } 0 < r < r_1.$$

It follows that

$$\int_0^r s^{N-1} f(u(s))ds \leq (N-2)\varepsilon \int_0^r s^{N-3} ds = \varepsilon r^{N-2}$$

for $0 < r < r_1$. From (2.4) we have

$$-r^{N-1}u'(r) = \int_0^r s^{N-1} f(u(s))ds \leq \varepsilon r^{N-2} \quad \text{for } 0 < r < r_1,$$

which implies that $u'(r)+\varepsilon/r \geq 0$ for $0<r\leq r_1$. Observe that

$$(r^{\varepsilon}e^{u(r)})' = (e^{u(r)+\varepsilon\log r})' = \left(u'(r)+\frac{\varepsilon}{r}\right)e^{u(r)+\varepsilon\log r} \geq 0 \quad \text{for } 0<r\leq r_1.$$

Then $r^{\varepsilon}e^{u(r)}$ is nondecreasing for $0<r\leq r_1$, and hence $r^{\varepsilon}e^{u(r)} \leq r_1^{\varepsilon}e^{u(r_1)}$ for $0<r\leq r_1$. Thus we obtain $e^{u(r)} = O(r^{-\varepsilon})$ as $r\to 0$. From (2.4) and (2.17) we have

$$-r^{N-1}u'(r) = \int_0^r s^{N-1}f(u(s))ds \leq C^* \int_0^r s^{N-1}e^{u(s)}ds \leq C\int_0^r s^{N-1-\varepsilon}ds = C'r^{N-\varepsilon}$$

with some constants $C, C' > 0$. Thus $u'(r) = O(r^{1-\varepsilon})$ as $r\to 0$. Since $\varepsilon<1$, we have $u'(r)\to 0$ as $r\to 0$, and hence $\lim_{r\to 0}u(r)<\infty$. This is a contradiction. Thus we obtain $\limsup_{r\to 0}(u(r)+2\log r) > -\infty$. $\square$

3 Asymptotic behavior of singular solutions

In this section, we will show the following proposition, which specifies the behavior of the singular solution as $r\to 0$.

Proposition 3.1. *Let u be a positive solution of* (1.1) *for $0<r\leq r_0$ with some $r_0>0$.*

(i) *Assume that* (F1) *holds. If u satisfies $\limsup_{r\to 0} r^{2/(p-1)}u(r) > 0$, then*

$$u(r) = Ar^{-2/(p-1)}(1+o(1)) \quad \textit{as } r\to 0,$$

where A is the constant defined by (1.2).

(ii) *Assume that (F2) holds. If u satisfies $\limsup_{r\to 0}(u(r)+2\log r) > -\infty$, then*

$$u(r) = -2\log r + \log 2(N-2) + o(1) \quad \textit{as } r\to 0.$$

Combining Proposition 3.1 and Lemmas 2.2 and 2.4, we obtain Theorem 1.2. In the remaining of this section we will prove Proposition 3.1. For a solution u of (1.1), define $E(u)$ by

$$E(u)(r) = \frac{1}{2}u'(r)^2 + F(u(r)), \tag{3.1}$$

where

$$F(t) = \int_0^t f(s)ds \quad \text{for } t\geq 0. \tag{3.2}$$

Observe that

$$\frac{d}{dr}E(u)(r) = \big(u''(r)+f(u(r))\big)\,u'(r) = -\frac{N-1}{r}u'(r)^2 \leq 0.$$

This implies that $E(u)(r)$ is nonincreasing in $r>0$.

When (F1) holds, by L'Hospital's rule we have

$$\lim_{u\to\infty} \frac{F(u)}{u^{p+1}} = \lim_{u\to\infty} \frac{u^p + g(u)}{(p+1)u^p} = \frac{1}{p+1}. \tag{3.3}$$

By a similar way, in the case (F2), we have

$$\lim_{u\to\infty} \frac{F(u)}{e^u} = 1. \tag{3.4}$$

Lemma 3.1. *Assume that either* (F1) *or* (F2) *holds. Let u be a solution of* (1.1) *for $0 < r \le r_0$ with some $r_0 > 0$. If $\limsup_{r\to 0} u(r) = \infty$ then $\lim_{r\to 0} u(r) = \infty$.*

Proof. Assume to the contrary that $\liminf_{r\to 0} u(r) = c < \infty$. Then there exist sequences $\{r_k\}$ and $\{\rho_k\}$ with $r_k \to 0$, $\rho_k \to 0$ such that

$$u'(r_k) = 0 \quad \text{and} \quad u(r_k) \to \infty \quad \text{as } k \to \infty$$

and

$$u'(\rho_k) = 0 \quad \text{and} \quad u(\rho_k) \to c \quad \text{as } k \to \infty.$$

Define $E(u)$ by (3.1). From (3.3) and (3.4), we have

$$E(u)(r_k) = F(u(r_k)) \to \infty \quad \text{as } k \to \infty$$

in the both cases (F1) and (F2). Since $E(u)(r)$ is nonincreasing in $r > 0$, we obtain $E(u)(r) \to \infty$ as $r \to 0$. On the other hand, we see that

$$E(u)(\rho_k) = F(u(\rho_k)) \to F(c) < \infty \quad \text{as } k \to \infty.$$

This is a contradiction. Thus we obtain $\lim_{r\to 0} u(r) = \infty$. □

In the proof of Proposition 3.1, we consider the following ordinary differential equation

$$w'' - aw' + H(w) + G(t, w) = 0 \quad \text{for } t \ge t_0, \tag{3.5}$$

where $a > 0$ is a constant, $t_0 \in \mathbf{R}$, $H \in C(\alpha, \beta)$ and $G \in C([t_0, \infty) \times (\alpha, \beta))$ with some $-\infty \le \alpha < \beta \le \infty$. We assume that there exists $\gamma \in (\alpha, \beta)$ such that

$$(w - \gamma)H(w) > 0 \quad \text{for all } w \in (\alpha, \beta) \setminus \{\gamma\}. \tag{3.6}$$

The condition (3.6) implies that $H(\gamma) = 0$.

Lemma 3.2. *Let $w \in C^2[t_0, \infty)$ be a solution of* (3.5) *satisfying $\alpha < w(t) < \beta$ for all $t \ge t_0$. Assume that w satisfies*

$$\alpha < \limsup_{t\to\infty} w(t) < \beta \tag{3.7}$$

and

$$G(t, w(t)) \to 0 \quad \textit{as } t \to \infty. \tag{3.8}$$

If $\alpha = -\infty$, *assume in addition that*

$$\frac{G(t, w(t))}{e^{w(t)}} \to 0 \quad \textit{as } t \to \infty. \tag{3.9}$$

Then $\lim_{t\to\infty} w(t) = \gamma$.

Remark 3.1. Assume that $\alpha > -\infty$. In this case, if w satisfies (3.7) and (3.8), then w satisfies (3.9) also. In fact, from (3.7) we have $\liminf_{t\to\infty} w(t) \geq \alpha$. From (3.8) we obtain

$$\frac{|G(t, w(t))|}{e^{w(t)}} \leq \frac{|G(t, w(t))|}{e^{\alpha}} \to 0 \quad \text{as } t \to \infty.$$

Proof of Lemma 3.2. We divide the proof into the following two cases:

(i) $w'(t)$ is nonoscillatory at $t = \infty$, that is, $w'(t) \geq 0$ or $w'(t) \leq 0$ for t sufficiently large.

(ii) $w'(t)$ is oscillatory at $t = \infty$, that is, the sign of $w'(t)$ changes infinity many times as $t \to \infty$.

First we consider the case where $w'(t) \geq 0$ or $w'(t) \leq 0$ for t sufficiently large. Since (3.7) holds, there exists a constant $c \in (\alpha, \beta)$ such that $w(t) \to c$ as $t \to \infty$. We will show that $c = \gamma$. Assume to the contrary that $c \neq \gamma$. First we show that

$$\liminf_{t\to\infty} |w'(t)| = 0. \tag{3.10}$$

In fact, if $\liminf_{t\to\infty} |w'(t)| > 0$, then there exist $c_* > 0$ and $t_* \geq t_0$ such that $|w'(t)| \geq c_*$ for all $t \geq t_*$. This implies that $|w(t)| \to \infty$ as $t \to \infty$. This is a contradiction. Thus (3.10) holds. Next we show that

$$\limsup_{t\to\infty} |w'(t)| = 0. \tag{3.11}$$

Assume to the contrary that $\limsup_{t\to\infty} |w'(t)| > 0$. Then, from (3.10), there exists a sequence $t_n \to \infty$ of local minimum point of $w'(t)$ such that

$$w''(t_n) = 0 \quad \text{and} \quad w'(t_n) \to 0 \quad \text{as } n \to \infty.$$

Put $t = t_n$ in (3.5), and let $n \to \infty$. Since $G(t_n, w(t_n)) \to 0$ as $n \to \infty$ from (3.8), we have $H(w(t_n)) \to 0$ as $n \to \infty$. On the other hand, since $w(t) \to c \neq \gamma$ as $t \to \infty$, from (3.6) we have $H(w(t_n)) \to H(c) \neq 0$ as $n \to \infty$. This is a contradiction. Thus we obtain (3.11).

Combining (3.10) and (3.11), we obtain $\lim_{t\to\infty} w'(t) = 0$. Let $t \to \infty$ in (3.5). Then, from (3.6) and (3.8), we have

$$-aw'(t) + G(t, w(t)) \to 0 \quad \text{and} \quad H(w(t)) \to H(c) \neq 0 \quad \text{as } t \to \infty.$$

This implies that $w''(t) \to H(c) \neq 0$ as $t \to \infty$. Then we have $|w'(t)| \to \infty$ as $t \to \infty$. This is a contradiction. Thus we obtain $w(t) \to \gamma$ as $t \to \infty$.

Next we consider the case where the sign of $w'(t)$ changes infinity many times as $t \to \infty$. Put c_* and c^*, respectively, by

$$\liminf_{t\to\infty} w(t) = c_* \quad \text{and} \quad \limsup_{t\to\infty} w(t) = c^*. \tag{3.12}$$

Since w satisfies $\alpha < w(t) < \beta$ for all $t \geq t_0$ and (3.7), we have $\alpha \leq c_* \leq c^* < \beta$. We will show that $c_* = c^* = \gamma$. From (3.12), there exist $\{t_i\}$ and $\{\tau_i\}$ with $t_i < \tau_i$ such that, for $i = 1, 2, \ldots$,

$$\lim_{i\to\infty} w(t_i) = c^*, \quad w'(t) < 0 \quad \text{for } t_i < t < \tau_i, \quad \text{and} \quad w'(t_i) = w'(\tau_i) = 0.$$

Multiplying (3.5) by $w'(t)$, and integrating it on $[t_i, \tau_i]$, we obtain

$$\int_{t_i}^{\tau_i} w''w' - a(w')^2 dt + \int_{t_i}^{\tau_i} (H(w) + G(t,w))w'(t)dt = 0$$

for $i = 1, 2, \ldots$. From $w'(t_i) = w'(\tau_i) = 0$, we have

$$-a\int_{t_i}^{\tau_i} (w')^2 dt + \int_{t_i}^{\tau_i} (H(w) + G(t,w))w'(t)dt = 0.$$

We will show that

$$\int_{t_i}^{\tau_i} G(t,w)w'(t)dt \to 0 \quad \text{as } i \to \infty. \tag{3.13}$$

Recall that (3.9) holds even if $\alpha > -\infty$ by Remark 3.1. We see that

$$\begin{aligned}\left|\int_{t_i}^{\tau_i} G(t,w)w'(t)dt\right| &\leq \int_{t_i}^{\tau_i} \left|\frac{G(t,w)}{e^{w(t)}} e^{w(t)} w'(t)\right| dt \leq \sup_{t\geq t_i} \frac{|G(t,w(t))|}{e^{w(t)}} \int_{t_i}^{\tau_i} e^{w(t)}(-w'(t))dt \\ &= \sup_{t\geq t_i} \frac{|G(t,w(t))|}{e^{w(t)}} (e^{w(t_i)} - e^{w(\tau_i)}).\end{aligned}$$

From (3.9) it follows that

$$\sup_{t\geq t_i} \frac{|G(t,w(t))|}{e^{w(t)}} \to 0 \quad \text{as } i \to \infty.$$

Since $\alpha < w(t) < \beta$ for $t \geq t_0$ and (3.12) holds, we have

$$\limsup_{i\to\infty} (e^{w(t_i)} - e^{w(\tau_i)}) \leq e^{c^*} - e^{\alpha} < \infty.$$

Thus we obtain (3.13).

Since $w'(t) < 0$ on (t_i, τ_i), there exists an inverse function $t = \eta_i(w)$ on $[w(\tau_i), w(t_i)]$. Putting $w'(t) = w'(\eta_i(w)) = q_i(w)$, we obtain

$$\int_{w(\tau_i)}^{w(t_i)} H(w)dw - \int_{t_i}^{\tau_i} G(t,w))w'(t)dt = a\int_{w(\tau_i)}^{w(t_i)} q_i(w)dw \leq 0. \tag{3.14}$$

Put $\tilde{c}_* = \liminf_{i\to\infty} w(\tau_i)$. Then there exists a subsequence, which we denote by $\{w(\tau_i)\}$ again, such that $w(\tau_i) \to \tilde{c}_*$ as $i \to \infty$. Letting $i \to \infty$ in (3.14), from (3.13) we obtain

$$\int_{\tilde{c}_*}^{c^*} H(w)dw = \lim_{i\to\infty} a \int_{w(\tau_i)}^{w(t_i)} q_i(w)dw \leq 0.$$

From (3.6), we have $\tilde{c}_* < \gamma$. Since $c_* \leq \tilde{c}_*$, we obtain

$$\int_{c_*}^{c^*} H(w)dw \leq \lim_{i\to\infty} a \int_{w(\tau_i)}^{w(t_i)} q_i(w)dw \leq 0. \tag{3.15}$$

From (3.12), there exist $\{s_i\}$ and $\{\sigma_i\}$ with $s_i < \sigma_i$ such that, for $i = 1, 2, \ldots$,

$$\lim_{i\to\infty} w(s_i) = c_*, \quad w'(t) > 0 \quad \text{for } s_i < t < \sigma_i, \quad \text{and} \quad w'(s_i) = w'(\sigma_i) = 0.$$

Multiplying (3.5) by $w'(t)$, and integrating it on $[s_i, \sigma_i]$, we obtain

$$\int_{s_i}^{\sigma_i} w''w' - a(w')^2 dt + \int_{s_i}^{\sigma_i} (H(w) + G(t,w))w'(t)dt = 0$$

for $i = 1, 2, \ldots$. From $w'(s_i) = w'(\sigma_i) = 0$, we have

$$-a \int_{s_i}^{\sigma_i} (w')^2 dt + \int_{s_i}^{\sigma_i} (H(w) + G(t,w))w'(t)dt = 0.$$

By the similar argument as above, we obtain

$$\int_{s_i}^{\sigma_i} G(t,w)w'(t)dt \to 0 \quad \text{as } i \to \infty. \tag{3.16}$$

Since $w'(t) > 0$ on (s_i, σ_i), there exists an inverse function $t = \tilde{\eta}_i(w)$ on $[w(s_i), w(\sigma_i)]$. Putting $w'(t) = w'(\tilde{\eta}_i(w)) = \tilde{q}_i(w)$, we obtain

$$\int_{w(s_i)}^{w(\sigma_i)} H(w)dw + \int_{s_i}^{\sigma_i} G(t,w)w'(t)dt = a \int_{w(s_i)}^{w(\sigma_i)} \tilde{q}_i(w)dw \geq 0. \tag{3.17}$$

Put $\tilde{c}^* = \limsup_{i\to\infty} w(\sigma_i)$. Then there exists a subsequence, which we denote by $\{w(\sigma_i)\}$ again, such that $w(\sigma_i) \to \tilde{c}^*$ as $i \to \infty$. Letting $i \to \infty$ in (3.17), from (3.16) we obtain

$$\int_{c_*}^{\tilde{c}^*} H(w)dw = \lim_{i\to\infty} a \int_{w(t_i)}^{w(\sigma_i)} \tilde{q}_i(w)dw \geq 0.$$

From (3.6), we have $\tilde{c}^* > \gamma$. Since $c^* \geq \tilde{c}^*$, we obtain

$$\int_{c_*}^{c^*} H(w)dw \geq 0 \tag{3.18}$$

Combining (3.15) and (3.18), we obtain

$$\int_{c_*}^{c^*} H(w)dw = 0 \tag{3.19}$$

and

$$\lim_{i\to\infty}\int_{w(\tau_i)}^{w(t_i)} q_i(w)dw = 0. \tag{3.20}$$

From (3.6) and (3.19), we have $c_* \le \gamma \le c^*$. We will show that $c_* = \gamma = c^*$. Assume to the contrary that $c^* > \gamma$. Put

$$L = \int_{(c^*+\gamma)/2}^{c^*} H(w)dw > 0.$$

We will show that

$$\limsup_{i\to\infty} q_i(w) \le -\sqrt{2L} \quad \text{for all } w \in [\gamma, (c^*+\gamma)/2]. \tag{3.21}$$

Take any $w_0 \in [\gamma, (c^*+\gamma)/2]$. Since $\tilde{c}_* < \gamma$, for sufficiently large i, there exists $\hat{t}_i$ satisfying $t_i < \hat{t}_i < \tau_i$ and $w(\hat{t}_i) = w_0$. Multiplying (3.5) by $w'(t)$, and integrating on $[t_i, \hat{t}_i]$, we obtain

$$\int_{t_i}^{\hat{t}_i} w''w' - a(w')^2 dt + \int_{t_i}^{\hat{t}_i} (H(w) + G(t,w))w' dt = 0.$$

From $w'(t_i) = 0$, we obtain

$$\frac{w'(\hat{t}_i)^2}{2} - a\int_{t_i}^{\hat{t}_i} (w')^2 dt + \int_{t_i}^{\hat{t}_i} (H(w) + G(t,w))w' dt = 0.$$

Then it follows that

$$-\frac{q_i(w_0)^2}{2} - a\int_{w(\hat{t}_i)}^{w(t_i)} q_i(w)dw + \int_{w(\hat{t}_i)}^{w(t_i)} H(w)dw - \int_{t_i}^{\hat{t}_i} G(t,w)w'(t)dt = 0. \tag{3.22}$$

By the similar argument as above, we obtain

$$\int_{t_i}^{\hat{t}_i} G(t,w)w'(t)dt \to 0 \quad \text{as } i \to \infty.$$

From (3.20) and $q_i(w) < 0$, we have

$$0 \ge \int_{w(\hat{t}_i)}^{w(t_i)} q_i(w)dw \ge \int_{w(\tau_i)}^{w(t_i)} q_i(w)dw \to 0 \quad \text{as } i \to \infty.$$

Then, letting $i \to \infty$ in (3.22), we have

$$\lim_{i\to\infty} \frac{q_i(w_0)^2}{2} = \int_{w_0}^{c^*} H(w)dw \ge \int_{(c^*+\gamma)/2}^{c^*} H(w)dw = L.$$

Since $q_i(w_0) < 0$, we obtain (3.21).

For sufficiently large i, there exist $\tilde{\tau}_i$ and $\tilde{t}_i$ satisfying $t_i < \tilde{t}_i < \tilde{\tau}_i < \tau_i$ and $w(\tilde{t}_i) = (c^* + \gamma)/2$ and $w(\tilde{\tau}_i) = \gamma$. Then, from (3.21), it follows that

$$\begin{aligned}\limsup_{i\to\infty} \int_{w(\tau_i)}^{w(t_i)} q_i(w)dw &\leq \limsup_{i\to\infty} \int_{w(\tilde{\tau}_i)}^{w(\tilde{t}_i)} q_i(w)dw \\ &= \limsup_{i\to\infty} \int_{\gamma}^{(c^*+\gamma)/2} q_i(w)dw \leq -\frac{c^*-\gamma}{2}\sqrt{2L} < 0.\end{aligned}$$

This contradicts (3.20). Thus we obtain $c^* = \gamma$. By (3.19), we obtain $c_* = \gamma$. $\square$

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. (i) By Lemma 3.1, we have $\lim_{r\to 0} u(r) = \infty$. By Lemma 2.1, we see that $r^{2/(p-1)}u(r)$ is bounded above. Define w by (2.10). Then $w(t)$ is bounded above. Since $u(r) = e^{\theta t} w(t)$ with $\theta = 2/(p-1)$, we obtain

$$e^{\theta t} w(t) \to \infty \quad \text{as } t \to \infty. \tag{3.23}$$

Put $a = N - 2 - 2\theta$, $H(w) = -A^{p-1}w + w^p$ and $G(t, w) = e^{-p\theta t} g(e^{\theta t} w)$. Then, from (2.12), w satisfies (3.5). Furthermore, H satisfies (3.6) with $\alpha = 0$, $\beta = \infty$ and $\gamma = A$. By the assumption, we have

$$\limsup_{t\to\infty} w(t) > 0.$$

From (3.23) and (1.3), we obtain

$$G(t, w(t)) = e^{-p\theta t} g(e^{\theta t} w(t)) = \frac{g(e^{\theta t} w(t))}{(e^{\theta t} w(t))^p} w(t)^p \to 0 \quad \text{as } t \to \infty.$$

Applying Lemma 3.2, we obtain $w(t) \to A$ as $t \to \infty$. Thus we obtain (i).

(ii) By Lemma 3.1, we have $\lim_{r\to 0} u(r) = \infty$. By Lemma 2.3, we see that $u(r) + 2\log r$ is bounded above. Define w by (2.19). Then $w(t)$ is bounded above. Since $u(r) = w(t) + 2t + \kappa$, we obtain

$$w(t) + 2t + \kappa \to \infty \quad \text{as } t \to \infty. \tag{3.24}$$

Put $a = N - 2$, $H(w) = 2(N-2)(e^w - 1)$ and $G(t, w) = e^{-2t} g(w + 2t + \kappa)$. Then, from (2.21), w satisfies (3.5). Furthermore, H satisfies (3.6) with $\alpha = -\infty$, $\beta = \infty$, and $\gamma = 0$. By the assumption, we have

$$\limsup_{t\to\infty} w(t) > -\infty.$$

From (3.24) and (1.4), we obtain

$$\frac{G(t, w(t))}{e^{w(t)}} = \frac{g(w(t) + 2t + \kappa)}{e^{w(t)+2t+\kappa}} e^{\kappa} \to 0 \quad \text{as } t \to \infty.$$

Applying Lemma 3.2, we obtain $w(t) \to 0$ as $t \to \infty$. Thus we obtain (ii). $\square$

4 Uniqueness of the singular solution

In this section we show the following proposition.

Proposition 4.1. *Assume that either* (F1) *or* (F2) *holds. Then there exists at most one singular solution of* (1.1)*.*

We recall the following lemma by [22, Lemma 4.2].

Lemma 4.1. *Suppose that $a(t)$ and $b(t)$ are continuous functions satisfying $\lim_{t\to\infty} a(t) = a > 0$ and $\lim_{t\to\infty} b(t) = b > 0$. Let $z(t)$ be a solution of $z'' - a(t)z' + b(t)z = 0$. If $z(t)$ is bounded as $t \to \infty$, then $z(t) \equiv 0$.*

Proof of Proposition 4.1. Let $u_1(s)$ and $u_2(s)$ be singular solutions of (1.1) for $0 < r \le r_0$. First we consider the case (F1). Define $w_i(t) = r^{2/(p-1)}u_i(r)$ with $t = -\log r$ for $i = 1, 2$. Then, for $i = 1, 2$, $w = w_i(t)$ satisfies (2.12) with $t_0 = -\log r_0$. Combining Lemma 2.2 and Proposition 3.1 (i), we obtain

$$\lim_{t\to\infty} w_1(t) = \lim_{t\to\infty} w_2(t) = A. \tag{4.1}$$

Define $z(t) = w_1(t) - w_2(t)$, and put $G(t, w) = e^{-p\theta t} g(e^{\theta t} w)$. Then z satisfies $\lim_{t\to\infty} z(t) = 0$ and

$$z'' - az' + b(t)z = 0 \quad \text{for } t \ge t_0, \tag{4.2}$$

where $a = N - 2 - 2\theta$ and

$$b(t) = -A^{p-1} + \frac{w_1(t)^p - w_2(t)^p}{w_1(t) - w_2(t)} + \frac{G(t, w_1(t)) - G(t, w_2(t))}{w_1(t) - w_2(t)}.$$

From (4.1) we have

$$\lim_{t\to\infty} \frac{w_1(t)^p - w_2(t)^p}{w_1(t) - w_2(t)} = pA^{p-1}.$$

By the mean value theorem, we have

$$G(t, w_1(t)) - G(t, w_2(t)) = e^{-(p-1)\theta t} g'(e^{\theta t}\eta(t))(w_1(t) - w_2(t))$$

for some $\eta(t)$, which is between $w_1(t)$ and $w_2(t)$. From (4.1), we have $\eta(t) \to A$ as $t \to \infty$. By (1.3), we obtain

$$\lim_{t\to\infty} \frac{G(t, w_1(t)) - G(t, w_2(t))}{w_1(t) - w_2(t)} = \lim_{t\to\infty} \frac{g'(e^{\theta t}\eta(t))}{(e^{\theta t}\eta(t))^{p-1}} \eta(t)^{p-1} = 0.$$

We therefore obtain $\lim_{t\to\infty} b(t) = (p-1)A^{p-1} > 0$. By Lemma 4.1 we obtain $z(t) \equiv 0$. Thus (1.1) has at most one singular solution.

Next we consider the case (F2). Define $w_i(t) = u_i(r) + 2\log r + \kappa$ with $t = -\log r$ for $i = 1, 2$. Then, for $i = 1, 2$, $w = w_i(t)$ satisfies (2.21) with $t_0 = -\log r_0$. Combining Lemma 2.4 and Proposition 3.1 (ii), we obtain

$$\lim_{t\to\infty} w_1(t) = \lim_{t\to\infty} w_2(t) = 0. \tag{4.3}$$

Define $z(t) = w_1(t) - w_2(t)$, and put $G(t, w) = e^{-2t} g(w + 2t + \kappa)$. Then z satisfies $\lim_{t\to\infty} z(t) = 0$ and (4.2) with $a = N - 2 > 0$ and

$$b(t) = 2(N-2)\frac{e^{w_1(t)} - e^{w_2(t)}}{w_1(t) - w_2(t)} + \frac{G(t, w_1(t)) - G(t, w_2(t))}{w_1(t) - w_2(t)}.$$

From (4.3) we have

$$\lim_{t\to\infty} \frac{e^{w_1(t)} - e^{w_2(t)}}{w_1(t) - w_2(t)} = 1.$$

By the mean value theorem, we have

$$G(t, w_1(t)) - G(t, w_2(t)) = e^{-2t} g'(\eta(t) + 2t + \kappa)(w_1(t) - w_2(t))$$

for some $\eta(t)$, which is between $w_1(t)$ and $w_2(t)$. From (4.3), we have $\eta(t) \to 0$ as $t \to \infty$. By (1.4), we obtain

$$\lim_{t\to\infty} \frac{G(t, w_1(t)) - G(t, w_2(t))}{w_1(t) - w_2(t)} = \lim_{t\to\infty} \frac{g'(\eta(t) + 2t + \kappa)}{e^{\eta(t)+2t+\kappa}} e^{\eta(t)+\kappa} = 0.$$

We therefore obtain $\lim_{t\to\infty} b(t) = 2(N-2) > 0$. By Lemma 4.1 we obtain $z(t) \equiv 0$, and hence (1.1) has at most one singular solution. □

5 Proofs of main results

5.1 Proof of Theorem 1.1

Following the idea by Lin [21], we show the following lemma.

Lemma 5.1. *Let $f \in C^1[0, \infty)$, and define F by (3.2). Assume that there exist constants $q > (N+2)/(N-2)$ and $\hat{u}_0 \geq 0$ such that*

$$0 < (q+1)F(u) < uf(u) \quad \text{for } u \geq \hat{u}_0. \tag{5.1}$$

(i) *Let $\alpha > \hat{u}_0$. Assume that $u(r, \alpha) \geq \hat{u}_0$ for $0 \leq r \leq r_0$ with some $r_0 > 0$. Then*

$$0 < -ru'(r, \alpha) < \frac{2N}{q+1} u(r, \alpha) \quad \text{for } 0 < r \leq r_0. \tag{5.2}$$

(ii) *Put*

$$\gamma = \frac{1}{2}\left(1 - \frac{2N}{(q+1)(N-2)}\right).$$

Take any $\beta \geq \hat{u}_0$*, and define* r_β *by*

$$r_\beta = \left(\frac{2N\beta}{f_M(\beta/\gamma)}\right)^{1/2},$$

where $f_M(r) = \max\{f(s) : 0 \leq s \leq r\}$*. If* $\alpha > \beta/\gamma$ *then* $u(r,\alpha)$ *satisfies*

$$u(r,\alpha) > \beta \quad \textit{for } 0 \leq r \leq r_\beta. \tag{5.3}$$

To prove Lemma 5.1, we first recall a Pohozaev identity which was obtained by Ni and Serrin [31].

Lemma 5.2. *Let* $u(r)$ *be a solution of* (1.1) *in* $(r_1, r_2) \subset (0,\infty)$ *and let* μ *be an arbitrary constant. Then, for each* $r \in (r_1, r_2)$*, we have*

$$\begin{aligned} &\frac{d}{dr}\left\{r^N\left(\frac{1}{2}u'(r)^2 + F(u(r)) + \frac{\mu}{r}u(r)u'(r)\right)\right\} \\ &= r^{N-1}\left\{NF(u(r)) - \mu u(r) f(u(r)) + \left(\mu + 1 - \frac{N}{2}\right)u'(r)^2\right\}. \end{aligned} \tag{5.4}$$

Proof of Lemma 5.1. (i) Letting $\mu = N/(q+1)$ in (5.4), we obtain

$$\begin{aligned} &\frac{d}{dr}\left\{r^N\left(\frac{1}{2}u'(r)^2 + F(u(r)) + \frac{N}{q+1}\frac{u(r)u'(r)}{r}\right)\right\} \\ &= r^{N-1}\left\{NF(u(r)) - \frac{N}{q+1}u(r)f(u(r)) + \left(\frac{N}{q+1} - \frac{N-2}{2}\right)u'(r)^2\right\}. \end{aligned} \tag{5.5}$$

Put $u = u(r,\alpha)$ in (5.5). From (5.1) and $q > (N+2)/(N-2)$, we see that the right-hand side of (5.5) is negative for $0 \leq r \leq r_0$. Integrating (5.5) on $[0,r]$ with $0 < r \leq r_0$, we obtain

$$\frac{1}{2}u'(r,\alpha)^2 + F(u(r,\alpha)) + \frac{N}{q+1}\frac{u(r,\alpha)u'(r,\alpha)}{r} < 0.$$

Note that $u(r,\alpha) \geq \hat{u}_0$ for $0 < r \leq r_0$ and $f(u) > 0$ for $u \geq \hat{u}_0$ from (5.1). Then we obtain $u'(r,\alpha) < 0$ for $0 < r \leq r_0$. From (5.1) we have $F(u(r,\alpha)) > 0$. Thus we obtain (5.2).

(ii) Assume to the contrary that there exists $r_* \in (0, r_\beta]$ such that

$$u(r,\alpha) > \beta \quad \text{for } 0 \leq r < r_* \quad \text{and} \quad u(r_*,\alpha) = \beta. \tag{5.6}$$

Note that $\beta \geq \hat{u}_0$. Then, it follows from (5.2) that $u'(r,\alpha) \leq 0$ for $0 \leq r \leq r_*$. Put $B = \beta/\gamma$. Since $\alpha > B > \beta$, there exists $R \in (0, r_*)$ such that

$$u(R,\alpha) = B \quad \text{and} \quad u(r,\alpha) \leq B \quad \text{for } R \leq r \leq r_*. \tag{5.7}$$

Let v be a solution of the initial value problem

$$\begin{cases} -(r^{N-1}v')' = r^{N-1}f_M(B), & R < r < r_*, \\ v(R) = u(R,\alpha) \quad \text{and} \quad v'(R) = u'(R,\alpha). \end{cases} \tag{5.8}$$

We will show that

$$v(r) \le u(r,\alpha) \quad \text{for } R \le r \le r_*. \tag{5.9}$$

In fact, put $w(r) = v(r) - u(r,\alpha)$. Then w satisfies

$$-(r^{N-1}w')' = r^{N-1}(f_M(B) - f(u(r,\alpha))) \quad \text{for } R \le r \le r_* \tag{5.10}$$

and $w(R) = w'(R) = 0$. From (5.7), we have $f_M(B) - f(u(r,\alpha)) \ge 0$ for $R \le r \le r_*$. Integrating (5.10) on $[R,r]$ with $r \le r_*$, we obtain $-r^{N-1}w'(r) \ge 0$. Then $w'(r) \le 0$ for $R \le r \le r_*$. Since $w(R) = 0$, we have $w(r) \le 0$ for $R \le r \le r_*$, which implies that (5.9) holds.

Integrating the equation in (5.8), we have

$$-r^{N-1}v'(r) = -R^{N-1}v'(R) + \int_R^r s^{N-1}f_M(B)ds < -R^{N-1}v'(R) + \frac{f_M(B)}{N}r^N.$$

This implies that

$$-v'(r) < -\frac{R^{N-1}v'(R)}{r^{N-1}} + \frac{f_M(B)}{N}r.$$

Integrating the above on $[R,r_*]$, we obtain

$$-v(r_*) + v(R) < -\frac{R^{N-1}v'(R)}{N-2}(R^{2-N} - r_*^{2-N}) + \frac{f_M(B)}{2N}(r_*^2 - R^2).$$

Since $v'(R) \le 0$, $f_M(B) \ge 0$ and $r_* \le r_\beta$, we obtain

$$-v(r_*) + v(R) < -\frac{Rv'(R)}{N-2} + \frac{f_M(B)}{2N}r_\beta^2 = -\frac{Rv'(R)}{N-2} + \beta.$$

From $v(R) = u(R,\alpha)$, $v'(R) = u'(R,\alpha)$ and (5.2), we obtain

$$v(r_*) > u(R,\alpha) + \frac{Ru'(R,\alpha)}{N-2} - \beta \ge \left(1 - \frac{2N}{(N-2)(q+1)}\right)u(R,\alpha) - \beta.$$

Recall that $u(R,\alpha) = B$ and $\gamma B = \beta$. Then we obtain $v(r_*) > 2\gamma B - \beta = \beta$. Since (5.9) holds, we obtain $u(r_*,\alpha) > \beta$, which contradicts (5.6). Thus (5.3) holds. $\square$

The next results follow immediately from Lemmas 2.1 and 2.3.

Lemma 5.3. (i) *Assume that* (F1) *holds. Let u_0 be a constant in* (2.1), *and let $\alpha > u_0$. Assume that $u(r,\alpha) \ge u_0$ for $0 \le r \le r_0$. Then*

$$u(r,\alpha) \le C_0 r^{-2/(p-1)} \quad \textit{and} \quad 0 \le -u'(r,\alpha) \le C_1 r^{-2/(p-1)-1} \quad \textit{for } 0 < r \le r_0,$$

where C_0 and C_1 are positive constants independent of α.

(ii) *Assume that* (F2) *holds. Let u_0 be a constant in* (2.16)*, and let $\alpha > u_0$. Assume that $u(r, \alpha) \geq u_0$ for $0 \leq r \leq r_0$. Then*

$$u(r) \leq -2 \log r + C_1 \quad \textit{and} \quad 0 \leq -u'(r) \leq \frac{C_2}{r} \quad \textit{for } 0 \leq r \leq r_0,$$

where constants $C_1 \in \mathbf{R}$ and $C_2 > 0$ are independent of α.

Proof of Theorem 1.1. In the case of (F1), take q such that $(N+2)/(N-2) < q < p$. Then, from (3.3) we obtain

$$\lim_{u \to \infty} \frac{(q+1)F(u)}{uf(u)} = \frac{q+1}{p+1} < 1.$$

In the case of (F2), take $q > (N+2)/(N-2)$. Then, from (3.4) we obtain

$$\lim_{u \to \infty} \frac{(q+1)F(u)}{uf(u)} = 0.$$

Thus, in the both cases, there exists $\hat{u}_0 > 0$ such that (5.1) holds. We may assume that $\hat{u}_0 \geq u_0$ in (5.1), where u_0 is the constant in (2.1).

Let $\{\alpha_k\}$ be a sequence such that $\alpha_k \to \infty$ as $k \to \infty$. Then $u(r, \alpha_k)$ and $u'(r, \alpha_k)$ are uniformly bounded in $k \in \mathbf{N}$ on any compact subset of $(0, r_0]$ by Lemma 5.3 (i) and (ii) in the cases (F1) and (F2), respectively. Since $f \in C^1[0, \infty)$ in (1.1), $u''(r, \alpha_k)$ and $u'''(r, \alpha_k)$ are also uniformly bounded in $k \in \mathbf{N}$ on any compact subset of $(0, r_0]$. By the Arzelá-Ascoli theorem with the diagonal argument, there exist $u^* \in C^2(0, r_0]$ and a subsequence, which we denote again by $\{u(\cdot, \alpha_k)\}$, such that

$$u(r, \alpha_k) \to u^*(r) \quad \text{in } C^2_{\text{loc}}(0, r_0] \quad \text{as } k \to \infty. \tag{5.11}$$

It is clear that u^* solves (1.1) for $0 < r \leq r_0$. Take any $\beta > \hat{u}_0$. Lemma 5.1 (ii) implies that $u(r_\beta, \alpha_k) > \beta$ if $\alpha_k > \beta/\gamma$. Letting $k \to \infty$, we obtain $u^*(r_\beta) \geq \beta$. Since $f_M(\beta/\gamma) \geq f(\beta/\gamma)$ and $f(u)/u \to \infty$ as $u \to \infty$, we have

$$0 < r_\beta \leq \left(2N\gamma \frac{\beta/\gamma}{f(\beta/\gamma)} \right)^{1/2} \to 0 \quad \text{as } \beta \to \infty.$$

This implies that $u^*(r) \to \infty$ as $r \to 0$, and hence u^* is a singular solution. Combining Proposition 3.1 and Lemmas 2.2 and 2.4, we obtain (1.6) and (1.7). By Proposition 4.1, the singular solution of (1.1) is unique. Thus, for any sequence $\alpha_k \to \infty$, there exists a subsequence such that (5.11) holds with a unique function u^*. This implies that $u_\alpha \to u^*$ in $C^2_{\text{loc}}(0, r_0]$ as $\alpha \to \infty$. $\square$

5.2 Proof of Corollaries 1.1 and 1.2

Let u^* be the singular solution of (1.1) obtained in Theorem 1.1. We denote by r_0^* the first zero of $v^*(r)$ if it exists.

Proof of Corollary 1.1. (i) Since $f(u) > 0$ for $u \geq 0$, $u^*(r)$ has the first zero r_0^*. Put

$$\lambda^* = (r_0^*)^2 \quad \text{and} \quad v^*(r) = u^*(r_0^* r). \tag{5.12}$$

Then $v^*(r)$ with $r = |x|$ solves (1.10) with $\lambda = \lambda^*$. It is clear that $v^*(r) \to \infty$ as $r \to 0$. Thus (1.10) has a singular solution (λ^*, v^*).

Assume that (1.10) has another singular solution $(\hat{\lambda}, \hat{v})$. Put $\hat{u}(r) = \hat{v}(r/\sqrt{\hat{\lambda}})$. Then $\hat{u}(r)$ also satisfies (1.1) and $\hat{u}(\sqrt{\hat{\lambda}}) = 0$. Since the singular solution of (1.1) is unique by Theorem 1.1, we obtain $u^*(r) \equiv \hat{u}(r)$, and hence $r_0^* = \sqrt{\hat{\lambda}}$. Then we obtain $\lambda^* = (r_0^*)^2 = \hat{\lambda}$ and $v^*(r) \equiv \hat{v}(r)$. As a consequence, (λ^*, v^*) is the unique singular solution of (1.10).

(ii) We extend the domain of f on $(-\infty, \infty)$ such that $f(u) > 0$ for all $u \in (-\infty, \infty)$. As a typical example, we define $f(u) = f(0)$ for all $u \leq 0$. Since $f(u) > 0$ for all $u \geq 0$, $u(r, \alpha)$ has the first zero, which we denote by $r_0(\alpha)$. Let $\delta > 0$. We extend the domain of $u^*(r)$ and $u(r, \alpha)$ to $(0, r_0^* + \delta)$ and $(0, r_0(\alpha) + \delta)$, respectively. Since $f(u) > 0$ for al $u \in (-\infty, \infty)$, $u^*(r)$ and $u(r, \alpha)$ are decreasing in $r > 0$. Then $u^*(r) < 0$ for $(r_0^*, r_0^* + \delta)$ and $u(r, \alpha) < 0$ for $(r_0(\alpha), r_0(\alpha) + \delta)$.

We see that

$$v(r, \alpha) = u(r_0(\alpha) r, \alpha) \quad \text{and} \quad \lambda(\alpha) = r_0(\alpha)^2.$$

From (1.5) we obtain $r_0(\alpha) \to r_0^*$ as $\alpha \to \infty$. Then, it follows from (5.12) that $\lambda(\alpha) \to \lambda^*$ as $\alpha \to \infty$. Observe that

$$\begin{aligned} |v(r, \alpha) - v^*(r)| &= |u(r_0(\alpha) r, \alpha) - u^*(r_0^* r)| \\ &\leq |u(r_0(\alpha) r, \alpha) - u^*(r_0(\alpha) r)| + |u^*(r_0(\alpha) r) - u^*(r_0^* r)|. \end{aligned}$$

By Theorem 1.1, we obtain $v(\cdot, \alpha) \to v^*$ in $C^2_{\mathrm{loc}}(0, 1]$ as $\alpha \to \infty$. Thus (1.11) holds. □

Proof of Corollary 1.2. Since the singular solution of (1.1) is unique by Theorem 1.1, it is clear that (1.12) has a singular solution if and only if $u^*(r) > 0$ for all $r \in (0, \infty)$.

First assume that $u^*(r)$ has a zero $r_0^* > 0$. Then (1.12) has no singular solution, and (1.10) has the the unique singular solution (λ^*, v^*) defined by (5.12). Thus (i) holds.

Next assume that $u^*(r) > 0$ for all $r \in (0, \infty)$. Then (1.12) has a singular solution. Assume that (1.10) has a singular solution $(\hat{\lambda}, \hat{v})$. Put $\hat{u}(r) = \hat{v}(r/\sqrt{\hat{\lambda}})$. Then, by the argument in the proof of Corollary 1.1, $\hat{u}(r) \equiv u^*(r)$ and $u^*(\sqrt{\hat{\lambda}}) = 0$. This is a contradiction. Thus (1.10) has no singular solution. Thus (ii) holds. □

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