Doctoral Dissertation

Conceptualizing Pre-service Mathematics Teachers' Knowledge for Teaching Probability in Egypt from the Perspective of Probabilistic Reasoning

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DEDICATION

To Allah, who gave me the power and strength to reach this stage, I dedicate my work. I hope it will add to my good deeds scale, and may he forgive me for any shortages. My lovely parents: GAMAL and AMAL, I’m very grateful for your extreme love, encouragement, support, and inspiration; I always keep in my mind your words about who I am. Thank you so much for being with me during each minute throughout this whole journey. My brother: AHMED, you cannot imagine how your words supported me and pushed me ahead; thank you so much. My friends: SAMIA, DINA, SHIMAA, FATMA, KARIMA, NORA, and ESRAA, thank you for what you all did for me; you spent a lot of your time taking care of me and trying to talk, advise, and reduce my stress. SAMIA and DINA: I never felt alone because of you; Allah blesses you and your families. Also, my foreign friends: NANAE, OTGONBAATAR, PIERRE, NOFI, MAYU, and YOSHINORI, thank you so much for making me feel that I am home. My uncles and aunties: EBRAHIM, ELSAID, MABROKA, LEILA, HANAN, FATME, and ENTSAAR, thank you for continuously being close. Extraordinarily, my big supporter and the one I love the most: MY GRANDMOM; without your prayers, I couldn’t have done anything, your deep love embraced me everywhere. Finally, with my sincere appreciation, I'm very thankful to everyone who supported me: family members, friends, colleagues, and teachers.

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LIST OF ABBREVIATIONS

CERME: Congress of European Research in Mathematics Education
CCIMD: Curriculum Center for Instructional Materials Development
CCSSM: Common Core State Standards for Mathematics
COACTIV: Cognitive Activation in the Mathematics Classroom
GAISE: Guidelines for Assessment and Instruction in Statistics Education
IASE: International Association for Statistical Education
ICME: International Congresses in Mathematical Education
ICOTS: International Conference on Teaching Statistics
KCC: Knowledge of Content and Curriculum
KCS: Knowledge of Content and Students
KCT: Knowledge of Content and Teaching
KoP: Knowledge of Probability
KoPL: Knowledge of Probability Language
KoSPK: Knowledge of Students' Probability Knowledge
KoTP: Knowledge of Teaching Probability
MKT: Mathematical Knowledge for Teaching
NCTM: National Council of Teachers of Mathematics standards
NZ: New Zealand
OECD: Organization for Economic Co-operation and Development
OSA: Onto- Semiotic Approach
PCK: Pedagogical Content Knowledge
PCMI: Park City Mathematics Institute
PISA: Programme for International Student Assessment
PoPR: Perspective of Probabilistic Reasoning
PSMTs: Pre-service Mathematics Teachers
SCK: Specialized Content Knowledge
SDG: Sustainable Development Goals
SMK: Subject Matter Knowledge
TEDS-M: Teacher Education and Development Study in Mathematics
TIMSS: Trends in International Mathematics and Science Study
UNAIDS: Joint United Nations Programme on HIV/AIDS
UNESCO: United Nations Educational, Scientific, and Cultural Organization
WSC: World Statistics Congress

ABSTRACT

Probability knowledge is needed for all citizens to reason on every day uncertain situations; besides, it is also required to train many professionals at the university level. Furthermore, an essential value for learning probability contributes to the formation of a specific type of reasoning: probabilistic reasoning, which helps learners formally structure their vague thinking about random phenomena. For such reasons, the probability was recognized by educational authorities and included in the curricula of many countries at different levels, from the primary stage to teacher education. However, several concerns have been discussed in the literature regarding the deficiency of probability education, which may negatively impact learners' acquisition of probability knowledge and the development of their probabilistic reasoning.

Among other issues, the following ones have been raised: (a) the probability curriculum was criticized in terms of acquiring a too narrow view of probability; it strengthens the statistical side that is relevant to the objective mathematical rules rather than the epistemic side that interprets probability as a personal degree of belief. And (b) there is an inadequate preparation of Pre-Service Mathematics Teachers (PSMTs) to teach probability efficiently. Both issues were found in the Egyptian context where, on one hand, ideas of independence and conditional probability, probability distribution and expectation, and convergence and the law of large numbers were disregarded from the intended curriculum; and, on the other hand, the implemented curriculum emphasized merely theoretical and axiomatic approaches to probability. Moreover, for the PSMTs, only about 9% of all subjects, which they were studying throughout the whole duration of their preparation program, were assigned to statistics and probability. Besides, there was no particular discussion concerning probability instruction. These issues cause various learning difficulties, and they also create further challenges for teachers, notably because of the distinct characteristics of probability that are not often found in other mathematics areas.

From this aspect, and acknowledging the influence of teachers’ knowledge on students’ learning and achievement, this study highlighted PSMTs’ knowledge for teaching probability; as argued by Dollard (2011), “One way to improve this situation is to ensure that new teachers graduating from teacher education programs have a good understanding of the fundamental concepts of probability” (p. 27). Moreover, this study embraced the perspective of probabilistic reasoning (PoPR) to approach such knowledge. This was decided in the light of reviewing the historical development of probability education research wherein the PoPR was assumed to

construct “a more unified development of the classical, frequentist, and subjective approaches to probability” (Jones et al., 2007, p. 949), and to clarify issues of subjective probability that did not receive much attention in the field of mathematics education.

Based on the previous discussion, this study was intended to *conceptualize PSMTs' knowledge for teaching probability in Egypt from the PoPR* through answering these questions:

RQ1. What is the current status of “statistics and probability” education in Egypt?

RQ2. What is the definition of mathematics teachers’ professional knowledge for teaching probability from the PoPR?

RQ3. What are the characteristics of PSMTs' knowledge for teaching probability in Egypt from the PoPR?

To answer the first research question, issues of statistics and probability education in Egypt were outlined locally and, then, internationally. From a local perspective, several documents were reviewed. As a result, it was evident that the Egyptian government advocated the need to enhance PSMTs' professional competence to meet pupils’ needs, especially to teach contents of statistics and probability. This content constituted a less emphasized area of study during PSMTs’ preparation; furthermore, Egyptian pupils’ achievement in the content area of Data and chance stayed the lowest among other mathematics areas, as reported by TIMSS 2003 and 2007. More specifically, **probability** denoted a core concept for which most of the textbooks' activities aimed, which is promoting pupils' probabilistic understanding.

Additionally, both the intended and implemented Egyptian school curricula of probability were analyzed from an international viewpoint. The intended curriculum was compared with the New Zealand curriculum, where ideas of independence and conditional probability, probability distribution and expectation, and convergence and the law of large numbers appeared to be ignored within the Egyptian curriculum. Moreover, analyzing the implemented curriculum exhibited a lack of addressing experimental probability interpretation; also, the subjective probability approach was revealed to be neglected until grade 9.

To answer the second research question, two essential steps were performed. While the first step outlined mathematics teachers' professional knowledge for teaching probability as defined in the literature (knowledge for practice), the second step manifested the psychological facet of teachers' knowledge; it was exemplified through their conceptions (knowledge in practice) and reasoning processes.

For the first step, aspects of KoP, KoTP, KoSPK, and KoPL were explored. Moreover, the **KoP** was sharpened in this study since it determined the heart of mathematics teachers’ professional knowledge for teaching probability. It also indicated teachers' epistemological reflection on the meaning of probability, which requires an understanding of its (a) objective facet that emphasizes the mathematical rules that govern random processes, and (b) subjective facet that sharpens the information available to the person assigning that probability. Despite this, these components disregarded the dynamic aspect of teachers' knowledge; they neither considered teachers' reasoning processes nor the cognitive biases that shape their pedagogical practices. Thus, to exhibit these aspects, the second step was recognized through introducing the study premises. These premises were: (a) conceptions represent knowledge in evolution, (b) reasoning indicates an individual cognitive process to interpret the acquired knowledge, and (c) there is a reciprocal relationship between conceptions and reasoning.

In light of both steps, the study framework was represented. Accordingly, mathematics teachers' professional knowledge for teaching probability was redefined from the PoPR to include these aspects: **R(in)P**, **R(in)PL**, **R(in)TP**, and **R(in)SPK**, which determine their reasoning in a situation that involves knowledge of probability, probability language, teaching probability, and students probability knowledge, respectively.

Finally, answering the third research question incorporated characterizing **PSMTs' R(in)P** that is related to (a) *simple unconditional* and (b) *conditional* probabilistic situations. Both issues of (a) and (b) were handled through a field study, in which a sample of sixty-eight PSMTs, who studied the mathematics teachers' preparation program during the academic year 2018-2019 at the Faculty of Education, Tanta University, Egypt, was engaged in this study.

The data were collected using a questionnaire; it included six items that were developed in terms of acknowledging (a) the value of adopting a social problem, (b) the school curriculum viewpoint, (c) the pupils' perspective, and (d) issues of previous research. As a result, PSMTs' reasoning in a *simple unconditional probabilistic situation* was characterized within four major categories: mathematical [M], subjective [S], outcome [O], and intuitively [I] oriented thinkers. Besides, several cognitive biases emerged (e.g., equiprobable bias, insensitivity to the prior probability of outcomes, representativeness heuristic, overgeneralization heuristic, prediction bias, dependence conception, Allah's will, prediction conception, and causal conception). Furthermore, and essentially, the three factors of variability, randomness, and contextual recognition emerged in all manners of reasoning.

Additionally, PSMTs' reasoning in a *conditional probabilistic situation* was also characterized within two broad categories of intrasubjective and intersubjective thinkers. The intrasubjective included those who shared the overgeneralization heuristic, confusion between joint and conditional probability, the combination of the confusion between conditioning and conditioned event and independence conception, and the illusion of validity. Besides, the intersubjective thinkers incorporated those who practiced the availability heuristic; the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome; the combination of the confusion between the conditioned event and another event, and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome; the combination of the confusion between conditioning and conditioned event, and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome; unawareness of basic probability axioms; the gambler fallacy; the causal conception; the fallacy of transposed conditional; and the confusion between the conditioned event and another event in the experiment.

Based on such findings, some directions for future research were proposed. For example, similar studies might be conducted over different groups (e.g., PSMTs in other universities, in-service teachers) to get a broad and profound understanding of the current state of mathematics teachers' knowledge for teaching probability in Egypt. Besides, to validate this study's results, different investigations might be carried out using more specific questions.

Furthermore, while the study findings exposed, on one hand, some misconceptions that mismatch with the probability theory, and on the other hand, concepts of variability, randomness, and contextual recognition as crucial factors to reason probabilistically, more areas for future research can be adopted. For example, how to improve PSMTs' conceptual knowledge of probability through a pedagogical treatment; also, how to change the traditional way of teaching probability and instead focus on concepts of variability, randomness, and contextual recognition.

CHAPTER 1: INTRODUCTION AND PROBLEM STATEMENT

This chapter outlines the study rationale and problem statement, objectives and questions, significance, delimitations, and ends with a description of its whole structure.

1.1 The study rationale and problem statement

Probability signifies a substantial part of our daily life. Starting from simple questions like, "Is it going to rain tomorrow?" to more complicated inquiries like, "Will the volcano erupt?" These are instances of some situations that may occur (Savard, 2014). Furthermore, probability knowledge and reasoning are needed in everyday settings, for all citizens in decision-making situations (e.g., medical diagnosis, voting, environmental consequences, research reports), and for professionals' training (e.g., engineers, doctors) at the university level (Batanero, Chernoff, Engel, & Sánchez, 2016; Gal, 2005; Jones, 2005).

In light of Borovcnik and Peard's (1996) study, there are two principal purposes for learning probability. The first purpose implies forming a specific type of reasoning that is probabilistic reasoning, in which learners can formally structure their vague thinking regarding random phenomena. Since there is a growing number of events described in terms of risk, the underlying concepts of probability reasoning must be learned in school, and its understanding should also be clarified (Martignon, 2014; Pange & Talbot, 2003). That is consistent with the need to overcome our deterministic thinking and accept the existence of chance in nature (Batanero et al., 2016). Besides, deep thinking is also required to understand probability, which contributes to the development of students' mathematical reasoning (Gürbüz, 2006). On the other hand, the second purpose admits the importance of probability applications (e.g., Poisson, Binomial, Normal distribution) to model various daily life phenomena. Accordingly, the value of probability knowledge and reasoning has been recognized by educational authorities in many countries, and probability has been included in the official curriculum at different levels, from primary to teacher education (Batanero, Burril, & Reading, 2011; Franklin et al., 2007; Jones, Langrall, & Mooney, 2007; the National Council of Teachers of Mathematics [NCTM], 2000; Torres & Contreras, 2014).

Despite the usefulness of probability in handling most of our daily practices and shaping individuals' probabilistic reasoning, several issues were addressed in the literature regarding the deficiency of probability education, which may impact learners' acquisition of probability knowledge and developing their probabilistic reasoning. For example, the curriculum was criticized for performing a too narrow view of probability (Batanero, Godino, & Roa, 2004;

Carranza & Kuzniak, 2008; Ortiz, Cañizares, Batanero, & Serrano, 2002). That narrow view usually refers to strengthening the statistical side of probability relevant to the objective mathematical rules rather than the epistemic side that interprets probability as a personal degree of belief (Hacking, 1975). Similar limitedness was identified in the Egyptian context. While some essential ideas (i.e., independence and conditional probability, probability distribution and expectation, and convergence and law of large numbers) were disregarded in the intended curriculum, the implemented curriculum emphasized the classical and axiomatic approaches (Elbehary, 2019). Such a situation affects teaching and learning processes, starting with teachers who prioritize discussing textbook activities, especially in the context of developing countries (Elbehary, 2019), leading to students who tend to form conceptions based on deterministic reasoning, when probability teaching predominantly employs a theoretical approach (Konold, 1995).

Another widely debated issue in the literature is the inadequate preparation of teachers to teach probability; particularly, the Pre-Service Mathematics Teachers (PSMTs), those university students who learn how to teach intentionally and systematically (Morris, Hiebert, & Spitzer, 2009). As reported, one pedagogical difficulty stems from mathematics teachers' lack of specific preparation in probability (Ainley & Monteiro, 2008; Batanero et al., 2011; Franklin & Mewborn, 2006; Pecky & Gould, 2005). Even when teachers have broad statistical knowledge, it is not sufficient to effectively teach probability (Batanero et al., 2004). That inadequate preparation appeared evident in the Egyptian context, where only about 9% of all subjects during the whole duration of the four-years mathematics teachers' preparation program was assigned to study statistics, including probability (Elbehary, 2019). Additionally, there was no particular discussion concerning the instruction of probability, which was also observed in other contexts (e.g., in Colombia by Torres, 2014).

Such issues about the probability curriculum and teachers' preparation cause various learning difficulties at different grades, from elementary up to university level (Batanero & Sanchez, 2005; Fischbein & Schnarch, 1997; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Stohl, 2005; Tarr, Lee, & Rider, 2006). It also creates further challenges for teachers, especially in terms of the distinct characteristics of probability that are not usually encountered in other mathematics areas (e.g., multifaceted view, lack of reversibility) (Batanero et al., 2016; Jones, 2005; Sharma, 2016).

Acknowledging the value of fostering teachers as professionals (Kunter et al., 2013; Ponte & Chapman, 2006), the current study highlights the notion of PSMTs' knowledge for teaching probability, which signifies the core of professionalism (Baumert & Kunter, 2013; Kaiser et al., 2017). In that regard, the impact of teachers' knowledge on students' learning has been extensively recognized (Danişman & Tanişli, 2017; Darling-Hammond, 2000; Darling-Hammond & Sykes, 2003; Feiman-Nemser, 2001; Mosvold & Fauskanger, 2014; Ojimba, 2013; Rivkin, Hanushek, & Kain, 2005; Schacter & Thum, 2004; Stigler & Hiebert, 1999). More specifically, about the deficiencies of probability education, Dollard (2011) has reported that "One way to improve this situation is to ensure that new teachers graduating from teacher education programs have a good understanding of the fundamental concepts of probability" (p. 27). This emphasis on PSMTs' knowledge may also contribute to the existing literature, wherein more research is needed to clarify essential components in the preparation of teachers for teaching probability (Batanero, Contreras, Fernandes, & Ojeda, 2010; Callingham & Watson, 2011; Ives, 2007; the 10th Congress of European Research in Mathematics Education [CERME10], 2017; Torres, Batanero, Díaz, & Contreras, 2016). Still, what perspective should be adapted to address PSMTs' knowledge for teaching probability requires clarification. For this, the historical development of probability education research was reviewed as follows:

Recently, Chernoff and Sriraman (2014, 2015) have classified the probability education research into four periods: *(1) The Piagetian period,* which was dominated by Piaget and Inhelder's (1975) investigations of people's probabilistic reasoning. *(2) Post-Piagetian period*, in which the probabilistic reasoning was investigated by Fischbein's (1975) research that focused on primary and secondary intuitions; and Tversky and Kahneman's (1974) psychological research regarding judgmental heuristics of adults when they reason under uncertainty. *(3) Contemporary research period*, which witnessed a significant shift toward investigating curriculum, instruction, and learning difficulties that were carried out by a group of researchers of mathematics education (e.g., Falk, 1986; Konold, 1989, 1991). *(4) Assimilation period* after 2000, in which the research continued to develop theories, models, and frameworks associated with intuition and learning difficulties in probability, in line with the previous period. At this stage, the probability education research has been shifted smoothly from importing research findings of other fields (e.g., psychology) to develop its specific interpretations of results stemming from learning and teaching difficulties under the umbrella of mathematics education. Nevertheless, recent investigations have gone back to their proverbial roots by incorporating Tversky and Kahneman's ideas.

Based on this historical development of probability education research, some directions for future investigations were defined. One such area that is embraced by this study is advocating for "a more unified development of the classical, frequentist, and subjective approaches to probability" (Jones et al., 2007, p. 949); alternatively stated, "involves modeling several conceptions of probability" (Shaughnessy, 1992, p. 469), to address the remarkable distinction between both mathematical and philosophical facets of probability theory (Gillies, 2000). From that aspect, to respond to such a challenge, the study utilized probabilistic reasoning, which has a psychological nature and focuses on how individuals reason under uncertainty, as a perspective to investigate PSMTs' knowledge for teaching probability. That is, describing PSMTs' reasoning processes and clarifying their cognitive biases when they reason under uncertainty matches the renaissance period of psychology research in mathematics education. Curiously, issues of subjective probability that consider psychical origins of mathematical probability (Dewey, 1964, as cited in Gierdien, 2008; Hawkins & Kapadia,1984) have not undertaken much deliberation in mathematics education (Chernoff, 2008; the International Conference on Teaching Statistics [ICOTS], 2014). As stated by Jones et al. (2007), "it is timely for researchers in mathematics education to examine subjective probability and the way that students conceptualize it" (p. 947). Ultimately, that may help to pave the way "for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard, Van Dooren, Schaeken, & Verschaffel, 2009, p. 13), wherein the study results may serve as a foundation to develop pedagogical interventions and didactical activities. Accordingly, and more precisely, the current study has conceptualized PSMTs' knowledge for teaching probability in Egypt from a cognitive psychological perspective that is probabilistic reasoning.

1.2 The study objective and questions

Based on what was discussed prior, this study's main objective is to *conceptualize PSMTs' knowledge for teaching probability in Egypt from the perspective of probabilistic reasoning*. Thus, to fulfil such an objective, the following research questions were constructed:

RQ1. What is the current status of "statistics and probability" education in Egypt?

RQ2. What is the definition of mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning?

RQ3. What are the characteristics of PSMTs' knowledge for teaching probability in Egypt from the perspective of probabilistic reasoning?

1.3 Significance of the study

Since this study is tackling the arena of teachers' professional knowledge for teaching probability, focusing on a case of PSMTs in the Egyptian context, with the principal intention of committing to competence models' creation for prospective mathematics teachers, it may be significant in multiple areas, as follows:

In a broad sense, probability indicates the least addressed content by statistics educators, which is reflected in the limited number of studies that are focused on probability education. Among ninety-five published papers in the *Journal of Statistics Education* during the period from 2017 until 2020, only two articles entitled Symbulate: Simulation in the Language of Probability (2019) and Development of an Informal Test for the Fit of a Probability Distribution Model for Teaching (2020) were relevant to probability. Similarly, four papers out of one hundred and five (Pre-Service Mathematics Teachers' Use of Probability Models in Making Informal Inferences about a Chance Game (2017); Quintile Ranking of Schools in South Africa and Learners' Achievement in Probability (2019); Game Invention as Means to Stimulate Probabilistic Thinking (2020); and Students' Informal Hypothesis Testing in a Probability Context with Concrete Random Generators (2020)) were published in the *Statistics Education Research Journal*. Hence, the whole argumentation of the current study may help clarify such specific content; mainly, how PSMTs do reason in a probabilistic situation, which can ultimately give insights to structure their pedagogical preparation and promote their professional competence.

Theoretically, regarding the study perspective of probabilistic reasoning described as a psychological perspective, it acknowledges that the existing conceptions determine the starting point of guiding students toward normatively correct procedures. As clarified by Van Dooren (2014), "understanding of reasoning mechanisms and the origins of prior conceptions may also lead to an engineering of these mechanisms and conceptions" (p. 125). Thus, embracing such a perspective to approach mathematics teachers' professional knowledge for teaching probability advocates three vital issues.

The first issue denotes admitting teachers' reasoning processes and conceptions as essential features to consider, which strengthens the *process knowledge* rather than content knowledge that cannot be neglected, particularly for statistics and probability education (Garfield & Ben-Zvi, 2008; Guidelines for Assessment and Instruction in Statistics Education [GAISE], 2016; Shaughnessy, 1992). That is reflected in this study through characterizing teachers' knowledge by how they reason to transmit and manipulate such knowledge, wherein these manners of reasoning affect their pedagogical practices. Alternatively, instead of describing mathematics teachers' knowledge for teaching probability as sufficient or even inadequate (e.g., Danişman & Tanişli, 2017), there is a need to explore what causes underpin those practices, relying on their conceptions shaped by reasoning processes.

Such an idea may contribute to the studies concerned with the creation of competence models for prospective mathematics teachers (Krainer & Llinares, 2010), and it provides directions for further investigations to evaluate the quality of teacher education. After all, it responds to recommendations regarding the need to define the essential components in PSMTs' preparation to overcome the lack of specialized pedagogical training in probability (e.g., Batanero et al., 2004; Batanero et al., 2010; Contreras, Batanero, Díaz, & Fernandes, 2011; Dollard, 2011; Estrella & Olfos, 2010; Greer & Mukhopadhyay, 2005; Franklin & Mewborn, 2006; Ives, 2007; Stohl, 2005; Torres, 2014).

While the first issue for why the probabilistic reasoning perspective may contribute to the literature implies matters of teachers' knowledge and, further, competence models' creation that can be generalized to other domains of mathematics, both second and third issues are more specific to the case of probability.

In detail, the *second value* underpins utilizing such perspective signifies developing a schema that involves theoretical, experimental, and subjective probability interpretations together (Jones et al., 2007; Shaughnessy, 1992), to respond to the unachieved challenge of connecting the three approaches (Chaput, Girard, & Henry, 2011). Since operating probabilistic reasoning advocates acknowledging several individuals' conceptions (without classifying them in terms of conceptual understanding), it sharpens how their minds work under uncertainty. These ideas were not adequately covered in previous studies. Besides, this study attempts to clarify subjective probability issues as a general classifier, which indicates the *third value* of employing the perspective of probabilistic reasoning. This may contribute to the literature wherein many researchers have reported that subjective probability signifies a neglected and vague area in probability education research, and further, in the curriculum (Chernoff, 2008; Chernoff & Russell, 2014; ICOTS 9, 2014; Torres, 2014; Torres & Contreras, 2014). Moreover, since subjective probability denotes an opportunity for researchers in mathematics education to relate probability with its psychological origins (Chernoff, 2014), characterizing it for PSMTs helps to respond to the need for further investigation that connects the mathematics education research on probability with its proverbial psychological roots (Chernoff & Sriraman, 2015; Gillard et al., 2009).

Indeed, the three above reported issues describe a theoretical value that this study may contribute; they are related to the literature gap on probability education research. However, practically, because researchers in mathematics education are more interested in designing instructional activities to promote learners' understanding of probability, strengthening PSMTs' reasoning under uncertainty can support adapting efficient pedagogical courses.

Primarily, the didactics of probability, including issues of the curriculum, are undervalued in PSMTs' preparation (e.g., Leviatan, 2010; Viali, 2010). In that sense, exploring PSMTs' knowledge for teaching probability focusing on their reasoning processes helps to define their shared conceptions, which may have a mathematical root, or originate from psychological problems (Shaughnessy, 1977); thus, "mere exposure to the laws of probability may not be sufficient to overcome some misconceptions of probability" (Ibid. p. 295). From this viewpoint, clarifying the possible discrepancies between PSMTs' conceptions and probability concepts gives teacher educators information about what conceptions PSMTs bring into the mathematics classroom. Accordingly, effective interventions, which consider both the mathematical obstacles and the psychological roots of such difficulties, can be performed. That, ultimately, may impact their pupils' understanding, since teachers' knowledge is associated with higher quality instruction, which in turn has a positive effect on pupils' learning; that is widely admitted, as detailed in the Study Rationale section.

Within this aspect that highlights the pedagogical preparation of PSMTs, it is valuable to note that the study regarded the importance of grounding probability instruction in different contexts with more attention to realistic social situations that individuals experience in their daily life. That may help, again, to eliminate learners' fallacious preconceptions (beliefs, heuristics, misconceptions, or biases) when they study formal probability theory. Additionally, it contributes to recent investigations that reveal the significance of contextualizing probability (e.g., Gusmão, Santana, Cazorla, & Cajaraville, 2010) and how it can be cultivated through authentic situations, instead of traditional formula-based approaches (Batanero & Díaz, 2012).

In a narrow sense of how the current study may contribute to the national context represented by the Egyptian (and Arab) community, it responds to the need for enhancing Egyptian graduate students' professional competence by characterizing their knowledge for teaching probability. That, sequentially, helps achieve a high quality of education and training systems (Sustainable Development Strategy: Egypt Vision 2030, 2016). More concretely, in teacher education, illuminating PSMTs' knowledge for teaching probability works as a catalyst for reforming the curriculum. Besides, it supports teacher educators, who undoubtedly have a significant role in preparing PSMTs, understand the expected learning difficulties which PSMTs may encounter during learning this content (as reported beforehand). Thus, the intended lectures and pedagogical activities can be modified to help them overcome these difficulties. Mapping such efficient training symbolizes one plausible approach to overcoming the fact that Egyptian pupils' achievement in *Data and chance* remains the lowest among other mathematics areas, as revealed by the results of the Trends in International Mathematics and

Science Study (TIMSS) in 2003 and 2007 (Mullis, Martin, Gonzalez, & Chrostowski, 2004; Mullis et al., 2008).

Closely related to that, defining PSMTs' knowledge for teaching probability in terms of their reasoning processes may help overcome some deficiencies of the Egyptian school curriculum; specifically, the emphasis on the objective probability interpretations and neglecting the subjective side. In other words, if PSMTs only relied on what the curriculum provides, pupils tend to develop conceptions based on deterministic reasoning (Konold, 1995). From this aspect, PSMTs' manners of reasoning are significant to investigate; that helps them critically interpret curriculum activities and promote their pupils' probabilistic reasoning. Such argumentation is consistent with what Forbes (2014) remarked concerning the success of the statistics curriculum that depends on the quality of teacher education and development.

Finally, the current study sustains the research of statistics education in the Arab context, wherein the majority of undergraduate statistics courses are provided in Egypt. As Hijazi and Alfaki (2020) noted, while sixty-four universities are offering seventy-three undergraduate programs in statistics across nineteen of the twenty-two Arab countries, around 60% of the universities that provide statistics programs are placed in Egypt, Iraq, and Sudan. Furthermore, it responds to Innabi's (2014) recommendation regarding the need to activate the study of statistics in the Arab world, since very little research was conducted, and the value of statistics has not yet been fully recognized (Hijazi & Alfaki, 2020). Accordingly, characterizing PSMTs' knowledge for teaching probability within the sphere of statistics education paves the way toward strengthening their pedagogical content knowledge that should be consistent with their needs and current state. Subsequently, it contributes to the movement of establishing the local accreditation and quality assurance system that oversees the quality of graduates in some Arab countries (e.g., United Arab Emirates, Saudi Arabia, Egypt) (Hijazi & Zoubeidi, 2017).

1.4 The study delimitations

The analytical part of this study was limited to the case of PSMTs in Egypt. The reason is that, according to the Egyptian context, accomplish a university degree is a pre-requisite to practice the teaching profession, particularly in light of the absence of systematic training concerned with in-service mathematics teachers' practices (Mullis et al., 2004). Besides, in-service teachers often do not have enough time to participate in similar studies, especially with the teaching burden and the curriculum load. Furthermore, placing the focus on PSMTs helps clarify their original conceptions during the development stage before the influence of working practices (experience) that may shape their reasoning in a prevalent dogmatic manner.

Another issue signifies the focal scope to characterize PSMTs' knowledge for teaching probability in Egypt. That is, although the study framework exhibits four different aspects that define mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning (see Figure 11), the investigation sharpened the aspect of R(in)P; it describes how PSMTs reason in a situation that involves knowledge of probability (i.e., simple and conditional probability). Such focus on R(in)P is originated from the value of investigating Knowledge of Probability (KoP), which corresponds to Subject Matter Knowledge (SMK) in the Mathematical Knowledge for Teaching (MKT) model; it outlines the heart of teachers' knowledge and reflect their deep understanding of the subject (Shulman, 1986). The SMK also appears in the stage of transformation at which mathematics teachers represent the probability through various techniques to facilitate their pupils' understanding. Moreover, several recommendations about teachers' knowledge for teaching probability stressed the importance of the SMK. That is detailed in Papaieronymou's (2009) study, which analyzed recommendations of the four professional organizations of the American Mathematical Society, American Statistical Association, Mathematical Association of America, and the NCTM) to describe teachers' knowledge for teaching probability. As a result, it revealed that 66% of these recommendations are relevant to teachers' SMK compared to 24% for the Pedagogical Content Knowledge (PCK).

1.5 Structure of the study

This section draws the structure of this study. At first, Chapter 2 describes the overall methodology (research logic) of how the research questions were answered to fulfil the study's principal objective. Next is Chapter 3 that outlines the status of "statistics and probability" education in Egypt; thus, it answered the first research question. After clarifying particular issues of Egyptian intended and implemented curriculum of probability, such ideas were complemented by (a) reviewing the current themes of research on probability education that manifests the theoretical gap at which the study has tackled, and (b) stating the study premises, to define mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning[1]. That is the argumentation of Chapter 4; it presents the

[1] Henceforth, the expression of "perspective of probabilistic reasoning" will be replaced by the abbreviation "PoPR" excepts in tables and figures' headlines, titles, and subtitles.

definition of mathematics teachers' professional knowledge for teaching probability from the PoPR, which responded to the second research question. Accordingly, Chapter 5 details the procedures and results of the field study at which PSMTs' knowledge for teaching probability in Egypt was approached from the PoPR to answer the third research question. In the end, the research study was summarized; also, some recommendations and directions for future research were given within Chapter 6. That is displayed in the following figure:

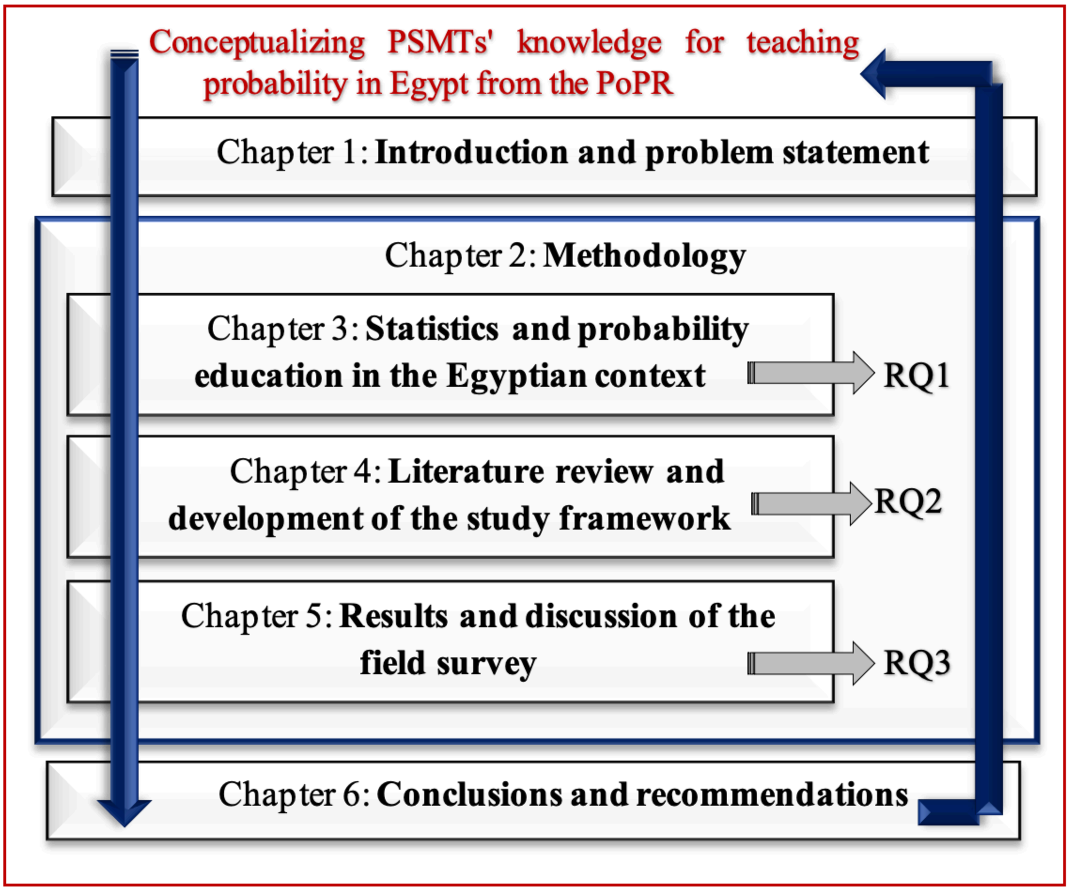


Figure 1. Structure of the current study

CHAPTER 2: METHODOLOGY

This chapter describes the overall methodology of how the research questions were answered to fulfil the study's principal objective.

2.1 Research logic and interrelationships among its questions

The logic of this research can be defined in terms of *abduction reasoning*; it signifies "selecting or inventing a provisional hypothesis to explain a particular empirical case or data set better than any other candidate hypotheses, and pursuing this hypothesis through further investigation" (Kennedy & Thornberg, 2018, p. 52). This study employed abduction reasoning because of the importance of the context and the influence of socio-cultural factors on learning probability. Levin-Rozalis (2010) reported that abduction reasoning helps connect the local with the universal at which more profound and context-related findings can be reached. Furthermore, while deduction works to evaluate a hypothesis and induction helps justify it, the abduction logic aims to generate new ideas (Peirce, 1960, as cited in Åsvoll, 2014). That matches the status of this study wherein, on one hand, the existing research on teachers' knowledge for teaching probability has not yet established a hypothesis that explains how various probability interpretations can be mapped together, to be examined in the Egyptian context. On the other hand, "it seems doubtful whether a high number of inductive cases can verify hypotheses" (Åsvoll, 2014, p. 293); alternatively, it is a little tricky to determine how many different cases can be engaged in the study to reach a hypothesis that explains Egyptian PSMTs' knowledge for teaching probability. Accordingly, the abduction reasoning logic was utilized; it defines a middle ground that helps overcome the classical distinction between induction and deduction (Delputte & Orbie, 2018).

While the abduction reasoning processes were described by Delputte and Orbie (2018) as in Figure 2, these processes were adopted in the current study and exhibited through Figure 3.

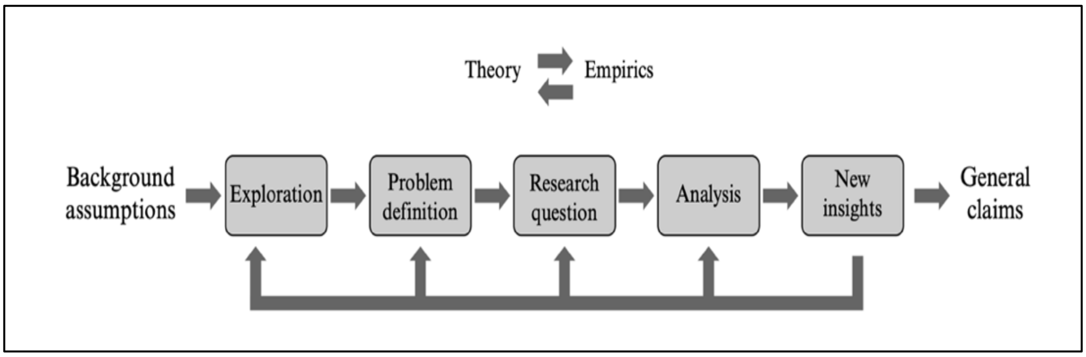


Figure 2. Utilizing abduction reasoning as a research method. Retrieved from Delputte and Orbie (2018, p. 296)

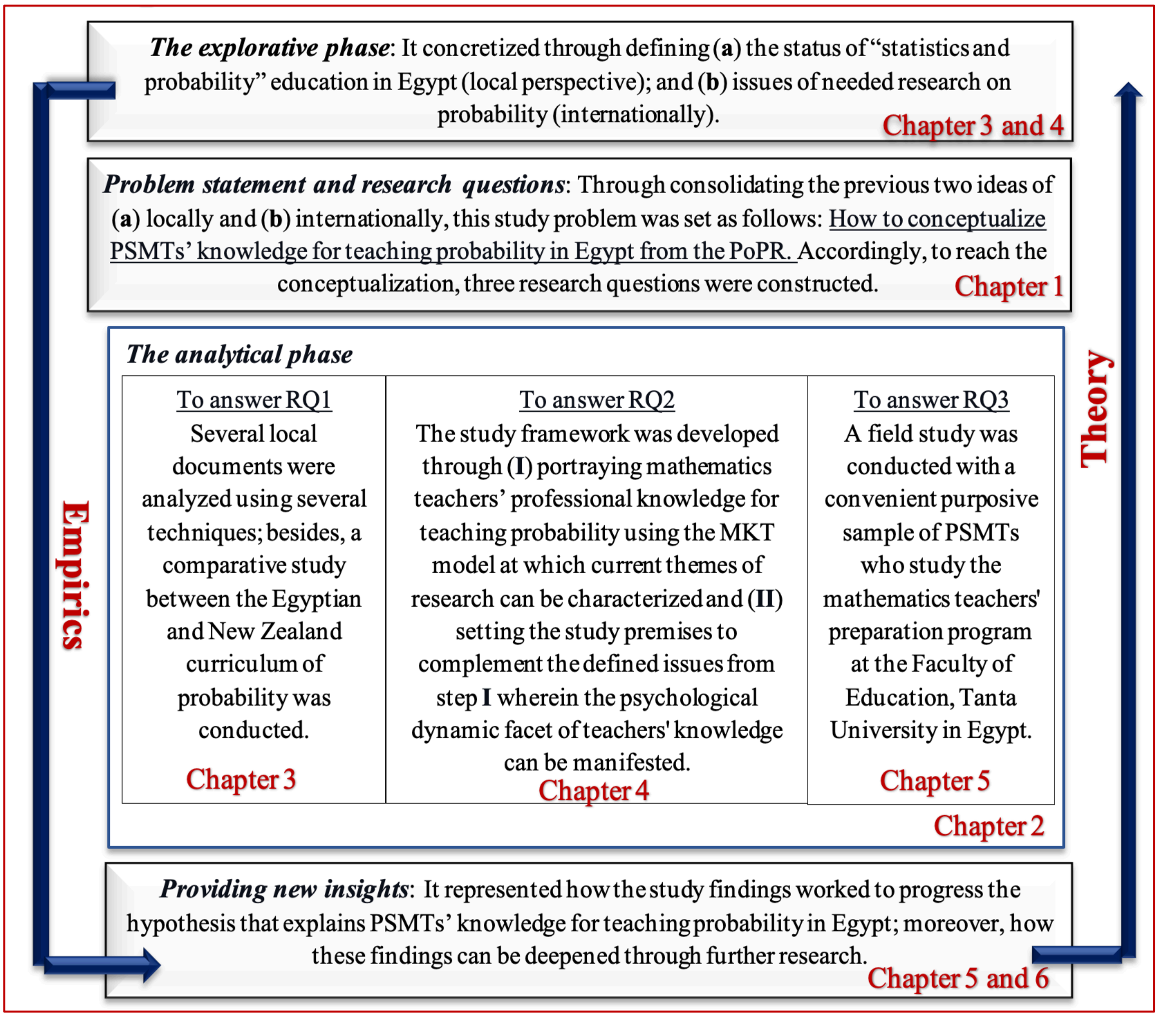


Figure 3. Research logic

According to Figure 3, the three research questions' results operated to fulfil the study objective of *conceptualizing PSMTs' knowledge for teaching probability in Egypt from the PoPR*; that process signified an interplay between theory and empirics.

Concretely, at first, the empirical evidence regarding weaknesses of the Egyptian school curriculum of probability worked with the theoretical argumentations about teachers' knowledge for teaching probability to define (a) what aspects of knowledge mathematics teachers need to acquire to teach probability effectively and (b) what perspective may help to approach such knowledge in a way that fills the research gap. Accordingly, a framework was proposed to redefine mathematics teachers' professional knowledge for teaching probability from the PoPR; and a field investigation was prepared to explore one aspect of that framework in the Egyptian context. As a result, practically, the findings revealed some factors that, on one hand, maintained conceptualizing PSMTs' knowledge for teaching probability in Egypt, and, on the other hand, provided some new insights on how the existing research on probability instruction can be modified to respond to the remaining needed issues in the literature.

2.2 Processes of answering the research questions

2.2.1 Processes of answering the first research question

The current status of “statistics and probability” education in Egypt was outlined from local and international perspectives. At first, locally, some documents were reviewed to decide which area of study has the priority to consider for research; thus, the **probability** was focused. Second, to reflect precisely on the probability content issues, both intended and implemented curricula were analyzed from an international viewpoint. Accordingly, several techniques were utilized to respond to these issues depending upon the available treated subject and the analysis' purpose. That is summarized in Table 2.

In detail, by reviewing Egyptian *vision 2030*, this study considered teacher education to respond to the insufficient skills for graduates that represent a national challenge. Afterward, *the academic program of PSMTs' preparation at the Faculty of Education*[2] that lasts for four-years was analyzed quantitatively in terms of the assigned hours to study each subject using Grossman's (1990) model. Grossman is the first researcher who systematized the seven categories of teachers' knowledge proposed by Shulman (1987); accordingly, he presented four broad categories of SMK, General Pedagogical Knowledge, Knowledge of Context, and the PCK that occupies the central position of his model (see Fernandez, 2014). Although there are some other frameworks of teachers' knowledge in the literature (e.g., Shulman, 1987; Carlsen, 1999), Grossman's framework stays simple, practical, and fits the general structure of preparation programs.

Depending upon the results of analyzing *the academic program of PSMTs* and *Egyptian pupils' achievement in TIMSS 2003 and 2007* (see Chapter 3), **Statistics and probability** was highlighted. However, another technique was operated to determine which area should be precisely sharpened within this domain. This technique implied analyzing the *implemented activities*[3] *stated in units of Statistics and probability in primary and lower secondary school textbooks*[4] in light of Burrill and Biehler's (2011) list of Fundamental Statistical Ideas[5]. That is because (a) these ideas are essential to implement either in the school curriculum or teacher education and relevant to clarify the discipline's specific characteristics (Burrill & Biehler, 2011). Moreover, (b) the assigned opportunities for students to learn school curriculum denotes

[2] This analysis considered only the academic program of PSMTs at the Faculty of Education, Tanta University. Although these studied subjects may vary slightly from a university to another, the structure of the program itself is almost alike across all governmental universities.
[3] This analysis covered the raised activities not only within the lessons but also exercises and revisions.
[4] There is only one official national series of textbooks that are used by all governmental schools.
[5] A brief description of these ideas is given in Appendix 2, before listing the textbooks' analysis results.

one approach to stipulate teachers' knowledge (Stylianides & Ball, 2004); notable, there is research evidence that revealed teachers lack a fundamental understanding of school mathematics (e.g., Ball, 1990; Ma, 1999; Simon, 1993). As detailed in Chapter 3, this analysis revealed that **probability** signifies a core statistical idea in the curriculum wherein the majority of the implemented activities intend to promote pupils' understanding of this concept (Elbehary, 2020); thus, **probability** has been precisely sharpened in this study.

Additionally, to define issues in the Egyptian school curriculum of probability, two other techniques were performed, as follows:

First, the *intended curriculum* was compared with New Zealand (NZ) curriculum through summative content analysis (see details in Chapter 3). Two reasons supported conducting such a comparison. One is NZ's high SDG achievement, with a global ranking of 11 out of 162 compared to 92 for Egypt (SDG, 2019). The other reason is more specific to the study of statistics and probability, wherein "when the discussion is placed into the field of statistics education research, it is worthwhile to mention that the New Zealand curriculum has developed to the stage where it now serves as a working model for other countries to adapt to fit their particular circumstances" (Elbehary, 2020, p. 4). Alternatively, the NZ curriculum considers a resource for pedagogical changes in other countries (Forbes, 2014).

Second, the *implemented curriculum* was characterized through (a) operating the Onto-Semiotic Approach (OSA) that serves as a practical, semiotic, and anthropological approach to analyze the subject through its symbols at an institutional level (Godino, Batanero, & Font, 2007). It was first operationalized as in Table 1 and utilized to define probabilistic Situations, Propositions, Procedures, and Terms, which appeared within textbooks' discourses. Later, (b) these defined probabilistic entities were assigned to Batanero et al.'s (2016) list of probability interpretations (see Table 17 in Chapter 4). These procedures were simulated depending upon previous studies (e.g., Gusmão et al., 2010) that attempted to clarify what probability interpretations are usually emphasized in the classroom discussion.

Table 1. The operational definition of the OSA entities. Based on Elbehary, 2019

Situation	*Propositions*	*Procedures*	*Terms*
Probabilistic activities, tasks, and problems discussed within the discourse.	Underlined properties, relationships, and theories that connect the probabilistic concepts.	Applied algorithms and techniques used to perform a given situation.	Embedded terms, expressions, notations, and concepts that appeared implicitly or explicitly through the discourse.
e.g., The experiment of tossing a coin	e.g., The relationship between events and sample space	e.g., P (H) = n (H)/ n (S)	e.g., Randomness, H, P (A), theoretical probability

To sum up, the following table compiles all the employed techniques to answer the first research question.

Table 2. Brief on how the current status of "statistics and probability" education in Egypt is outlined

The reviewed contents	***Method***	***Main result***
• **First, to determine issues of statistics and probability education in Egypt locally:**		
Sustainable Development Strategy: Egypt vision 2030	The main objectives raised in the titles and subtitles and the current local challenges that may hinder achieving these objectives were reviewed and summarized.	– One essential goal of the 2030 strategy in Egypt is to enhance graduate students' professional competence to meet pupils' needs.
The academic program of preparing PSMTs at the faculties of education	A quantitative analysis of the subjects that PSMTs study during the entire preparation program based on the assigned number of hours allocated to each subject was conducted using Grossman's (1990) model of teachers' knowledge.	– Learning the subject matter took proper consideration, with about 64% of the studied hours throughout the whole program. However, only 9.2% were assigned to learn statistics, including probability.
TIMSS reports	The shared documents by TIMSS 2003 and 2007 were reviewed, focusing on issues of Egyptian pupils' achievement and teachers' requirements.	– Egyptian pupils' achievement in *Data and chance* remains the lowest among other mathematics areas. – Earning a university degree is sufficient to practice teaching in Egypt, and around 99% of in-service teachers possess only this degree.
The school content of statistics for the basic education sector (grade 1 to 9)	The declared activities within the statistics school content (including probability) were analyzed by exploring the correspondence between these activities' objectives and the seven fundamental statistical ideas introduced by Burrill and Biehler (2011).	– Probability indicates a core statistical idea within the Egyptian school curriculum where the majority of the implemented activities are intended to promote pupils' understanding of probability.
• **Second, to determine issues of probability education in Egypt internationally:**		
The intended curriculum of probability	The Egyptian curriculum was compared with the NZ curriculum through summative content analysis. It combined utilizing the OSA and the fundamental probabilistic ideas listed by Batanero et al.'s (2016) to reach a specific conclusion regarding the weaknesses of the intended curriculum of probability in Egypt.	– The ideas of independence and conditional probability, probability distribution and expectation, and convergence and law of large numbers were discussed in NZ but were not considered in Egypt.
The implemented curriculum of probability	Textbooks' activities were analyzed through the OSA and later structured based on Batanero et al.'s (2016) classification of various interpretations of probability to determine the most emphasized probability interpretation in the Egyptian classroom discussion.	– The implemented curriculum placed more emphasis on operating theoretical and axiomatic interpretations. Also, the subjective probability seemed to be neglected until grade 9.

2.2.2 Processes of answering the second research question

To define mathematics teachers' professional knowledge for teaching probability from the PoPR and exhibit it through the study framework, these two steps were, basically, conducted:

- **The first step** involved mainly literature review; it combined two minor steps of **(a)** outlining current themes of research on probability through analyzing the contributed papers of ICOTS 8 (2010), 9 (2014), and 10 (2018), and **(b)** reviewing previous studies on teachers' knowledge for teaching probability (*knowledge for practice)* and crystallizing it in light of the MKT model. While outlining ICOTS' papers intended, generally, to expose the research gap on probability, reviewing, precisely, previous studies on teachers' knowledge aimed at determining the initial entities of the study framework.
- **The second step** placed the study premises; it attempted to fulfil the research gap, which was inferred from the first step. That symbolized exhibiting the dynamic psychological facet of probability as exemplified by teachers' conceptions and reasoning processes. Thus, it explained interrelationships among knowledge (*knowledge for practice*), conceptions (*knowledge in practice*), and reasoning.

In detail, regarding the **first step**, as indicated above, in the beginning, current themes of research on probability were drawn through analyzing the contributed papers of ICOTS 8, 9, and 10. Under the International Association for Statistical Education (IASE), considered one of the most influential communities that support statistics education at all levels around the world, ICOTS, IASE Satellite, and IASE Roundtables are usually held. Besides, the IASE also supports some other conferences (e.g., World Statistics Congresses of the ISI [WSC] and the International Congresses in Mathematical Education [ICME]). However, as stated on the IASE website (IASE, n.d.), ICOTS conferences held every four years since 1982 stay the most important events on the international statistics education calendar. That describes why papers of the latest three ICOTS were selected and analyzed through the following detailed procedures, which were developed by the researcher to manifest the research gap.

I. All the contributed papers were first retrieved and reviewed from the official websites of https://icots.info/8/, https://icots.info/9/, and https://icots.info/10/. As a result, 15 (out of 127), 8 (out of 127), and 2 (out of 79) papers were found in ICOTS 8*: Data and context in statistics education: towards an evidence-based society*, ICOTS 9: *Sustainability in statistics education,* and ICOTS 10*: Looking back, looking forward*, respectively. Thus, in total, 25 papers were selected to analyze (see Table 3); all showed either the term probability or probabilistic in their titles.

Table 3. Summary of the contributed papers on probability education that were shared by ICOTS 8, 9, and 10

ICOTS	*Total number of papers*	*Number of papers on probability*	*The selected papers related to probability education*
8 (2010)	127	15	Batanero et al.; Caldeira and Mouriño; Theis and Savard; Grenon, Larose, Bourque, and Bédard, 2010; Larose, Bourque, and Freiman; Papaieronymou; Gusmão, Santana, Cazorla, and Cajaraville; Savard; Gundlach, Kuntze, Engel, and Martignon; Viali; Chadjipadelis and Anastasiadou; Estrella and Olfos; Kapadia and Borovcnik; Borovcnik and Kapadia; and Leviatan.
9 (2014)	127	8	Eckert; Primi, Morsanyi, and Chiesi; Edwards; Torres and Contreras; Torres; Díaz, Mier, Alonso, and Rodríguez-Muñiz; Kuzmak; and Moreno and Cardeñoso.
10 (2018)	79	2	Levy and Stukalin; and Takagi.
Number	*333 (100%)*	*25 papers (7.51%)*	

II. Two initial categories were set to characterize the 25 selected papers. That is, based on the historical development of research on probability, it was assumed that these papers could be assigned to two broad classes of (a) mathematics education perspective and (b) psychological perspective; wherein Shaughnessy (1992) differentiated between researchers in psychology and mathematics education as describers versus interveners. That matches Watson (2014), who described that purpose of the proposed interventions, within the view of mathematics education, is to enhance learners' understanding of probability. Similarly, Ejersbo and Leron (2014) reported that while researchers' goal in cognitive psychology is constructing an understanding of how the mind works, in mathematics education, they are more concerned with what can be done through education.

III. A preliminary review of a sample of 6 (out of the 25) randomly selected papers was conducted to decide whether the previously proposed categories (observers vs. interveners) could help to classify these papers. As a result, the researcher found that this classification gave full attention to research logic without considering context issues. More precisely, although several approaches were utilized to enhance learners' understanding of probability, these approaches relied on various theoretical backgrounds and contexts. Therefore, to recognize the context in the analysis, multiple viewpoints on probability education were reviewed from the book *Probabilistic Thinking: Presenting Plural Perspective*s, edited by Chernoff and Sriraman in 2014. As detailed in the book, probabilistic reasoning research can be classified under four perspectives of historical-philosophical, psychological, stochastics, and mathematics education. Accordingly, a developed matrix (see Table 14 in Chapter 4) was proposed alternatively (instead of observers vs. interveners) to characterize the selected papers from ICOTS.

In addition to characterizing ICOTS paper and as a part of **the first step**, previous studies on teachers' knowledge were also analyzed using the MKT model. It signifies a well-defined practice-based framework utilized by many organizations to drive the improvement of teaching (Kleickmann et al., 2013). According to the MKT, teachers' knowledge comprises SMK and PCK. Furthermore, the PCK involves Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC), as displayed in Figure 4. Thus, this framework worked significantly as a lens to categorize previous research on teachers' knowledge for teaching probability of which the initial entities of the study framework could be determined (see details in Chapter 4).

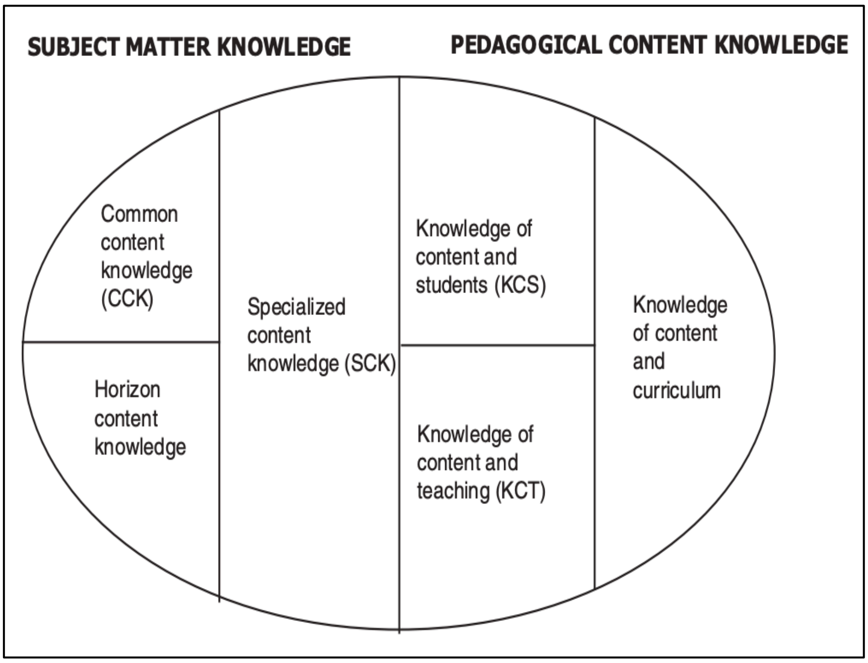


Figure 4. The MKT framework. Retrieved from Ball, Thames, and Phelps (2008)

About the **second step**, three premises were set (see Section ***4.3.3.***). It outlines the researcher's viewpoint on how the research gap can be fulfilled, and findings of the first step can be complemented; thus, ultimately, the framework that defines mathematics teachers' professional knowledge for teaching probability from the PoPR was developed. These premises were reflected in the current study as follows:

I. "Conceptions represent knowledge in evolution" was the first premise; it manifested two issues: (a) It is not reasonable to pretend that a specific conception might exactly explain a certain level of understanding. Accordingly, it was argued that employing the PoPR would support admitting PSMTs' various probabilistic conceptions without classifying them as levels of conceptual understanding. Besides (b) the socio-cultural influence on learners' conceptions of probability. Hence, all PSMTs emerged conceptions relevant to their daily experiences were acknowledged.

II. "Reasoning defines an individual cognitive process to interpret the acquired knowledge" was the second premise. (a) It recognized that probabilistic reasoning is the essential goal that underpins learning probability, wherein it should be the ground for all educational practices. (b) It admitted that to overcome the distinct characters of the probability, it should be addressed through an approach that looks at concepts from a non-mathematical perspective. Accordingly, (c) the PoPR was considered a possible perspective to deal with the duality of the probability concept; it defeated the conventional approach of teachers' knowledge that pays more attention to the statistical side, and instead, it accepted the subjectivity (subjective reasoning) as a reasonable way to interpret a probabilistic situation. Furthermore, (d) applying the PoPR responded to several recommendations about grounding the probability instruction in experiences that help learners overcome their misconceptions and develop an understanding based on probabilistic reasoning. Besides, it fulfilled the need for research that exhibits the psychological perspective on probability and connects it with the mathematics education perspective.

III. "The hypothetical relationship between conceptions and reasoning" was the third premise. It was interpreted as follows: Depending upon how we reason in an uncertain situation that contains probability knowledge (theoretical constructs), our conceptions can be clarified. Thus, (a) it acknowledged a possible existing gap between *knowledge for practice* (professional knowledge for teaching probability) and *knowledge in practice* (how a PSMT perceives probability). (b) It considered such a gap to have originated from PSMTs' various ways of reasoning. Moreover, (c) it served to consolidate PSMTs' reasoning and probability conceptions in one model.

By integrating the previously described first and second steps, mathematics teachers' professional knowledge for teaching probability was defined from the PoPR through these aspects: **R(in)P**, **R(in)PL**, **R(in)TP**, and **R(in)SPK** (see the details in chapter 4).

2.2.3 Processes of answering the third research question

The third research question was intended to characterize PSMTs' knowledge for teaching probability in Egypt from the PoPR. Accordingly, a field study was conducted in light of the study framework to respond to such a question. This section summarizes how this field study was conducted as defined by its participants, tools, data collection, and analysis processes.

I. Participants in the field study

To answer the third research question, a purposive sample of PSMTs[6] was selected based on two criteria: (a) Their availability (e.g., access, location, time) and willingness to participate (Lopez & Whitehead, 2013), and (b) prior knowledge of primary probability concepts (i.e., theoretical, experimental, and conditional), whether in school or during the preparation program. Employing such a criterion helped in exploring PSMTs' biases and conceptions that persisted even under the formal education of probability theory. Accordingly, sixty-eight PSMTs, who study a four-year preparation program of mathematics teachers at the Faculty of Education, agreed to participate in this study, as clarified in Table 4. That faculty belongs to Tanta University that is the only governmental university in the Gharbia Governorate, which ranks the eighth governorate (among twenty-seven) in terms of the Egyptian population.

Table 4. The study sample and population

The study population	PSMTs at the Faculty of Education, Tanta University, Egypt.				
Enrolled number of PSMTs during the academic year 2018–2019	***1st year***	***2nd year***	***3rd year***	***4th year***	***Total***
	107	99	92	102	400
Participants	*Not available*	***32***	***23***	***13***	***68***

II. Tools: Contents of the study questionnaire

Based on the study framework, a questionnaire was developed to characterize PSMTs' R(in)P; that involved (a) determining three different probability contexts, and (b) adjusting one of these contexts and adding a calculation problem; that is detailed as follows:

[A] Determining the probability contexts

Primarily, it was acknowledged that both intuitive assessments and formal knowledge of probability are probably available during the process of probabilistic reasoning; yet, which one is applied is a function not only of individuals' knowledge but also of situation variables (Konold, 1989). Alternatively stated, "As the demands of probability problems become more

[6] Although the field study has focused primarily on PSMTs to respond to the third research question, a convenient sample of pupils was also engaged intentionally to determine one aspect of the study questionnaire (see Table 5).

sophisticated the reasoning brought to them by students may change” (Watson, 2005, p. 145). From this aspect, the variability of the questionnaire items, which addressed the characteristics of PSMTs' R(in)P, took much attention. Nevertheless, the intention was not to see how PSMTs' responses might vary over problem-type but rather to investigate the response-type that may persist throughout these problems. Therefore, to determine such various contexts, the following three issues were considered:

- The first issue implied acknowledging the **Egyptian curriculum perspective**

Since PSMTs in this study are being prepared to teach the primary and lower secondary pupils, the school content of probability was analyzed inductively to explore probability settings from a national viewpoint [7]. In other words, the researcher tried to categorize probability tasks that indicate a similar context together. As a result, seven different contexts at which the probability can be used were inferred (see Table 18), and accordingly, *the activity of throwing a die was regarded in the study questionnaire.*

- The second issue signified acknowledging the **Egyptian pupils’ viewpoint**

Based on the defined contexts of probability resulting from analyzing the school curriculum, a survey was prepared (see Appendix 6) and administered over pupils at various grades in the same province where PSMTs were practicing their practicum and most probably to be employed after graduation. This way, PSMTs’ interpretations could be connected with the viewpoint of their prospective pupils. That is recommended by Garfield and Ahlgren (1988) as teachers should “create situations requiring probabilistic reasoning that correspond to students’ views of the world” (p. 48). Moreover, probability instruction should be contextualized by drawing on pupils’ daily social practices (Grenon et al., 2010). Accordingly, as detailed in Table 5, in total, 359 pupils were asked to determine which probability setting is more applicable in our daily life, in which they should select three contexts and prioritize them. This is consistent with what was reported by the Park City Mathematics Institute ([PCMI], 2017) regarding teaching probability; it incorporates finding probabilistic knowledge that values applying the concept of probability in real life. As a result of implementing such a survey, environmental concerns were the most commonly relevant context of probability to everyday situations from the pupils’ perspective. *Thus, the task of weather predictability was also considered in the questionnaire.*

[7] The total number of the reviewed tasks equals (106). That included all the discussed activities within lessons' content of both primary and lower-secondary grades, starting from grade 3 at which the probability is first introduced until grade 9 (Revisions and exercises were not addressed).

Table 5. The statistics of pupils who were engaged in the study to define the most applicable probability context

Grade	**G[8] 4**	**G 5**	**G 6**	**G 7**	**G 8**	***Total***
Number of Classes	2	3	2	3	2	***12***
Number of pupils	59	87	67	85	61	***359***

– The third issue was **adapting one social problem**

The difficulty of reasoning in a probabilistic situation is determined in terms of (a) sample space clarity, (b) apparent chance factors, and (c) cultural prescription toward viewing the phenomena statistically (Nisbett, Krantz, Jepson, & Kunda, 1983); accordingly, the activity of throwing a die, which reflected the curriculum perspective, is simple to estimate because randomizing devices are usually designed so that the sample space is evident and the repeatability of trials is notable (Nisbett et al., 1983). Besides, based on these criteria, Konold (1989) judged the task of weather predictability, which mirrored pupils' viewpoint, as an intermediate level of difficulty. On the contrary, in the social domain, the sample space is often obscure, and repeatability is hard to imagine (Nisbett et al., 1983). From this view, to accommodate different levels of difficulties, *the problem of giving birth was also considered.*

It is also relevant to note that utilizing the gender context was further validated by analyzing PSMTs' responses to a survey (see Appendix 7). That survey was similar to the one that was applied before to pupils. However, it was distributed among PSMTs who were engaged in this study, and accordingly, they were asked to decide the suitability of several contexts (based on the curriculum analysis) to address each probability concept of theoretical, experimental, and conditional[9]. As a result, from PSMTs' viewpoint, the gender implied one probabilistic context that can be used to approach all these three concepts; particularly, the conditional probability (see Figure 13 in Chapter 5).

Based on the three above discussed issues, *tasks of (a) giving birth, (b) throwing a die, and (c) weather predictability were adapted in the questionnaire to* characterize PSMTs R(in)P.

[B] Adjusting one of the probability contexts and adding a calculation problem

Additionally, for further clarification of the notion of subjective probability in this study, three more tasks were considered. On one side, one task was modified by restricting the problem of giving birth by specifying a condition. This modification was inspired by Díaz and Batanero's (2008) description of the diachronic situation that signifies a series of sequential experiments carried out over time. On the other side, the other two tasks involved calculation

[8] The term G refers to the grade; for example, G 3 symbolizes grade 3.
[9] The participants were informed that some contexts could be adapted to approach more than one concept.

problems that required computing conditional probabilities from a two-way table. They were adapted from Díaz and Batanero (2009). Moreover, as reported by Watson and Kelly (2007), the usage of two-way tables with convenient frequencies (rather than probabilities, Gigerenzer & Hoffrage, 1995) has been advocated in recent years (e.g., Díaz & Batanero, 2009; Díaz & de la Fuente, 2007; Pfannkuch, Seber, & Wild, 2002; Reaburn, 2013) to designate learners' conceptual difficulties in conditional probability.

Depending upon the above steps **[A]** and **[B]**, the study questionnaire was constructed, and it was also divided into two parts (see Appendix 8). The first part included Items **A** and **B** that recognized the context of gender; one defined a simple probabilistic situation while the other signified a conditional one. Additionally, Items **C**, **D**, **E1**, and **E2** were included in the second part. While **C** and **D** reflected two simple probabilistic situations, **E1** and **E2** were proposed to be two equivalent items that required conditional probability calculations. As the local educators recommended, instead of providing only one question that demands calculating the conditional probability from a two-way table, it is better to formulate two similar questions; and distribute them interchangeably among the participants (see details in p. 85). Accordingly, while Items **A**, **C**, and **D** worked together to define PSMTs' reasoning in simple unconditional probabilistic situations, **B**, **E1**, and **E2** aimed at exploring how PSMTs do reason in conditional probabilistic situations. Ultimately, all items were intended to characterize PSMTs' R(in)P that is the focus of the current investigation, as summarized in the following table:

Table 6. Contents of the study questionnaire

Parts	*Items*	*Context*	*Number of questions*	*Intention Explore PSMTs' R(in)P that is related to*	*Relation to areas of needed research*
First part	***A***	Giving birth	2	a simple unconditional probabilistic situation	Develop a unified schema that involves different probability conceptions
	B		2	a conditional probabilistic situation	
Second part	***C***	Throwing a die	1	a simple unconditional probabilistic situation	
	D	Weather predictability	1		Clarify the notion of subjective probability
	E1	Calculation problems using two-way tables	4	a conditional probabilistic situation	
	E2		4		

III. Procedures of data collection and analysis[10]

According to the Egyptian academic calendar, the research data were collected during the second semester of the school year 2018–2019. The process of collecting these data was

[10] All details (including several examples) were provided within Chapter 5

conducted as a part of the micro-teaching course that PSMTs study at the Faculty of Education, Tanta University. It took four sessions with each group of second, third, and fourth-year students (PSMTs), once a week. That is twelve classes, each class lasted one hour, conducted three times per week. Besides, twelve classes were carried out with pupils during the mathematics class to answer the probability context survey. Those twelve classes involved two, three, two, three, and two classes with grades 4, 5, 6, 7, and 8 pupils, respectively; each class took forty-five minutes (see Table 7). Furthermore, a brief on what was done is given next.

Table 7. Duration of the data collection

	PSMTs			***School pupils***				
Place	Faculty of Education, Tanta University			The same province where PSMTs are practicing their practicum				
Course	During the micro-teaching course			During the mathematics class				
Participants	2^{nd} year	3^{rd} year	4^{th} year	G 4	G 5	G 6	G 7	G 8
Number of sessions	4	4	4	2	3	2	3	2
Duration	1 hour for each session			45 minutes for each class				
	4 weeks (3 sessions per week)			1 week				

[A] First stage: Initial arrangement and preparation

This stage took two weeks that included (a) six sessions with PSMTs (two with each group), (b) an interview with two teachers, (c) twelve classes with pupils, and (d) an interview with three university lecturers, as follows:

Regarding the first week, the researcher first discussed with PSMTs the purpose and content of the current investigation and confirmed their availability and willingness to cooperate. Besides, an interview with two teachers who got a master's degree, one in teaching geography while the other in teaching mathematics, was conducted. They both expressed their enthusiasm to help to collect the research data. Accordingly, the interview was concerned with the process of implementing the probability context survey.

During the second week, a warmup session with PSMTs was conducted; it involved a specific discussion about probability interpretations. Furthermore, an interview with some PSMTs and three university lecturers was also done. The PSMTs were informed about the probability context that the school curriculum stressed and what pupils' perspectives were. Then, the questionnaire was handed to them to discuss its items; that served to verify (a) PSMTs' understanding of the presented inquires and (b) the consistency between that understanding and the researcher's intention. Additionally, the questionnaire items were reviewed by three university lecturers; two are specialists in teaching mathematics and one in science education. That included issues of language clarity, objectives, what problems may be difficult for PSMTs, and what alternatives can handle such a situation. Consequently, some of the questionnaire items were rephrased or simplified afterward (see details in Chapter 5).

[B] Second stage: Implement the study questionnaire

As described earlier, the questionnaire was divided into two parts, and it took two sessions (six classes, two with each group of PSMTs) to be implemented. During the implementation, still, some questions were raised by the participants as detailed in Chapter 5. Yet, it is reliable to declare that not all PSMTs responded to all items when they had a schedule in conflict; for a case, because of time constraints for third-year PSMTs, only three (out of 23) answered Item **D** (i.e., the task of weather predictability). Besides, regarding Items **E1** and **E2**, since they were distributed interchangeably among the participants, as reported earlier, only thirty-four (out of 68) PSMTs responded to each item. This is summarized in the following table.

Table 8. Number of respondents to items of the study questionnaire

Parts	***Items***	***Context***	***Number of respondents***			
			2nd year	***3rd year***	***4th year***	***Total***
First part	***A***	Giving birth	32	23	13	68
	B		32	23	13	68
Second part	***C***	Throwing a die	32	23	13	68
	D	Weather predictability	32	3	13	48
	E1	Calculation problems using two-way tables	16	12	6	34
	E2		16	11	7	34

[C] Third stage: data coding and analysis processes

The processes of data analysis are displayed in the following figure:

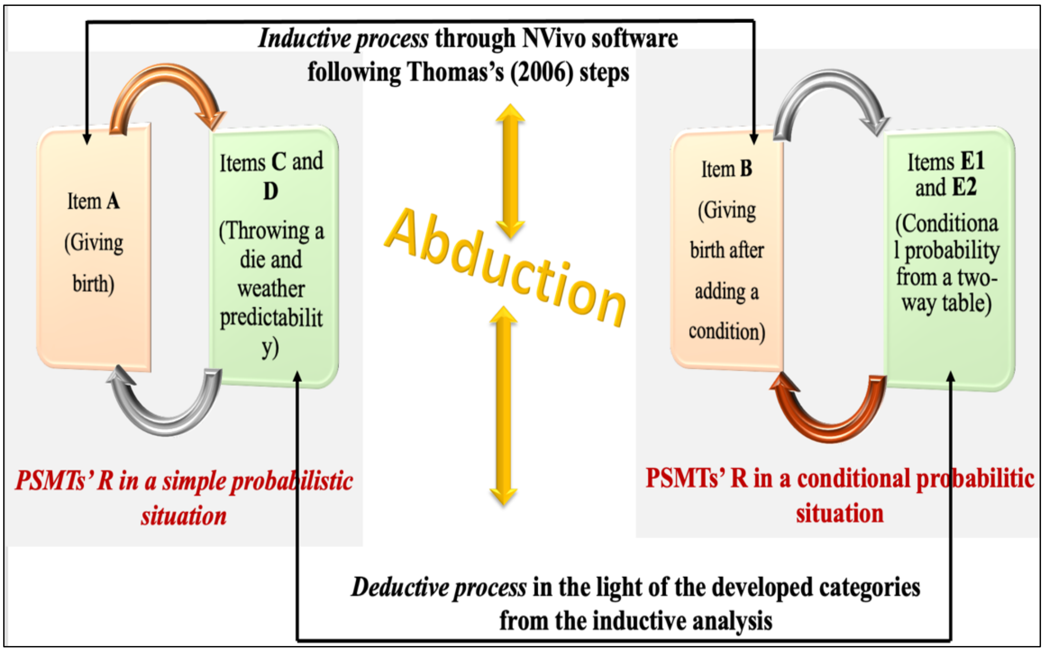


Figure 5. Processes of data analysis in light of the logic of abduction research

According to Figure 5, the data analysis processes involved two types of coding. First was *inductive coding;* it was performed through NVivo software and following Thomas's (2006) steps to analyze PSMTs' responses to the first part of the questionnaire (i.e., answers to Items **A** and **B**). While the second coding incorporated *deductive analysis* to interpret PSMTs' responses to the second part of the questionnaire (i.e., answers to Items **C**, **D**, **E1**, and **E2**); that was done in terms of the developed categories from the first inductive process. Admittedly, drawing on both inductive and deductive data procedures meets the research logic of abduction (Alvesson & Kärreman, 2007; Graebner, Martin, & Roundy, 2012; Pierce, 1978); it helped to invent the most plausible hypothesis that could explain PSMTs' R(in)P. As clarified by Delputte and Orbie, "the strength of abduction is that it uses both inductive and deductive reasoning tactics: 'instead of trying to impose an abstract theoretical template (deduction) or "simply" inferring propositions from facts (induction)', the researcher aims to reason 'at an intermediate level (abduction)' (Friedrichs and Kratochwil 2009, p. 709)." (2018, p. 249). That is further illustrated as follows:

- Analyzing PSMTs' responses to the first part of the questionnaire (Inductively)

In the beginning, it is necessary to note that PSMTs' responses to Items **A** and **B** included both numerical and textual explanations. Although their numerical answers were reported (see Chapter 5), the textual responses through which PSMTs stated reasons and, further, conditions that could change their probabilistic judgment were considered entirely to develop the intended categories. Such textual responses were *inductively* analyzed since there is a strong tradition in qualitative research concerning developing codes directly from data, rather than using prior understandings of the researcher (Linneberg & Korsgaard, 2019). Accordingly, although, at first, the researcher tried to focus more on PSMTs' responses to see how it can be modeled, during the last stage and to generate categories that speculate how PSMTs do reason in a probabilistic situation, a theoretical reflection was needed. This is described by Thomas's (2006) steps, which guided the *inductive* data analysis procedures for Items **A** and **B**.

In that regard, it is valuable to note that although PSMTs' responses to Item **B** were also analyzed in light of Thomas's (2006) steps, the last step of *Incorporating the emerged categories into a model* that often requires a theoretical reflection was not operated. The reason is that even though learners' conceptions of conditional probability were explained in the literature, there was no comprehensive categorization that describes how the individuals perceive the provided condition. That mirrors Gioia, Corley, and Hamilton's (2013) view of the *inductive* approach, which is relevant when no theoretical concepts are immediately available to help grasp the studied phenomenon.

– Analyzing PSMTs' responses to the second part of the questionnaire (Deductively)

As reported earlier, the process of analyzing the second part of the study questionnaire was performed *deductively* in light of the emerged categories from analyzing Items **A** and **B**. In such a manner, these categories were operated as an analytical framework to structure PSMTs' responses to Items **C**, **D**, **E1**, and **E2** (Miles, Huberman, & Saldana 2013). Drawing on the *deductive* approach responded to the abduction research logic and helped to validate the emerged categories (i.e., types of PSMTs' reasoning) across different probabilistic contexts (Rowley, 2002). That satisfies Paavola's (2004) explanation concerning the need to constantly compare the candidate hypotheses (i.e., the first developed categories that described PSMTs' reasoning) with the empirical cases (i.e., PSMTs' responses to the second part of the questionnaire) to explore the most plausible hypothesis that explains the phenomenon under study in abduction research.

The whole discussion above reviewed (**I**) who participated in the field study, (**II**) the tool through which the data were collected, and (**III**) how data were collected and analyzed; to answer the third research question. Still, more details are explained in Chapter 5. It mainly focuses on interpreting PSMTs' responses to the study questionnaire; however, features about participants, the questionnaire development, and data analysis processes are declared at first.

CHAPTER 3: STATISTICS AND PROBABILITY EDUCATION IN THE EGYPTIAN CONTEXT

This chapter provides an extensive overview of statistics and probability education in Egypt, not only from a local perspective but also from an international viewpoint. It starts by describing the local policy of sustainable development goals in Egypt and moves specifically to the status of both PSMTs preparation programs, pupils' achievement, and school curriculum. Lastly, it analyzes the school content of probability for both primary and lower-secondary levels. That answers the first research question.

3.1 Statistics and probability education in Egypt from a local perspective

It is worthwhile to consider the following issues to get a comprehensive picture regarding statistics and probability education in Egypt:

3.1.1 The implementation of Sustainable Development Goals (SDG) (Governmental level)

In Egypt, one principal goal of the 2030 strategy is that a high quality of education and training system should be available to all, without discrimination, within an efficient, just, sustainable, and flexible institutional framework (Sustainable Development Strategy: Egypt Vision 2030, 2016). That matches the fourth pillar of the global agenda of the SDG, which is proposed by the United Nations concerning qualifying the educators and aspiring to ensure inclusive and equitable quality education and promote life-long learning opportunities for all (SDG 4, 2019). Accordingly, improving the education system's quality to fit with global systems has been admitted as a principal target. In terms of the challenges that may hinder achieving the national goals and the proposed strategies to address these challenges (see Figure 6), a significant area to consider is teacher education.

Challenges

- The absence of a professional license, which is the prerequisite for employment.
- The deficiencies of the current assessment system for pre-service teachers.
- The insufficient skills of graduate students to be qualified teachers.

Goals

- Establish a mechanism for the evaluation processes.
- Enhance graduate students' professional competence to meet pupils' needs.

Figure 6. National challenges and proposed strategies to achieve the quality of education, in light of the 2030 vision

From the above-presented figure, it is evident that the graduates' insufficient skills signify one challenge that should be overcome by enhancing their professional competence, especially, since the university students directly get the teaching license after their graduation. Yet, what does the current mathematics teacher education program look like? Alternatively stated, how do national faculties of education prepare PSMTs to practice the profession of teaching (to be professionals)? The answer to this question is presented in the next section.

3.1.2 *The status of the mathematics teacher preparation program in faculties of education (University level)*

The faculties of education are the national institutions responsible for preparing university students to be mathematics teachers; thus, it is valuable to analyze what those policymakers recommend prospective teachers learn in teacher preparation programs to provide insights into what is required knowledge for teaching (Stylianides & Ball, 2004). From this viewpoint, as reported in Chapter 2 (see p. 13), courses that PSMTs study during the entire four-years of their preparation were reflected and categorized in light of Grossman's (1990) model of teachers' knowledge. It helped to classify these courses drawing on the consistency between each course's declared purpose and Grossman's defined aspects of teachers' knowledge. For example, because the SMK in Grossman's model indicates learning the content itself, all Mathematics, Statistics, and Sciences disciplines were listed under this category. Similarly, courses related to community, district, or school environment (e.g., school and community) were assigned to the category of Knowledge of Context (see Appendix 1).

As a result, aspects of *SMK*, *General Pedagogical Knowledge*, *Knowledge of Context,* and *PCK* were exemplified to represent the national view on main requirements to practice the profession of teaching mathematics. Although learning subjects took proper regard with about 64% of the studied hours throughout the entire preparation program, only 9.2% were assigned to learn statistics and probability (see Table 9), which indicates how this content area is less emphasized (compared to mathematics) during the preparation of PSMTs. Besides, from the researcher's experience, such limited consideration not only involves statistics and probability as a discipline but rather as content that should be pedagogically manipulated during microteaching sessions or teaching practicum (i.e., PCK). As shown in Table 9, the courses aimed at promoting PSMTs' PCK occupied only about 13%. Most of the discussion during these courses focuses on numbers, algebra, or geometry.

Table 9. The distribution of knowledge base aspects in PSMTs' preparation program by the assigned hours for each study subject. Based on Elbehary, 2019

Grossman's (1990) aspects of teachers' knowledge	*The defined aspects of knowledge in the Egyptian PSMTs' preparation program*	*Percentages*
SMK	Advanced Mathematics (47.5%), Statistics (9.2%) and Physics (7.6%)	64.37%
General Pedagogical Knowledge	General pedagogical subjects (e.g., Curriculum, Educational Psychology)	14.94%
Knowledge of Context	General cultural subjects (e.g., School and community, Human rights)	7.66%
PCK	Teaching methods and practicum training	13.03%
Total	***Four aspects of knowledge***	***100%***

3.1.3 The status of pupils' achievement and school content of statistics (School level)

Indeed, there is an international agreement regarding how both the quality of teachers' preparation and their professional development affect pupils' achievement. Although research findings on factors impacting that achievement are mixed, the evidence of teachers having a substantial influence is increasing (Fennema & Franke, 1992; Hiebert & Grouws, 2007; Nye, Konstantopoulos, & Hedges, 2004; Schwille & Dembélé, 2007). As acknowledged by the United Nations Educational, Scientific, and Cultural Organization (UNESCO, 2004), teachers significantly influence learning. The experience of countries that have achieved high learning outcomes clearly shows that investment in teachers is critical to any educational reform and education quality. Similarly, and more specifically about teacher preparation, the Organization for Economic Co-operation and Development (OECD, 2015) reported that future teacher quality is affected not only by the in-service teachers' knowledge and skills but also by the quality of new entrants to the profession. That matches what Feiman-Nemser (2001) argued; if we want schools to produce more valuable learning for pupils, we have to offer more powerful learning opportunities to teachers in their training. From this aspect and considering what was reported previously regarding the limited emphasis on learning statistics and probability during the preparation of PSMTs, it is relevant to reflect on Egyptian pupils' achievement in the content area of *Data and chance*.

In light of TIMSS 2003 and 2007 results (Mullis et al., 2004; Mullis et al., 2008), Egyptian pupils' achievement in *Data and chance* remains the lowest among other mathematics areas (see Table 10). That may indicate that the lack of PSMTs preparation in statistics and probability (pedagogically) signifies one factor that affected pupils' achievement; especially the university degree is regarded as sufficient to practice the teaching profession, which encourages most PSMTs to work at governmental or private schools directly after graduation.

That was reported by Mullis et al. (2004) in which the requirement of being a mathematics teacher in the Egyptian context is to have a university degree or another equivalent diploma, and around 99% of the in-service teachers possess only this degree (see Table 11). Besides, there is no regular training to promote the quality of in-service mathematics teachers. In such a situation, since (a) K-12 school practices, (b) teacher education programs, and (c) teaching experiences represent the potential sources of teachers' knowledge (Friedrichsen et al., 2009), most Egyptian teachers develop their knowledge during the initial preparation. That explains why this study emphasizes PSMTs' knowledge. Taking into concern that although there are two different preparation programs in the Faculty of Education (one for those who are going to work at primary school level and the other at secondary school level), the majority of graduates are, first, employed at primary schools; later, through time and experience, they can apply to be promoted to work at higher grades.

Table 10. Average achievement in the mathematics content areas for 8th Grade Egyptian pupils according to the TIMSS 2003 and 2007

	Number	*Algebra*	*Measurement*	*Geometry*	*Data and chance*
2003	421 (3.0)	408 (3.9)	401 (3.3)	408 (3.6)	393 (3.2)
2007	393 (3.1)	409 (3.3)	-	406 (3.4)	384 (3.1)

Table 11. Mathematics teachers' requirements and educational levels in Egypt according to TIMSS 2003

Requirements for being a Mathematics Teacher					
	Pre-practicum and supervised practicum	*Passing an examination*	*University degree or equivalent*	*Completion of a probationary teaching period*	*Completion of an induction program*
Egypt	No[11]	No	Yes	No	No
The Highest Educational Level of Mathematics Teachers					
	Beyond Initial University Degree	*Finished University or Equivalent*	*Finished Post-Secondary Education but not University*	*Finished Upper Secondary Schooling*	*Did not Complete Upper Secondary Schooling*
Egypt	1 %	99 %	0%	0%	0%

According to the Curriculum Center for Instructional Materials Development (CCIMD) accountable for developing pre-university curricula in Egypt, Numbers and operations; Algebra, relationships, and functions; Geometry; Measurement; and Statistics and probability incorporate the five content areas of mathematics curriculum, which are explicitly involved in

[11] A weekly course of a supervised practicum is conducted in the third and fourth years of the preparation program.

each level. Moreover, pupils start to learn statistics from grade 1 until the end of lower-secondary school (Elbehary, 2020). Hence, as reported in Chapter 2, to determine the specific area for research within the Statistics and probability domain, textbooks' activities (from grade 1 to 9) were analyzed in the light of Burrill and Biehler’s (2011) list of Fundamental Statistical Ideas.

The analysis process relied on assigning each activity to the corresponding statistical idea based on this activity’s objective. For example, Figure 7 shows an activity for third-grade pupils at which they were asked to represent given numbers by bar graphs. Hence, that activity was classified under the idea of Representation; it includes graphical or other representations that reveal stories in the data, including the notion of transnumeration (Burrill & Biehler, 2011). Considering that, during the analysis process, some activities consolidated more than one objective (e.g., both Data and Representation), in this case, the assigned code was given to both areas (e.g., one code for Data and the other for Representation). Following such processes, the number of activities committed to each statistical idea was determined (see Appendix 2), and **probability** was selected to be the subject of this study (see Table 12).

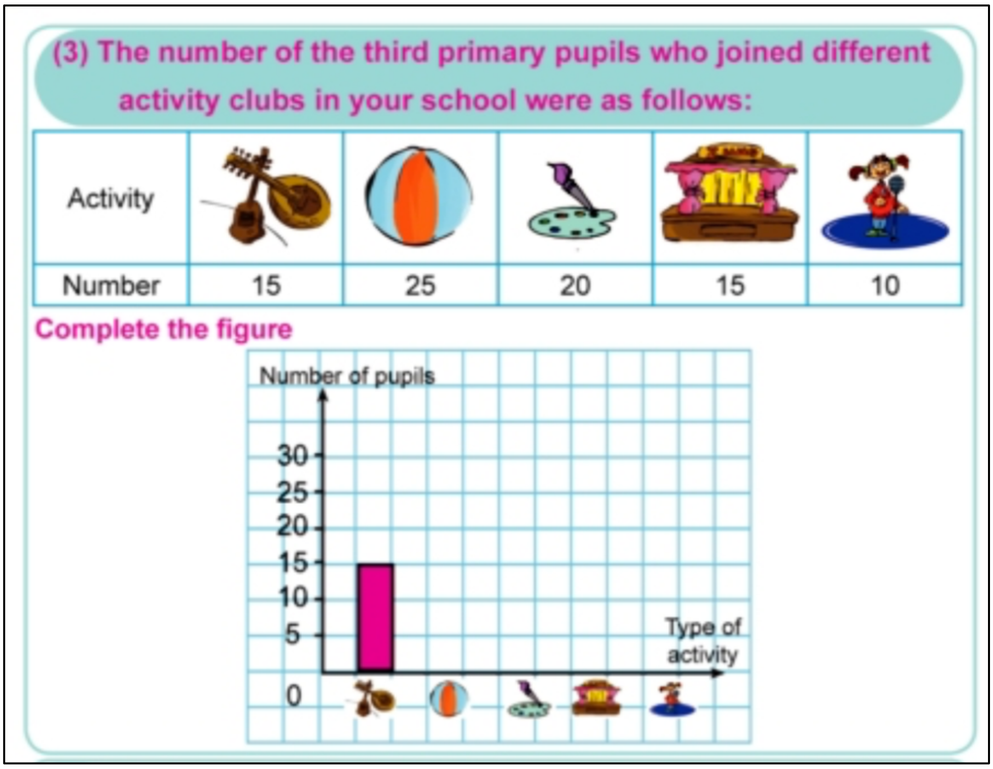


Figure 7. An example from third-grade pupils’ textbook in Egypt

Table 12. The distribution of fundamental statistical ideas in Egyptian school textbooks for the basic education sector. Based on Elbehary, 2020

The fundamental statistical ideas as determined by Burrill and Biehler (2011)	*Primary school level*						*Lower secondary school level*			*Total*
	G 1	*G 2*	*G 3*	*G 4*	*G 5*	*G 6*	*G 7*	*G 8*	*G 9*	
Data	4	4	3	8	6	23	1	16	0	65
Variation	0	0	0	0	0	0	0	0	15	15
Distribution	0	0	0	0	0	0	7	13	5	25
Representation	5	3	5	5	11	30	0	13	4	76
Association and modelling relations between two variables	0	0	0	0	0	0	0	0	0	0
Probability models for data-generating processes	0	0	10	9	26	18	9	12	27	111
Sampling and inference	0	0	0	0	0	0	10	4	8	22
Total	***9***	***7***	***18***	***22***	***43***	***71***	***27***	***58***	***59***	***314***

From the preceding, the points were summarized as follows:

- The need to investigate (and later enhance) PSMTs' knowledge not only because it signifies a national goal but also because its influence on pupils' learning and achievement has been reported and validated internationally.
- There is less emphasis on learning statistics (including probability) as a subject according to the academic program of preparing prospective Egyptian mathematics teachers. Besides, this limited regard of statistics involves its pedagogy, in which most of the discussion is focused on mathematics, particularly domains of Numbers and Algebra. Furthermore, Egyptian pupils' achievement in *Data and chance* stayed the lowest among other content areas, as recorded in TIMSS 2003 and 2007.
- Among fundamental statistical ideas, probability denotes a core concept according to the Egyptian school textbooks in both primary and lower-secondary school levels.

Such issues constituted a rationale for why this study highlighted PSMTs' knowledge for teaching probability. Additionally, the following section tries to precisely review the status of probability education reflected in the intended and implemented school curriculum. According to Stylianides and Ball (2004), both teacher education and school curricula designate directions to define teachers' knowledge. However, the researcher emphasized issues of the school curriculum. For the probability, most teachers have little experience with many of its topics in K-12 schooling, and teacher preparation programs mainly presented probability from a purely theoretical perspective (Stohl, 2005). That also reflects the situation in the Egyptian context, which appeared during the introductory discussion with PSMTs who participated in this study at which they raised issues of disconnection between learning statistics as a discipline regularly done in the Faculty of Science using teacher education curriculum and teaching it as a school content that they need to acquire. Furthermore, as Lortie (1975) stated in an agreement with Kleickmann et al. (2013), professional knowledge begins to develop even before the applicants enter teacher education. In other words, PCK is significantly shaped by teachers' school experiences. In such a case, analyzing the school content of probability is beneficial to speculate teachers' knowledge, especially in developing countries where many teachers prioritize discussing curriculum activities (Elbehary, 2019).

3.2 Probability education in Egypt from an international perspective

Based on what has been discussed, this section outlines issues in the intended and implemented school curricula of probability for the basic education sector, as follows:

3.2.1 The intended curriculum of probability in Egypt in comparison with NZ curriculum

Since probability defines a crucial concept in school and out of school settings (Franklin et al., 2007; Gal, 2005; Kazima, 2007; Nacarato & Grando, 2014; Paul & Hlanganipai, 2014; Watson, 2006), it has emerged as a mainstream strand within the school curriculum, worldwide (Burrill & Biehler, 2011; Jones et al., 2007). Thus, learning probability has been acknowledged by the CCIMD through embedding its content within the domain of statistics and probability in the Egyptian mathematics school curriculum. Although pupils have to learn statistics from grade 1, as noted previously, they are confronted with probability as a necessary concept to acquire from grade 3 continuously until grade 9 (Elbehary, 2020).

This study is motivated by the 2030 agenda for sustainable development, which provides a shared schema for peace and prosperity for people now and in the future, and Egypt and NZ are committed to achieving SDG. The proposed probability curriculum in Egypt was compared with the NZ curriculum. As reported earlier in Chapter 2, this comparison involved a summative content analysis; it helped determine what probabilistic ideas were disregarded from the Egyptian curriculum (compared to NZ) in a way that helps to strengthen these ideas when discussing teachers' knowledge.

The process of the analysis involved these procedures: (1) Select the proposed sample to be analyzed; this sample represented the intended curriculum of probability that is shared by the official websites of CCIMD (2012) and the NZ curriculum online (2007). (2) Define the categories to be applied; this was performed through employing the OSA (see Chapter 2) that helped in inferring the primary entities of probability (i.e., Situations, Propositions, Procedures, and Terms) (Gusmão et al., 2010; Torres & Contreras, 2014), which were declared in both curricula. (3) These determined entities were assigned to the Fundamental Probabilistic Ideas listed by Batanero et al.'s (2016) and included Randomness; Events and sample space; Combinatorial enumeration and counting; Independence and conditional probability; Probability distribution and expectation; Convergence and the law of large numbers; Sampling and sampling distribution; and Modeling and simulation. Lastly, (4) Issues of trustworthiness were explained by operating two rounds of coding in a time difference of two months (see all the procedures and results in Elbehary, 2020). Accordingly, several issues were revealed and summarized as follows:

- The name of the study domain at which probability was assigned is *Statistics and probability* in Egypt compared to S*tatistics* in the NZ curriculum. The Egyptian situation reflects the tendency to organize learning statistics and probability within a hierarchical structure, rather than maintaining the complementarity between them (Steinbring, 1991). In such a conventional approach, the conceptual link between probability and statistics is not accessible until the discussion of statistical inference (Kazak & Confrey, 2006).
- The learning objectives of statistics in Egypt aligned with the NZ curriculum in declaring the study of statistical investigation, statistical literacy, and the probability that was given notable attention as appeared earlier during analyzing textbooks' activities (see Table 12).
- As exhibited in the following table,

Table 13. The fundamental probabilistic ideas discussed in both Egypt and NZ intended curricula. Retrieved from Elbehary (2020, p. 10)[12]

	Situations	Language	Procedures	Propositions
Randomness	Investigate situations that involve elements of chance (both)	Probability, likelihood, chance, and uncertainty (both)		
Events and sample space		*Fractions, ratios and percentages (NZ)* Possible, impossible, and certain events (Egypt)	Calculate the probability of an event (both) *Use fractions, ratios and percentages to describe the probability (NZ)*	Relationship between two events (Egypt) Relationship between the event (certain, possible, impossible) and its probability (Egypt)
Combinatorial enumeration and counting				
Independence and conditional probability	*Acknowledge variation and independence (NZ)*		*Calculate the probability of independent, combined, and conditional events (NZ)*	
Probability distribution and expectation		*Theoretical, Normal, Binomial, Poisson, and Experimental distribution (NZ)*		*Compare the variation between theoretical and experimental distributions (NZ)*
Convergence and law of large numbers	*Acknowledge the variation between experimental and expectations (NZ)*			*Comparing experimental results with expectations from models of all the outcomes (NZ)*
Sampling and sampling distribution				
Modeling and simulation	Identify some kinds of software that include simple trials (both)		Use suitable simulations to calculate probability (both)	

[12] The dark-colored cells indicate the probabilistic ideas emphasized in both Egypt and NZ. However, normal and italic letters were used to highlight the ideas discussed in either the Egyptian or the NZ curriculum, respectively.

The Egyptian intended curriculum was compatible with the NZ curriculum in emphasizing the probabilistic ideas of *Randomness*, *Events and sample space*, and *Modeling and simulation*. Nonetheless, *Independence and conditional probability*, *Probability distribution and expectation*, and *Convergence and law of large numbers* were discussed within the NZ curriculum but not yet considered in Egypt. The absence of such ideas affects pupils' understanding of probability (Elbehary, 2020), expressly in terms of emphasizing learning these ideas as listed in various literature. For instance, notions of independence and conditional probability are widely recommended to be included within the curriculum (Borovcnik & Kapadia, 2009; Chernoff, 2014; Franklin et al., 2007; Heitele, 1975; NCTM, 2000).

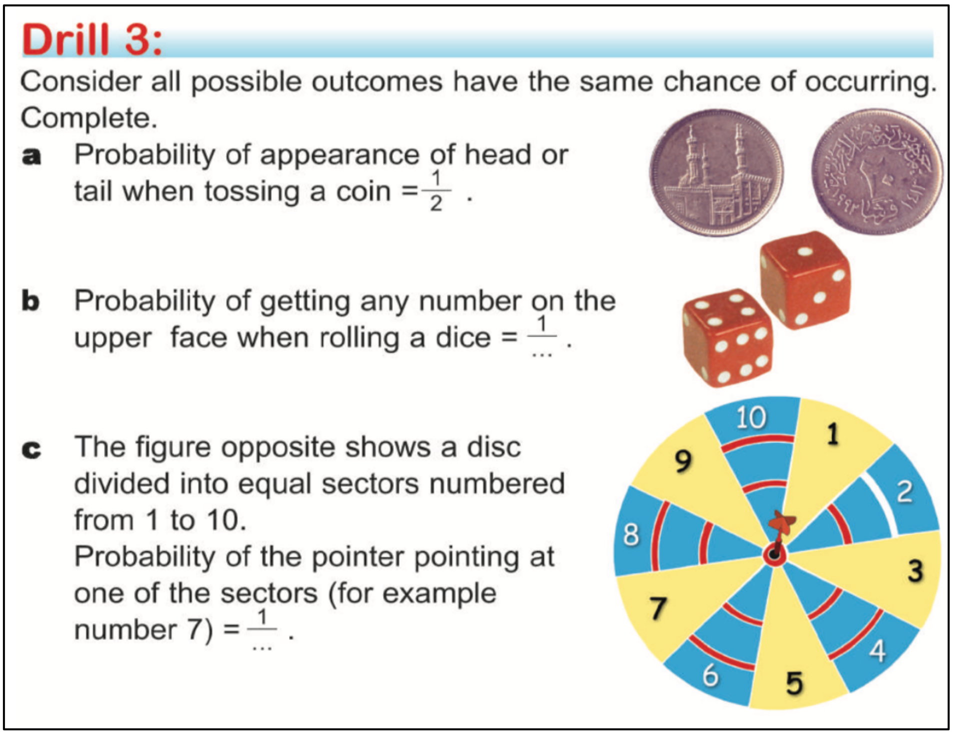
3.2.2 *The implemented curriculum of probability in Egypt*

Textbooks are described as a "significant factor in determining students' opportunity to learn and their achievement" (Robitaille & Travers, 1992, p. 706). As stated by Houang and Schmidt (2008), data from TIMSS suggest that textbooks are found in almost every classroom and are regularly utilized in instruction. They constitute a means for learning and signify the potentially implemented curriculum (Schmidt, McKnight, Valverde, Houang, & Wiley, 1997b; Tran, 2016). Nevertheless, for the case of probability instruction, some textbooks were criticized for presenting a too narrow view of probability (Batanero et al., 2004). That seems to be critical for teachers' knowledge since curricula provide teachers with the required tools and methods to perform their jobs (Shulman, 1987), wherein curriculum, here, indicates the implemented curriculum outlined by textbooks' activities.

To define issues of the Egyptian implemented curriculum of probability, the textbooks' activities were reflected through utilizing the OSA and Batanero et al.'s (2016) list of various probability interpretations, as noted earlier in Chapter 2. For example, to classify the presented activity in Figure 8, (a) <u>Situations</u> of tossing a coin, throwing a die, and spin a spinner; (b) <u>Procedures</u> for calculating the probability of an event; and (c) <u>Terms</u> of possible, chance, and probability were assigned to the <u>Classical Interpretation</u> that defines probability through dividing the number of favorable outcomes by all possible outcomes (Batanero et al., 2016). Accordingly, the results were shown in Appendix 3, and the following concerns were raised:

- Although the implemented curriculum places more emphasis on operating theoretical and axiomatic probabilities, intuitive and experimental interpretations were also considered. One good point is approaching the intuitive probability during early grades, which makes the probability concept meaningful to young pupils (Batanero et al., 2016). On the other side, the analysis revealed a lack of experimental probability; moreover, the subjective probability defined formally by the notions of conditional probability and Bayes theorem seems to be neglected until grade 9.

- Such bias toward the objective probability negatively impacts teachers' knowledge and ultimately influences pupils' probabilistic reasoning. That is, emphasizing the axiomatic approach is not appropriate to pupils at the elementary level; being too formal should be for those who follow pure mathematics studies at the post-secondary level (Batanero et al., 2016). Besides, the ignorance of the conditional probability that considers a prerequisite for understanding the subjective approach (Jones et al., 2007) hinders pupils from attaining the multi-structural and rational levels of conditional probabilistic reasoning (Mooney, Langrall, & Hertel, 2014). Furthermore, from an instructional viewpoint, when teaching probability predominantly utilizing a theoretical approach rather than a frequentist, students tend to develop conceptions based on deterministic reasoning (Konold, 1995).
- Another issue that indicates the relationship between the intended and implemented curriculum was observed. Concretely, the disconnection between theoretical and experimental probability interpretations in the implemented curriculum (there was a lack of textbooks' activities that aim to connect both concepts [Elbehary, 2020]) reflects the absence of the fundamental idea of Convergence and the law of large numbers in the intended curriculum.


Drill 3:

Consider all possible outcomes have the same chance of occurring.
Complete.

a Probability of appearance of head or tail when tossing a coin = $\frac{1}{2}$.

b Probability of getting any number on the upper face when rolling a dice = $\frac{1}{\dots}$.

c The figure opposite shows a disc divided into equal sectors numbered from 1 to 10.
Probability of the pointer pointing at one of the sectors (for example number 7) = $\frac{1}{\dots}$.

Figure 8. An example from fourth-grade pupils' textbook in Egypt

Such reported results reflect the limitedness of statistics and probability education in Egypt; and, particularly, teacher training and school content of probability. Nonetheless, similar findings were shared by some researchers in other countries. For example, regarding teacher preparation programs, after analyzing eleven mathematics teachers' bachelor programs in Colombia, Torres (2014) reported that these programs spent only a few hours on probability education, and such specific pedagogical training either in statistics or probability was absent at many universities. In a consensus with Ainley and Monteiro (2008), Batanero et al. (2004) and Pecky and Gould (2005) reported that graduates from mathematics departments have some basic knowledge in probability and statistics but are not always prepared to teach these contents. Additionally, in Brazil, Fernandes, Ferreira, Kataoka, Souza, and Gonçalves (2008), as translated in Kataoka et al. (2008), considered the absence of subjects related to probability and statistics within major mathematics courses as a critical deficiency in teachers' preparation.

Furthermore, and concerning the probability curriculum, it has been argued that the school documents and textbooks do not offer enough support for teachers. They sometimes present a too narrow view of probability, applications are mostly limited to games of chance, and some definitions of concepts are inaccurate (Batanero et al., 2004; Ortiz et al., 2002). For a case, Carranza and Kuzniak (2008) noted that the official curriculum of probability in France supports the experimental interpretation to be the only necessary approach to solve problems. Again, in Brazil, it was remarked that some elementary schools' textbooks address only the theoretical interpretation of probability (Lopes & Moran, 1999, as translated in Kataoka et al., 2008). Similarly, the Colombian curriculum explicitly acknowledged both theoretical and experimental interpretations; however, the subjective approach took a marginal emphasis with some suggested contents of conditional probability and independence (Torres, 2014). Despite that, the Spanish context exposed a good status, wherein the analysis of the intended primary school curriculum of probability in Spain revealed that it was in line with the suggestions for improving probability literacy, which was listed by Gal (2005). As detailed by Torres and Contreras (2014), primary Spanish pupils usually experience intuitive, classical, experimental, and subjective interpretations of probability.

To summarize (see Table 2 in Chapter 2), the whole argumentation raised within this chapter, on one side, provided an answer to the first research question (what is the current status of "statistics and probability" education in Egypt?); and, on the other side, established a rationale, from a local standpoint, for why this study had to be conducted. Concretely, because of the reported constraints that influenced the quality of probability education in the Egyptian context, more precisely, the status of teacher education and current school curriculum, this study has focused on PSMTs knowledge for teaching probability as one plausible way to overcome such constraints.

CHAPTER 4: LITERATURE REVIEW AND DEVELOPMENT OF THE STUDY FRAMEWORK

This chapter presents a review of the historical development of research on probability complemented by discussing current themes of probability education research, which helped in interpreting the research gap addressed through this study. Additionally, the study premises were argued to define mathematics teachers' professional knowledge for teaching probability from the PoPR. That responded to the second research question.

4.1 Historical development of research on probability

Jones and Thornton (2005) provided a historical overview of research on probability teaching and learning, with the clarification of the *Piagetian*, *Post-Piagetian*, and *Contemporary research periods*. Moreover, Chernoff and Russell (2014); and Chernoff and Sriraman (2015) proposed a fourth phase of the *Assimilation period* and explained some directions for future research. That is presented as follows:

- *First phase: The Piagetian period (the 1950s and 1960s)*

During this period, the study of probability was dominated by Piaget and Inhelder's (1975) research that focused on the developmental growth of people's probabilistic thinking.

- *Second phase: The Post-Piagetian period (the 1970s and 1980s)*

This period was governed by Fischbein's (1975) seminal research on primary and secondary probabilistic intuitions as a progressive work of Piaget and Inhelder. Besides, Tversky and Kahneman's (1974) analysis of psychological heuristics and biases in thinking under uncertainty. As a result, the ideas of Tversky and Kahneman's investigations were transmitted into the field of mathematics education through prominent researchers such as Shaughnessy (1977, 1981), Falk (1981), and Konold (1989, 1991). Later on, when the *Post-Piagetian period* came to a close, the field of mathematics education began to see an increasing amount of research on intuitions and learning difficulties, which is well synthesized in Shaughnessy's (1992) extensive chapter on research in probability and statistics education. In that regard, Shaughnessy (ibid.) reported the difference between researchers in psychology and mathematics education, as observers or describers versus interveners, respectively.

- *Third phase: Contemporary Research period (the 1990s and 2000s)*

During this period, there was a significant shift toward studying curriculum, instruction, and learning difficulties in mathematics education. It was carried out by a particular group of researchers, such as Falk (1981) and Konold (1991), who began to develop their theories, frameworks, and models regarding students' responses to situations that involve uncertainty. Notably, the study of Konold et al. (1993); it contributed not only to the resettlement process but also shifted the focus from heuristics to the informal conceptions of probability (informal reasoning). Furthermore, these theories were well-structured through the exploration of various constructs in probability (e.g., randomness, sample space, and probabilistic reasoning) in numerous further studies (e.g., Batanero & Serrano, 1999; Falk & Konold, 1997; Fischbein, Nello, & Marino,1991; Fischbein & Schnarch, 1997; Jones et al., 1999; Lecoutre, 1992; Pratt, 2000).

- *Fourth phase: Assimilation period (after 2000 and current research)*

In line with the prior period, researchers maintained to develop theories and models associated with intuitions and learning difficulties. More than fifty years of research had passed since the initial days when researchers attempted to replicate and import research findings from different fields such as psychology. Mathematics education researchers continued forming their interpretations of results stemming from the intuitive nature and difficulties associated with teaching and learning probability. However, recent investigations regarding the probability instruction have proceeded back to the proverbial roots toward integrating such recent studies of psychological heuristics and biases. That was recommended by Chernoff (2012b), who noted that the mathematics education research literature has, until recently, ignored subsequent research results deriving from the field of cognitive psychology.

- *Directions for future research*

Following the above-listed periods of research on probability, the state of future research is concentrating around two arenas, which are presented as follows:

- **First**, some scholars have strengthened the potential new shift from heuristics and informal reasoning to fallacious reasoning (e.g., Chernoff, 2012b).

While in the past, the focus was on normatively incorrect responses to probabilistic tasks (e.g., determine which sequence of coin flips less likely than another) (e.g., Thompson, 2008), recent investigations are moving away from utilizing the traditional notions of heuristic as a framework to analyze the incorrect responses toward logically fallacious reasoning (Chernoff & Sriraman, 2015). For example, Chernoff and Russell (2011a, 2012a) reported that some PSMTs utilize a particular logical fallacy, the fallacy of composition, in which the subjects assume something to be true about the whole based on facts associated with their parts. Accordingly, because the coins [the parts] are equiprobable, and the events [the whole] are comprised of coins; then, such events are equiprobable, which is not necessarily true.

Following this trend, there is a need for further studies that focus on fallacious reasoning as one area of future research; especially, since many of these fallacies still account for both correct and incorrect responses. As Konold et al. (1993) argued, interpreting why a particular answer was provided can reveal that a correct answer was given for the wrong reason. In such a case, this direction sheds light on individuals' justification and reasoning processes rather than their typical normative answers.

- **Second**, another direction of research tries to clarify two contested areas that exist in probability education research; (a) the different interpretations of probability and (b) the dispute over the term heuristic (Chernoff & Sriraman, 2015).

The first controversial area is concerned with the discussion on probability, which has both mathematical and philosophical facets. Still, there is a remarkable distinction between the two. "While an almost complete agreement exists about the mathematics, there is a wide divergence of opinions about the philosophy" (Gillies, 2000, p. 1). As a result, the probability education research resumes advocating "a more unified development of the classical, frequentist, and subjective approaches to probability" (Jones et al., 2007, p. 949). Alternatively stated, it "involves modeling several conceptions of probability" (Shaughnessy, 1992, p. 469), which was also reported by Chaput et al. (2011) as the challenge to connect three approaches of probability is not yet achieved. Despite that, the debated nature regarding the concept of probability will forever remain at the very core of research about teaching and learning probabilistic reasoning.

The second area of dispute considers the research on heuristics, which has two grounds; one is the work of Kahneman, Tversky, and colleagues (e.g., Kahneman, Slovic, & Tversky, 1982), while the other is Gigerenzer and his colleagues (e.g., Neth, Meder, Kothiyal, & Gigerenzer, 2014). Although Kahneman and Tversky's studies on heuristics and biases remain seminal to investigate teaching and learning probabilistic reasoning, the developments of such studies are not reflected in mathematics education literature, despite some exceptions; mainly, Chernoff's research (2012a) that regarded the "arrested development of the representativeness heuristic" (p. 951). On the other side, Gigerenzer's research is trying to stop the continuation of this arrested development of heuristics in mathematics education (Chernoff & Sriraman, 2015). Consequently, as Chernoff and Sriraman (2015) reported, both research trends shed light on a renaissance period of psychological research in mathematics education. In other words, the probability education research is seeking to pave the way "for theories about mathematics education and cognitive psychology to recognize and incorporate achievements from the other domain of research" (Gillard et al., 2009, p. 13).

Following the identification of such various directions of research on probability and considering what was raised regarding the deficiency of probability instruction in the Egyptian context (see Chapter 3), the current study attempted to embrace the PoPR to articulate PSMTs' knowledge for teaching probability in Egypt. In such a discussion at which the focal point is how the individuals think in a probabilistic situation, this study tried to fulfil two essential issues: (a) connect the mathematics education perspective on probability with its psychological roots of learners' reasoning under uncertainty, and (b) develop a unified schema that incorporates various probabilistic conceptions together with much focus on the subjective probability interpretation. Both issues are related and can be operated through the PoPR. It strengthens how PSMTs reason in a probabilistic situation (psychologically), representing their different conceptions of probability in a consolidated schema and, later, serves to design and engineer better instructional interventions (educationally) that may contribute to promote their professional knowledge.

The above argumentation raises these questions: (a) What is the current research state on probability? (b) Why cannot it provide an answer to the previously addressed issues of needed research? Furthermore, (c) how has the current study utilized the PoPR as an alternative approach to fulfil these issues? The next sections were arranged to approach these questions.

4.2 Themes of research studies on probability and research gap

4.2.1 Current research on probability

In light of the detailed procedures in Chapter 2, 25 papers were selected from ICOTS 8, 9, and 10 to outline current research themes on probability and expose the research gap. These papers were classified depending upon the presented matrix in Table 14. That is, the primary purpose of each paper was clarified based on what the author explicitly wrote (e.g., in the introduction or the conclusion), and, accordingly, it was assigned to category A, B, C, or D. For example[13], Theis and Savard's study (2010) was categorized under D because there was an intervention (i.e., mathematics education perspective or educational) and an emphasis on simulation processes (i.e., stochastics or mathematical). Following this technique, the 25 papers of ICOTS were distributed among A, B, C, D, and E that was emerged during the analysis to indicate irrelevant studies, as shown in Table 15.

[13] A list of all coded texts at which ICOTS' papers were classified is provided in Appendix 4

Table 14. Assumed criteria to classify ICOTS' papers that focused on probability

<table>
<tr><th>*Logic* / *Context*</th><th>***Psychological perspective Observational studies***
How learners think in a given context</th><th>***Mathematics education perspective Interventional studies***
What kind of pedagogical approaches can be used to promote learners understanding of probability</th></tr>
<tr><td rowspan="2">***Historical-philosophical (contextual)***
The usage of paradoxes (historical origins of probability), puzzles, and games of chance in the development of probability theory and its understanding (Sriraman & Lee, 2014)</td><td>**A**</td><td>**B**</td></tr>
<tr><td colspan="2">More focus on exhibiting the subjective side of probability</td></tr>
<tr><td rowspan="2">***Stochastics (mathematical)***
The connection between probability and statistics that appears in discussing the frequentists approach, random process, simulations, sampling techniques, and probability models and distributions (Shaughnessy, 2014)</td><td>**C**</td><td>**D**</td></tr>
<tr><td colspan="2">More focus on exhibiting the objective side of probability</td></tr>
</table>

Table 15. Classification of ICOTS' papers on probability

A (Psychological contextual)	***B (Educational contextual)***	***C (Psychological mathematical)***	***D (Educational mathematical)***	***E (Irrelevant studies)***
(Larose et al., 2010; Gusmão et al., 2010;Torres & Contreras, 2014; Torres, 2014; Kuzmak, 2014; Moreno & Cardeñoso, 2014)	(Batanero et al., 2010; Savard, 2010; Eckert, 2014; Levy & Stukalin, 2018)	(Papaieronymou , 2010; Viali, 2010; Leviatan, 2010; Primi et al., 2014; Díaz et al., 2014)	(Theis & Savard, 2010; Grenon et al., 2010; Chadjipadelis & Anastasiadou, 2010; Estrella & Olfos, 2010; Takagi, 2018)	(Caldeira & Mouriño, 2010; Gundlach et al., 2010; Kapadia & Borovcnik, 2010; Borovcnik & Kapadia, 2010; Edwards, 2014)
6 papers	***4 papers***	***5 papers***	***5 papers***	***5 papers***

Before exploring category **A** papers that maintained similar characteristics (psychological contextual) like the current study, concerns in other categories were detailed, as follows:

In the beginning, category **E** incorporated irrelevant studies, which aimed at addressing other features that differ from cognitive aspects (i.e., knowledge, reasoning, or understanding). On one hand, *Caldeira and Mouriño (2010*) and *Gundlach et al. (2010)* studied students' opinions and PSMTs' motivation and self-efficacy that are related to the subject of probability and statistics, respectively. On the other hand, various literature reviews on probability research, electronic publications, and currently used mobile Apps were provided by *Kapadia and Borovcnik (2010), Borovcnik and Kapadia (2010),* and *Edwards (2014),* respectively.

Moving to both **B** and **D** that designated interventional studies with much focus on the mathematical side of probability for class **D** papers. The common trait among all these papers emphasizes issues of enhancing teachers' knowledge and students' understanding of probability through several didactical activities.

In detail, some researchers focused on in-service teachers' knowledge of probability. For example, *Batanero et al. (2010)* employed paradoxical games that revealed a positive change in some of the teachers' initial misconceptions of probability. Furthermore, *Eckert (2014)* emphasized the social interaction between teachers and students in the classroom; and, accordingly, he highlighted the potentiality of the grounded theory approach to study teachers' knowledge, wherein this interaction could be analyzed. Again, to promote teachers' knowledge for teaching probability, particularly the objective side, *Theis and Savard (2010)* trained in-service lower secondary school teachers to implement activities rooted in a gambling context and represented by computerized simulators. Hence, the results showed that teachers faced some difficulties when approaching probabilistic concepts via simulation software.

Additionally, both *Chadjipadelis and Anastasiadou (2010)* and *Takagi (2018)* addressed pre-service teachers' knowledge. While the former investigated the impact of a student-centered environment on improving their understanding of probability distribution, the latter proposed a syllabus to promote the status of teaching statistics and probability, with much focus on the statistical charts such as bar and line graphs, pie charts, and histograms.

On the other side, four papers centered around students' understanding of probability. While *Savard's (2010)* study endeavored to describe primary school students' probabilistic thinking in fake gambling situations and showed that they operated deterministic reasoning to predict the outcome, *Levy and Stukalin (2018)* examined first-year undergraduate biology students' understanding of conditional probability. Hence, the results showed that students who experienced the intuitive explanation of a problem performed better than those who applied mathematical procedures. Moreover*, Grenon et al. (2010)* and *Estrella and Olfos (2010)* were more concerned with the mathematical rules of probability theory. Thus, the former described the usefulness of the computerized simulators as a teaching tool to motivate students in building probability knowledge. Similarly, the latter reported the effectiveness of a proposed sequence of lessons in developing talented children's understanding of probability, in which they were able to justify their procedures using formal arguments of probability theory.

Finally, papers of **A** and **C** categories attempted to discuss current practices in teaching and learning probability, including curriculum issues. At first, regarding **C**, which characterized observational studies that strengthen the statistical side of probability, among its five papers, four analyzed programs of statistics and probability (i.e., *Papaieronymou, 2010; Viali, 2010; Leviatan, 2010;* and *Díaz et al., 2014*). Nonetheless, there were some differences among them in terms of the regarded stage. While Papaieronymou (2010) studied the educational reform taking place in Cyprus and provided some implications for the teaching of statistics and probability at the secondary level, Viali (2010); Leviatan (2010); and Díaz et al. (2014) emphasized undergraduates' courses and concerns in tertiary education. These studies shed light on (a) the university curriculum's limitedness in addressing issues of statistics and probability; further, its pedagogy (as reported in Chapter 3). Moreover, (b) the importance of clarifying principles and strategies rooted in probability axioms at which this curriculum can be restructured, and, at the same time, it stressed operating such axioms in realistic social situations.

The fifth paper of *Primi et al. (2014)* shared a similar concern of observing for promoting objective probability knowledge. It endeavored to develop a scale that measures the basics of probabilistic reasoning ability. Although it referred to the probabilistic reasoning in its title, the proposed questionnaire has not intended to address students' cognitive biases but rather identify those who may struggle at the introductory courses to provide them with extra activities. Thus, the scale afforded typical mathematical tasks (e.g., a ball was drawn from a bag containing ten red, thirty white, twenty blue, and fifteen yellow balls. What is the probability that it is neither red nor blue?), to measure low levels of probabilistic reasoning ability.

Now, it is time to explore the status of category **A**, which inspired this study. The following argumentation focuses on placing the study among category **A** papers at which similarities and differences are explained. Accordingly, by the end of this section, the current state of research on probability education is manifested. Still, why it cannot answer the previously addressed issues of needed research requires clarification; that will be handled in the next section of the *Research gap and the study perspective.*

First, two studies among the assigned six papers to category **A** involved curriculum analysis. While *Torres (2014)* analyzed undergraduate programs of teaching probability in some Colombian universities, *Torres and Contreras (2014)* defined the probability concept in the Spanish primary school curriculum. Both studies were concerned with the epistemological meaning of probability and showed that the subjective probability is neglected. That differs from what was discussed earlier about curriculum analysis in category **C** studies, of which they aimed at exhibiting weaknesses of statistics and probability education generally (outlined by the number of hours). Similar to this in indicating the meaning of probability, *Gusmão et al. (2010)* employed the OSA to analyze the mathematical objects that in-service mathematics teachers use during teaching Monica's random walk activity. Hence, the results showed the value of contextualizing probability education to address probabilistic concepts and figuring out semiotic conflicts between teachers' prior knowledge and formal probability theory.

Second, while the prior three studies incorporated issues of current intended and implemented curriculum and revealed the significance of contextualizing probability instruction in displaying teachers' conceptions, the articles of *Larose et al. (2010)* and *Kuzmak (2014)* maintained the focus on probability conceptions held by students. On one hand, Larose et al. (2010) reflected on middle and high school students' conceptions of probability; on the other hand, Kuzmak (2014) reported college students' immature understanding of random phenomena of which there was a discrepancy between their conceptions and the researchers' developed schema that represented formal knowledge of randomness.

Finally, and closely related to the current study context, *Moreno and Cardeñoso (2014)* investigated prospective mathematics teachers' probabilistic thinking. As a result, the study exposed four hierarchical levels labeled as deterministic, personalistic, uncertainty, and contingency. It also confirmed a certain distance between teachers' mental models and the standard conceptual models in probability theory.

4.2.2 *Research gap and the study perspective*

In light of the above discussion, it is evident that there is (a) a balance between the interventional studies (**B** and **D**, 9 papers) that aimed primarily to implement several interventions to develop in-service or pre-service teachers' understanding of the probability, which, ultimately, would impact their students, and the observational studies (**A** and **C**, 11 papers) that investigated the current state of probability education. Moreover, there was (b) a similar poise between emphasizing the objective mathematical facet of probability (**C** and **D**, 10 papers) and the subjective side (**A** and **B**, 10 papers).

Despite that, to infer some critical points that may exhibit the research gap and expose the study's uniqueness, the next argumentation sharpens category **A** that embodies the scope of this study. Because **A** papers intersect both **C** (observational) and **B** (contextual), this argumentation, first, illustrates what is still required in the observational studies, then what are the needed issues from a contextual viewpoint.

- Regarding observational studies represented by **A** and **C** papers, <u>more attention was given to curriculum analysis with six studies (out of eleven). Yet, merely one article was explicitly concerned with PSMTs, but it stayed different from this study</u>, as follows:

As outlined before, studies of Papaieronymou (2010), Viali (2010), Leviatan (2010), Díaz et al. (2014), Torres (2014), and Torres and Contreras (2014) addressed issues of curriculum. Besides, both Larose et al. (2010) and Gusmão et al. (2010) defined the probability conceptions held by pupils and in-service teachers. Furthermore, Primi et al. (2014) emphasized correct normative responses to typical probability tasks to assess students' basic low probabilistic reasoning levels (not PSMTs). That is, all these studies were not focused on PSMTs. On the other side, while Kuzmak (2014) studied college students' understanding of random phenomena, without particular emphasis either on probability or PSMTs, Moreno and Cardeñoso (2014) proposed a hierarchical order of probabilistic thinking levels modeled by prospective teachers. In that sense, although Moreno and Cardeñoso's (2014) study seems to be the only research that shared with this study issues of PSMTs' reasoning processes, it deviates from the current argumentation, as detailed in the next paragraph.

According to Moreno and Cardeñoso (2014), PSMTs' reasoning can be arranged into levels. That deviates from the current study premises, wherein defining learners' conceptions based on conceptual understanding levels does not admit the value of individuals' reasoning to make sense of phenomena (see Section ***4.3.3.***). Moreover, they have ordered PSMTs' probabilistic thinking by analyzing to what extent their responses reflect randomness and subjectivity. For instance, while the lowest level of deterministic thinking was characterized by denying randomness and accept subjective criteria, the best recognition of randomness with the minimum dependence on subjectivity was recognized for the uncertainty level. That raises two extra critical features of (a) the identification of these levels intended to describe thinking processes separated from teachers' knowledge; accordingly, there was neither discussion about biases (or conceptions) embedded in such levels nor reflecting on the three principal interpretations of probability (i.e., theoretical, experimental, and subjective). Besides, (b) how they define *subjectivity* in their research differs from the current study perspective; for them, *subjectivity* signifies one factor that negatively affects probabilistic reasoning and further causes a low level of thinking. However, within this study's context, the term subjective reflects one plausible approach that individuals rely on to reason probabilistically; further, the study aimed to explore how PSMTs conceptualize it.

It is worthy to remark that such lack of research *on PSMTs' knowledge for teaching probability*, which appeared from analyzing the observational studies of ICOTS, has also been reported in various other studies (e.g., Ainley & Monteiro, 2008; Batanero et al., 2004; Batanero et al., 2010; Dollard, 2011; Estrella & Olfos, 2010; Franklin & Mewborn, 2006; Greer & Mukhopadhyay, 2005; Ives, 2007; Pecky & Gould, 2005; Stohl, 2005; Torres, 2014). These studies recommended much more research to clarify the essential components in PSMTs' preparation at which prevailing difficulties in learning and teaching probability can be overcome. From this aspect, the current study attempted to approach such an area; particularly, after what was raised concerning the limitedness of the Egyptian curriculum of probability to enhance pupils' probabilistic reasoning (see Chapter 3).

- More critical, and by providing insights into the contextual studies (**A** and **B**), <u>(a) there was neither a clear identification of subjective probability as a type of reasoning that students may manipulate to think of random phenomena nor regard to learners' cognitive biases in reasoning under uncertainty as a framework to explain such subjectivity</u>. Moreover, (b) <u>the conditional probability concept that signifies a prerequisite for understanding the subjective interpretation has not taken much deliberation.</u>

That is, among all the contextual studies of ICOTS that attempted to figure out issues of subjective probability, there were few instances of learners' cognitive biases (e.g., gambler fallacy, the personalistic interpretation). Besides, only Levy and Stukalin's (2018) research reflected on students' understanding of conditional probability. Such limited recognition of subjective probability is interpreted as ignorance of both psychical roots and precursor of formal mathematical probability (Dewey, 1964, as cited in Gierdien, 2008; Hawkins & Kapadia,1984). Also, Wilson and Berne (1999) defined mathematical probability as a bounded process that neglects the subjective side, which was explained by Gierdien (2008) as a separation between subject matter from its method (i.e., being skillful in computing formal probabilities without understanding why and how probability formulas work).

With a similar matter, it is relevant to acknowledge that among invited papers of ICOTS 9, there was an essential topic entitled "Bayesian inference (probability) goes to school: meanings, tasks, and instructional challenges"; it sharpened the value of exhibiting the subjective facet of probability. As stated on the website, in our everyday life, it is seldom that one can implement objective probability at which it is impossible in many cases to ensure the equiprobability or repeat the experiment infinitely many times. Accordingly, there is a need to develop thinking in line with subjective probability and Bayesian theory; especially, in light of the limited attention given to this concept in both school and research (Chernoff & Russell, 2014; ICOTS 9, 2014). That is also reported by Chernoff (2014) of when Bayesian probability goes to school, new areas of investigation will be opened for researchers in mathematics education; it signifies an opportunity for those researchers to reconnect with the psychological roots of their field. Furthermore, Jones et al. (2007) declared that "it is timely for researchers in mathematics education to examine subjective probability and the way that students conceptualize it" (p. 947). Again, that coincides with what was reported earlier (see Chapter 3) regarding the Egyptian school curriculum that neglected subjective probability, except for some examples of the intuitive interpretation at the beginning of the primary level.

The whole analysis exposes why current investigations on probability have not yet responded to some needed areas of research; specifically, what was raised first regarding (a) generating a unified schema of theoretical, experimental, and subjective interpretations of probability, and (b) incorporate findings of cognitive psychological research into mathematics education. The fulfillment of such concerns requires a perspective that admits several individuals' conceptions and sharpens how the mind functions to balance their cognitive structures when dealing with uncertain situations. From this aspect, and to address such issues, the current study embraced the PoPR to define mathematics teachers' professional knowledge for teaching probability and conceptualize it for PSMTs in Egypt.

In that regard, it is valuable to note that probabilistic reasoning, in this study, signified a psychological perspective. It did not merely explain how individuals think, but it further incorporated how our beliefs and social practices may influence probability knowledge and reasoning. Alternatively stated, the current study describes probabilistic reasoning as being a psychological perspective because it (a) acknowledges that the existing conceptions should be the starting point to guide students toward normatively correct procedures; (b) reflects the educational perspective, since "understanding of reasoning mechanisms and the origins of prior conceptions may also lead to an engineering of these mechanisms and conceptions" (Van Dooren, 2014, p. 125); (c) exhibits the fact that the worse performance in probabilistic tasks is explained not only in cognitive terms but also in the affective domain (e.g., superstitious thinking); and (d) admits the sociocultural influence on students' reasoning (Ibid.).

Such attention to reasoning processes leads to strengthening the process knowledge rather than content knowledge, which cannot be neglected, particularly for probability education. As explained by Shaughnessy (1977), while some misconceptions have a mathematical root (e.g., students' inexperience with mathematical laws of probability), there is considerable evidence that these misconceptions are sometimes of a psychological sort; then "mere exposure to the laws of probability may not be sufficient to overcome some misconceptions of probability" (p. 295). Moreover, in 1992, Shaughnessy claimed that challenges underpinning probability instruction are related to teaching problem-solving.

4.3 Utilization of the probabilistic reasoning perspective and development of the study framework

As stated earlier in Chapter 2, the process of utilizing the PoPR has embedded two steps to define mathematics teachers' professional knowledge for teaching probability. That is the argumentation of this section (in ***4.3.2*** and ***4.3.3***). However, before going into details, issues concerning the term professional knowledge are described first (in ***4.3.1***).

4.3.1 The notion of professional knowledge to explain teachers' success

In light of the growing importance of international comparative studies on learning outcomes (e.g., TIMSS and the Programme for International Student Assessment (PISA) study), teachers' professional knowledge and its influence on the instructional quality and students' achievement has taken much attention.

The determination of teachers' success and its effect on students' achievement was progressed smoothly from the concept of *bright person*, passing to the *knowledgeable person*, and reaching the notion of *professional competence* (Kunter et al., 2013). The *bright person hypothesis* considers that "the best teachers are bright, well educated, people who are smart enough and thoughtful enough to figure out the nuances of teaching in the process of doing it" (Kennedy, Ahn, & Choi, 2008, p. 1248). On the other side, the *knowledgeable teacher hypothesis* reflects Shulman's (1987) ideas about the specialized type of knowledge shared among a community of professionals when practicing the teaching profession. Operating this hypothesis in various proximal studies revealed that teacher knowledge is associated with higher quality instruction, which, in turn, has a positive effect on students learning and achievement.

Later, the concept of *professional competence* admits that in addition to knowledge, teachers' beliefs, motivation, and self-regulation represent several aspects involved in determining teachers' success. That is, the term *professional competence* refers to the application of the concept to working life, particularly in highly complex and demanding professions, in which mastery of situations depending upon the interplay of knowledge, skills, attitudes, and motivation (Epstein & Hundert, 2002; Weinert, 2001). More specifically, in mathematics education, the Teacher Education and Development Study: Learning to Teach Mathematics (TEDS-M) and the Cognitive Activation in the Mathematics Classroom and Professional Competence of Teachers (COACTIV) study proposed two frameworks that define mathematics teachers' professional competence.

Regarding the TEDS-M study, a conceptual model of mathematics teachers' professional competence was developed based on extensive international discussion among all participating countries. This model incorporated two essential aspects of (a) professional knowledge (content knowledge, PCK, and general pedagogical knowledge) and (b) affective-motivational characteristics (beliefs, motivation, and self-regulation) (see the framework in Kaiser et al., 2017). Similarly, the COACTIV framework aimed at interpreting central concepts of teachers' professionalism; it outlined the interplay among teachers' professional knowledge that is the core of professionalism, values and beliefs, motivational orientations, and self-regulation (see the framework in Baumert & Kunter, 2013).

Although the current study matches both frameworks in defining teachers' knowledge as the heart of their professional competence (Ball, Lubienski, & Mewborn, 2001; Shulman, 1986) wherein its influence on students' learning is widely acknowledged, it does not intend to assess such knowledge but rather explore, then, conceptualize it. That conceptualization focuses on probabilistic reasoning that denotes a kind of thinking associated with probability as a school content area. That may contribute to the competence models' creation for the prospective teachers (Krainer & Llinares, 2010) in the domain of *Data and chance*, which has not received much interest because at the time of conducting the TEDS-M, the probability was unequally implemented in school and teacher education curricula of the participating countries (Li & Wisenbaker, 2008). Nonetheless, nowadays, there is a growing interest in probability in many countries due to its relevance for applications in everyday life and sciences (see Chapter 1); besides, it is also incorporated in the NCTM standards from kindergarten to the secondary level (NCTM, 2000). Acknowledging that creation of such competence models in statistics, and specifically, probability requires much focus on thinking processes. As detailed in the GAISE report, college students should learn statistical thinking to cook creatively instead of merely following traditional recipes (GAISE, 2016). Furthermore, Garfield and Ben-Zvi (2008) pointed out that the main challenge in teaching and learning statistics is to ensure that students have not only obtained mechanics of statistical methods but also concepts underlying statistical reasoning.

Since the purpose of the whole discussion is defining mathematics teachers' professional knowledge for teaching probability, it is proper to move one step further beyond discussing the professional competence and proceeding, specifically, to teachers' knowledge.

Indeed, the specificity of the probability as a content area was explained in several studies. For a case, Batanero et al. (2004) stated that broad statistical knowledge, even when essential, is not enough for teachers to teach probability. Also, Batanero et al. (2016) further reported that although the probability is authorized in different stages from primary school to the teacher education curriculum, its inclusion into the curriculum does not automatically assure accurate teaching and learning. Especially for probability, because it has some specific characteristics, such as a multifaceted view and the lack of reversibility of random experiments, which are not usually encountered in other mathematics areas.

On one hand, such specificity creates several challenges for students. As many studies detailed, learners at different grades vary from elementary up to the college level have difficulties in learning probability (Batanero & Sanchez, 2005, Jones et al., 1999; Memnun, Ozbilen, & Dinc, 2019; Sharma, 2016). On the other hand, for the university students and prospective mathematics teachers, many researchers reported their insufficient understanding of probability, wherein without specific training in probability, preservice and practicing teachers (and perhaps some teacher educators) may rely on their beliefs and share similar misconceptions with their students (Batanero et al., 2016; Fischbein & Schnarch, 1997; Konold et al., 1993; Pratt, 2005; Prodromou, 2012; Shaughnessy, 1977; Stohl, 2005).

The above argument exposes the value of research on teachers' knowledge. Notably, although research concerning teachers' knowledge for teaching mathematics is abundant, studies related specifically to probability are rare (Callingham & Watson, 2011; Torres et al., 2016). Moreover, many teachers still approach statistics and probability lessons like other mathematical topics; they focus only on procedures and results rather than thinking and reasoning processes (the 10th Congress of European Research in Mathematics Education [CERME10], 2017). More specifically, for the PSMTs, there is a lack of research on PSMTs' knowledge for teaching probability, as revealed previously from analyzing ICOTS papers. Also, Dollard (2011) reported that "One way to improve this situation is to ensure that new teachers graduating from teacher education programs have a good understanding of the fundamental concepts of probability" (p. 27). Consequently, clarifying PSMTs' knowledge is necessary; to develop effective probability instruction, teacher educators need to identify what conceptions PSMTs bring into the mathematics classroom (Shaughnessy, 1992).

Additionally, it is worthy to remark that professional knowledge, in this study, reflects *knowledge for practice* that "depends on the assumption that the knowledge teachers need to teach well is produced primarily by university-based researchers and scholars in various disciplines" (Cochran-Smith, 1999, p. 255). Besides, it assumes that "it is possible to be explicit about a formal knowledge base rather than relying on the conventional wisdom of common practice, which some have referred to as natural, intuitive, or normative" (ibid., p. 255). Such formal knowledge is expressed in studies that use several quantitative or qualitative scientific methods to "yield a commonly accepted degree of significance, validity, generalizability, and intersubjectivity" (Fenstermacher, 1994, p. 8). This conforms to what Stylianides and Ball (2004) reported about reviewing researchers' findings, which implies one possible approach to scrutinize mathematical knowledge for teaching. From this aspect and as stated at first (see Chapter 2), to determine the initial entities of the study framework that defines mathematics teachers' professional knowledge for teaching probability, several previous studies were reviewed and categorized in light of the MKT model. This is further detailed in the next section.

4.3.2 The initial entities of the study framework

The process of operating the MKT to determine the initial entities of the framework has followed these procedures: (a) The MKT sub-constructs were first defined (see Figure 4). (b) The primary recommendations of ICOTS papers and other reviewed studies on teachers' knowledge for teaching probability were highlighted. (c) Each recommendation was assigned to the relevant sub-constructs. Some examples are provided in Table 16. Following such procedures, aspects of mathematics teachers' professional knowledge for teaching probability were explored and termed by **KoP**, **KoTP**, and **KoSPK**, which corresponds to SMK, KCT, and KCS in the MKT model, respectively. Moreover, a distinct component of **KoPL** that has not explicitly been displayed in the MKT was found, which is detailed in the next sub-sections.

Table 16. Some examples of how the initial entities of the study framework were determined in light of the MKT model

Definition	***Related discussion in the literature of probability (relevant quotes)***	***Criteria for judging the relevancy***
SMK: A deep understanding of the content to be taught.	– "Epistemological reflection on the meaning of concepts to be taught (e.g., reflection on the different meaning of probability)" (Batanero et al., 2004, p. 3). – "The Colombian curriculum explicitly considers classical and frequentist approaches to probability while the subjective approach is only implicit. For good classroom performance a mathematics teacher should know these approaches" (Torres, 2014, p. 2). – "Teacher should not only present different probabilistic concepts and their applications but be aware of the different meanings of probability and philosophical controversies around them (Batanero et al., 2004)" (Batanero et al., 2016, p. 23).	The given argumentations define what probability concepts mathematics teachers should understand, specifically, emphasized concepts in the school curriculum.

KCT: The knowledge that combines knowing about teaching and content.	– "These two approaches should not be separated if we want students to develop a good understanding of probability, and apply it in practical situations (Chaput, Girard, & Henry, 2011). School students are expected to explore and contrast the theoretical and empirical/experimental approaches to probability" (Prodromou, 2012, p. 855). – "Using computerized simulators as a teaching medium, is effective in motivating pupils and in building knowledge" (Grenon et al., 2010, p. 1).	The discussed ideas are relevant to how the probability concepts can be manipulated in the classroom, including designing effective activities or strategies to promote students' understanding of probability.
***KCS*:** The knowledge that combines knowing about students and content.	– "Prediction of students' learning difficulties, errors, obstacles and strategies in problem solving" (Batanero et al., 2004, p. 3) – "Teachers' knowledge about students is discussed under three sub-themes: students' prior knowledge, their misconceptions and difficulties, and student development" (Danişman & Tanişli, 2017, p. 24).	The provided concerns express the essential issues that mathematics teachers need to know about students' understanding of probability.

- **Knowledge of Probability [KoP] (The essence of professional knowledge)**

Since this study has embraced the PoPR that implicitly involves teachers' conceptions, it is necessary to admit what Batanero et al. (2004); Batanero et al. (2010); Godino, Batanero, Roa, and Wilhelmi (2008); and Torres (2014) reported regarding the epistemological reflection on the meaning of probability, which corresponds to the SMK in the MKT framework.

According to Hacking (1975), the probability is conceived from two perspectives, statistical and epistemic, at which both can be legitimately claimed to be correct. Thus, it is defined as a *Janus-faced* concept (Brase, Martinie, & Castillo-Garsow, 2014; Chernoff & Russell, 2014). While the statistical facet is relevant to objective mathematical rules that govern random processes, the epistemic side views probability as a personal degree of belief that depends on the information available to the person assigning that probability. Hence, both approaches were reflected in the work of many authors; and recently, Batanero et al. (2016) summarized the various interpretations of probability in the literature as follows:

Table 17. Various interpretations of probability as defined by Batanero et al. (2016)

Probability interpretation	***Explanation***
Intuitive	The intuitive interpretation of probability defines the probability as a formal encapsulation of intuitive views of chance that leads to the idea of assigning numbers to uncertain events. Besides, it appears in young children's argumentations when using qualitative expressions (e.g., probable, unlikely) to expose their degrees of belief in the occurrence of random events.

Theoretical, classical, or Laplacian	The theoretical interpretation defines the probability as a fraction of the number of favorable cases of a particular event divided by the total number of all possible outcomes, under the premise that all these outcomes are equally likely to occur.
Experimental, empirical, or frequentist interpretation	The experimental interpretation explains the probability as the hypothetical number that the relative frequency tends to stabilize when a random experiment is repeated infinitely many times under identical conditions. It signifies the limit of relative frequencies of an event when the experiment is repeated a large number of times (Batanero, Henry, & Parzysz, 2005).
Propensity	The propensity is one interpretation of the probability concept; it denotes the tendency of a given type of a random system (or physical situation) to behave in a certain way or yield a particular outcome.
Logical	That logical interpretation incorporates classical probability. However, the possibilities may be assigned unequal weights. Thus, probability denotes a rational degree of confirmation of one hypothesis, H, given some evidence E, a conditional probability that depends entirely on H and E's logical properties and relations between them.
Subjective	In this view, the probability symbolizes a personal degree of belief that depends on a person's knowledge or experience. Thus, the probability of an event can be revised in light of the newly available data of which initial (prior) probability can be transformed into a posterior probability through utilizing such new data.
Axiomatic	Probability is a function defined from A in the interval of real numbers [0,1] that meets the following three axioms: (1) $0 \leq P(a) \leq 1$, for every $a \in A$. (2) $P(S) = 1$. (3) For a finite sample space S and disjoint events A and B (i.e., $A \cap B = \emptyset$), $P(A \cup B) = P(A) + P(B)$; besides, for an infinite sample space S and a countable collection of pairwise disjoint sets A_i, i = 1, 2, ..., $P(\bigcup_{i=1}^{\infty} Ai) = \sum_{i=1}^{\infty} P(Ai)$.

Although Batanero et al. (2016) have regarded the seven above-presented definitions of probability, the theoretical, experimental, and subjective probabilities imply the basic performed interpretations in the K-12 curriculum that mathematics teachers should understand and further incorporate in their teaching (Batanero et al., 2005; Brase et al., 2014; Borovcnik, 2012; Dollard, 2011; Eichler & Vogel, 2014; Torres, 2014; Torres & Contreras, 2014; Kapadia & Borovcnik, 2010; Kazak & Confrey, 2006; Kvatinsky & Even, 2002, 2010; Sharma, 2016; Torres et al., 2016). Hence, and by the school curriculum, while theoretical and experimental represent the objective side of the probability, the subjective side can be accommodated through the intuitive interpretation and the concept of conditional probability. Nonetheless, such distinction is still under revision in the mathematics education literature (e.g., Chernoff, 2008). Both approaches are further detailed as follows:

– *The objective side of probability*

There are two main interpretations in the objective school: theoretical and experimental. On one hand, *theoretical probability* indicates a fraction whose numerator is the number of favorable cases and whose denominator is the number of all equally likely cases (Batanero et al., 2005; Konold, 1991; Laplace, 1995). Still, this interpretation is criticized in terms of equiprobability; it hinders applying the concept to several daily life situations (e.g., weather events, accident risks) at which this assumption may not be valid (Sharma, 2016). Consequently, the theoretical interpretation of probability is difficult to be performed outside games of chance (Torres & Contreras, 2014; Torres et al., 2016). Because of equiprobability, this interpretation is mostly connected with symmetrical random generators (e.g., fair die, equivalent spinning wheel) since there is no need to perform any experiments, and the probability can be calculated deductively (Sharma, 2016). Also, from a mathematical viewpoint, it cannot be applied for an infinite set of possible outcomes of a random process.

Furthermore, as per the school curriculum, the theoretical probability remains the most commonly practiced interpretation in the classroom (Kvatinsky & Even, 2002). It can be easily applied to random devices (e.g., dices), in which the sample space outcomes are assumed to be equally likely. Besides, it enables teachers to avoid the uncertainty of real random phenomena (Dollard, 2011). As Stohl (2005) noted, many teachers prefer the theoretical interpretation of probability because of counting techniques that lead to a definitive answer.

On the other hand, the *experimental probability* signifies a hypothetical number toward which the relative frequency tends during the stabilization process when random sequences are regarded (Sharma, 2016; Mises, 1957). Since it is required to gain data (frequencies) of the outcomes for estimating the corresponding experimental probability, the term posterior is assigned to this interpretation (Chernoff, 2008). It indicates that the probability is determined through experimentation to define the observed relative frequencies of an event throughout several identical trials (Borovcnik, Bentz, & Kapadia, 1991).

Yet, the experimental interpretation has a practical drawback, of which we only obtain an estimation that varies from one series of repetitions to another. Besides, it is not appropriate when it is not possible to repeat an experiment under the same conditions. Thus, within the real-world phenomena, this is often neither possible nor practical; as argued by Sharma (2016), it is impossible to conduct repeated trials to estimate the probability that someone's apartment will be stolen within a year. Moreover, no number can be fixed to ensure an optimal estimation for the probability. Similarly, Kvatinsky and Even (2002) stated that for a one-time daily situation, the subjective interpretation is more appropriate to utilize, for example, to determine the chance of having successful surgery for a specific patient or winning an election. Despite that, the experimental probability stays beneficial in explaining some contexts like a rain chance; since a 30 % chance of rain describes a model of past weather events, in which it has rained in three out of the ten previous days that have similar circumstances (Brase et al., 2014).

Regarding the school curriculum, because of the growing interest in using technology and simulation software in teaching probability to quickly generate random experiments and exhibit the effect of sample size, the experimental interpretation is receiving special treatment (Batanero et al., 2005; Batanero et al., 2016). As explained by Andrew (2009), through the experimentation, students can evaluate their prior judgments; further, eliminate some of their misconceptions and beliefs that may contradict with probability theory (e.g., it is most likely to appear, because it is my favorite color). This is explicitly declared in multiple curricula standards documents such as the NCTM (2000) and the Common Core State Standards for Mathematics (CCSSM, 2010).

After all, it is necessary to understand that neither of the interpretations (i.e., theoretical or experimental) is suitable to address every situation; instead, the appropriate approach should be operated depending upon the context (Kvatinsky & Even, 2002; Torres & Contreras, 2014). Moreover, some pedagogical activities can be approached through both interpretations. For example, to determine the probability of rolling a six on a regular six-sided die, the theoretical probability interpretation can be utilized to represent the complexities of the physics and visible symmetry of the die. Hence, the probability of rolling number six equals 1/6 (Stohl, 2005), and also, if the same die has been rolled in a large number of identical independent trials, the experimental interpretation can be utilized to provide a judgment. Similarly, Konold (1989) clarified that the probability of getting a head in an experiment of flipping a coin equals 1/2 based on the theoretical interpretation, which could match the experimental estimation when the relative frequency of heads (after a large number of trials) approaches 1/2. That resembles what Torres and Contreras (2014) argued about the quantity zero that embodies impossible events; it can be defined as there are (a) no favorable events, or (b) no observable outcomes, for both theoretical and experimental interpretations, respectively.

– *The subjective side of probability*

The epistemic side (subjective) treats probability as a language for describing the level of uncertainty that one feels (Liberman & Tversky, 1996). Moreover, the term subjective reflects an individual judgment, in which the probability does not have measurable characteristics. Thus, different people may assign different probabilities to the same event (e.g., election results) if they have different information or scope of view (Dollard, 2011; Kvatinsky & Even, 2002). For example, a 1% chance that the earth will be destroyed within ten years depends primarily on the individual's beliefs, where there are no past events (Brase et al., 2014). Consequently, Batanero et al. (2005) stated that it is impossible to treat the probability, within the subjective view, as a physical magnitude and to measure it accordingly in an objective way. It is not an intrinsic characteristic of an object, but a degree of belief given by the individual to a proposition (Carranza & Kuzniak, 2008). That matches Borovcnik's (2012) argumentation regarding the subjective probability that is still closer to the concept of provability: “it is the personal expression of a degree of credibility of a statement, which forms the subjectivist counterpart of an event” (Ibid., p. 9).

Formally and by the school curriculum, the subjective approach can be implemented through the intuitive explanation and the concept of conditional probability. Nevertheless, within the subjective view, all probabilities can be considered conditional probabilities in which even unconditional probabilities are conditioned by the sample space (Lindley, 1994). Despite that, there is a consensus among many researchers that the school curriculum (as well as mathematics education research, as detailed earlier) seems to ignore the subjective side of probability, which is widely practiced today in applications of statistics.

The *intuitive probability* reflects an encapsulation of intuitive views of chance that leads to the idea of assigning numbers to uncertain events; it utilizes qualitative expressions (e.g., probable, possible) to express the degree of confidence in the occurrence of an event (Batanero et al., 2005; Torres et al., 2016). Later, when students reach secondary school, they perceive subjective probability through the conditional probability concept and Bayes theorem.

Additionally, the *conditional probability* describes an update of the predictor’s knowledge of a particular event when new additional information is available (Kvatinsky & Even, 2002; Torres & Contreras, 2014). Hence, Borovcnik (2012) reported that the concept of conditional probability keeps the inherent *dual object-subject* character of probability; in other words, it stands at the border between the *objectivist conception*, mainly the frequentist interpretation, and the *subjectivist conception* at which the probability defines a degree of confidence (Kapadia & Borovcnik, 2010). Because of that, the conditional probability is regarded by many researchers as a crucial concept to learn (Díaz & Batanero, 2008; Díaz & de la Fuente, 2007; Kapadia & Borovcnik, 2010). For example, Heitele (1975) has included it within the list of fundamental stochastic ideas. Jones et al. (2007) also described acquiring the conditional probabilities as a prerequisite for learning the subjective probability.

Repeatedly, despite such value of the *subjective probability*, it stays an obscure area of research within mathematics education. As Chernoff (2014) declared, the state of the term subjective probability is subjective; it includes “the inconsistent use of multiple terms, such as “subjective,” “Bayesian,” “intuitive,” “personal,” “individual,” “epistemic,” “belief-type,” “epistemological” and others” (p. 3). Furthermore, although subjective probability has a dual meaning of general classifier and specific theory, explaining it as a *general classifier* that incorporates various philosophical differences is often neglected in mathematics education literature. Instead, the research defines it as a specific theory wherein a complete consensus exists about mathematics (Chernoff, 2008, 2014; Gillies, 2000). This is consistent with Steinbring's (1991) argument regarding subjective probability, which many researchers interpret as a personal degree of belief that depends on the amount of knowledge accessible to the individual, as it is not just a matter of opinion; yet, it should be checked through experimentation. Moreover, what Konold (1989) refers to by the idea of *calibration*; as he reported, although, within the personalist view, different individuals could validly allocate several values to the probability based on their beliefs about multiple factors, some mechanisms are required to handle these initial values and process the new information when formalizing such personalistic view (*calibration*).

From that aspect, this study relied on PSMTs' reasoning in a conditional probabilistic situation to *characterize the subjective probability concept as a general classifier*, that is grounded in Chernoff's (2008) argument where subjective probability takes the status of a general classifier that corresponds with belief-type probabilities. Accordingly, Chernoff (2008) classified subjective probability into intrasubjective (*personal belief-type*) and intersubjective probability (*interpersonal belief-type*). Moreover, he interpreted Jones et al.'s (2007) statement of "it is timely for researchers in mathematics education to examine subjective probability and the way that students conceptualize it" (p. 947) to "it is timely for researchers in mathematics education to examine *subjective probability* and the way that students conceptualize [*intrasubjective* and *intersubjective* probability]" (Chernoff, 2008, p. 21).

More specifically, in the current investigation, PSMTs were asked to reason on two situations that involve the conditional probability concept, which is the key to the subjectivist theory of probability (Borovcnik, 2012): one is relevant to the context of giving birth, and the other contains a two-way table, as an attempt to explore notions of *intrasubjective* and *intersubjective* probability. The distinction between the two depends on Chernoff's (2008) discussion of whether the latter (*intersubjective*, *interpersonal, a sense of objectivity is implied*) is more objective than the former (*intrasubjective*, *personal, less of an element of objectivity*). As stated in Borovcnik's (2012) report about the educational perspective on conditional probability, the conditional probability, on one hand, fulfils probability axioms for objectivists (i.e., *intersubjective*), and, on the other hand, it reflects that any probability is conditional to available information and is related to the idea of updating it in light of the new evidence (i.e., *intrasubjective*).

- **Knowledge of Teaching Probability [KoTP]**

There are many aspects relevant to teaching probability that were raised in the literature; the following discussion summarizes these aspects:

– *Warm up the probability lesson:*

The topic of probability should be approached through real-life examples, wherein students can utilize their intuitive understanding of uncertainty to capture the formal concept of probability (Kataoka et al., 2008). That matches the school curriculum at which the probability lesson is often introduced to students through the intuitive interpretation.

– Access the probability activities (implemented curriculum concerns):

Teachers' capacity to access the implemented activities is highlighted in the literature as a basic repertoire for teaching probability (Cordani & Wechsler, 2006; Gusmão et al., 2010; Kataoka, Trevethan, & Silva, 2010; Kvatinsky & Even, 2002; Theis & Savard, 2010; Torres et al., 2016). In detail, to operate textbooks activities, teachers should define simple, compound, and conditional probability; understand concepts of variability, expectation, randomness, and independence; distinguish between mutually exclusive (exclusion concept), joint, and independent events (independence concept); and draw inferences about a population from random samples (Batanero & Sanchez, 2005). Moreover, teachers are also expected to differentiate a mathematical problem from the statistical one. For a case, assume a coin is fair, and we tossed the coin five times; how many heads will we get? It represents a mathematical problem. On the other hand, you pick up a coin; is this a fair coin? It outlines a statistical one, in which the mathematical probability model can be used as a tool to seek a solution (Franklin et al., 2007). In this regard, the connection between mathematics, statistics, and probability is essential. That is, many mathematical concepts (e.g., numbers, proportions, ratios, combinatorics) still exist when working with probability (Batanero et al., 2010). Furthermore, the notion of fairness, which allows us to reason through observing experimental results and comparing them with the theoretical calculations, defines a central idea to deal with probability (PCMI, 2017).

Closely connected to such a discussion, although many teachers tend to draw on textbooks' sequences in their instruction, especially in developing countries, some researchers recommended starting with experimentations. For example, Andrew (2009) believes that students better understand probability concepts if they perform experiments in advance; it encourages them to develop understandings grounded in actual concrete events, compared to merely getting results based on algorithms.

– *Connect and differentiate among various probability interpretations:*

On one hand, Batanero et al. (2016) regarded the value of the differentiation between the theoretical model of probability and frequency data from reality to help students model real-life phenomena. On the other hand, the connection between both theoretical and experimental probability is acknowledged to enhance students' probabilistic reasoning (Chaput et al., 2011; Eichler & Vogel, 2014; Gusmão et al., 2010; Jones et al., 2007; Steinbring, 1991; Torres & Contreras, 2014). Such a connection leads the discussion toward the law of large numbers (Dollard, 2011; Kapadia & Borovcnik, 2010; Sharma, 2016), in which the absence of such a law leads students to see both interpretations as separate entities (Theis & Savard, 2010).

The law of large numbers recognizes that the difference between the experimental and the theoretical probability limits to zero as more trials are performed (Stohl, 2005). Hence, it helps students compare their inferences from theoretical and empirical work, then judge and modify their initial hypotheses (Sharma, 2016). Nevertheless, teachers sometimes describe this law to their students as if a necessary convergence between both probabilities with a large number of trials should be observed. Although the concept of limit implies that it is not possible to have an experimental probability that is significantly divergent from a theoretical probability when a large number of experiments are conducted, teachers may slightly modify the words to *get closer*. Consequently, students may misunderstand such law and expect a convergence between theoretical and experimental probabilities, even with a small sample size (Stohl, 2005).

In that sense, it is valuable to report what Nilsson (2013) argued regarding the movement between theoretical and experimental interpretations, which depends primarily on whether the underlying sample space is known by or hidden from students. According to Nilsson (2013), two methodological directions emphasize such connection; (a) the mapping direction that starts with the theoretical probability ⇒ experimenting ⇒ deduce the empirical probability, and (b) the inference direction that begins with experimenting ⇒ identify the empirical probability ⇒ deduce the theoretical probability. Hence, teachers' knowledge to adapt appropriate tasks for each interpretation through accommodating a cycle of "data, theoretical model, simulation, data" is essential for providing the students with adequate understanding to interpret phenomena (Prodromou, 2012; Serradó, Mavrotheris, & Paparistodemou, 2017).

– *Utilize various representations of probability:*

Teachers' familiarity with multiple representations that provide students with a substantial understanding of probability is crucial (Danişman & Tanişli, 2017; Even & Kvatinsky, 2010; Theis & Savard, 2010). For instance, teachers may employ tables; area models; Venn, pipe, or tree diagrams to clarify some probability concepts. While the tree diagram is useful to calculate probabilities associated with series of events, the area model is convenient for computing conditional probability because the ratio between areas of rectangles can be visualized (Kvatinsky & Even, 2002). Furthermore, the simulation process has been extensively discussed in the literature, either through concrete materials (e.g., spinners) or via computerized simulators (e.g., Grenon et al., 2010; Kapadia & Borovcnik, 2010; Savard, 2010). It helps to overcome students' deterministic reasoning by comparing the observed outcomes with prior predictions; besides, it reduces technical calculations and, instead, sharpening learners' understanding of probability concepts (Kapadia & Borovcnik, 2010).

- **Knowledge of Students' Probability Knowledge [KoSPK]**

According to Danişman and Tanişli (2017), teachers' knowledge about students includes recognizing their prior knowledge, misconceptions that were also stressed by Stohl (2005) at which teachers should perceive students' conceptions of probability, difficulties, and various levels of cognitive development. For instance, students' understanding of ratios, proportions, percentages, fractions, and rational number concepts related to probability is crucial to investigate. That requires strong curriculum knowledge (horizontal and vertical curriculum), wherein mathematics teachers connect what students learn in previous grades with essentials to understand current probability concepts. Similar concerns were advocated by Fischbein (1975), Steinbring (1991), and Pratt (2005) regarding the necessity for instruction to be built on students' existing knowledge of probability. For such issues, Batanero et al. (2016) advised mathematics teachers to be aware of research results that explain students' probabilistic reasoning and misconceptions; and further the appropriate instructional approaches that can help develop that reasoning.

- **Knowledge of Probability Language [KoPL]**

The consideration of language has been strengthened not only in case of probability but also for the whole statistics education, in which the approach to statistics content by the teacher who is conscious of statistical words, positively affects students' understanding (Otani, Fukuda, Tagashira, & Iwasaki, 2018). More specifically, about probability, many researchers highlighted the probability language as a fundamental aspect of teachers' knowledge (Batanero et al., 2016; Brijlall, 2014; Danişman & Tanişli, 2017; Dollard, 2011; Gal, 2005; Gusmão et al., 2010; PCMI, 2017; Torres & Contreras, 2014; Watson, 2005). As reported, the usage of probabilistic expressions and suitable vocabularies for students draws a necessary condition to warm up the probability lesson (Skoumpourdi & Kalavassis, 2003). Particularly because the probability concept is related to many expressions that we use in everyday language (e.g., probable, likely, possible, risky, sure). Therefore, connecting students' daily intuitions of chance manifested in their natural conversation with the academic language of probability signifies a further challenge for mathematics teachers (Batanero et al., 2016; PCMI, 2017). This may help to overcome what Green (1984) called linguistic weaknesses, which expose students' difficulties in utilizing the probability language; for a case, as he clarified, some middle school students define a 50/50 chance as anything can happen, rather than two equally likely events. Accordingly, mathematics teachers should develop pedagogical pathways that promote students' formal knowledge of probability, relying on their informal natural language expressed in daily life situations. For instance, progressing from the natural expression of "*it is most likely to rain tomorrow*" to the formal one of "*the probability of rain tomorrow equals 90% or 0.9*" (PCMI, 2017).

Additionally, teachers' awareness of differences between both languages is needed (Kazima, 2007; Nacarato & Grando, 2014; Paul & Hlanganipai, 2014; Watson, 2006). Sharma (2014) noted that sometimes, the usage of the probability words during formal instruction differs from how these words are practicing in everyday life. For example, while the language of *fairness* in daily life situations reflects rules of equity (unbiased), it indicates the same theoretical chance of an event occurring inside the formal probability education (Sharma, 2014; Watson, 2006). Therefore, the probability language represents one fundamental aspect that should be reinforced within the discussion of mathematics teachers' professional knowledge for teaching probability. This exemplifies a particular matter related to probability, which differs from other areas of mathematics. Besides, it signifies one unique feature of the study framework, which has not been explicitly figured out in the MKT model. That may contribute to competence models' creation, specifically, within the domain of probability.

Based on the above discussion, mathematics teachers' professional knowledge for teaching probability consolidates **KoP** that outlines the heart of teachers' knowledge and indicates their deep understanding of the subject (Shulman, 1986). It crosses with knowledge of the language, knowledge of teaching, and knowledge of students, to construct **KoPL**, **KoTP**, and **KoSPK**, respectively. That is represented in Figure 9.

According to Figure 9, the exhibited interplay among the four aspects of teachers' knowledge can overcome two reservations regarding the MKT model representation. The first reservation denotes using the term PCK that did not appear as an appropriate name to identify the right side of the MKT framework. As Hurrell (2013) reported, "Perhaps Pedagogical

Knowledge (PK) may have been a better term to employ as there is a strong argument to be stated that PCK only occurs at the overlap between the SMK and PK" (p. 59). The second reservation, which is closely connected with the first one, implies that interactions among knowledge domains were not displayed. More precisely, the SMK and PCK, which are intimately associated with each other; as Marks (1990) described, such an obstacle faces any attempt to categorize teachers' knowledge, wherein ambiguities between content knowledge and PCK always exist. Considering that, the **KoP** component in the proposed framework was defined to exemplify the core of teachers' professional knowledge, instead of being a distinct aspect by itself.

Additionally, such **KoP**, which resembles SMK in the MKT model, intersects knowledge of the language, teaching, and students, to define **KoPL**, **KoTP**, and **KoSPK**, respectively, as stated earlier. Perhaps this interpretation acknowledges Shulman's original idea about the PCK that should embody the intersection between content knowledge and pedagogy (e.g., Lowery, 2002; Marks,1990; Niess, 2005). That is, PCK is a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p. 8). Furthermore, it appears during the stage of transformation when teachers represent the subject in various forms that their students can understand, which meets what Dewey (1964) declared that separating content from method distorts teachers' knowledge. This was also reported in some previous studies on probability. For instance, Brijlall (2014) explored PCK for teaching probability in the South African context; accordingly, a strong relationship between teachers' content knowledge and their teaching practices was exposed.

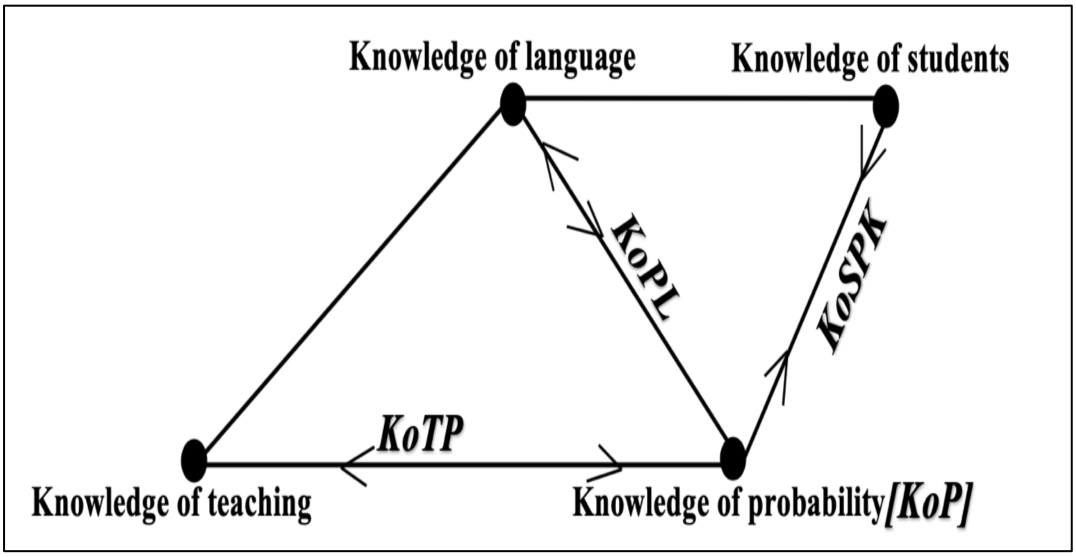


Figure 9. Initial entities of the study framework, building upon a literature review on probability education research

In that regard, it is reliable to remark that beyond ICOTS' papers, several other studies were conducted to investigate teachers' knowledge for teaching probability; it adopted the MKT model to approach this area. For example, in the case of in-service teachers, Danişman and Tanişli (2017) explored secondary school teachers' PCK of probability. As a result, the study revealed that their knowledge was insufficient; besides, teachers' beliefs were identified as the most influential factors affecting their PCK. Similarly, Brijlall (2014) conducted a case study research to explore the PCK for South African middle school teachers. Accordingly, the study reported that the Specialized Content Knowledge (SCK) of probability includes knowledge of "1) definition of probability, 2) recall of the probability scale and attaching meaning to a number on this scale, 3) thorough understanding of fractions and their operations and 4) translating correct English vocabulary usage into mathematical notions relevant to probability tasks" (Brijlall, 2014, p. 725). Additionally, Chick and Baker (2005) detailed multiple issues about the content knowledge and PCK for two teachers who taught probability lessons to fifth-grade students. Hence, they highlighted that the probability concepts embedded in the curriculum and further appeared during the implementation, should be understood by the teachers themselves.

Also, a similar trend of research concentrated around PSMTs' knowledge for teaching probability. For instance, Birel (2017) examined PSMTs' SMK defined by procedural and conceptual knowledge of basic probability concepts. Accordingly, the results proved that although PSMTs showed a high achievement in procedural knowledge, most of them had difficulties to solve questions that required conceptual knowledge. From this aspect, the study recommended much more research is needed to explore why PSMTs' conceptual knowledge of probability was less developed compared to their procedural knowledge. While Birel's (2017) study was conducted in the Turkish context, Contreras et al. (2011) assessed prospective Spanish primary school teachers' common content knowledge and SCK of probability. Repeatedly, the results confirmed the inadequate knowledge of elementary probability and the need to strengthen preparation of prospective teachers to teach probability.

One critical point of such previous studies is that they centered on assessing teachers' practical knowledge and merely described it as insufficient or inadequate. More precisely, the majority of these investigations *neither regarded teachers' reasoning processes nor their cognitive biases that underpin such insufficient knowledge or practices*. Perhaps that is because of the general tendency concerning teachers' knowledge research to strengthen the content knowledge more than process knowledge. Apart from that is the study of Torres et al. (2016) that utilized the MKT framework; yet, it has also addressed PSMTs' biases.

In detail, Torres et al. (2016) developed a questionnaire to assess prospective primary school teachers' content knowledge of probability. As a result, the study has not only described their knowledge as insufficient but also reported that a "high proportion of the participants demonstrated poor combinatorial reasoning, made errors in computing conditional probability and in interpreting frequentist probabilities, and evidenced use of common heuristics and biases" (Torres et al., 2016, p. 210). Despite that, again, the researchers strengthened evaluation issues, which was detailed in their conclusion as follows: "the study shows that both advanced and specialized content knowledge need to be strengthened among prospective primary school teachers. In particular, specialized content knowledge related to probability was very low among the participants" (Ibid., p. 211).

After all, and as reported in several sections, besides ignoring PSMTs' reasoning processes and cognitive biases embedded in such reasoning, issues related to subjective probability, specifically, have not taken enough attention. That pulls us back to the direction of future research that calls for connecting mathematics education perspective on probability with its roots of psychological research. Because of such concerns, the study framework has not only relied on what was raised in the previous studies (i.e., the first employed step to develop the framework), but it also attempted to express a new angle that may exhibit the psychological facet of teachers' knowledge for teaching probability, which is represented by their reasoning processes and conceptions. These ideas are further detailed in the next section; it endeavors to discuss the study premises (i.e., the second step to fix the framework).

4.3.3 The study premises

As stated before, to develop the study framework, these assumptions were acknowledged:

- **Premise 1: Conceptions represent knowledge in evolution**

Conception is knowledge produced by the interaction between an individual and his/her milieu (Brousseau, 1998; Gras & Totohasina, 1995). It is formed based on individuals' personal experiences. Hence, the conception signifies a mental filter to interpret a situation, for making sense of it (Giordan & Pellaud, 2004). As Piaget (1974) noted, these conceptions are the answers to regulations developed by a person to balance his/her cognitive structures at the time of the adaptation.

Although conceptions are valid in certain circumstances, they cannot be generalized across all contexts. For such reason, the term misconceptions may be unsuitable to describe individuals' conceptions because they still work for them under some conditions. These conceptions were defined by Giordan (1998) as operating or inefficiency, rather than being correct or incorrect. Thus, Moreno and Cardeñoso (2014) preferred to use the term *mental models* instead of either conception or misconceptions to reflect on how individuals perceive the world to maintain their cognitive systems free of contradictions. They also explained that such models might incorporate irrelevant, inaccurate, or conflicting elements, but they must be functional. Similalry, Savard (2014) used the term *alternative conception*; it indicated the validity of a particular conception in some contexts and its inadequacy outside the domain of its validity. These *alternative conceptions* were not seen as a deviation of norms or rationality, nor as illogic, but rather knowledge in evolution (alternative understandings). Hence, to clarify the relationship among conceptions, conceptual understanding, and reasoning, it is not possible to pretend that a specific type of conceptions might exactly explain a certain level of understanding because classifying these conceptions as levels of conceptual understanding does not recognize the value of individuals' reasoning to make sense of phenomena (Savard, 2014). From this aspect, utilizing the PoPR would support admitting learners' various conceptions. That is crucial since individuals' world is full of diverging personal probabilistic conceptions (Kapadia & Borovcnik, 2010). Besides, these conceptions also signify a necessary component for the process of knowledge construction (Smith, diSessa, & Roschelle, 1993).

Perhaps that is one implicit reason under Shaughnessy's (1992) indication of thinking processes (instead of conceptual understanding) when he claimed that PSMTs lack the opportunity to develop their stochastic thinking during the university preparation. In agreement with Birel (2017), who reported that the offered courses for the PSMTs are recipe-like or rule-bound, they only deal with calculations, work in the direction of memorizing the subject, and underestimating the logic behind it; also, the conceptual knowledge of probability was less developed compared to procedural knowledge.

Although the elementary probability is often determined through limited techniques, several deep conceptual issues (e.g., variation, randomness, fairness) stay essential to investigate (Chick & Baker, 2005). In this view, the complexity of probability conceptual understanding remains a fundamental obstacle for developing teachers' knowledge. Such complexity is originated from counterintuitive issues in probability; as reported by Borovcnik and Peard (1996), the counterintuitive results in probability are found even at very elementary levels, while they are encountered in other branches of mathematics when students work at a high degree of abstraction. These distinctive traits of probability explain why many conceptions and learning difficulties persist up to the university level (Batanero & Sanchez, 2005; Fischbein et al., 1991; Fischbein & Schnarch, 1997; Kapadia & Borovcnik, 1991; Konold et al., 1993; Shaughnessy, 1992; Stohl, 2005; Torres & Contreras, 2014).

Additionally, about the probability conceptual knowledge, it is valuable to note that such relevance of probability to daily life experiences has provoked a trend of research that recognizes socio-cultural influence on learners’ conceptions of probability. For example, Amir and Williams (1999) concluded that students' cultural experiences impact their probability knowledge, in which some of them reveal superstitions of attributing random events to God. Similarly, Chassapis and Chatzivasileiou (2008) reported the influence of religious beliefs and social values on students' conceptions of chance and probability, which may confirm or contradict mathematics education. Such studies acknowledge that students come to classrooms with previously formed beliefs and knowledge of probability (Fischbein, 1987). This is consistent with what Konold (1991) argued regarding students’ construction of knowledge; the acquired knowledge is incorporated in their existing knowledge fabric. Notably, what students learn from the classroom experiences remains limited and is probably shaped by what they already know; accordingly, the acquired concepts are not freely formulated, but rather, they are subjected to restrictions of the existing concept-relations (Ibid.).

- **Premise 2: Reasoning defines an individual cognitive process to interpret the acquired knowledge**

Generally speaking, and from the perspective of teachers’ knowledge, it is meaningful to note that across all teaching practices (e.g., figuring out what students know, manipulating representations, modifying textbooks), teachers’ reasoning is always involved (Ball, Lubienski, & Mewborn, 2001). Such argumentation stays significant for the probability instruction in which psychological interpretation feels at home (Van Dooren, 2014). Accordingly, and about teaching probability, Kapadia and Borovcnik (2010) regarded the time to replace Heitele’s (1975) ideas, which resemble probability textbooks’ chapters, with an approach that looks at concepts from a non-mathematical perspective, to overcome such distinct characters of the probability teaching, wherein it is not always sensible to seek a closed solution as expected in mathematics. This non-mathematical perspective is displayed within this study by probabilistic reasoning, which has a cognitive psychological nature and focuses on how the mind works.

Indeed, various researchers have interpreted probabilistic reasoning to be the essential goal that underpins learning probability, in which it should be the ground for all educational practices. For example, Gürbüz (2006) and Batanero et al. (2016) described that probability provides an important reasoning mode on its own that contributes to the development of students' mathematical reasoning; it is not just a precursor of inferential statistics. Furthermore, probabilistic reasoning signifies one primary reason for why probability is involved in the school curriculum. As Borovcnik and Peard (1996) stated, the study of probability sustains the creation of probabilistic reasoning that supports learners to formally structure their vague thinking about random phenomena. Additionally, because of the growing number of events described in terms of risk, relevant concepts to reason under uncertainty must be learned in school, and its understanding should also be investigated (Martignon, 2014; Pange & Talbot, 2003). This meets the need to overcome individuals' deterministic thinking and accept the existence of chance in nature (Batanero et al., 2016).

Another critical issue for why probabilistic reasoning is appreciated in this study is the duality of the probability concept of which it has statistical and subjective facets (Carranza & Kuzniak, 2008; Hacking, 1975). In that sense, the conventional approach to address teachers' knowledge, which focuses on leveling their conceptual understanding through paying more attention to the statistical side, may remain unsuitable to employ; because the subjectivity itself is one plausible approach to interpret a probabilistic situation. That is well described by Brase et al. (2014) in which "having two different conceptions of probability can lead to two people having different answers to the same question yet both believing they are rational and correct" (p. 162). Moreover, strengthening the statistical facet, which reduces teaching probability to formula-based computational procedures with few instances of real applications, as a unique basis to judge a probabilistic phenomenon deepens the gap between both facets (Batanero & Díaz, 2010; Carranza & Kuzniak, 2008).

The aforementioned argumentation exposes the significance of the PoPR to conceptualize PSMTs' knowledge for teaching probability. It explores their conceptions and cognitive biases that should not be ignored; particularly, if they are not objectively acceptable, they must be eliminated, and alternative representations must be developed instead (Fischbein & Gazit, 1984). In other words, because intuitions about probability could impede its learning, it is crucial to investigate learners' reasoning and biases (Chiesi & Primi, 2009), which explains Sharma's (2016) recommendation of grounding the instruction in experiences that help learners overcome their misconceptions and develop an understanding based on probabilistic reasoning.

That is valuable to teacher education because the world of personal intuitions signifies a source of success or failure of teaching, then, conceptualizing PSMTs' knowledge for teaching probability from the PoPR advocates clarifying whether they accept (or ignore) what they learned (Kapadia & Borovcnik, 2010). It also stimulates their awareness of probability conceptions, which helps them assess these misconceptions later in their students (Batanero et al., 2010). Ultimately, it impacts pupils' reasoning wherein "the success of any probability curriculum for developing students' probabilistic reasoning depends greatly on teachers' understanding of probability" (Stohl, 2005, p. 345). Moreover, as highlighted before, employing the PoPR meets the need for further research that exhibits the psychological perspective on probability and connects it with the mathematics education perspective. Such a connection indicates that the defined conceptions and biases can be operated, later, as a foundation to reform PSMTs' pedagogical preparation. The following paragraphs summarize the properties of probabilistic reasoning.

Probabilistic reasoning implies judgments and decision-making under uncertainty (Falk & Konold, 1992); it considers two concepts of variability and randomness (Chick & Baker, 2005).

Variability locates at the heart of statistics, and it designates why it is so difficult to make decisions under uncertainty (Garfield & Ben-Zvi, 2005; Pfannkuch & Wild, 2004). Moreover, to construct a deep understanding of variability, learners have to acquire multiple ideas, which were outlined by Garfield and Ben-Zvi (2005). From these ideas, learners have to (a) recognize that variability is everywhere (i.e., the omnipresence of variability; Moore, 1997), (b) explain the different reasons and sources for such variability, and (c) use variability to predict random samples or outcomes, which is relatively linked to probability. In other words, "There is variability in outcomes of chance events. We can predict and describe the variability for random variables" (Garfield & Ben-Zvi, 2005, p. 95). Hence, about outcomes of a random experiment, variability indicates that the outcome is not determined; it varies depending upon favorable cases (theoretical probability), frequencies (experimental probability), or some evaluation criteria (subjective probability).

Additionally, randomness includes uncertainty and independence; while the former reflects that the outcome cannot be predicted with certainty, the latter indicates no correlation between what happened before and the new outcome (Green, 1993; Sari & Hermanto, 2017; Savard, 2014). On the relationship between randomness and probability, Batanero (2015) has reflected on Hacking's (1975) argument about the two complementary views of probability (i.e., epistemic and statistical) to describe multiple perspectives on random events; accordingly, she detailed three different perspectives. First, Randomness as Equiprobability, at which an event denotes random if it has the same probability to occur as any other event in the experiment. This conception of randomness is related to theoretical probability wherein all the possible outcomes are assumed to be equiprobable. Second, Randomness as Stability of Frequencies, at which an event considers random if "we could select it through a method providing a given a priori relative frequency in the long run to each member of this class" (Batanero, 2015, p. 6). This is connected with the experimental probability interpretation, wherein it is necessary to ensure that the successive trials are independent. Third, Subjective

View of Randomness, wherein randomness defines a subjective judgment at which what is random to one person might be non-random for another.

Accordingly, probabilistic reasoning differs from deterministic reasoning that (a) leads to look at one definitive answer, and (b) seeks for a correlation using present and past information to explain a phenomenon, where the dependency or causality still exist (Savard, 2010, 2014; Shaughnessy, 1992). On the contrary, in a probabilistic situation, (a) there is more than one possible outcome, (b) the occurrence of an exact outcome is unpredictable, and (c) the sequence of obtained results lacks a pattern; it cannot be controlled or predicted, and the only thing to be done is to critically choose the event most likely to occur (Batanero, Green, & Serrano, 1998; Tsakiridou & Vavyla, 2015). This meets Borovcnik and Peard's (1996) differentiation between the probabilistic and logical reasoning that designates a true or false proposition. Nevertheless, we have no complete certitude concerning a random event in the case of probability.

- **Premise 3: The hypothetical relationship between conceptions and reasoning**

In light of the previously reported premises, the relationship between the individual's probabilistic reasoning and his/her conceptions was interpreted within the context of this study as follows: Depending on the way we reason in an uncertain situation that contains probability knowledge (theoretical constructs), our conceptions can be clarified. Some researchers implicitly declared such a connection; that is, probability conceptions are rooted in various epistemologies, those epistemologies themselves are underlined by the reasoning employed to think about probabilistic phenomena. For example, Konold (1989) noted that reasoning about uncertainty involves two types of cognition: formal knowledge of probability and intuitive assessments (heuristics). Later, these types were redefined by Savard (2014) as probabilistic versus deterministic reasoning, and she used them to classify the commonly described conceptions of probability in the literature, as displayed in Figure 10.

Based on that, and under the umbrella of probabilistic reasoning, describing such conceptions and biases is essential for the current investigation. Hence, several studies in both fields of cognitive psychology and mathematics education were reviewed (e.g., Amir & Williams, 1999; Batanero & Sanchez, 2005; Díaz, Batanero, & Contreras, 2010; Díaz & de la Fuente, 2007; Díaz & Batanero, 2008, 2009; Dollard, 2011; Falk,1986; Garfield & Ben-Zvi, 2005; Green, 1983; Kazak & Pratt, 2017; Konold, 1989; Lysoe, 2008; Nicolson, 2005; Savard, 2014; Tversky & Kahneman, 1974; Watson & Moritz, 2003). Accordingly, characteristics of (a) the individual who holds such conceptions (misconceptions, heuristics, or biases) and (b) the principal probability interpretations that the individual might rely on to reason in a situation were summarized as in Figure 10 (see the details in Appendix 5). It worked as a lens to interpret PSMTs' responses at which their knowledge for teaching probability could be characterized.

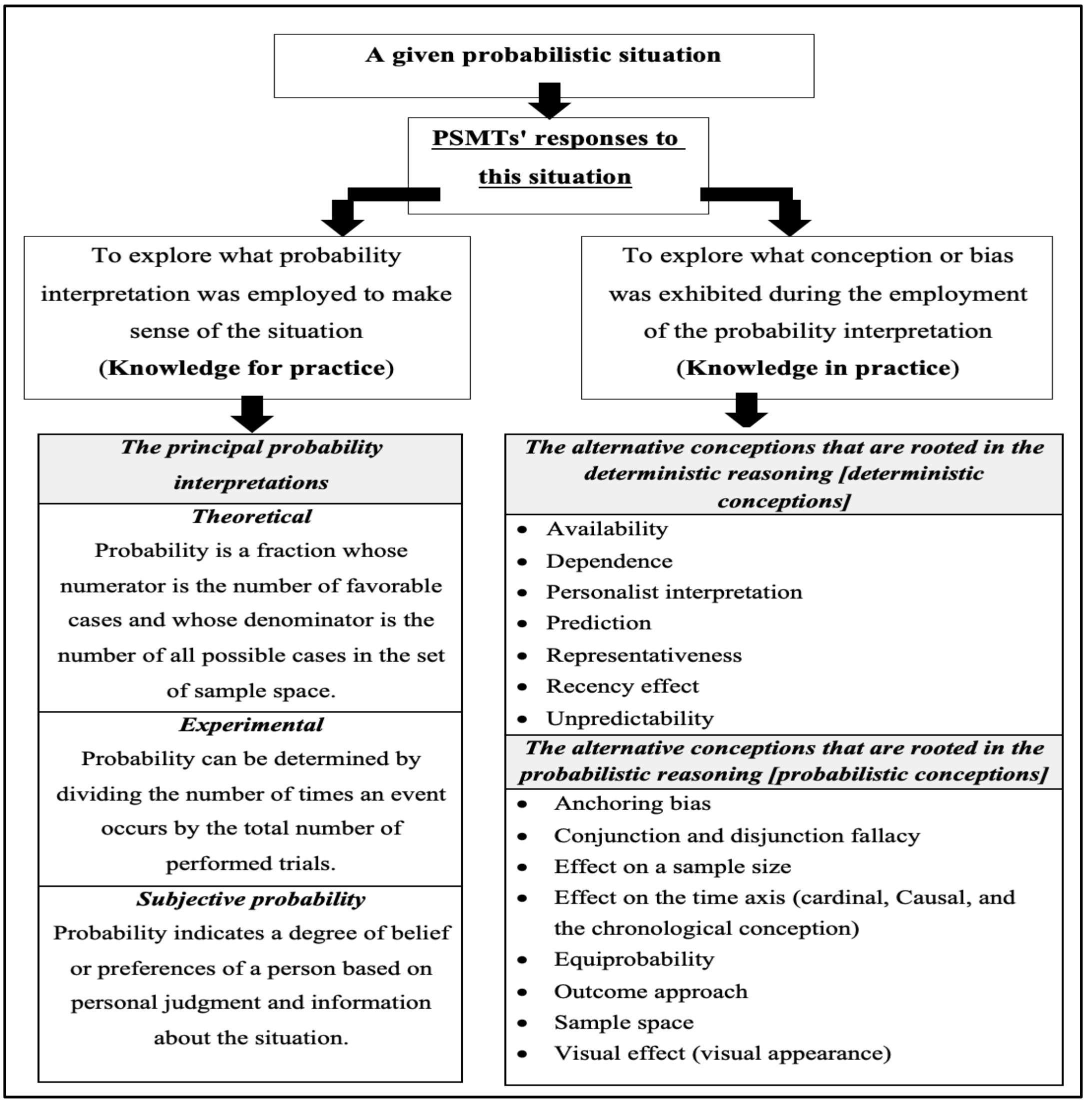


Figure 10. Essentials to characterize PSMTs' knowledge for teaching probability

Indeed, admitting such a relationship not only helped to define the framework, but it may also contribute to the literature through consolidating PSMTs' reasoning and probability conceptions together in one model. Although several studies have shown that adults (including university students) hold various conceptions about probability and relevant biases in reasoning under uncertainty (e.g., Dollard, 2011; Kazak & Pratt, 2017; Konold, 1989), there is no further discussion that connects PSMTs' reasoning with associated probabilistic conceptions in such a way to prototype both in a unified schema. From this aspect, and as stated first, this study acknowledged that *learners' conceptions are underlined by their way of reasoning toward a certain phenomenon* to be an essential hypothesis. In other words, *one way to identify PSMTs' conceptions of probability is to explore how they reason under uncertainty.*

4.3.4 Description of the study framework (skeleton of the study) and definition of its terms

- **Description of the study framework**

In light of the whole preceding discussion that (a) determined the initial entities of the framework, and (b) acknowledged the study premises, the study framework is displayed in Figure 11. It defines mathematics teachers' professional knowledge for teaching probability from the PoPR, which embodies *interrelationships among professional knowledge, conceptions, and reasoning processes.*

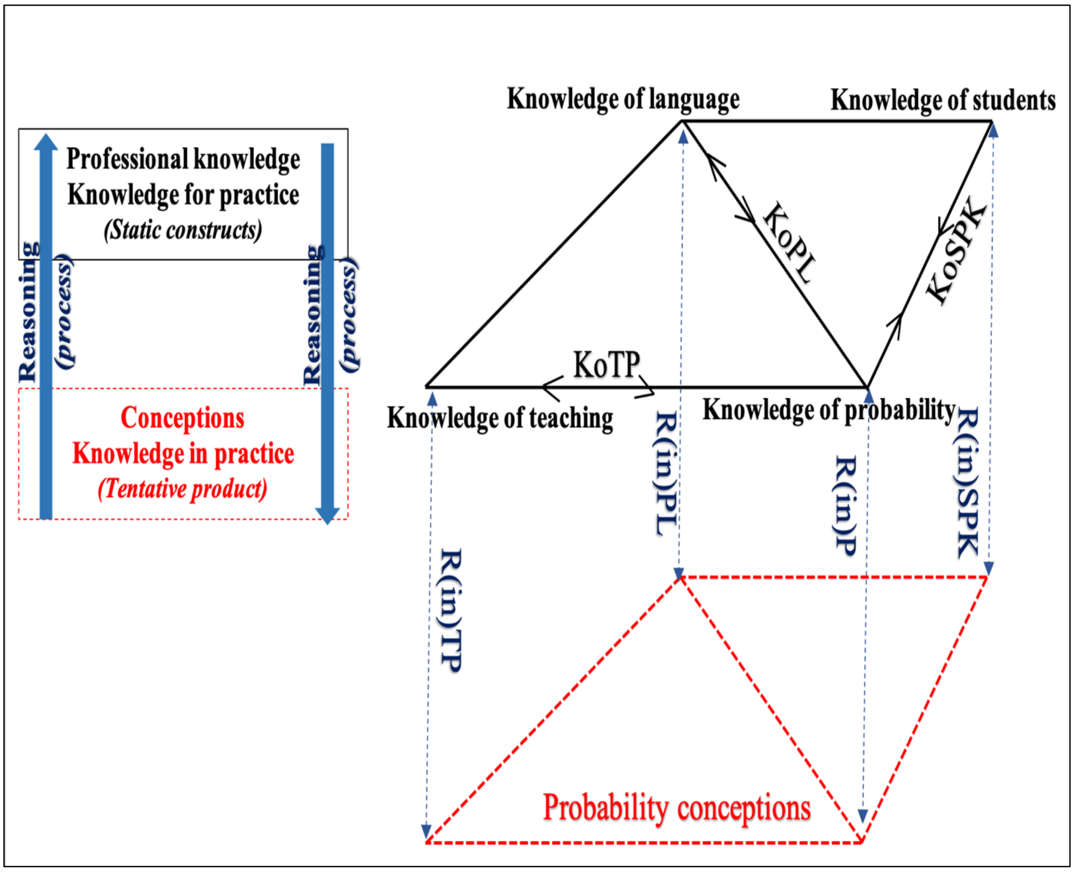


Figure 11. Mathematics teachers' professional knowledge for teaching probability from the perspective of probabilistic reasoning

According to the presented model, mathematics teachers' professional knowledge for teaching probability (*knowledge for practice*), which is acquired through either formal teacher education or professional development training, signifies the static black parallelogram. It consolidates knowledge of probability (**KoP**) that outlines the essence of this parallelogram, which crosses with knowledge of the language, teaching, and students to assemble knowledge of probability language (**KoPL**), knowledge of teaching probability (**KoTP**), and knowledge of students' probability knowledge (**KoSPK**), respectively. Nonetheless, practically, during the actual teaching, each teacher transmits this knowledge through his lens; that is probability conceptions, which are represented by the red parallelogram. This red parallelogram describes teachers' practical knowledge (*knowledge in practice*); it could match the black parallelogram when teachers' conceptions agree with scientific knowledge (theoretical static constructs). Still, there is a gap between how a teacher perceives (then implements) probability knowledge and professional knowledge for teaching probability if his/her conceptions do not fully fulfil the probability theory.

The existence of such a gap reflects teachers' various ways of reasoning under uncertainty. That means after each teacher utilizes his/her own reasoning in a situation that contains standardized probability knowledge (i.e., **KoP**, **KoPL**, **KoTP**, and **KoSPK**), he/she develops a particular distinct type of knowledge (i.e., knowledge in practice, knowledge in evolution, teachers' conceptions of concepts embedded in an instructional activity). In this way, placing the focus on reasoning processes helps to characterize that gap. Alternatively, acknowledging the PoPR may respond to what was raised regarding the needed research that indicates founding probability instruction (the perspective of mathematics education) in its psychological roots. Concretely, it (a) manifests the influence of teachers' reasoning under uncertainty in shaping their probability knowledge (conceptions), and (b) reflects the possibly existed distance between these conceptions and what the educational community recommends mathematics teachers comprehend for teaching probability efficiently. Accordingly, effective instructional interventions can be organized to minimize such a distance.

It is also worthy to perceive that knowledge and conceptions are not a linear relationship that always begins with knowledge. Instead, the opposite direction still exists since such resultant conceptions are not isolated but integrate into a complex system (knowledge system). In other words, new knowledge does not destroy existing knowledge; instead, it will be connected to existing concepts to reorganize and keep the individual's cognitive structure balanced (Savard, 2014; Vosniadou & Verschaffel, 2004).

Finally, regarding the PSMTs (the focus of this study), managing such a discussion in teacher education denotes that PSMTs have to learn probability theory to teach their pupils effectively. In that sense, what they have to recognize and match the probability theory defines professional knowledge for teaching probability. However, that knowledge is not acquired directly; instead, PSMTs employ their reasoning to make sense. Accordingly, they develop various conceptions that may or may not match the probability theory. From this aspect, and through the lens of probabilistic reasoning, PSMTs' knowledge for teaching probability includes these redefined aspects: **R(in)P**, **R(in)PL**, **R(in)TP**, and **R(in)SPK**, which symbolize their reasoning in a situation that involves knowledge of probability, probability language, teaching probability, and students' probability knowledge, respectively (see Figure 11). Despite that, and as previously justified in Chapter 1 (see the study delimitations), the current investigation has sharpened the aspect of **R(in)P** that corresponds to SMK in the MKT model.

- **Definition of the key terms**

Based on the defined framework, to characterize **PSMTs' R(in)P** that incorporates professional knowledge, reasoning, and conceptions, these definitions were set:

– *Professional knowledge for teaching probability*

It (a) indicates what mathematics teachers need to perceive to implement the probability lesson effectively, (b) is defined by four aspects of KoP, KoPL, KoTP, and KoSPL, and (c) is termed by the term *knowledge for practice*.

– *Knowledge of probability (KoP)*

It defines the three primary interpretations of theoretical, experimental, and subjective probability to approach a probabilistic situation, as determined by the educational community.

– *Reasoning under uncertainty*

It (a) defines PSMTs' ways of reasoning when they encounter an authentic probabilistic situation, and (b) to ensure such authenticity, various contexts that draw on the curriculum and pupils' viewpoints and a reflection on a social phenomenon was admitted, as follows:

While the school curriculum reflects one plausible way to stipulate teachers' knowledge (Stylianides & Ball, 2004), the reflection on a realistic probabilistic situation denotes an essential idea that supports this study; it helped to exhibit the psychological facet of PSMTs' probabilistic reasoning. As explained earlier, emphasizing the objective side of probability and disregarding the subjective side (roots of mathematical probability, which is manifested in most individuals' conceptions) implies a critical point regarding probability education. This is discussed by many researchers, for example:

Generally speaking, for statistics education, Gal (2005) advocated that real-world situations should be considered for teaching and assessing statistical knowledge. This is also emphasized in the GAISE college report in which integration of real data with a context is recommended in the introductory statistics courses for college students (GAISE, 2016). More precisely, in the case of probability, it provides a tool to link mathematics with the real world through modeling random situations (Borovcnik, 2008; Chaput et al., 2011). Nevertheless, Theis and Savard (2010) declared that teaching probability rarely builds upon authentic contexts and predominantly uses a theoretical approach. This is exemplified in textbooks and curriculum documents, which sometimes perform a too narrow conceptual view of probability;

besides, many applications are limited to games of chance and are not based on real data. Such fact is apparent, specifically, in the Egyptian context at which the statistical side of probability is stressed compared to the epistemic side. As detailed in Chapter 3, most of the probabilistic tasks in the Egyptian school curriculum were like tossing coins or rolling dice; they have well-defined quantifiable sample spaces where only the classical and frequentists interpretations can be manipulated.

Such conventional tasks do not provide an adequate basis for understanding subjective probabilities (Stohl, 2005); they also lead to believing that probability indicates empirical properties of a situation, rather than a measure of our knowledge of outcomes (Devlin, 2014). More particularly, in teacher education, Musch and Ehrenberg (2002) asserted that types of learning activities proposed for teachers during their preparation are generally stereotyped; it brings the concept of probability to the notion of calculating the relative frequencies. Besides, probability instruction is rarely based on exploiting authentic circumstances, leaving the field open for erroneous reasoning regarding daily life situations that affect school success and daily practices. Consequently, Larose et al. (2010) argued that only the perspective of real practices would make it possible to deal with the gap between erroneous conceptions about school education of probability and the individuals' implicit unrealistic theories (e.g., controllability of chance). This is also consistent with what Grenon et al. (2010) reported regarding the consequences of teaching probability from a purely mathematical viewpoint (relative frequency), in which they have recommended contextualizing probability education by drawing on pupils' daily social practices. For such reasons, this study implemented some real probabilistic situations to exhibit the subjective probability interpretation; it represents PSMTs' beliefs and previous experiences about some relevant probabilistic contexts.

- *Probability conceptions*

It (a) refers to what PSMTs understand about the three primary interpretations of theoretical, experimental, and subjective probability, (b) is characterized by the way how they reason in a probabilistic situation, and (c) is coined by the term *knowledge in practice.*

In this study, mathematics teachers' conceptions of probability were termed by *knowledge in practice (practical knowledge)*; it designates PSMTs' various perspectives on knowledge for and about teaching (Cochran-Smith, 1999), and is mostly related to classroom practices when teachers face dilemmas and strive to achieve educational purposes (Carter, 1990). As Fenstermacher (1994) reported, such *practical knowledge* describes what teachers know from their experience, and it differs from theoretical research-based knowledge. Nonetheless, since the current study focuses on PSMTs, *knowledge in practice* indicates what PSMTs perceive upon their individual reasoning in a given situation; furthermore, these situations mirror what they will teach their pupils. In that view, describing PSMTs' conceptions as *knowledge in practice* that implies an association between their conceptions and pedagogical practices was reflected in some previous studies. For instance, Ives (2007) revealed a relationship between pre-service teachers' understanding of randomness and probability conceptions, both of which influence their pedagogical decisions (e.g., interpreting students' works and answers).

Accordingly, and to summarize before proceeding to the next chapter, the answer to the third research question, aimed at characterizing PSMTs’ knowledge for teaching probability in Egypt from the PoPR, indicates exploring their R(in)P that denotes one essential aspect of such knowledge. Furthermore, R(in)P includes three interrelated features of (a) the way PSMTs reason in an authentic probabilistic situation (simple unconditional and conditional), (b) the theoretical constructs and probability theory they rely on to interpret such a situation, and (c) the conceptions and cognitive biases embedded in their reasoning.

CHAPTER 5: RESULTS AND DISCUSSION OF THE FIELD SURVEY

This chapter outlines the characteristics of PSMTs' knowledge for teaching probability in Egypt from the PoPR; precisely, their R(in)P that denotes one essential aspect of such knowledge. However, some methodological details are presented first, and the ethical considerations are discussed in the end. That answered the third research question.

5.1 Details of participants, tools, and processes of data collection and analysis

Considering what was stated in Chapter 2 about the processes employed to answer the third research question, more details are first presented in this section before progressing to the results.

- **Participants** and what they learned during the preparation program

As mentioned in Chapter 2, the answer to the third research question depends primarily on interpreting the responses of 68 PSMTs in the mathematics teachers' preparation program at the Faculty of Education, Tanta University, during the academic year of 2018–2019. They all had prior knowledge regarding theoretical, experimental, and conditional probability. That is, according to the teacher education curriculum, in their second year, PSMTs have to study (a) basic concepts of probability, which include random experiments, sample space, mutually exclusive and exhaustive events, probability of an event, equally likely principle, probability function and axioms, conditional probability, independent events, and the Bayes theorem; (b) random variables, such as discrete and continuous random variables, density function, and mathematical expectation; and (c) probability distributions, such as normal, Bernoulli, binomial, Poisson, gamma, and exponential distributions. Furthermore, during the third and fourth years, PSMTs take advanced courses wherein statistics and probability are combined. For instance, the statistical mechanics course, which is often taught in the fourth year, starts with an extensive review of probability concepts, and then the probability function is discussed before moving to the Maxwell distribution and Kinetic theory of gases.

- **Study questionnaire** used to collect the data

As stated in Chapter 2, in light of the study framework that defines mathematics teachers' professional knowledge for teaching probability from the PoPR, a questionnaire was developed to approach the aspect of R(in)P. This involved [**A**] determining three different probability contexts in which PSMTs' reasoning that may occur in a simple unconditional probabilistic situation can be characterized, and [**B**] adjusting one of these contexts and adding a calculation problem to designate PSMTs' emerged reasoning in a conditional probabilistic situation.

In detail, [**A**] *determining three different probability contexts* involved acknowledging both the curriculum and pupils' viewpoints, as well as adapting a probabilistic social problem.

Regarding the curriculum, although it considered seven different circumstances in which the probability could be utilized, it focused more on treating traditional activities (see Table 18)[14]. Thus, the activity of **throwing a die**, a typical task category, was considered in the study questionnaire to explore PSMTs' R(in)P. Nevertheless, because such traditional activities cannot provide an adequate foundation to explore PSMTs' subjective reasoning, other probability contexts were also defined. This explains why the pupils' viewpoints and the social problem were reflected.

Table 18. Probability contexts within the Egyptian school curriculum

Identified contexts	***Examples***	***Number of activities***		***Total***
		Primary	***Lower secondary***	
Environmental issues	Rain, sun, day and night, and weather forecast	5	0	***5 (4.7%)***
School experiences	Grades, success, and results of a competition	5	4	***9 (8.5%)***
Gender	Boys and girls, and giving birth	4	1	***5 (4.7%)***
Life expectancy	Life expectancy and insurance concerns	1	2	***3 (2.8%)***
Preferences	Family visits, preferable food, language, sport, newspaper, and transportation	4	5	***9 (8.5%)***
Manufacturing	Production and feasibility study	1	8	***9 (8.5%)***
Conventional activities	Draw a ball, toss a coin, throw a pin, roll a die, spin a spinner, draw a card of two-digit numbers, and throw a stick	40	26	***66 (62.3%)***
Total		***60***	***46***	***106***

To define Egyptian pupils' viewpoints, as explained before in Chapter 2, they were asked to determine which probability setting is more applicable to daily life. This was done through a survey (see Appendix 6), which provided the probability contexts that were inferred from the curriculum, and asked pupils to select three contexts and prioritize them. For example, if a pupil arranged three situations as in Table 19, it means, for him, the probability was frequently

[14] Interestingly, the identified probability contexts in the Egyptian curriculum resemble the seven specific events (i.e., success in school exams, rainy weather, ace in throwing a die, win in football, head in tossing a coin, beginning of war, and road accident) that were utilized by Chassapis and Chatzivasileiou (2008) to explore children's conceptions of probability.

used to explain weather conditions. Then, predicting the gender signified the second context to employ the probability, while operating games of chance indicates the third situation in which the probability was involved. In this way, frequencies of the pupils' choices were calculated by assigning the values of 3, 2, and 1 to the first, second, and third choices, respectively; the results are displayed in Figure 12. These results reveal that environmental concerns are the most relevant context of probability to everyday situations from the pupils' viewpoint. *Thus, the task of weather predictability was also considered to explore PSMTs' R(in)P.*

Table 19. Example of a pupil's choices

The situation	*An example*	*Pupil's choices*
1. To predict weather conditions	It is most probable that it will rain tomorrow	1
2. To predict the result of a handball match for your school team	……………. See appendix 6	
3. To predict the gender of a newborn baby	The probability of giving birth to a girl equals 50%	2
4. To express the condition of a sick person	……………. See appendix 6	
5. To express what we prefer	……………. See appendix 6	
6. To predict the quality of some products	……………. See appendix 6	
7. To predict the winner in chance games	The probability of getting number 4 when throwing a die equals 33%.	3

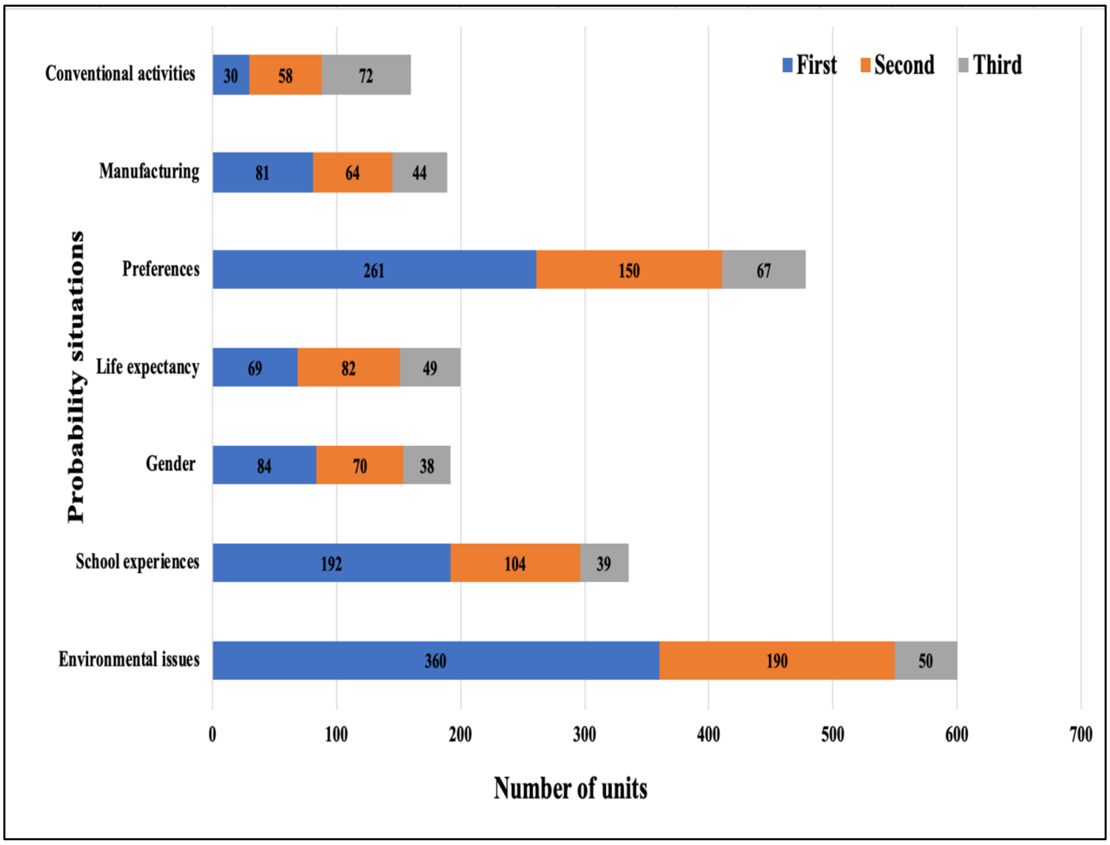


Figure 12. The priority of probability contexts from pupils' viewpoint

Moreover, to accommodate different levels of difficulties, *the problem of giving birth was considered.* It defines a probabilistic social problem wherein the sample space is often obscure, and repeatability is hard to imagine (Nisbett et al., 1983). Furthermore, when PSMTs were asked through a survey (see Appendix 7) to select which among the curriculum's contexts is suitable to approach each theoretical, experimental, and conditional probability, the gender context was highlighted. Although PSMTs judged that all settings could be employed to address multiple probability concepts, the context of gender stayed as the one wherein (a) the conditional probability was strongly manifested and (b) different concepts could be approached through it, since balanced choices across the three probability concepts appeared (around 46%, 32%, and 22% for theoretical, experimental, and conditional probability, respectively), as explicit in Figure 13.

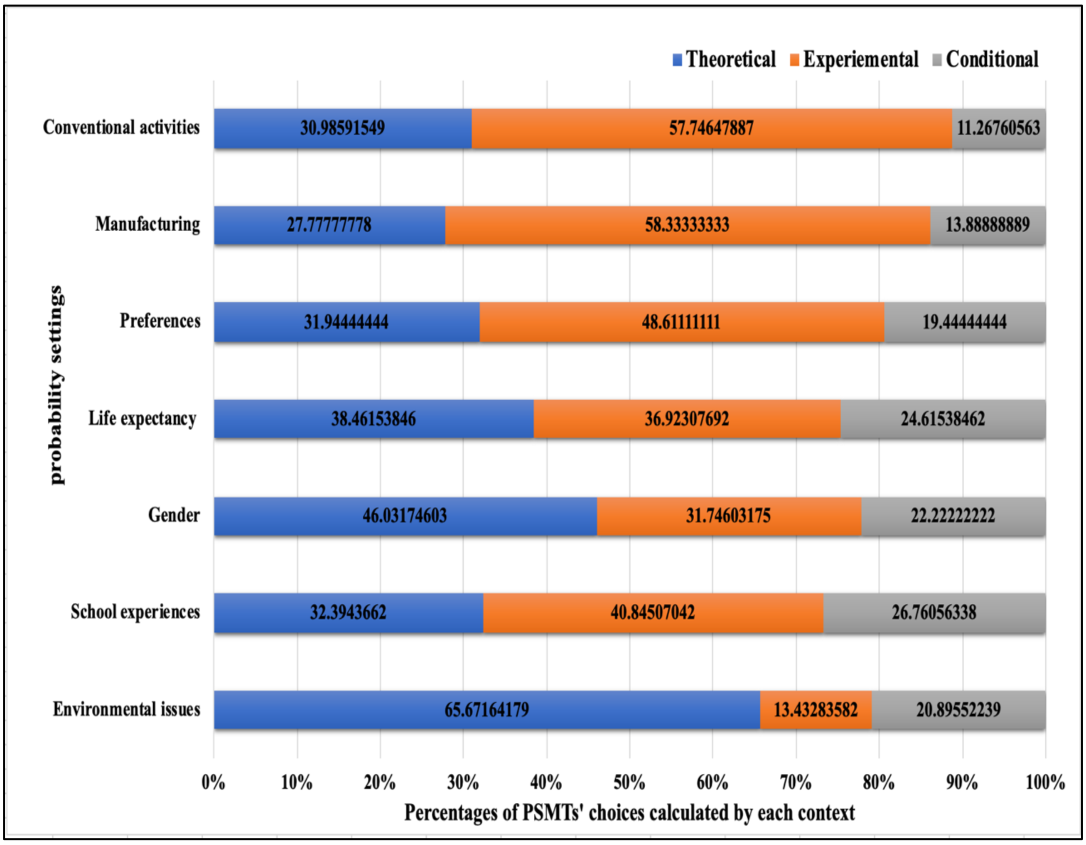


Figure 13. PSMTs' determination of the probability context

Accordingly, to characterize PSMTs' reasoning that may occur in a simple unconditional probabilistic situation, *the three tasks of giving birth (Item **A**), throwing a die (Item **C**),* and *weather predictability (Item **D**)* were adapted. These items varied not only from a contextual viewpoint but also based on what PSMTs were required to do. Although PSMTs were asked to estimate the probability (determine a percentage) in both problems of giving birth and throwing a die, they had to interpret a given numerical estimation for the task of weather predictability.

Nonetheless, PSMTs were informed that numerical answers are of less concern than their reasoning that leads to these answers.

Regarding [**B**], which involves *adjusting one of the previously identified contexts and adding a calculation problem*, the aim is to characterize PSMTs' reasoning in a conditional probabilistic situation, and based on that reasoning, the notion of subjective probability, which considers a vague area in mathematics education, can be clarified.

For doing so, Chernoff's (2008, 2014) argument about Bayesian inference was considered and applied in this study. As stated before (see Chapter 4), the term "subjective probability" constitutes multiple terms (e.g., intuitive, personal, epistemic, and beliefs), all applied as descriptors for Bayesian probability (Chernoff, 2008; Chernoff & Russell, 2014). Moreover, based on Gillies' (2000) viewpoint, Chernoff (2008) claimed that subjective probability has a dual meaning, that is, it can be manipulated as *a general classifier* or *a specific theory*. Nevertheless, in mathematics education, subjective probability is often discussed as a specific theory wherein "an almost complete consensus and agreement exists about the mathematics" (Gillies, 2000, p. 1). In contrast, the definition of subjective probability as *a general classifier*, wherein various philosophical differences may appear, is usually ignored. ("there is a wide divergence of opinions about the philosophy" (ibid.). In this respect, to further clarify the notion of subjective probability, three more items were considered with the previously reported tasks of giving birth, throwing a die, and weather predictability.

As explained in Chapter 2, one item (*Item **B***) relied on the context of gender but with a little modification that includes adding one condition to the problem of giving birth to explore how PSMTs may interpret such a condition. That is, fundamentally, how they do reflect on their prior judgments (probability estimation) to incorporate the newly provided information? The other two items (*Items **E1** and **E2***) approached the subjective probability from a narrow sense that is represented by calculating the conditional probability from a two-way table; it displays the frequency distribution in a population or sample that is classified according to two categorical variables (Contreras et al., 2011; Watson, 2011). From this discussion and as reported in Chapter 4, it can be understood that this study approached *subjective probability as a general classifier* by characterizing how PSMTs reason in two conditional probabilistic contexts. The first is a social descriptive context that pays more attention to their argumentations (*Item **B***), while the other is a more mathematical context that focuses on their calculations (*Items **E1** and **E2***).

In that regard, it is valuable to note that the variability of the study's questionnaire items mirrors what Watson (2005) argued about *probability in context*. As she reported, on one hand, the school curriculum mostly addresses probability as a part of pure mathematics in which the presented examples are based on finite sample spaces. Hence, such questions that involve dice, coins, or cards consider one approach to explore learners' understanding (mathematics education perspective). On the other hand, another possible approach to interpret this understanding is inspired by the early research on probability by psychologists (e.g., Tversky & Kahneman, 1974); it focuses on descriptive social settings wherein the relevant questions are neither necessarily numerical nor require calculations (psychological perspective). Moreover, Watson (2005) advocated for exploring students' understanding in both contexts during classroom practices and further in the research: one context includes explicitly defined sample spaces, while the other is relevant to the obscure sample spaces of social contexts.

Finally, based on the details reported in steps **[A]** and **[B]**, the study questionnaire was constructed (for its contents, see Appendix 8 and the summary in Table 6).

- **<u>Data collection and analysis procedures</u>**

As summarized in Chapter 2, these procedures included [**A**] the first stage of initial arrangement and preparation, [**B**] the second stage of implementing the study questionnaire, and [**C**] the third stage of data coding and analysis. These stages are detailed as follows:

– [**A**] The stage of the initial arrangement and preparation

This stage took two weeks. During the first week, the researcher interviewed three PSMT groups who studied in the second, third, and fourth years of the preparation program. First, there was a general discussion regarding which statistics and probability courses they study at the Faculty of Science and how these courses differ from the school curriculum. Accordingly, almost all participants expressed the disconnection between their pedagogical preparation in the Faculty of Education and the academic one that defined what they learned in the Faculty of Science. They also criticized how the general pedagogies, which they learn at the Faculty of Education, cannot help them in classroom practices, as some experienced during the teaching practicum (see Elbehary, 2019, for details). Later, after PSMTs were asked their willingness to participate in this study, 68 of them agreed to cooperate (see Table 4). In addition, an interview with two teachers was conducted. The interview discussion focused on issues about the intention of the probability context survey, questions anticipated from pupils, the role of the teacher who is going to implement that survey, additional examples that may help pupils understand probability contexts (especially in the early grades of 4 and 5), and, finally, the average time for pupils to answer such a survey. As a result, we agreed to have 45 minutes for each class to get pupils' answers and collect the whole data within one week (see Table 5).

During the second week, a warmup session was conducted with the study participants (68 PSMTs). It involved a specific discussion about probability lessons that pupils learn in each grade. For example, the researcher raised questions about when pupils begin studying probability as the content: Do you think it is an appropriate time to begin learning such content? Why do they have to learn it? If you knew that Egyptian pupils' achievement in *Data and chance* domain is the lowest, according to the international assessment, could you explain why specifically such content has a low achievement? Is the reason related to students, teachers, curriculum, or the content itself? What do you think about the interrelationships among mathematics, statistics, and probability? How can we define probability? Although most of the participants did not recognize precisely when pupils first started to learn probability, they stated that it is often addressed within the units of statistics at the end of the algebra course, which is true. Moreover, after they were told by the researcher that probability must be learned from grade 3, they did not have a clear view on why studying it in these early grades was necessary. Nevertheless, they identified various situations in which probability can be used in everyday discussions (e.g., "I'm not sure about going to school tomorrow," "I have checked the forecast on my mobile app," and "It's going to be sunny today"). Thus, they highlighted how learning probability is valuable, because not only does it define a domain of study but also considers a practiced language in our daily conversations.

Regarding the interrelationships among mathematics, statistics, and probability, PSMTs stated that pupils must understand ratios, rational numbers, and percentages when calculating probability. Nevertheless, they could not identify the association between statistics and probability or express why probability was addressed within the statistics domain. Furthermore, to explain why pupils showed low achievement in *Data and chance*, PSMTs identified two issues: (a) There is less emphasis on learning statistics and probability during the initial preparation of mathematics teachers, which causes a perception that statistics is merely a unit within the mathematics curriculum. (b) The statistics unit is positioned at the end of the entire school curriculum; thus, it is always ignored in light of the limited time for teachers to address all curriculum topics. Lastly, regarding the meaning of probability, almost all PSMTs defined it as the number of favorable outcomes divided by all the possible elements in the sample space. Then, the researcher raised some issues regarding experimental and conditional probabilities, which PSMTs had studied before. Later, they were invited to respond to the probability context survey to determine which circumstances could be applied to handle each probability concept (theoretical, experimental, and conditional).

In addition to the warmup session, the researcher interviewed a small sample of 10 PSMTs and 3 university lecturers. This was intended to validate the study's questionnaire items; consequently, some of these items were rephrased or simplified. For a case about the context of gender, the question "*are there any conditions to determine that probability?*" (see Appendix 8) was not clear to PSMTs. During the interview, they raised several issues about traditional beliefs they encounter in daily situations (e.g., the woman's belly shape); therefore, they asked whether that can be a possible condition to judge the probability. For this, the researcher confirmed that they must state every condition that they may think would affect their judgment, whether it is scientifically accepted or not. Besides, another student argued, "*but in some cases, we do not know because it is a matter of Allah's will.*" Again, the researcher encouraged them to write what they think. Thus, to clarify the intention of that question, the following statement was added to Item **A**: *in other words, explain the reasons because of which you have decided the proposed probabilistic ratio (try to reflect and state the criteria that helped you to judge, or any conditions that you may think may change your estimation)*.

Additionally, in the interview with the lecturers, they recommended rearranging the questionnaire items to ensure that the first part includes questions about the gender context (i.e., problems of giving birth), and the second part covers other traditional probability contexts. Besides, regarding Item **E1**, which aimed to clarify PSMTs' conceptual difficulties in calculating the conditional probability, it was recommended to simplify the given numbers to focus on the procedures more than the mathematical errors as well as to provide a similar question to minimize the sided discussion and cheating among participants. Accordingly, Item **E2** was considered as being interchangeable with **E1**. However, these items were described by PSMTs as being difficult questions, as they declared that both required a revision on how to calculate the conditional probability; thus, they said that "unless we remember the formula, it is hard for us to solve these questions."

- [**B**] The stage of implementing the study questionnaire

As noted in Chapter 2, the questionnaire took two sessions for implementation. Besides, during the implementation, some more questions were raised by the participants. For example, regarding Item **B**, PSMTs asked the following about the second question (*do you think that your expectation in the first situation is the same as in the second one?*): (a) What did the researcher mean by the first situation? (b) Is there a relationship between Items **A** and **B**? (c) Is she the same woman? Accordingly, the answer was yes; the researcher thus answered the following: "When you tried to respond to Item **A**, you did not have any information concerning the pregnant woman. However, in Item **B**, you were informed that this woman gave birth to two boys before (i.e., it is not her first time to deliver a baby). Will you retain your initial judgment? Are you going to change it? What do you think of that, and why in both cases?"

Moreover, regarding Items **E1** and **E2**, fourth-year students asked the researcher to remind them of the formula to calculate the conditional probability; then, the following example was discussed: Suppose that you have a survey of smokers; it includes a sample of 100 girls and 200 boys. There were 25 smokers among the girls compared with 150 among the boys. How can you represent such information through a two-way table? After PSMTs constructed the table correctly, these additional questions were posed: (a) What is the probability that a person is a smoker? (b) What is the population in that case? (c) What is the probability that a girl is a smoker? (d) What is the population in that case? Lastly, they were asked to apply this example for reflecting on and solving the given questions. Accordingly, all PSMTs' responses to the questionnaire items were collected (see Table 8) and prepared for the coding process.

- [**C**] The stage of data coding and analysis

Following the logic of abduction research, the data analysis processes were visualized in Figure 5 (see Chapter 2). These processes involved both inductive and deductive analyses that are detailed in the following discussion.

  - *Inductive coding procedures*

Inductive coding was performed using the NVivo software and following Thomas's (2006) steps to analyze PSMTs' responses to the first part of the questionnaire (i.e., their answers to Items **A** and **B**). These answers included both numerical and textual arguments. Figure 14 exhibits a case of a PSMT's responses to Item **A** questions (see Appendix 8); as shown, the PSMT determined the probability of giving birth to a girl as one-fifth because the sample space had the five elements of {one boy, one girl, twin boys, twin girls, and twin boy and girl}, and the favorable outcome was one event of a girl. Nonetheless, for the second question asking "are there any conditions to determine that probability," the PSMT's response was "the probability might differ if we knew that such a woman always gives birth to girls." Furthermore, the following points detail the process of employing Thomas's (2006) steps to analyze the Item **A** responses.

- *Determine a label for each node that is a short phrase to refer to it.*

Through the NVivo software, several nodes were developed based on PSMTs' presented expressions and phrases. In the initial stage, all the given responses were highlighted to reflect the two cases of (A) there are no specific conditions to determine the probability except the mathematical calculation that we operated, and (B) there are some conditions to be reflected for modifying the probability. The label for case A was "No, there are no conditions," while that for B was "Yes, there are some circumstances."

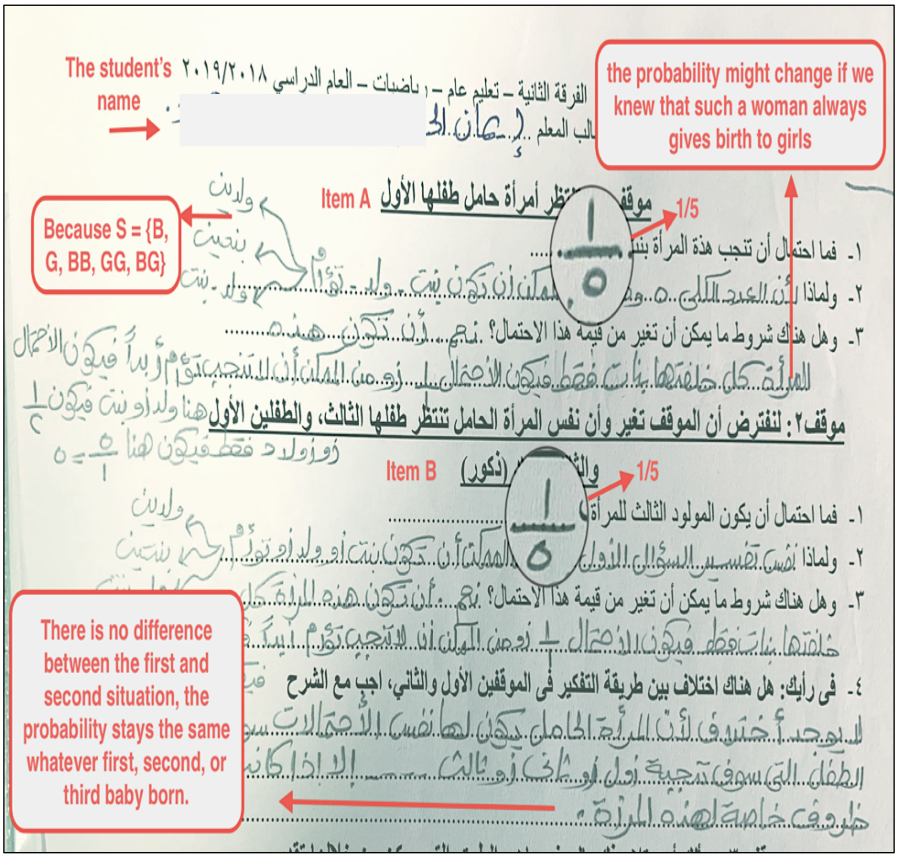


Figure 14. A case of PSMTs' responses to Items A and B

- *Describe the scope of each node.*

Case A represented claims of PSMTs who agreed that there are no restrictions to determine the probability or did not mention any conditions; for them, it was just a matter of mathematical calculation. It also included those who merely calculated the probability theoretically without justifying why such algorithms were performed. On the other hand, if PSMTs acknowledged that some criteria (e.g., sample space, woman's belly shape, and knowing the results of the ultrasound scan) limited their judgment, such criteria were coded under case B. In addition, the coded responses for case B were further separated into b1 and b2, which altered in terms of the nature of the given criteria. While b1 symbolized PSMTs who maintained the mathematical analysis (e.g., it depends on the number of sample space elements, because of the ratio between the sample space elements and favorable outcome), b2 expressed those who declared any non-mathematical reason (e.g., it depends on medical checkup results or Allah's willingness).

- *Illustrate some examples of texts associated with nodes*

For instance, the common response of PSMTs who were assigned to case A was that there are no conditions to decide the probability except the provided formula. On the other hand, for case b1, they answered that it depends on n(S): if S has two elements, then P(G) = 50%, or, if S has three elements, P(G) = 33.3%. Furthermore, for case b2, there were several raised conditions in which the probability may change, such as whether spontaneous abortion was considered as a possible outcome or not, or information regarding the issues of X and Y chromosomes (more examples are presented within the sections of data interpretation).

- *Create links among several nodes*.

During this stage, PSMTs' responses were further revised; and accordingly, two issues were found: First, nodes A and b1 stayed related to each other since all PSMTs who belonged to both relied on the sample space and favorable outcomes to determine the probability. Although the PSMTs in A stated that there are no restrictions, they maintained the mathematical calculation by dividing the number of favorable outcomes [n(G)] by the number of sample space elements [n(S)]. Accordingly, A and b1 were combined and labeled as one category of *mathematically oriented thinkers,* which was identified using the letter **M**.

The second issue was about b2 that characterized PSMTs who admitted several non-mathematical criteria, according to which the probability may change. For b2, it was found that some PSMTs provided these criteria to illustrate what issues could change the probability, while others thought of similar circumstances not to expose the probability, but rather to interpret the favorable outcome. In detail, as reported earlier, the common response for case b2 was that the probability of giving birth to a girl depends on information regarding the issues of X and Y chromosomes or results of the ultrasound scan. Still, such types of responses concentrated around the probability of an event (giving birth to a baby girl)—precisely, the circumstances for which the estimation may change. Nevertheless, among b2, some respondents stated that we could know that this woman is going to give birth to a girl by observing her belly's appearance: if it is rounded, she is expecting to deliver a girl. Similarly, in strengthening the outcome, others declared that if this woman always gives birth to girls, she is more likely to deliver a girl because some women give birth only to girls. Thus, to admit both types of responses, b21 and b22 were developed. They reflected cases of PSMTs who utilized such non-mathematical conditions to find the probability and to anticipate the outcome, respectively.

- *Incorporate the emerged categories into a model.*

Following the emergence of **M**, b21, and b22, the principal interpretations of theoretical, experimental, and subjective probability (see Figure 10) were applied as a framework in which these cases can be consolidated and theorized. For further illustration, the category of **M** represents individuals who determined the probability of giving birth to a girl by dividing n(G) by n(S); this exhibits their utilization of the theoretical interpretation. Besides, cases of b21 revealed those who raised several non-mathematical conditions to set the probability that, for them, reflected a certain degree of belief regarding the validity of these conditions to modify their judgment. These cases were noted by **S** and involved PSMTs who modeled the situation through the subjective probability lens. Finally, category **O** was introduced to indicate cases of b22 who reported non-mathematical conditions to predict the outcome. That term was decided while considering Konold's (1989) description of the outcome approach; it describes a model of informal reasoning under uncertainty, which conflicts with experimental probability. Such reasoning is compatible with the responses of b22's cases, in which individuals focused on the favorable outcome and judged their predictions based on whether that outcome would (or would not) occur in a particular trial. Hence, Figure 15 outlines the steps for inferring the categories of PSMTs' reasoning in the context of giving birth (Item A).

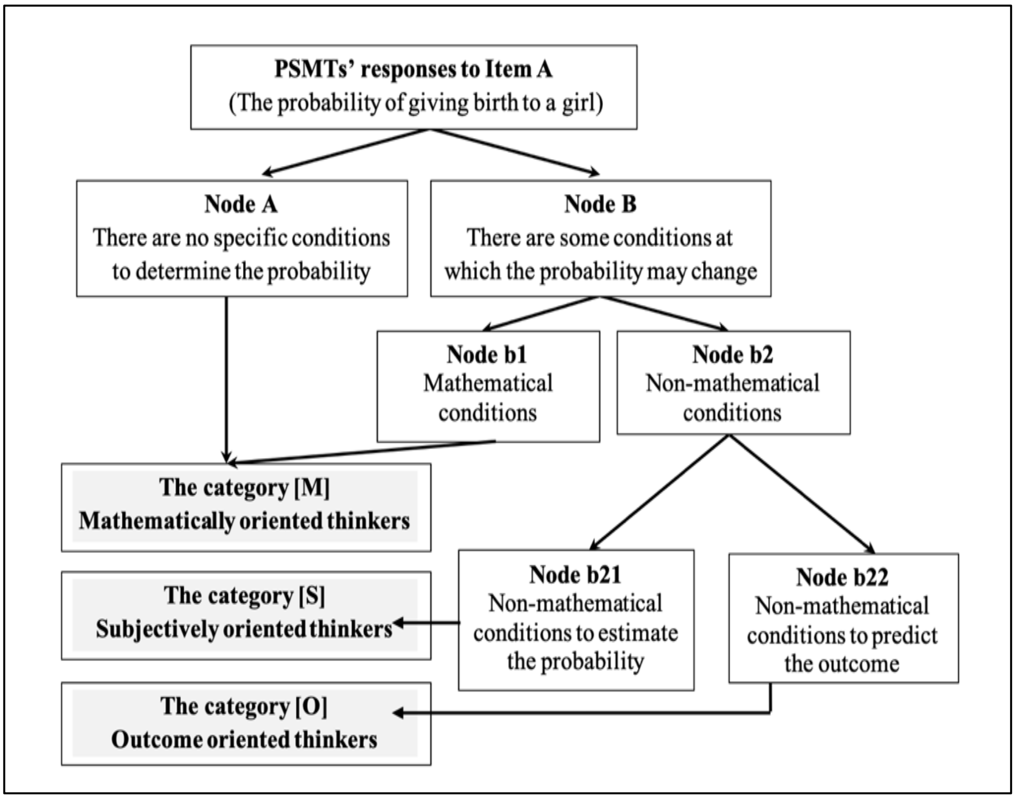


Figure 15. The process of developing the categories of PSMTs' reasoning in the context of giving birth based on their responses to Item A

Similar procedures were followed to portray PSMTs' responses to Item B that described a conditional probabilistic situation. Yet, as noted in Chapter 2, the last step of *incorporating the emerged categories into a model* was not implemented. This explains what was declared earlier about the lack of research on subjective probability as a general classifier wherein several philosophical differences may appear. Alternatively, this exposes the lack of research on how PSMTs manipulate a condition to estimate probability. The conducted steps are summarized in Figure 16, which shows two broad groups of PSMTs who (a) disregarded and (b) utilized the given condition in their analysis; while the former is represented by both the generalizer (case A) and conservative thinkers (case b1), the latter incorporates correlational (case b21) and rational thinkers (case b22); see the details in Section ***5.3.1.***

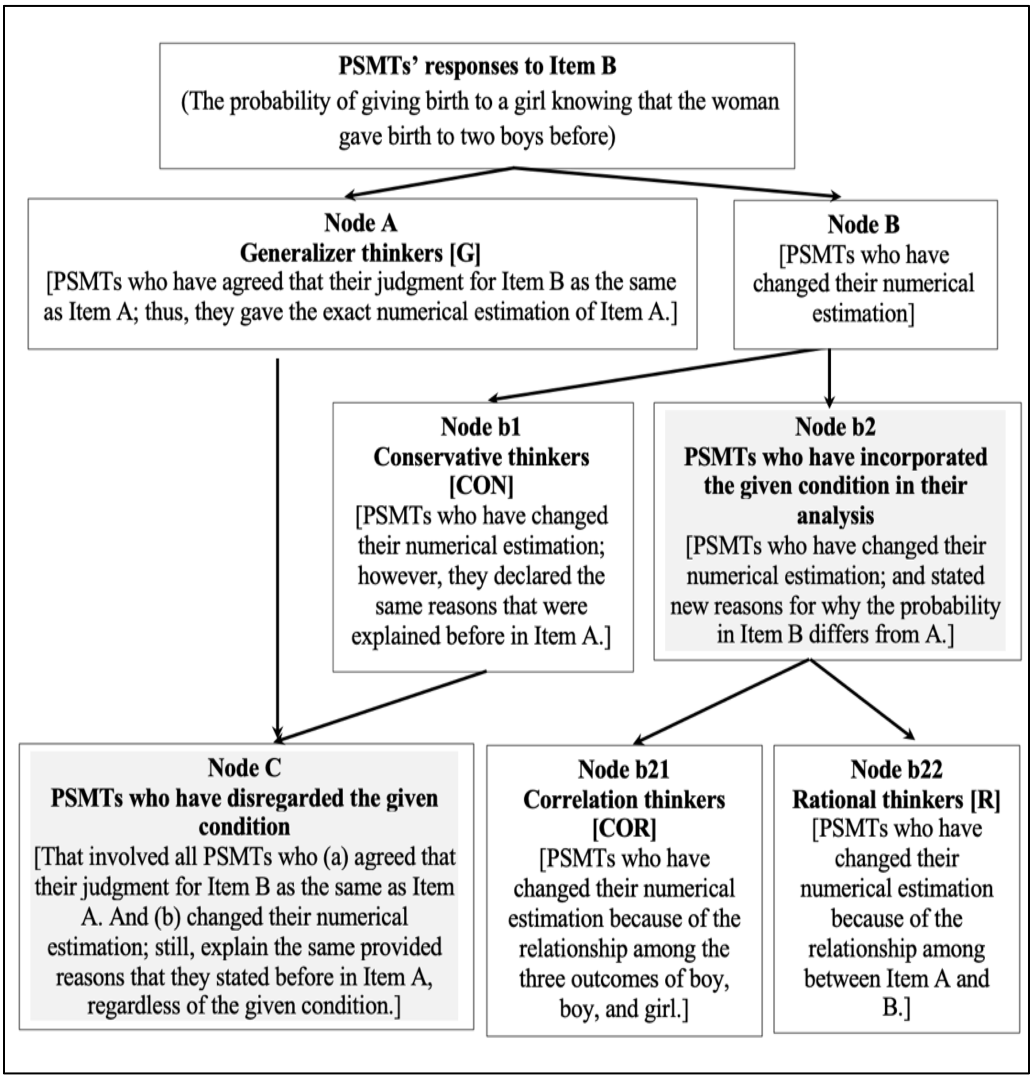


Figure 16. The process of developing the categories of PSMTs' reasoning in the context of giving birth after conditioning it, based on their responses to Item B

- *Deductive coding procedures*

As noted in Chapter 2, in light of the categories developed from analyzing Items A and B, PSMTs' responses to the second part of the study questionnaire (i.e., Items C, D, E1, and E2) were categorized. Concretely, for Items C and D, wherein PSMTs were asked to determine the probability of getting number 5 in an experiment of throwing a die and to interpret the meaning of a 60% chance of rain, respectively (see Appendix 8), the developed categories of **M**, **S**, and **O** worked as a lens to characterize PSMTs' responses to both items. As a result, for the activity of throwing a die, similar manners of **M** and **O** reasoning emerged compared with **M**, **S**, and **O** for the task of weather predictability. Furthermore, new sub-categories of **m**** and **o**** also appeared. While type **m**** reasoning arose only during the activity of throwing a die, **o**** emerged in both contexts. Additionally, another principal category of **I**, which reflected cases of PSMTs who utilized the intuitive interpretation of the probability, appeared in the context of weather predictability. This is illustrated in Figure 17.

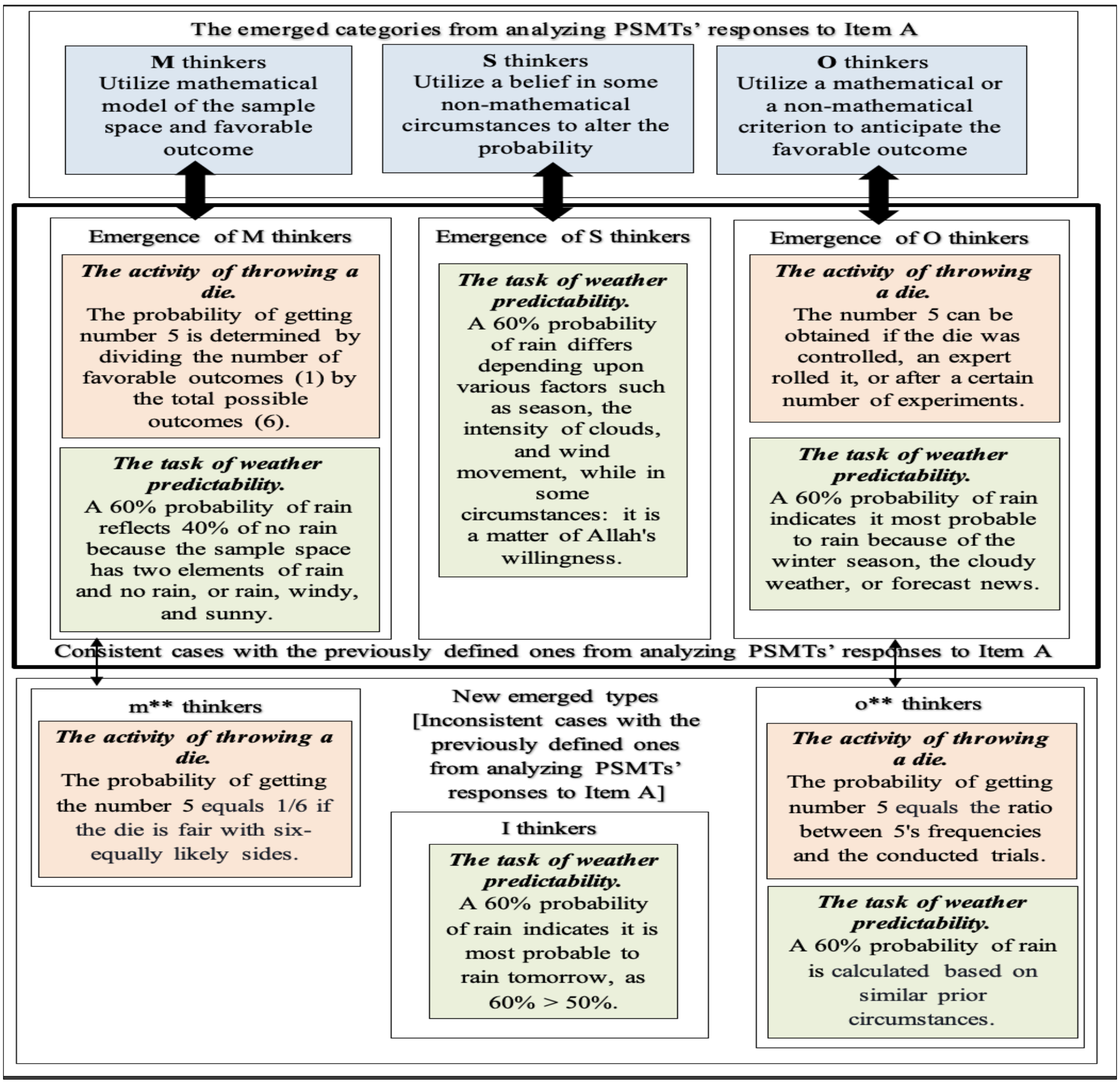


Figure 17. The process of categorizing PSMTs' responses to Item C and D

In that sense, it is relevant to clarify why **m**** and **o**** were judged as sub-categories of **M** and **O**, respectively, but **I** was admitted as a new major category. This is because PSMTs who were committed to **m**** and **o**** also modeled the given tasks through the theoretical and experimental probability, respectively; thus, they had the same features as **M** and **O** categories. On the other hand, **I** indicated PSMTs who relied on intuitive interpretation to explain a 60% chance of rain. Moreover, based on Batanero et al. (2016), the intuitive probability was considered as a principal approach to model probabilistic phenomena (see Table 17).

Additionally, it was assumed that PSMTs' responses to Items E1 and E2, which are equivalent and aimed to explore the conceptual difficulties in calculating the conditional probability from a two-way table, could be categorized under two groups of those who (a) dropped and (b) operated the condition when interpreting a conditional situation. While the first resembled case C that emerged from analyzing PSMTs' responses to Item B (see Figure 16), the second group corresponded to case b2. Such method of analysis made it possible to connect Item B with both E1 and E2 wherein PSMTs' reasoning in a conditional probabilistic situation could be captured. Accordingly, instead of classifying PSMTs' answers into correct and wrong, this way of categorization was preferred to analyze their responses to the conditional probability questions of both Items E1 and E2.

From this aspect, and in light of some previous studies (e.g., Contreras et al., 2011), the detailed conceptions in Table 20 were hypothesized to analyze PSMTs' responses to Q3 and Q4 in E1 and E2 (see Appendix 8):

Table 20. Expected numerical answers for Items E1 and E2[15]

<table>
<tr><th rowspan="2"></th><th rowspan="2">Expectations</th><th colspan="2">Item E1</th><th colspan="2">Item E2</th></tr>
<tr><th>Q3</th><th>Q4</th><th>Q3</th><th>Q4</th></tr>
<tr><td rowspan="4">Dropped the condition</td><td rowspan="2">Conception1
Independence</td><td>P(A|E) = P(A)</td><td>P(E|A)= P(E)</td><td>P(S|M)= P(S)</td><td>P(M|S)= P(M)</td></tr>
<tr><td>385/800</td><td>500/800</td><td>200/460</td><td>300/460</td></tr>
<tr><td rowspan="2">Conception 2
Confusion between joint and conditional probability</td><td>P(A|E) = P (A ∩ E)</td><td>P (E|A) = P (E ∩ A)</td><td>P(S|M) = P (S ∩ M)</td><td>P(M|S) = P (M ∩ S)</td></tr>
<tr><td colspan="2">195/800</td><td colspan="2">110/460</td></tr>
<tr><td rowspan="3">Operated the condition</td><td rowspan="2">Conception 3
Transposed conditional</td><td colspan="2">P(A|E) = P (E|A)</td><td colspan="2">P(S|M) = P(M|S)</td></tr>
<tr><td>195/385</td><td>195/500</td><td>110/200</td><td>110/300</td></tr>
<tr><td>Concept 4
Correct answer</td><td>195/500</td><td>195/385</td><td>110/300</td><td>110/200</td></tr>
</table>

[15] The terms A, E, S, and M refer to School A, ElAhly football team, Secondary level, and Mathematics class, respectively (see Appendix 8).

Based on all the above-mentioned details in Section **5.1**, the characteristics of PSMTs' reasoning in (a) simple unconditional and (b) conditional probabilistic situations are detailed in Sections **5.2** and **5.3**, respectively. Lastly, the issues of trustworthiness and ethical considerations are mentioned in Section **5.4**.

5.2 PSMTs reasoning in a simple unconditional probabilistic situation

As stated earlier, to characterize PSMTs' reasoning in a simple unconditional probabilistic situation, they were asked to respond to three items that reflect three different contexts in which probability can be utilized (i.e., giving birth, throwing a die, and weather predictability). Accordingly, PSMTs' arguments on these items are presented and discussed within Sections ***5.2.1*** and ***5.2.2***; further, Section ***5.2.3*** summarizes the whole discussion. Before going into detail, Table 21 compiles the PSMTs' numerical answers to these three items.

Table 21. PSMTs' numerical answers to the probability tasks[16]

	The probabilistic situation[1]				
	Giving birth (Item A)				
The given percentage (expected probability)	S = {B, G}, n(S) = 2, n(G) = 1; then, P(G) = 1/2	S = {B, G, twins}, n(S) = 3, n(G) = 1; then, P(G) = 1/3	S = {B, G, BB, BG, GG}, n(S) = 5, n(G) = 1; P(G) = 1/5	S = {B, G, BB, BG, GG}, n(S) = 5, n(G) =3; P(G) = 3/5	There are different possibilities; and the probability depends on many factors.
	50%	33.3%	20%	60%	It depends (no specific %)
Number of responses	***35***	***12***	***11***	***3***	***7***
	68 responses				

	Throwing a die (Item C)				***Weather predictability (Item D)***				
The given percentage (expected probability)	S = {1, 2, 3, 4, 5, 6}; and n (5) = 1; then, P (5) = 1/6	S = {side1, 2, 3, 4, 5, 6}; and n (side 5) = 1; then, P (5) = 1/6	It depends on the ratio between 5's frequencies and the conducted trials.	Number 5 can be obtained in many ways	60% probability of rain reflects 40% of no rain.	60% of rain does not reflect an absolute value; and, the probability depends on many factors.	It may rain tomorrow because of several reasons.	60% probability of rain was calculated based on similar prior circumstances.	It is most probable to rain tomorrow, as 60% > 50%.
	1/6	1/6	No %	No %	40%	No %	No %	No %	No %
Number of responses	***15***	***35***	***11***	***7***	***8***	***4***	***20***	***4***	***12***
	68 responses				***48 responses***				

[16] The symbols S, B, G, n, and P, refer to sample space, boy, girl, number, and probability, respectively.

5.2.1 PSMTs reasoning in the context of giving birth (inductive data analysis process)

Because the resultant categories that describe PSMTs' reasoning[17] in the context of giving birth have guided the analysis of their reasoning in the other two contexts (i.e., throwing a die and weather predictability), a detailed description of that reasoning is first presented and discussed in this section.

As shown in Appendix 8, two questions were raised in Item A that are related to the context of giving birth. First, what is the probability of giving birth to a girl? Second, explain the reasons because of which you have decided the proposed probabilistic ratio?

In general, the expected values students assigned to estimate the probability of giving birth to a girl and expressed as their numerical answers were 50%, 33.3%, 20%, 60%, and *it depends* (as displayed in Table 21). Moreover, to characterize students' reasoning in this situation, the analysis process concentrated on their stated criteria that revealed their reasons why such probabilities were judged. Accordingly, students' manners of reasoning in the context of giving birth were grouped into three main categories: **M**athematical [**M**], **S**ubjective [**S**], and **O**utcome oriented [**O**]. Moreover, each category included sub-categories. Table 22 summarizes the distribution of these types of reasoning among the participants.

Table 22. PSMTs' manners of reasoning in the context of giving birth

Major category	***Frequency***	***Percentage***
Mathematically oriented [**M** = m and m*]	20	29.4 %
Subjectively oriented [**S** = s, s*, and s**]	41	60.3 %
Outcome oriented [**O** = o and o*]	7	10.3 %
Sub-categories	***Frequency***	***Percentage***
m	15	22.1 %
m*	5	7.4 %
s	7	10.3 %
s*	2	2.9 %
s**	32	47 %
o	3	4.4 %
o*	4	5.9 %
Total	***68 responses***	***100%***

In detail, the following argumentation defines each type of reasoning. It starts with type **M** thinkers, that is, students whose common explanations were as regarded in Table 23.

[17] Henceforth, the term "PSMTs" will be replaced by the word "students" until the end of this chapter (except in table and figure titles and essential headlines)

– **Characteristics of types m and m* reasoning [M thinkers]**

Types **m** and **m*** reasoning have a common feature in which both depend on the *theoretical interpretation of probability* since they model the given situation through the notion of sample space (S) and expected outcome (G). They grasp the idea of *variability*, wherein the result is not determined; the result varies depending on the possible favorable cases (i.e., elements of S; Canada, 2006; Garfield & Ben-Zvi, 2005). Besides, these types maintained the *equiprobable bias* (Lecoutre & Fischbein, 1998; Lysoe, 2008; Savard, 2014) that appeared when the students judged the probability of giving birth to a girl to be equal to giving birth to a boy; moreover, when they considered twins as a possible outcome, they supposed that the possibility of giving birth to twins is the same as that of giving birth to a boy or girl.

Indeed, holding such a bias (i.e., *equiprobability)* prevented **M** thinkers from thinking of base rate frequency that symbolizes actual gender distribution. Consequently, they were insensitive to the prior probability of outcomes (Tversky & Kahneman, 1974) since there were no declared responses relevant to the population. The respondents ignored the fact that the possibility of giving birth to a girl is slightly less than giving birth to a boy, as the actual gender distribution in Egypt shows that the ratio of males to females equals 1.06 (Egypt Demographic Profile, 2019). Another plausible cause for such insensitivity is the form of the task itself. For example, no conditions were given to define the pregnant woman (is she an Egyptian woman?); further, there were no stated percentages of the gender distribution. Accordingly, the students were neither required nor expected to give a specific and correct ratio for the gender distribution throughout the country; it was preferably that they referred to the base rate frequency of the outcome as a necessary factor when judging such a probabilistic situation. That is, the notion of population, whether within a family or the whole country, was anticipated (by the researcher) to appear among the students' responses when estimating a reasonable value.

Table 23. Mathematically oriented thinkers' typical responses in the context of giving birth

	Mathematically oriented thinkers [M]	
	Type m	***Type m****
Students' typical responses	There are no specific conditions to judge the probability; it is a matter of mathematical calculation. Thus, because S contains two events of B and G; then, P = 50%.	The probability depends on the number of events in S. For example, if S = {B, G}, then P(G) = 50%. Similarly, if S = {B, G, twins}, then P(G) = 33.3%. Hence, based on the stated hypotheses, particularly the number of elements in S, the expected probability will vary.
N = 20	***15 responses***	***5 responses***

Additionally, there is a slight difference between **m** and **m*** in terms of *the essence of equiprobable bias*. Type **m** thinkers tend to think that random events are equiprobable by their nature, even when they are not (Lecoutre, 1992). This appeared clearly in their argumentation regarding the conditions in which the probability was judged. The following was a common response: "*there are no conditions for determining the assigned probability because all events have the same chance to occur*." This likely reflects the utilization of the *representativeness heuristic*; it indicates the judgment of the likelihood of an event according to how well such an event represents some aspects of the parent population, or how it resembles the process that generated it, which is the case in type **m** thinkers (Kazak & Pratt, 2017; Kustos & Zelkowski, 2013). That is, **m** thinkers emphasized the random process, in which, for them, the situation of giving birth implies a random experiment that always yields equiprobable outcomes (regardless of any conditions).

Type **m*** responses exposed a more abstract mindset that attempted to interpret the given social phenomenon of giving birth through the lens of theoretical probability, which may not entirely fulfil such a situation. Although **m*** thinkers admitted the limitations of S elements in restricting the chance of giving birth to a girl, they were reluctant to connect these mathematically stated limitations with the actual circumstances that may appear in reality. Their argumentations were typically algorithmic without any explanations on how (or, under what circumstances) such constraints on S elements (e.g., two or three outcomes) may occur. Accordingly, **m*** thinkers were judged to lack consideration of the realistic context because of the *overgeneralization; they sought an ideal mathematical abstract model.* This may explain why **m*** thinkers approached the giving birth problem like drawing a card or tossing a coin. In that sense, on one hand, the *equiprobable bias* originated from employing the *representativeness heuristic* that prevented **m** thinkers from confirming the required assumptions of theoretical probability (*Laplace's axioms*). On the other hand, it was inherent in the *overgeneralization heuristic* for the case of **m*** thinkers.

– **Characteristics of types s, s*, and s** reasoning [S thinkers]**

The second group of students' responses denotes subjectively oriented thinkers [**S**], who shared the common expressions that are detailed in Table 24.

The common trait among types **s**, **s***, and **s**** reasoning is that they are all rooted in the *subjective meaning of probability*, wherein the students relied on their personal information to estimate the chance of giving birth to a girl—more precisely, to express some factors that may affect their judgment. Consequently, they reported several experienced circumstances based on which the birth of a baby girl can be predicted, as exhibited in Table 24.

Table 24. Subjectively oriented thinkers' typical responses in the context of giving birth

	Subjectively oriented thinkers [S]				
Students' typical responses	Probability of giving birth to a girl alters depending upon				
	Type s		***Type s****	***Type s*****	
	our information about pregnancy ultrasound results (if we used the ultrasound scan to determine the baby's gender, the probability would change to 100%).	our information about the previous babies' genders (if we had some information regarding the first and second babies' genders, our estimation could change).	Allah's willingness.	our information concerning the possible outcomes. For example, considering miscarriage or spontaneous abortion as a possible outcome, knowing (through a medical checkup) that the woman may give birth to twins changes the probability from 1/2 to 1/3.	understanding the biological or genetic state of the woman, that is, scientific knowledge of the X and Y chromosomes.
N = 41	***6***	***1***	***2***	***28***	***4***
	7 responses			***32 responses***	

S thinkers understood the *notion of variability*, in which, for them, the expected outcome (i.e., a baby girl) differs depending upon several personal experienced contingencies, which are listed in Table 24. Although the students did not explicitly change their estimation (what they reported as an answer for the first question; see Table 21), their responses, when they were asked to reflect on and state the criteria that helped them to judge, were expressed in the common form of *it depends*. This is closely related to *Bayesian reasoning* that allows updating our estimation (or revising a prior probability) by processing new information for estimating a posterior probability (Batanero et al., 2016; Dollard, 2011; Sharma, 2016). Even though there were no new given data for Item A, **S** thinkers reviewed their own available information that may affect the expectation of getting a baby girl.

Despite such commonality, type **s** thinkers differ from both **s*** and **s**** in terms of understanding the concept of *randomness*, which is a crucial element for reasoning probabilistically. In detail, students who argued that knowing the ultrasound scan results would change the expectation from 50% to 100% tended to remodel their estimation to certainty (100%), which contradicts the essence of *randomness* that demands *uncertainty.* Accordingly,

s thinkers were judged (by the researcher) to have the *prediction bias* because of which their prediction has the meaning of exact prediction (Savard, 2008).

Similarly, in omitting the notion of *randomness,* which also requires *independence without correlation* (Falk & Konold, 1992; Green, 1993; Savard, 2014), a student in the same category associated the previous babies' gender with the newborn's gender (see Table 24). Regardless of whether his idea is scientifically correct or not, such a response reflects the utilization of past information as a tool to predict the new outcome, which lacks the notion of *independence* and, consequently, *randomness*. This was termed as the *dependence conception*, which indicates the tendency to interpret the dependent relationship between two events as a causal relationship. According to Kelly and Zwiers (1986), "Events are independent when the occurrence (or nonoccurrence) of one of the events carries no information about the occurrence (or nonoccurrence) of the other event" (p. 97). Hence, they explained a common misconception related to students' understanding of independence that interprets a *dependent relationship between events as a causal relationship.*

For **s*** and **s**** thinkers, although their responses indicate an understanding of *randomness* (i.e., the baby girl cannot be predicted with a certainty of 100%) and *variability* (i.e., multiple factors explain why the resultant baby's gender varies), the nature of their stated reasons that may alter the outcome stayed quite different. As shown in Table 24, while the type **s**** criteria and stated conditions remained cognitive and practical, type **s*** thinkers were inspired by the religious conception of "*Allah's willingness*." Nonetheless, this conception did not restrict them from determining the probability; further, they relied on *Allah's will*, not as a cause that affects the baby gender, rather to reveal some out-of-control circumstances in which the actual outcome may change. This was reflected in the responses of two students who reported that "*the probability of giving birth to a girl equals 50%; still, we cannot certainly anticipate a baby girl because the actual baby's gender may alter depending upon Allah's will.*"

In this respect, it is worthwhile to remark that the animism attribution of the phenomena to God was judged by various researchers to be a *personalist interpretation* (Amir & Williams, 1999; Garfield & Ben-Zvi, 2005; Kissane & Kemp, 2010; Sharma, 2014; Watson & Kelly, 2004; Watson & Moritz, 2003). Amir and Williams (1999) reported that some secondary school students believe that the outcomes of certain events depend on a force that is beyond their control (e.g., *God* commands everything that happens in the world). Nevertheless, in the current study, this interpretation was defined as a specific type of probabilistic reasoning, since the students embraced the concept of *Allah's will* not as a cause to explain why the variability of the outcomes occurred, but rather as a factor that may intervene in the situation.

Such judgment concerning type **s*** is a level of probabilistic reasoning that acknowledges the influence of socio-cultural factors on students' conceptions of probability (e.g., Amir & Williams, 1999; Larose et al., 2010; Sharma, 2016), which was reported before in Chapter 4. More precisely, Chassapis and Chatzivasileiou (2008) considered that mathematics knowledge is culturally situated, which implicitly or explicitly involves social and cultural values. As they explained, beyond the mathematical aspect of each construct, another aspect exists that is associated with the practice of that construct in daily life. Following this argument, for type **s*** thinkers, the mathematical construct is the probability, which is connected to the religious belief of *Allah's will*; it is not only a religious belief but also a socially shared conception practiced by most Egyptian citizens.

On one hand, the results of Chassapis and Chatzivasileiou's (2008) study deny Amir and Williams's (1999) findings, according to which the students did not select luck (fate or superstition) when they were asked to attribute the cause of an unexpected event to chance, probability, fate, or God's will. Alternatively, in Chassapis and Chatzivasileiou's (2008) research, most Jordanian students (Arabian speakers and Muslims) assigned the unpredictability to God's will, compared with Greeks (Greek speakers and Christians) who attributed it to chance. On the other hand, Chassapis and Chatzivasileiou's (2008) findings support the claimed interpretation regarding type **s*** reasoning; more precisely, they stated that "beliefs in God's will and probabilistic thinking may be compatible in some cases leaving space to the formation of chance and probability conceptions" (Ibid., p. 204). This typically mirrors what was reported a little earlier regarding type **s*** thinkers who first tried to determine the probability of giving birth to a girl (mathematically by assigning some percentages; see Table 21). Moreover, they added the phrase "*Insha'Allah*" (if God wished it) to reflect the limitedness of human beings in providing an exact prediction. Hence, type **s*** thinkers did not manipulate the concept of *Allah's will* as a cause of the newborn's gender, but rather as a factor that may alter such probability, as explained previously. They maintained their understanding of randomness (without dependence or certainty) and the variability based on which the outcome varies upon *Allah's will*. Therefore, type **s*** reasoning was defined as a particular type of probabilistic reasoning rather than a belief.

- **Characteristics of types o and o* reasoning [O thinkers]**

The third type of students' reasoning was coded under the term outcome-oriented thinkers [**O**], who utilized the expressions exhibited in Table 25.

Table 25. Outcome oriented thinkers' typical responses in the context of giving birth

<table>
<tr><th></th><th colspan="3">Outcome oriented thinkers [O]</th></tr>
<tr><td rowspan="3">Students' typical responses</td><td colspan="3">We can predict that the woman will give birth to a girl by</td></tr>
<tr><td>Type o</td><td colspan="2">Type o*</td></tr>
<tr><td>Checking the outcome of the delivery process. Accordingly, if the woman already gave birth to a boy or twins, had a miscarriage, or passed away during the delivery process, the probability would change from 1/2 to 0.</td><td>Observing the woman's physical appearance (e.g., belly shape).</td><td>Recognizing whether this woman gives birth to the same gender always, or not; if yes, and she usually gives birth to girls (for example), the probability will be higher than 50%.</td></tr>
<tr><td rowspan="2">N = 7</td><td rowspan="2">3 responses</td><td>2</td><td>2</td></tr>
<tr><td colspan="2">4 responses</td></tr>
</table>

Indeed, both the **o** and **o*** sub-categories incorporate students who emphasized the favorable outcome (i.e., a girl, as provided in the question) more than the probability. They interpreted the task as if it asked, when will a woman give birth to a girl? (How to know? Or, under what circumstances?). Consequently, their response took the specific form of stating that this woman would give birth to a girl if something specific happened; likewise, if this thing did not occur, another gender would be expected. This implicitly indicates a dependence on some causes (see Table 25) because of which a baby girl can be assigned.

Such focus on the favorable outcome led the students toward evaluating their predictions as being *right or wrong*; besides, their responses took the form of *yes-no decisions* on whether that outcome will occur in a particular trial, which here denotes the baby's gender in the next delivery process (Batanero & Sanchez, 2005; Konold, 1989; Savard, 2014). Two students reported that if the woman's belly shape is round, she will give birth to a girl; similarly, if her belly shape is not round, she would not give birth to a girl, as displayed in Table 25.

This reasoning indicates a partial understanding of *the experimental probability*, which was applied by checking the posterior results (e.g., outcomes of the delivery process, in type **o**) or reflecting on similar previous situations (e.g., recognizing whether this woman gives birth to the same gender always, in type **o***). Nevertheless, the students used such information not to interpret the probability but rather to describe why (or why not) the next baby's gender will be a girl. Accordingly, instead of defining the probability of giving birth to a girl based on the distribution of occurrences in a series of events (i.e., a large number of previous similar cases), **O** thinkers limited their ideas to the case of the next expected event (not the probability). This mismatches with the experimental interpretation of probability in which "the probability is meaningful only with respect to repeatable event and is defined as the relative frequency of occurrence of an event in an infinite (or very large) number of trials" (Reichenbach, 1949; Mises, 1957, as cited in Konold, 1989, p. 62). Consequently, although **O** thinkers admitted the *variability* of the outcomes, such *variability* was not grounded on frequencies but instead on *one single trial* in which the baby's gender could be interpreted. Hence, they adjusted their expectation to satisfy particular causes, and further, their yes-no decision to be within two broad groups: one group contained the favorable outcome of a girl, while the other included all other expected events (i.e., the complementary set).

Additionally, although neither **O** thinker could apply the *experimental probability interpretation* successfully because of much focus on the favorable outcome (i.e., girl), their inadequate operation was exhibited in different objects. On one hand, type **o** thinkers, who tried to reflect on the posterior results of the delivery process, seemed not to understand the idea behind the prediction according to which the intention was quantifying the information regarding unknown phenomena. They stated, "*if the woman already gave birth to a boy or twins, or she had a miscarriage or passed away during the delivery process, the probability would change from 50% to 0%*." Such argumentation denies the fact that after the delivery process, the situation of giving birth will not be probabilistic anymore. Hence, there is no meaning in estimating the probability of giving birth to a girl in that case. This understanding of the prediction was termed as *prediction conception*, which was distinguished from the previously reported *prediction bias*.

On the other hand, although type **o*** respondents understood the idea of the prediction wherein the results are still unknown, they were less conscious about the distinction between *causality and conditionality*, in which distinguishing between both concepts signifies a crucial element of probabilistic reasoning (Batanero et al., 2016; Borovcnik, 2012). For the probability, the dependence characterizes a bi-directional relationship, and if an event B is the cause of another event A, then whenever B is present, A is present too (i.e., $P(A|B) = 1$); however, the two directions involved in conditional probabilities have a completely different connotation from a causal standpoint (Díaz et al., 2010). That is, although the conditional probability of having a baby girl based on having a positive result on an ultrasound test (or a round belly shape); P(positive ultrasound result| G) is causal, the backward direction from a positive ultrasound diagnosis to having a baby girl is merely indicative [P(G| positive ultrasound result)]. In other words, while the test is positive because the woman is pregnant with a girl, a baby girl is not caused by positive test results. Accordingly, type **o*** thinkers were judged to share the *causal conception*; they assumed a causal relationship wherein the conditioning event B (e.g., rounded belly shape) is the cause and A (i.e., girl) is the result (Gras & Totohasina, 1995, as cited in Batanero & Sanchez, 2005; Savard, 2014).

Based on the above description, Table 26 summarizes the characteristics of types of reasoning that were inferred upon considering students' responses to the problem of giving birth. This guided the analysis of the other two contexts (i.e., throwing a die and weather predictability).

Table 26. Characteristics of PSMTs' reasoning in the context of giving birth

Major category	***Shared biases and conceptions*** / ***Commonalities***	***Theoretical constructs on which PSMTs relied***
[M] Thinkers	– Equiprobable bias – Insensitivity to the prior probability of outcomes	– Theoretical probability – Laplace's theory
[S] Thinkers		– Subjective probability – Bayesian reasoning
[O] Thinkers		– Experimental probability
Sub-categories	***Differences***	
m	– Representativeness heuristic	
m*	– Overgeneralization heuristic	
s	– Prediction bias – Dependence conception	
s*	– Allah's will	
s**		
o	– Prediction conception	
o*	– Causal conception	

5.2.2 PSMTs reasoning in the contexts of throwing a die and weather predictability (deductive data analysis process)

As stated earlier, the emerged categories of **M**, **S**, and **O** reasoning were used as a basis to characterize students' responses in both the contexts of throwing a die and weather predictability. This is detailed in this section, as follows:

- **The emergence of type m and m* reasoning [M thinkers]**

The category of **M** thinkers emerged again in both contexts, as explicit in Table 27. First, for the activity of throwing a die in which students were asked to explain their various strategies to determine the probability of getting number 5 in a random experiment of rolling a die one time (see Appendix 8), both types **m** and **m*** reasoning emerged. That is, **M** thinkers approached the given situation through *theoretical probability* that relies on the notion of sample space and the favorable outcome of 5. As shown in Table 27, mathematically speaking, while **m** thinkers modeled the experiment using S = {side 1, side 2, side 3, side 4, side 5, side

6}, and A[18] = side 5; then, P(A) = n(A) / n(S) = 1/6, **m*** thinkers employed a more general formula of S = {1, 2, 3, 4, 5, 6} and A= {5}; then, P(A) = n(A) / n(S) = 1/6.

Similarly, for the task of weather predictability, when students were asked to interpret the meaning of a 60% probability of rain, they modeled this situation as if the sample space contained two mutually exclusive events of rain and no rain. Consequently, a 60% chance of rain reflects 40% of no rain (the complementary event). That is, mathematically speaking, S = {rain, no rain}, A = {rain}, and P(A) = 60%; then, P (A^c) = 40%. Furthermore, one student in this category interpreted a 60% chance of rain as if there were various possible outcomes regarding tomorrow's weather, such as rainy, windy, and sunny. Accordingly, if the probability of rain tomorrow equals 60%, this means that the sum of all other possible outcomes equals 40%. Alternatively, S = {rainy, windy, sunny, etc.} and P (A= rain) = 60%; then, P (A^c = wind + sunny + etc.) = 40%.

Table 27. Mathematically oriented thinkers' typical responses in the context of throwing a die and weather predictability

	Mathematically oriented thinkers [M]			
	The context of throwing a die			**The context of weather predictability**[19]
Reasoning type	***Type m***	***Type m****	***Type m** (newly emerged)***	***Type m***
Students' typical responses	Since the die has six sides and number 5 exposes only one side among them, the probability equals 1/6.	This experiment involves six numbers, and number 5 denotes only one number among them; then, the probability equals 1/6.	The probability equals 1/6 if the die is fair with six equally likely sides.	It means that the probability of no rain equals 40%; or, if rainy, windy, sunny, etc., are the possible outcomes, then the probability of all these outcomes (except rain) equals 40%.
N = 57	***10***	***35***	***5***	***7 responses***
	50 responses			

Although **M** thinkers did not calculate the probability of rain by dividing the number of favorable outcomes by all possible outcomes as what they did in the tasks of giving birth and throwing a die, their interpretation of a 60% chance of rain indicated a *reliance on the theoretical probability*, since this 60% was decided based on various plausible events in S.

[18] Symbol A refers to the favorable outcome of obtaining 5 in the experiment of throwing a die, or rain occurrence for the task of weather predictability, while A^c denotes the complementary event.
[19] Because of time constraints for third-year students, only three (out of 23) of them responded to the task of weather predictability (see Table 8 in Chapter 2).

Again, and as stated before, one essential thing to consider here is the form of the weather predictability task; it differs from both giving birth and throwing a die. While the former required an explanation of a particular percentage, the latter two problems demanded the determination of a certain percentage to express the probability.

In detail, regarding the shared conceptions and biases in the sub-categories of **m** and **m***, for the task of throwing a die, 10 students were judged to be type **m** thinkers. These students considered the physical structure of the die itself, in which the numbers symbolize the die's various facets; hence, the favorable outcome of 5 denotes one side among six sides (see Table 27). Based on what was argued earlier in the giving birth context, the *equiprobable bias* prevented **m** thinkers from confirming the required assumptions to apply *Laplace's theory*. Thus, they assumed that all die sides are equiprobable, even though there was no explicit information in the task that declares the die's regularity. Nonetheless, one thing to consider here is that **m** thinkers may think of the equiprobability assumption as a *premise for all chance games, in which there is no need to confirm it*. Such an idea was explained by Stohl (2005); she clarified that because of our inability to judge the complexity of the physical circumstances when we experiment with throwing a die (e.g., air resistance and speed), we cannot predict whether a particular outcome will occur or not. Hence, one possible way to approach this phenomenon is to utilize the *theoretical probability*; it helps to embody such physical complexities and apparent symmetry of the die. Her usage of "apparent symmetry of the die" indicates that when we operate die experiments, the fairness of dice is usually assumed.

Moreover, two students expressed an understanding of the *equally likely hypothesis.* While one of them declared that "*the reason why the probability of getting number 5 equals 1/6 is that the die has only one side that holds number 5 and there is no possibility to get two sides together*," the other student stated that "*the die has six equally likely sides*." Such reasoning involved a clear understanding of *theoretical probability*; this resembles what Savard (2010) identified in a study of students who were concerned about the fairness of a spinner for explaining variability. This case was recognized as unique among **M** thinkers who demonstrated adequate knowledge regarding *theoretical probability*. Accordingly, it was coded under the sub-category of **m**** thinkers, which had not emerged previously in the context of giving birth.

Additionally, three other students paid sufficient attention to the task formulation; accordingly, they preferred utilizing the *theoretical probability over the experimental*. As they reported, based on the given information, we must rely on the *theoretical interpretation because the experimental approach cannot be applied for one trial* (i.e., the term once that appeared in the task). In detail, two of them explained that only processing a one-time experiment of throwing a die is not enough to calculate the probability of getting number 5. While the other student stated that such a situation could be managed only if we already had the outcome, and we were asked to calculate its probability. Accordingly, such cases were also judged under **M** thinkers as they strengthened the theoretical probability interpretation to represent a phenomenon and did not give any alternatives for operating the experimental probability (e.g., by increasing the number of trials). Yet, they were classified under type **m**** thinkers since they decided to rely on theoretical probability, not because of the die itself (whether fair or not), but because of the situational circumstances in which the experimental interpretation could not be performed. This also indicates adequate understanding of the required assumptions to operate the experimental probability.

On the other hand, 35 students exhibited type **m*** reasoning that overgeneralizes the *theoretical probability* to model probabilistic situations. Accordingly, **m*** thinkers focused on the numbers assigned to the die facets in which the favorable outcome of 5 implies one number among six different numbers. In other words, they sought to develop an *abstract mathematical formula* to model the situation without giving much attention to *the meaning of the embedded numbers in that formula*, which resembles the emergence of ***m**** thinkers in the context of giving birth. Although the students admitted the sample space as one factor to determine the probability, their responses did not indicate a connection between such sample space elements and the die's physical structure. Consequently, as the data analysis revealed, the theme of overgeneralization for **m*** thinkers appeared clearly in their ignorance of the term *once*. Besides, most of them did not narrow their answers to the probability of getting number 5; they also stated that "*the probability of any number's occurrence equals 1/6 since the experiment represents a random process with six possible outcomes; then, each number has only one chance to occur.*"

Such an *overgeneralized* mindset led **m*** thinkers to think of the outcomes of any random experiment of throwing a die as being equally likely, which is not necessarily true. This idea was discussed in Pratt's (2005) study in which, from the experts' viewpoint, the random process can be biased and then regarded as unfair. As Pratt (2005) exemplified, for young students, a spinner with fair six equal-sized sectors is random. However, if that spinner were numbered and the sixth sector was twice the size of the other sectors, students might think of the spinner as unfair and, accordingly, non-random. Therefore, he further commented that both spinners "generate irregular results, and so in these respects the experiment with the non-uniform spinner might have been regarded as random too" (Ibid., p. 176). In this regard, Pratt

(2005) recommended that mathematics teachers should help their students *distinguish between fairness and randomness*. Yet, **m*** thinkers also seem to be unaware of such differences.

Dealing with such small discrepancies between **m** and **m*** was not that clear for the task of weather predictability in which all students' responses were assigned to type **m** reasoning, because they explicitly expressed *attention to the context*. Furthermore, the *equiprobable bias* did not emerge here; one likely reason for this is the formation of the task itself in which the probability of rain was already provided, as reported earlier. Nevertheless, the student who listed the outcomes of rainy, windy, sunny, etc., in one group to represent the sample space seemingly considered that all these events are equally likely to occur, which contradicts the given data regarding 60% of rain.

– **The emergence of types s, s*, and s** reasoning [S thinkers]**

Interestingly, the category of **S** thinkers that defined around 60% of students' reasoning in the context of giving birth did not appear in the activity of throwing a die. Nevertheless, some responses to the task of weather predictability showed such subjective reasoning, as explained in Table 28.

Table 28. Subjectively oriented thinkers' typical responses in the context of weather predictability

	Subjectively oriented thinkers [S]		
	Context of throwing a die	**Context of weather predictability**	
Reasoning type	***Type, s, s*, and s****	***Type s****	***Type s*****
Students' typical responses	Null	60% probability of rain indicates a 40% probability of no rain; still, such probability is a matter of Allah's will.	60% probability of rain does not reflect an absolute value. The probability depends on many factors such as time of the year, season, inclination and intensity of clouds, and wind movement.
N = 5	***0***	***1 response***	***4 responses***

As detailed in Chapter 4, one critical idea, which this study embraced, is the fruitfulness of implementing an authentic probabilistic situation to expose the subjective side of students' probabilistic reasoning. Such an idea appeared here, wherein no responses to the conventional activity of throwing a die were classified under the **S** category, except the two students' responses who emphasized the die regularity and were judged as exceptional cases of **M** thinkers because they did not share the *equiprobable bias*. This means that before the data analysis, it was expected that **S** reasoning might arise in forms such as observation of the die regularity (fairness), numbers attributed to its facets, person who is rolling the die, or the technique of rolling. Thus, different values could be assigned to the probability of rolling a number 5 based on students' beliefs or prior information regarding these issues (Dollard, 2011); students might also update their predictions considering such information (Kvatinsky & Even, 2002). However, only the two previously mentioned students did so, and, for them, the subjective interpretation that depends on the available amount of knowledge coincided with the objective probability.

On the other hand (see Table 28), for the task of weather predictability, while one answer indicated type **s*** reasoning, four responses were assigned to **s****; both resembled manners of reasoning that emerged before in the context of giving birth. In detail, one student persisted in manifesting the concept of *Allah's will* to reflect the uncertainty of the rain falling. He commented, "*first, a 60% chance of rain reflects a 40% chance of no rain; yet, we cannot expect rain to occur because the actual event may alter depending upon Allah's will*." Such reasoning mirrors **s*** in the context of giving birth in which the concept of *Allah's will* was utilized not as a cause to explain the variability but rather as a factor that may interfere with the situation. It also designates the existence of out-of-control circumstances wherein the actual outcome may vary. Again, because employing such a concept did not prevent **s*** thinkers from interpreting the given probability mathematically, their reasoning was regarded as a particular type of probabilistic reasoning rather than a belief.

Additionally, within the main category of **S** thinkers, four students were assigned to **s****. For them, a 60% chance of rain did not indicate a certain percentage; instead, it defined their various degrees of uncertainty regarding weather conditions (Liberman & Tversky, 1996). The **s**** thinkers reported: "*a 60% chance of rain may designate several circumstances, such as the temperature, season (winter or summer), movements and intensity of clouds, or flow inclination; this percentage was judged in light of all these circumstances.*" Accordingly, for **s**** thinkers, the given probability did not specify one issue of only clouds (as a case), but rather many factors that worked together to determine that probability. Hence, the prediction may vary upon what we know about all these criteria. Such reasoning resembles how some students judged the probability of giving birth to a girl that, for them, differs upon the available information about issues such as the woman's ultrasound results or her genetic state (see Table 24).

– **The emergence of types o and o* reasoning [O thinkers]**

As expected, some responses to the tasks of throwing a die and weather predictability reflected the same manner of reasoning that appeared first in the context of giving birth and coded under the category of **O** thinkers. These responses are displayed in Table 29.

Table 29. Outcome oriented thinkers' typical responses in the context of throwing a die and weather predictability

	Outcome oriented thinkers [O]				
	Context of throwing a die			**Context of weather predictability**	
Reasoning type	***Type o***	***Type o****	***Type o***** ***(newly emerged)***	***Type o****	***Type o***** ***(newly emerged)***
	We can get number 5 if		**We can calculate the probability of getting a 5 if**	**It may rain tomorrow because**	**A 60% chance of rain**
Students' typical responses	the die has been thrown on the floor; then, number 5 may appear in the first, second, or after six trials.	the die was controlled, an expert rolled it, or the number of trials was increased.	the number of trails has been increased; then, such probability equals the ratio between 5's frequencies and the conducted trials.	it is winter, the sky is dense, the weather is cloudy, or it was announced on the weather forecast.	has been calculated based on similar prior circumstanc es.
N = 42	***3***	***4***	***11***	***20***	***4***
	18 responses			***24 responses***	

Similar to what was explained earlier, type **O** reasoning reveals a partial understanding of *the experimental probability interpretation* in which the students focused on the outcome itself rather than its probability. **O** thinkers recognized the probability of getting number 5 as if the question were how to get number 5? In what way can the individual get 5? Or how many trials should be performed to obtain number 5? Similarly, in the context of weather predictability, 20 students interpreted a 60% chance of rain based on several causes due to which rainfall occurs. Moreover, they sharpened the favorable outcome (i.e., rain occurrence) as if it had already occurred, and they were discussing its causes (under what circumstances did the rain occur?)

In detail, for the activity of throwing a die, both sub-categories of **o** and **o*** emerged, compared with the emergence of merely **o*** for the task of weather predictability (see Table 29).

On one hand, **o** thinkers operated the *experimental probability* by reflecting on the expected posterior outcomes after experimenting with the die. Accordingly, two of them judged, with certainty, that the favorable event of 5 would appear in the first, second, or after six trials, as displayed in Table 29. Besides, one student stated that this given situation is not probabilistic because, in the random experiment, the outcome could not be known beforehand, but we knew it after the experimentation. Such an answer reveals the understanding of the experimental probability as a posterior expectation; yet, the student described that experimentation as a process to get the outcome of 5, and not to calculate its probability.

On the other hand, **o*** thinkers explained several techniques, likely based on their prior experience of obtaining numbers when experimenting with a die. Concretely, two students reported that the die might be controlled by the person who manipulates that die, like what many experts often do in *backgammon games*. The way in which the die is dominated and directed was described by another student using the expression *cheating*. Another student's response also indicates the idea of *chance controllability*: "*if the die itself was designed to hold number 5 for all sides, we could get such a number from the first trial*." Thus, those students supposed that such strategies reflect several causes one may rely on to explain the number 5 occurrence. A similar manner of reasoning was operated by 20 students to interpret a 60% chance of rain. They all based their predictions on a *causal analysis* of the situation. They declared that a 60% probability of rain reflects that the sky is overcast, it is cloudy without stars, there is 60% of water accumulation in the clouds, or there is humidity (12 responses); the season is winter (2 responses); the climate is cold or stormy with dust (3 responses); or the forecast announced that it will rain (3 responses). All these circumstances were utilized by **o*** thinkers to explain why there is a 60% chance of rain (i.e., possible causes for rainfall occurrence). This argumentation resembles what Konold (1989) reported regarding students who thought that humidity or cloudiness is a measure of the strength of factors that would produce rain.

As described above, for the activity of throwing a die, although **o** thinkers thought about the required number of trials in which number 5 surfaces, **o*** thinkers focused on performing techniques to obtain number 5, which mirrors students who emphasized the circumstances in which rainfall may occur for the task of weather predictability. This leads the discussion toward identifying shared conceptions and biases, which students exhibited in both **o** and **o***. For the activity of throwing a die, all students focused their approaches on the next expected event of the number 5 occurrence, which represents the explicitly given outcome in the activity. Further, **o*** thinkers especially tend to evaluate their predictions as valid or faulty. For example, if the die were controlled, then number 5 could be obtained; if not, another outcome will result. Similarly, while interpreting the task of weather predictability, they reported multiple causes for which their predictions could be assessed. For instance, if it was the winter season, there was a 60% probability of rain; inversely, it might not rain if it was summer.

Such reasoning defeats the *theory of experimental probability* that reflects the limit of relative frequencies of an event when an experiment is repeated a large number of times (Konold, 1989; Torres & Contreras, 2014). Moreover, because of **o** and **o*** thinkers' focus on the favorable outcome more than its probability, which affects their utilization of the experimental approach, several conceptions appeared. Although these conceptions agreed with what was explained previously in the giving birth context for **o*** thinkers, it differed slightly for type **o** reasoning. This is detailed in the following paragraphs.

First, **o** thinkers shared the *prediction bias* (not the *prediction conception* as in the case of the giving birth problem) wherein their expectation had the meaning of accurate prediction of whether number 5 will occur in a particular experiment or not. This appeared when they judged precisely that, for instance, number 5 may arise after two trials. As explained before, such *prediction bias* contradicts the essence of *randomness* that demands *uncertainty* in which the specific number of trials to obtain an outcome cannot be defined. This differs from what was discussed regarding the *prediction conception* that signifies a misunderstanding of the prediction purpose. As analysis of the giving birth problem revealed, some students did not fully understand the intention of the expectation as quantifying our information regarding unknown phenomena. Nevertheless, **o** thinkers (in the context of throwing a die) considered the experiment as an undiscovered situation; yet, they gave a precise prediction, which was determined by the number of performed trials to obtain a favorable outcome.

Second, for the sub-category of **o*** that emerged in both contexts of throwing a die and weather predictability, **o*** thinkers also lacked an understanding of the concept of *randomness* that requires *independence*. Exactly as in the giving birth problem, **o*** thinkers shared the *causal conception*, wherein they confused *causality* with *conditionality*. This was reflected in their arguments on how to obtain number 5 (e.g., by controlling the die) as well as on when it is going to rain (e.g., in the winter season or when the sky is cloudy). Thus, *die controllability* was the cause to get number 5; similarly, *fuzzy sky* denoted a reason for rainfall; alternatively, the declared concerns designated several causes for the favorable outcome to arise. In other words, for **o*** thinkers, the conditioning event (e.g., techniques of rolling or humidity) remains the cause, while the favorable outcome (i.e., number 5 or rain) signifies the consequence.

Similar to the exceptional case of **m**** that newly appeared in the context of throwing a die (see Table 27), a distinct sub-category from **O** thinkers emerged in both the contexts of throwing a die and weather predictability. That is, **o**** thinkers resemble **o** and **o*** in manipulating the *experimental probability* but nevertheless have a clear understanding of such interpretation without shared biases or conceptions.

For the task of throwing a die, **o**** thinkers focused on the probability that, for them, describes a posterior judgment since it is essential to obtain the data (frequencies) of outcomes to calculate the relevant probability (Chernoff, 2008). Hence, **o**** thinkers acknowledged the validity of the *experimental probability interpretation* to fulfil the situation of throwing a die, if and only if the experiment had been repeated many times. They also recognized the term *once* in the provided task, which led them to think of increasing the number of trials. Such reasoning reflects an awareness of the *law of large numbers.*

For illustration, two students reported that "*the probability of getting a 5 equals 1/6 through experimenting with the die. However, we cannot depend on one experiment; instead, the number of experiments should be increased.*" They continued, "*as more trials are conducted, a more precise probability estimation can be determined, in which precise means the prior theoretical expectation of 1/6*." Although this reasoning indicates an understanding of the variability concept in which the experimental probability varies upon the frequency of the occurrence of 5 among all trials, both students reported that the probability equals (not approaches) 1/6. Two reasons may explain this: (a) the students may be attracted toward thinking of the theoretical interpretation according to which the ambiguity can be avoided (Stohl, 2005), or (b) they may be careless regarding the probability language. Then, instead of stating that the experimental probability will approach 1/6 after many identical trials, they simply wrote that it equals 1/6.

Additionally, the other nine students among **o**** thinkers did not specify any particular percentage that defines the experimental probability, like the two previously mentioned cases. Instead, they explained that the probability depends on *the ratio of 5's frequencies in the total number of performed trials*. For them, utilizing the experimental approach leads to uncertainty regarding the judgment in which multiple percentages may express the probability of the occurrence of 5, based on how many 5s will appear in a large number of identical experiments. Furthermore, they agreed that as the number of trials increased, the experimental probability approximated the theoretical expectation (i.e., *the law of large numbers*).

In the task of weather predictability, four students utilized the experimental probability to describe a 60% chance of rainfall. They reported that a 60% probability of rain indicates that in the past 100 days that had similar weather and environmental circumstances, rainfall occurred 60 times. Such an explanation speculates based on adequate knowledge regarding the appropriate context in which the experimental probability can be operated. This matches Brase et al.'s (2014) determination of a 30% chance of rain; it describes a model of past weather events in which it rained on 3 out of the 10 previous days that had similar circumstances.

– **The emergence of the new category of the intuitively oriented thinkers [I]**

While the preceding discussion reported how such types of reasoning that characterized students' responses in the context of giving birth emerged in both contexts of throwing a die and weather predictability, this section outlines another type of reasoning that did not appear in the former but arose in the latter, principally, in the task of weather predictability. Note that the sub-categories of **m**** and **o****, which were described earlier, have not been outlined here because they employed *theoretical* and *experimental* probability, respectively. Thus, they both were regarded as sub-categories of **M** and **O** thinkers, and not distinct categories to address in this section.

For the activity of throwing a die, no separate categories emerged, wherein all students' responses to that activity were distributed among **M** and **O** thinkers, as detailed previously. On the other hand, the distinct **intuitively oriented thinkers [I]** category was developed to portray some of the students' responses to the weather predictability task. These responses are displayed in Table 30.

Table 30. Intuitively oriented thinkers' typical responses in the context of weather predictability

	Intuitively oriented thinkers [I]	
	Context of throwing a die	**Context of weather predictability**
Reasoning type	**Type I**	**Type I**
Students' typical responses	Null	It is most probable that it will rain tomorrow, as 60% > 50%.
N = 12	***0***	***12 responses***

The category of **I** thinkers incorporated 12 students who transformed the quantitative expression of a 60% chance of rain to the qualitative one: "*It does mean that: It is most probable that it will rain tomorrow*" (see Table 30). Moreover, while one student continued his answer by declaring, "*Because 60% is higher than 50%, I reported that: It is most probable that it will rain*," all other students wrote, "*Still, we are not sure whether it is going to rain or not*." Such qualitative expressions speculate a novice understanding of the probability that reflects an encapsulation of intuitive views of chance and leads to the idea of committing numbers to uncertain events, which implies the *intuitive probability interpretation* (see Table 17 in Chapter 4).

Although some researchers classified the *intuitive interpretation* under the subjective facet since the usage of qualitative idioms expresses the degree of individuals' confidence in the occurrence of an event (e.g., Torres & Contreras, 2014), **I** was not labeled as a sub-category of **S** thinkers. Instead, it was considered as a category by itself. The reason is that when **I** thinkers gave the expression of most probable, they judged it compared with 50%, whether the given percentage was higher or lower than 50%. Thus, for their case, the *variability* of outcomes (rain or no rain) did not speculate a subjective criterion, but rather a mathematical standard. Beyond that, **I** thinkers understood the idea of *randomness*; it appeared in nearly all replies in which the uncertainty adequately resembled when they reported, "*Still, because of 60% probability, we are not sure that it is going to rain*."

In this sense, it is relevant to clarify that **I** thinkers' responses were not also classified under the **O** thinkers' category. In detail, Konold (1989) judged that students who translated a 70% chance of rain into the definitive qualitative statement of *it is going to rain* are outcome-oriented thinkers. He further explained that such responses were usually accomplished by utilizing the range of 0% to 100% as a decision continuum, where 0% means no, 100% means yes, and 50% indicates *I do not know*. Furthermore, the intermediate values were ultimately associated with one of these three anchors. Nonetheless, in the current investigation, **I** thinkers' responses remained quite different in terms of the degree of certainty of rain occurrence. While all students (in this investigation and in Konold's study) compared the given probability percentage with the three decision points of 0%, 50%, and 100%, **I** thinkers did not employ such precise phrases, which Konold's study participants used. Instead, they adopted skeptical qualitative idioms (e.g., *it is most probable that it will rain tomorrow*) to indicate that uncertainty still exists, as described above.

Finally, the following table summarizes the distribution of emerged types of reasoning in both the contexts of throwing a die and weather predictability among the participants.

Table 31. PSMTs' manners of reasoning in the context of throwing a die and weather predictability

Major category	*For the activity of throwing a die*		*For the task of weather predictability*	
	Frequency	***Percentage***	***Frequency***	***Percentage***
Mathematically oriented [**M** = m + m* + m**]	50	73.5%	7	14.6%
Subjectively oriented [**S** = s + s* + s**]	0	0%	5	10.4%
Outcome oriented [**O** = o + o* + o**]	18	26.5%	24	50%
Intuitively oriented [**I**]	0	0%	12	25%
Sub-categories	***Frequency***	***Percentage***	***Frequency***	***Percentage***
m	10	14.7%	7	14.6%
m*	35	51.5%	0	0%
m**	5	7.3%	0	0%
s	0	0%	0	0%
s*	0	0%	1	2.1%
s**	0	0%	4	8.3%
o	3	4.4%	0	0%
o*	4	5.9%	20	41.7%
o**	11	16.2%	4	8.3%
I	0	0%	12	25%
Total	***68***	***100%***	***48***	***100%***

5.2.3 Characteristics of PSMTs' reasoning in a simple probabilistic situation

Based on the discussions in the preceding sections, in general, students have exhibited four essential types of reasoning under uncertainty; **M**, **S**, **O**, and **I**. Moreover, the utilization of each type varied not only among students themselves but also depending upon the given context. This section focuses on characterizing students' R(in)P, which is related to a simple probabilistic situation. First, Figure 18 displays the distribution of such manners of reasoning in the three contexts of giving birth, throwing a die, and weather predictability, as summarized before in Tables 22 and 31.

First, as exhibited in Figure 18, type **M** reasoning represented the most applied manner of reasoning that strongly appeared in the three contexts of giving birth, throwing a die, and weather predictability, in the percentages of 29.4%, 73.5%, and 14.6%, respectively. Such reasoning relies on utilizing the *theoretical probability* of modeling an uncertain situation. Accordingly, this reasoning was manipulated by defining sets of sample space and favorable outcomes, as follows: (a) the set of boy and girl (sometimes boy, girl, and twins) and the favorable outcome of girl; (b) the set of 1, 2, 3, 4, 5, and 6 (sometimes die sides) and the favorable outcome of five; and (c) the set of all complementary events of rain and the favorable outcome of rain, for the problems of giving birth, throwing a die, and weather predictability, respectively.

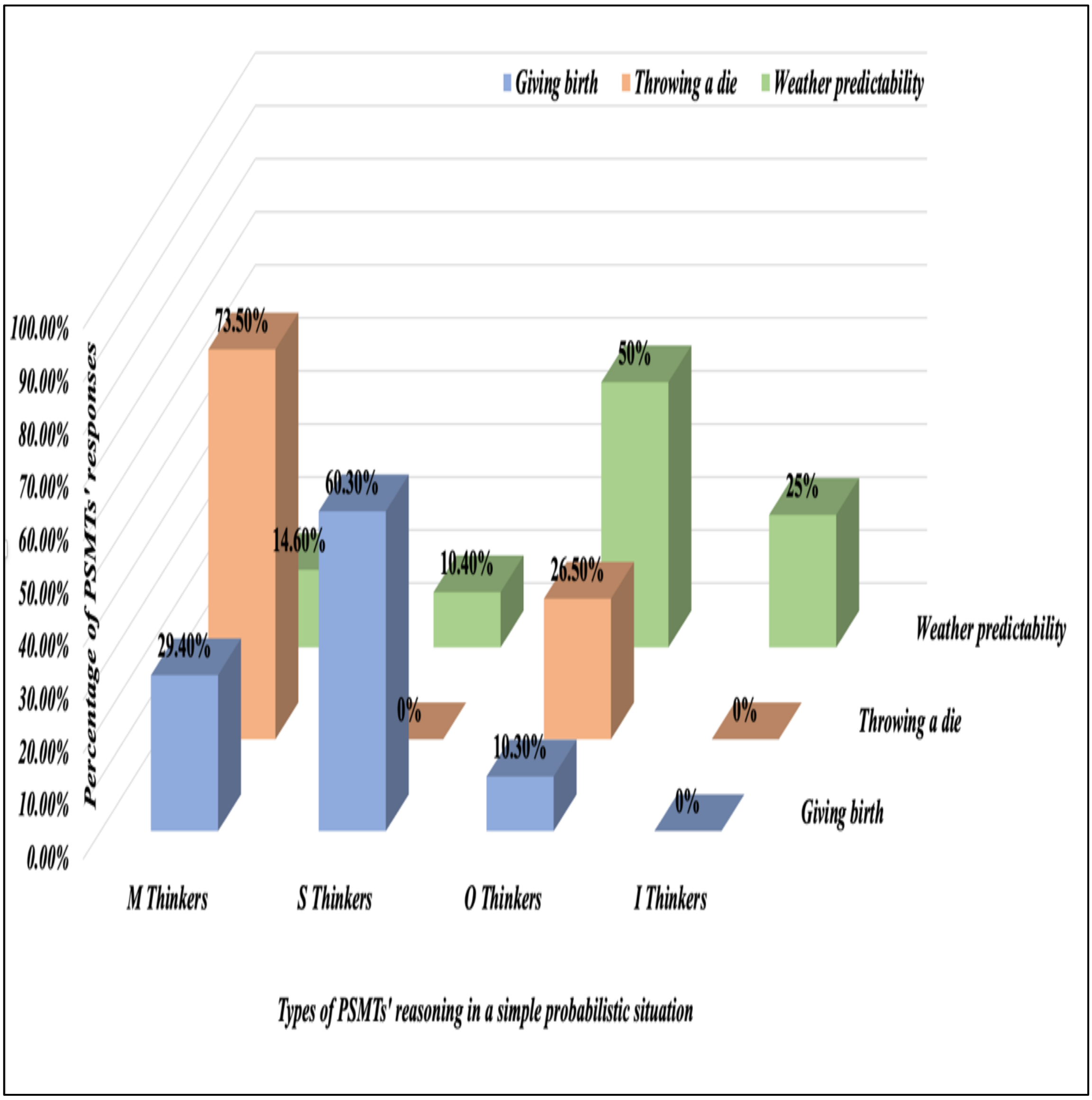


Figure 18. The distribution of reasoning types among PSMTs in the context of giving birth, throwing a die, and weather predictability

Indeed, the dominance of **M** reasoning on students' thinking was not surprising. As reported by Stohl (2005), many teachers prefer approaching the probabilistic situation through theoretical probability. This thinking relies on mathematical techniques and leads to one single answer. For example, regarding the task of throwing a die, as she explained, the experimental probability can also be employed to determine the probability of getting the number 5. Nonetheless, it requires a repeated set of die rolls that most likely will yield different percentages. Thus, employing the theoretical interpretation helps respondents overcome such uncertainty. This typically mirrors students' reasoning in the context of throwing a die wherein

73.5% of them preferred modeling the experiment through the theoretical approach while only 26.5% utilized the experimental interpretation.

Additionally, the **M** thinkers' category involved the three sub-categories of **m**, **m***, and **m**** that all employed *theoretical probability*. However, they differed in terms of shared conceptions and biases. These identified conceptions are discussed next in light of the concepts of *randomness* and *variability*. Both theoretical constructs define *two essential factors to reason probabilistically, that is, the determinants of the process of probabilistic reasoning* (see p. 70). This may lead to the unified schema that connects all emerged types of reasoning in which students' R(in)P can be characterized.

Regarding the *variability,* all **M** thinkers expressed an understanding of such an idea; for them, it indicated that the resultant outcome varies depending upon the possible events in the sample space. Nonetheless, they shared several conceptions related to their understanding of *randomness*. In detail, both **m** and **m*** thinkers shared the *equiprobable bias,* which made them *insensitive to the prior probability of outcomes*. Admitting that the meaning of *randomness* differs based on the individual's understanding of probability (Batanero et al., 1998), the emergence of that *equiprobability* reflects **m** and **m*** thinkers' conception of *randomness*. This conception is exemplified in Lecoutre's (1992) argument regarding the random nature of the experiment. It is considered by some learners as a sufficient indication for equiprobable outcomes; that is, *two events are equiprobable because it is all about chance*. Accordingly, if the favorable outcome is randomly generated, it has the same probability to occur as any other event in the sample space. This illuminates why these thinkers (a) ignored the actual gender distribution for the problem of giving birth, (b) did not confirm the die regularity in the activity of throwing a die, and (c) supposed that rainy, windy, and sunny are equally likely outcomes in the context of weather predictability.

Furthermore, the data analysis revealed another significant factor, beyond the variability and randomness, that should be considered when thinking probabilistically: *contextual recognition*. Drawing on Pfannkuch's (2011) study, *contextual recognition* here signifies the *data context*, which reflects the context of the real-world situation from which the problem arose. Following this argument, although **m** and **m*** thinkers shared the *equiprobability* that remained a common conception and reflected their understanding of *randomness*, they differed in terms of the *contextual recognition.*

On one hand, **m** thinkers were able to connect mathematical models of probabilistic situations with the related realistic conditions. This appeared in their responses to three tasks, wherein they described sample spaces of the (a) boy and girl, (b) die sides, and (c) environmental circumstances (e.g., rainy, windy, cloudy). They favored listing the authentic elements of the sample space over the general abstract formula. Consequently, the *representativeness heuristic was judged to be the origin of equiprobability* for **m** thinkers. Although they understood the realistic features of each situation, they overestimated the replicability of the experimental results (Lecoutre, Durand, & Cordier, 1990), in which any distribution must resemble the parent population. In this sense, such heuristic oriented **m** thinkers think of all the situational circumstances as equiprobable merely because of the random phenomenon, as is often reflected in the population. Thus, these thinkers were reluctant to confirm the required assumptions of utilizing theoretical probability.

On the other hand, **m*** thinkers showed an abstract mindset that attempted to drop the realistic circumstances to be able to theoretically interpret the probabilistic situation through the lens of Laplace's theory. This was termed as the *overgeneralization heuristic,* which defined the origin of *equiprobability* for **m*** thinkers and indicated individuals' mental orientation toward providing an ideal mathematical abstract model that works for any situation. In this sense, both *representativeness and the overgeneralization heuristic* are relevant to *contextual recognition*. While the former reflects an understanding of realistic circumstances to be equal in occurrence, the latter declines such practical obstacles and instead seeks a general formula.

It is worthy to report that both the *equiprobable bias and the representativeness heuristic* were discussed reciprocally in the literature (e.g., Dollard, 2011; Konold, 1989; Lecoutre et al., 1990; Pratt, 2000; Savard, 2010; Tversky & Kahneman, 1974). The *equiprobability* defines the tendency to believe that any random process produces a fair distribution with equal probabilities for each possible outcome (Savard, 2010). This bias was found in different contexts when determining the probability of simple, compound, and conditional events (Watson, 2005; Watson & Moritz, 2003). It was also reported for many learners in different grades. For example, Lecoutre (1992) described that college students exhibited the equiprobability bias when comparing the probability of rolling a five and a six on two dice with the probability of rolling two sixes. Furthermore, Pratt (2000) and Zawojewski and Shaughnessy (2000) presented similar findings for fifth- and twelfth-grade students.

Additionally, this study defined the *representativeness heuristic* as one plausible cause for *equiprobability*, which is quite different from the previous studies' arguments. In several previous studies, *representativeness* was addressed in a comparison context, such as judging which sequence of male and female births among MMMMMM and MFFMMF is less likely to occur (Kahneman & Tversky, 1973). Accordingly, in such a context, representativeness indicates the degree of similarity between the sample and population. Nonetheless, within the context of this study, it reflects the students' overestimation of the replicability of experimental results, as reported beforehand. Since students strongly relied on the randomness property to yield equally likely outcomes, all the events generated by that random system and relevant samples were judged to have the same probability.

While **m** and **m*** thinkers shared the above-reported biases that were mainly connected to their perception of *randomness*, **m**** thinkers exhibited an adequate understanding of the *theoretical probability* in terms of *variability, randomness, and contextual recognition.* Such reasoning appeared only in the task of throwing a die in which **m**** thinkers were able to differentiate between *randomness* and *fairness* (Pratt, 2005). Although they admitted the appropriateness of theoretical probability to model the experiment of throwing a die, they reported that the probability of obtaining number 5 equals 1/6 if and only if the die was fair. Similarly, acknowledging the required assumptions to manipulate the proper probability interpretation, some **m**** thinkers stated that the reason they relied on the theoretical probability was the conditions of the given activity, that is, the activity fixed one experiment of throwing a die in which the *experimental approach could not be operated.* Hence, the case of **m**** thinkers reveals an adequate recognition of the *contextual circumstances.* This helped clarify the meaning of *contextual recognition* as a factor required to reason probabilistically. It not only designates students' ability to connect the mathematical model with real circumstances, but it also indicates their recognition of the required assumptions to select and handle the appropriate probability interpretation that may fulfil the situation.

Second, **O** reasoning defined the second prevalent model of thinking after **M** reasoning; it also emerged in students' responses to the three problems of giving birth, throwing a die, and weather predictability, in the percentages of 10.3%, 26.5%, and 50%, respectively (see Figure 18).

According to such type of reasoning, *experimental probability* implies the preferred approach to model an uncertain situation; yet, it was not fully understood by the students. They thought about experimentations not to define the probability but to anticipate the favorable outcome (except the case of **o**** thinkers). This was reflected when they decided to (a) check the posterior results of the delivery process to judge the expected event of a girl's birth, (b) determine the number of trials or the possible techniques at which number 5 can be obtained, and (c) define several causes of why rainfall will occur, for the problems of giving birth, throwing a die, and weather predictability, respectively. Moreover, **O** thinkers (except **o****) interpreted these tasks as if the question was under what circumstances (a) the woman is going to deliver a baby girl, (b) the die rolling will produce number 5, and (c) rainfall will occur, respectively. Thus, their judgments were self-evaluated as being *right or wrong* on whether such favorable outcomes would occur in a particular trial or not. This means that, for them, the next expected event will or will not be (a) a baby girl, (b) number 5, and (c) rain occurrence, respectively. In this way, their given arguments reflected multiple *causes* because of which that outcome might occur if those causes were sustained.

As revealed, **O** thinkers' category included the three sub-categories of **o**, **o***, and **o****, which all operated the *experimental probability*. While both **o** and **o*** thinkers misunderstood the *experimental probability*, as described above, **o**** thinkers showed an adequate understanding of it since they recognized the *experimental probability* to be the relative frequency of occurrence of an event in a large number of trials (see Table 29). This appeared when **o**** thinkers (a) calculated the probability of getting a 5 by manipulating the die many times to get a precise estimation that approaches their prior theoretical expectation and (b) interpreted a 60% chance of rain in terms of previous similar environmental circumstances. In addition, the next discussion explains how the concepts of *randomness*, *variability,* and *contextual recognition* were perceived by **o**, **o***, and **o**** thinkers.

Regarding *variability,* although all **O** thinkers admitted it, the ways in which they perceived such variability were quite different. The *variability* for **o** and **o*** did not depend on the frequencies, but instead, on *one single trial* through which the favorable outcome can be interpreted. This explains why they adjusted their expectations to be within two sets: one contained the favorable outcome, while the other included all other outcomes (i.e., the complementary set). Their predictions were self-evaluated to be correct for the former and faulty for the latter. On the contrary, **o**** thinkers exposed sufficient knowledge of the concept of *variability* in which the estimation varies depending upon the frequencies in the total number of performed trials.

Additionally, other conceptions relevant to **o** and **o*** thinkers' understanding of *randomness* were also defined. For **o** thinkers, who thought of examining the next posterior result of the random process to operate the experimental probability, both the *prediction conception* and *prediction bias* emerged. The *prediction conception* signifies a misunderstanding of the expectation's intention and does not recognize it to be a way to quantify our information regarding unknown phenomena. This emerged when some **o** thinkers supposed that after the experiment occurred, we would still have to predict the outcome, which is not correct since the phenomenon will not be probabilistic anymore. On the other hand, although the *prediction bias* shows an understanding of the expectation's intention, this expectation is often judged precisely. Thus, because *randomness* requires *uncertainty* and *independence*, all students who shared the *prediction conception* and *prediction bias* declined such randomness, precisely the *uncertainty*.

Similarly, for **o*** thinkers, other conceptions that indicated various understandings of *randomness* appeared. Nonetheless, it was not related to the *uncertainty*, similar to **o** thinkers, but rather to the concept of *independence* as one factor associated with randomness. As detailed in the three contexts, **o*** thinkers confused *causality* with *conditionality*, wherein they judged the conditioning event that they self-reasoned (decided it by themselves) to be a cause for the favorable outcome that represents the result. This explains why they considered (a) the woman's appearance, (b) die controllability, and (c) environmental circumstances to be the causes of having a baby girl, obtaining number 5, and rainfall occurrence for the tasks of giving birth, throwing a die, and weather predictability, respectively. Hence, **o*** thinkers exposed their *causal conception*, which denies the *independence* that remains an essential feature of probabilistic reasoning. As reported earlier (see Chapter 4), in a probabilistic situation, there is neither dependence nor causality; moreover, the present information cannot provide enough evidence to explain the resultant outcome.

Again, **o**** thinkers exhibited an adequate understanding of *variability*, and they also displayed a tacit recognition of *randomness*. Such randomness appeared obviously in the activity of throwing a die for students who acknowledged *the law of large numbers to obtain a better judgment that approximates the theoretical expectation of 1/6*. For them, the favorable outcome of 5 could not be undoubtedly predicted; alternatively, when the number of trials increased, the prediction would be more accurate. Thus, *randomness* could generate a fair distribution in the long term if and only if the number of trials were increased.

In addition to the previous analysis that detailed **O** thinkers' conceptions of *variability* and *randomness*, this paragraph describes how they recognized the *contextual concerns* that were embedded in the three given tasks. On one hand, all **O** thinkers identified real-world conditions from which the problem arose. As revealed, in the context of giving birth, **o** and **o*** thinkers strengthened several circumstances they may encounter in daily situations such as miscarriage or women's bodily appearance (see Table 25). Similarly, their recognition of the data context was exhibited when they declared issues such as the following for the tasks of throwing a die and weather predictability, respectively: (a) the backgammon game that signifies one dominated traditional game is often played by Egyptian males and involves using two dice, and (b) overcast, cloudy, humidity, or stormy weather is the cause because of which rainfall may occur.

On the other hand, **o**** thinkers showed another type of *contextual recognition*: the *task context* (Pfannkuch, 2011). While the *data context* defines students' awareness of the real context from which the problem arose, and it was reflected in **o** and **o*** thinkers' responses, the *task context* denotes one of the learning experience contexts (i.e., historical, social, and task contexts). It includes identifying the task sequence and its motivating story (Hershkowitz, Schwarz, & Dreyfus, 2001, as cited in Pfannkuch, 2011). Thus, recognition of the *task context* was mainly expressed in **o**** thinkers' awareness of the term *once*; it motivated them to increase the number of trials to calculate the experimental probability of obtaining number 5 (the motivating story). Moreover, **o**** thinkers acknowledged the constraints of manipulating the experimental probability which requires conducting a very large number of identical trials. Similarly, their reliance on the experimental probability to approach the weather predictability task and explain a 60% chance of rain designates a clear understanding of the appropriate circumstances when that experimental probability works.

In general, the case of **O** thinkers (except **o****) mirrored Konold's (1989, 1995) identification of the outcome approach; it is the manner in which adults (undergraduate students) performed informal reasoning under uncertainty wherein they understood that their task was to decide what is going to occur. Consequently, their focus shifted to the favorable outcome itself, whether it is going to happen or not. Moreover, such type of thinking was described in some previous studies as *deterministic reasoning* that is usually employed in contexts where there is no uncertainty (Savard, 2010). As a pioneer, Konold (1989) stated that outcome-oriented thinkers' predictions represent a deterministic model of the situation. This model is mostly generalized by students to all other situations (Musch & Ehrenberg, 2002), affecting their probabilistic reasoning. Also, Engel and Sedlmeier (2005) described that secondary school students held a mechanistic-deterministic view of the world, which is difficult to change even when increasing years of schooling. Perhaps the position of probability within the mathematics school curriculum denotes one cause for why students still exhibit such deterministic reasoning that persists even in university students. PCMI (2017) declared that because statistics and probability are included in other courses (e.g., algebra), students often find it difficult to differentiate between deterministic and probabilistic reasoning. Such argument resembles the Egyptian context wherein statistics and probability are usually admitted as the last unit within the algebra course.

Additionally, other studies identified similar conceptions and biases that **O** thinkers shared (i.e., *prediction bias, chance controllability, causal conception,* and *prediction conception)*. For example, the *prediction bias* was described in different studies and was also associated with students' deterministic reasoning. As clarified in Savard's investigation (2010), some students utilized deterministic reasoning wherein they supposed that prediction means to determine the next outcome *certainly*. Besides, the conception of *chance controllability*—which appeared in some responses that affirmed managing the technique of die rolling to get number 5—was reported by Estrella and Olfos (2010). In addition, Theis and Savard (2010) termed it as the *illusion of control* to reflect individuals' belief that they control issues of chance games. Furthermore, Larose et al. (2010) stated that the reason for holding such a conception is that the probability instruction does not draw on students' real social practices (e.g., gambling games), of which only these practices can help them to overcome several erroneous conceptions, such as the *chance controllability;* it is inherited in individuals' everyday life psychology. This resembles what Konold (1995) declared about teaching probability that is rarely built upon authentic contexts; thus, students often exhibited deterministic conceptions of probability.

The *causal conception* often arises in reasoning under uncertainty wherein individuals have a strong natural tendency to search for specific causes (Wild & Pfannkuch, 1999). This agrees with what Konold (1989, 1991) affirmed concerning the outcome approach wherein individuals often base their predictions on a *causal analysis of the situation*. Although the outcome approach is inconsistent with the experimental probability interpretation, it is still reasonable in various everyday decisions (Konold, 1989) in which causality is practiced. As reported before, the tendency to think of causes to explain an event's occurrence devalues the concept of randomness, as chance stems from what is not attributable to linear causality (Larose et al., 2010).

Finally, the *prediction conception*, a term coined by the researcher, defines students' misunderstanding of the prediction purpose. Although there was no such recognized conception among the reviewed studies, Devlin's (2014) argumentation in the "Foreword" of the book *Probabilistic Thinking: Presenting Plural Perspectives* expressed a relevant idea. He argued that in a random situation like tossing a coin, if the coin was thrown and the outcome became already known by some students while it is still unknown to others, the distinction between the two groups is what they know about the outcome. In this sense, the probability quantifies our information about events, but not the events themselves. This idea was not fully grasped by some students in the current investigation.

The third manner of reasoning that the students exhibited is **S**. It appeared in both the contexts of giving birth and weather predictability in the percentages of 60.3% and 10.4%, respectively; however, it did not emerge in the task of throwing a die (see Figure 18). As stated before, the subjective interpretation of probability can lead different individuals to specify different probabilistic values for the same event. This may happen even within the context of throwing a die wherein the probability of getting number 5 changes based on the individual's information about factors such as fairness of the die or the technique of throwing itself (Konold, 1989). Similarly, when a coin is tossed, if any circumstance such as the way a person tosses, air movement, or peculiarity of the ground is altered, we might obtain other events (Rast, 2005). Nevertheless, the emergence of type **S** reasoning, specifically in the contexts of giving birth and weather predictability, matches the interpretation of many researchers regarding the value of authentic daily life situations in revealing the subjective side of probability and further probabilistic reasoning (e.g., Chassapis & Chatzivasileiou, 2008; Konold, 1995; Larose et al., 2010; Musch & Ehrenberg, 2010; Savard, 2008).

M and **O** reasoning reflect the objective side of probability, in which probability defines the property of an object, and they are often separated from a person's judgments (Borovcnik, 2012). However, **S** reasoning reveals the epistemic subjective side based on which the probability can be always revised and updated according to the individual's knowledge and experiences. This explains why **S** thinkers relied on various information regarding (a) issues of the woman genetic state or sonar results and (b) the factors of season, time of the year, inclination and intensity of clouds, or wind movement to judge the probability of giving birth to a girl and interpret a 60% chance of rain, respectively. Moreover, all these circumstances specified several sources of information that may alter their judgment; they did not work as causes or reasons that justify why a particular event occurred like in the case of **o*** thinkers. In this sense, **S** thinkers speculate what Borovcnik (2012) stated regarding subjectivists who consider probability to be the degree of credibility that is judged based on various types of information; this information may stem from relative frequencies, experts' knowledge, or can be formed by personal expectations and experiences, which is the current case of **S** thinkers.

As revealed, the **S** thinkers' category included three sub-categories of **s** that arose only in the context of giving birth; and **s*** and **s**** that emerged in both the contexts of giving birth and weather predictability. Besides, each sub-category outlined some of the students' conceptions and biases; they were characterized in terms of the factors of *variability*, *randomness*, and *contextual recognition*, as follows.

All **S** thinkers regarded the *variability* of outcomes; for them, it meant that the expected outcome alters depending upon the available information regarding the phenomenon under study. Thus, their responses were expressed in a common form of *it depends* (see Tables 24 and 28). As reported earlier, in the context of giving birth, although they determined the probability of giving birth to a girl mathematically (see Table 21), they also expressed that such given percentages (their numerical estimations) may change considering the available information about the pregnant woman. Moreover, for the task of weather predictability, they affirmed that a 60% chance of rain did not indicate an absolute value; instead, it designated several environmental circumstances that work together, based on which this percentage might alter. Nonetheless, the nature of the conditions in which the outcomes' variability was assigned varied between **s, s***, and **s****.

Both **s** and **s**** attributed the *variability* to several cognitive criteria, such as considering spontaneous abortion as a possible outcome or the issues of X and Y chromosomes to explain why the probability of giving birth to a girl may alter. Similarly, **s**** thinkers reasoned about multiple environmental circumstances (see Table 28) to describe a 60% probability of rain. On the other hand, **s*** thinkers emphasized the religious *conception of Allah's will* as a possible factor that may alter the outcome. Furthermore, in this study, the conception of *Allah's will* determined a specific type of probabilistic reasoning, which was judged in light of admitting the influence of socio-cultural factors on students' conceptions of probability. This is in contrast with some other studies that considered the animism attribution of phenomena to God to be a *personalist interpretation* or *superstitious reasoning*, as reported before.

While **s*** thinkers' understanding of the concept of *variability* remained different from both **s** and **s****, other differences relevant to the *randomness* were found between **s** and both **s*** and **s****. **s** thinkers shared the *prediction bias* that emerged only in the giving birth context; consequently, some students thought that the available information could help them judge the probability certainly. For example, if the ultrasound scan showed a baby girl, then the probability of giving birth to a girl would change to 100% (see Table 24). This resembled **o** thinkers' reasoning in the task of throwing a die, wherein they precisely determined the number of trials after which number 5 can be obtained. Thus, the *prediction bias* appeared in two different contexts, and it originated in various manners of reasoning; nonetheless, the form of that *bias* was a little different. The *exact prediction* was expressed as (a) 100% for **s** thinkers because they kept their focus on the probability and (b) a specific number of experiments to get a particular outcome for **o** thinkers who sharpened the favorable outcome. In addition to the *prediction bias*, some **s** thinkers maintained the *dependence conception* to interpret the dependent relationship between two events as a *causal relationship*. Thus, for **s** thinkers, the probability of giving birth to a girl was determined by previous babies' gender, which also eliminates the *randomness* that demands *independence*.

On the contrary, both **s*** and **s**** thinkers acknowledged the *randomness.* This emerged in the context of giving birth when they declared that the probability of giving birth to a girl could not be predicted with a 100% certainty; similarly, they claimed that a 60% chance of rain was not an exact judgment, but it might alter depending upon the interplay among several conditions (see Table 28). Because **s**, **s***, and **s**** thinkers relied on the *subjective probability* to explain both the contexts of giving birth and weather predictability, it is reasonable to share different conceptions about *randomness*. This was clarified by Batanero et al. (1998) that for subjectivists, *randomness is also subjective.* It is no longer an objective physical property, but rather a subjective judgment (Batanero, 2015), which means that what may be random to one person may not be random to another. This typically explained **s** and **s**** thinkers' decisions regarding the probability of giving birth to a girl; it was judged as a non-random phenomenon after knowing the ultrasound scan results for **s** thinkers, but it stayed an uncertain phenomenon for **s**** thinkers.

Also, *contextual recognition*, as revealed by the analysis, defined two issues of the (a) *data context*, which refers to students' understanding of the real context from which the problem emerged and (b) *task context*, which reflects an understanding of the task's motivating story using which the appropriate probability interpretation can be handled depending upon the situational circumstances of the random phenomena. In this regard, all **S** thinkers showed the *data context* wherein they all relied on several real conditions to explain both the situations of giving birth and weather predictability. Moreover, they showed a recognition of the *task context* in which these situations, for **S** thinkers, might be operated through the subjective probability interpretation.

According to De Finetti (1974), because "the degree of belief in the occurrence of an event attributed by a person at a given time with a given set of information is the subjective probability" (Rast, 2005, p. 21), situations such as election, winning a lottery or a chance game, gender of a child at birth, and the state of the weather should be approached through subjective probability. The reason is that such situations cannot be repeated under the same conditions (Rast, 2005). Following this argument, **S** thinkers were judged to have a kind of understanding of the probability context. This meant that each probabilistic situation represents a particular case wherein past information cannot help one attain a reasonable judgment regarding that situation. This explains why **S** thinkers did not rely on the experimental probability to approach both the contexts of giving birth and weather predictability.

Lastly, the fourth manner of students' reasoning in a simple unconditional probabilistic situation is **I**. It was applied by 25% of students who responded to the task of weather predictability (see Figure 18) wherein they interpreted a 60% chance of rain using the qualitative expression of "*it is most probable that it will rain tomorrow"* (see Table 30).

Although **I** reasoning appeared only in the task of weather predictability, it was practiced by a quarter of the respondents. Besides, as clarified before, **I** thinkers were neither classified as a sub-category of **S** (e.g., Torres & Contreras, 2014) nor **O** (e.g., Konold, 1989). They judged a 60% probability of rain compared with 50%, whether it is higher or lower, which means that their criterion was not subjective but mathematical (still, without relying on theoretical interpretation). **I** thinkers also focused on defining the probability without reflecting on similar prior circumstances; in other words, they did not intend to utilize the experimental probability. Based on this, **I** thinkers' conceptions of *variability*, *randomness*, and *contextual recognition* were interpreted as follows:

About *variability*, **I** thinkers recognized that the probabilistic situation involved more than one possible outcome. However, their alternatives included only two options of the favorable outcome or any other event, which is similar to the case of **o** and **o*** thinkers. Furthermore, their understanding of *randomness* was expressed in a common qualitative expression of "*Still, because of 60% probability, we are not sure that it is going to rain*." That is, for **I** thinkers, *randomness* reflected a quantification of their information about the situation; it was judged in light of a continuous decision line that rangs from 0% to 100%. Notwithstanding, **I** thinkers showed a novice recognition of the *task context* wherein the uncertain situation was explained qualitatively.

According to Lysoe (2008), four categories determine the usage of uncertain words (hedges in the language). Category 1 involves expressions such as "He is likely to come," which reflects the individual's awareness about the existence of other outcomes rather than the mentioned one (i.e., *the variability of outcomes*). Category 2 includes expressions as "It is less likely that she is going to use those shoes," which signifies the strengthening of Category 1 since the event in Category 2 was described to have a smaller (or greater) chance of occurrence. Category 3 includes expressions such as "The probability of being shot by a policeman is greater than the risk of being murdered by a professional killer," which contains a specific description of what alternatives may occur. Finally, Category 4 reflects high accuracy compared with the other categories because of the usage of numbers that quantify how probable some events are, such as "The police said that the probability of clearing up this case is 100%." Although almost all **M**, **O**, and **S** thinkers belonged to Category 4 since they quantified the random phenomena (the three given tasks) using specific mathematical percentages, **I** thinkers remained in level 2 because their interpretation of the probability included qualitative idioms (e.g., most probable). This illustrates why **I** thinkers were judged to have a novice recognition of the *task context;* they committed to the second level of Lysoe's (2008) categorization, which describes how individuals may develop their intuitive knowledge of probability.

Based on what was discussed about the four distinguished manners of students' reasoning and their shared conceptions characterized in terms of *variability*, *randomness*, and *contextual recognition*, Table 32 displays the developed schema that exemplifies students' R(in)P, which is related to simple unconditional probabilistic situations. It showed that from the PoPR, the *theoretical constructs* (*concepts*) students often rely on to model probabilistic phenomena include *variability*, *randomness*, and *contextual recognition*; however, students still share various conceptions of each construct depending upon their different manners of reasoning.

Table 32. A model of PSMTs' R(in) P that is related to a simple unconditional probabilistic situation

Reasoning types	***Theoretical constructs that are required to reason probabilistically***		
	Variability	***Randomness***	***Contextual recognition***
M reasoning: It models the uncertain situation through the theoretical probability			
m	The outcomes vary depending upon several possible events in the sample space. **[Variability by sample space elements]**	• Equiprobability • Insensitivity to the prior probabilities of the outcomes (the random nature of the experiment remains a sufficient indication of equiprobable outcomes) **[Randomness as Equiprobability]**	• Representativeness heuristic **[Context defines the realistic circumstances that are equal in occurrence to explain the uncertain situation]**
m*			• Overgeneralization heuristic (the practical obstacles must be declined, and instead, a general formula should be developed) **[Context defines a barrier against interpretation of the uncertain situation]**
m**		**[Randomness does not always yield a fair distribution]**	Depending upon the circumstances of the uncertain phenomenon, the appropriate probability interpretation should be utilized. **[Context defines the conditions of the task that may strengthen the utilization of a specific probability interpretation more than another]**
O reasoning: It models the uncertain situation through the experimental probability			
o	The outcomes vary between two alternatives of either the	• Prediction conception (misunderstanding of the expectation's intention) • Prediction bias	Several daily life situations resemble the given contexts, such as the backgammon game or the rainy weather in winter.

	favorable outcome or any other event. **[Variability by either a specific outcome or its complementary]**	(the prediction has the meaning of the exact prediction) **[Randomness can be judged certainly]**	**[Context defines realistic circumstances that resemble the uncertain situation]**
o*		• Causal conception (the conditioning event is the cause for the favorable outcome occurrence) **[Randomness does not always require independence]**	
o**	The outcomes vary depending upon the possible resultant frequencies in a large number of performed trials. **[Variability by the frequencies of many trials]**	Randomness generates a fair distribution in the long term if and only if the number of trials has been increased. **[Randomness as stability of frequencies]**	Depending upon the circumstances of the uncertain phenomenon, the appropriate probability interpretation should be utilized. **[Context defines the conditions of the task that may hinder the utilization of a specific probability interpretation]**
S reasoning: It models the uncertain situation through the subjective probability			
s	The outcomes vary depending upon the multiple available information about the phenomena. **[Variability by the available information]**	• Prediction bias • Dependence conception (if two events are dependent, then one is a cause for the other) **[Randomness as self-criterion based on the credibility of the available information]**	Several real situations may explain the probabilistic phenomenon. **[Context defines realistic circumstances that are known at the moment by a specific person to explain the uncertain situation; thus, each situation is restricted by the information available to the person who is judging it]**
s*	The outcomes vary depending upon Allah's will. **[Variability by Allah's will]**	Randomness still exists even after adapting the new information; that is, whatever we knew, the outcome could not be anticipated certainly. **[Randomness as a self-criterion based on the credibility of the available information]**	
s**	The outcomes vary depending upon the multiple available information about the phenomena. **[Variability by the available information]**		
I reasoning: It explains the uncertain situation using qualitative expressions			
	The outcomes vary between two alternatives of either the favorable outcome or any other event. **[Variability by either a specific outcome or any other event]**	Randomness reflects any percentage that lies on the continuous decision line ranging from 0% to 100%. **[Randomness as an expression of any percentage ranging from 0 to 100]**	**[Context defines the usage of qualitative expressions to explain the uncertain situation]**

5.3 PSMTs reasoning in a conditional probabilistic situation

5.3.1 PSMTs reasoning in the context of giving birth after adding a new condition

As mentioned, the intention of analyzing students' responses to Item **B**, which denoted a modification of the problem of giving birth by adding one condition, was characterizing their R(in)P that is related to a conditional probabilistic situation. This helped clarify the *notion of subjective probability as a general classifier*. In that sense, it is necessary to, again, highlight the epistemological difference between both Items **A** and **B**. Item **A** was designed to address students' reasoning in a simple probabilistic situation (as stated earlier); accordingly, it was formulated in a static form without any given conditions. On the other hand, Item **B** signified a diachronic situation that contains a series of sequential experiments carried out over time (Díaz & de la Fuente, 2007; Díaz et al., 2010)—that is, P(G|BB).

As detailed in Appendix 8, Item **B** exemplified this situation: If you knew that a woman had given birth to two boys before, and she will give birth to her third child, then (Q1) What is the probability of her giving birth to a girl in the new case (i.e., after incorporating the given condition)? and (Q2) Explain how you have determined such a probability. In other words, why do you think that your expectation in the first situation (i.e., firstborn) is the same or different than in the second one (i.e., third born)?

Before progressing into detail, Table 33 summarizes students' expectations regarding the probability of giving birth to a girl after knowing that the woman gave birth to two boys before. Their expectations varied between keeping the initial percentages (i.e., what they stated earlier in Item **A**) and adjusting it higher or lower. This represents their numerical answers given in response to the first question of Item **B**.

Table 33. PSMTs' expected probabilities of giving birth to a girl after knowing about the condition of giving birth to two boys before

<table>
<tr><th colspan="11">Typical responses</th></tr>
<tr><td></td><td colspan="10">Probability of giving birth to a girl</td></tr>
<tr><td></td><td colspan="4">is still as same as the first situation in Item A</td><td colspan="6">differs from the first situation in Item A. Thus, the estimated value will change</td></tr>
<tr><td>Student stated probability (numerical answers)</td><td>1/2 or 50%</td><td>1/3 or 33.3 %</td><td>1/5 or 20%</td><td>3/5 or 60%</td><td>from 50% to a lower value (33.3%; 25%; 20%)</td><td>from 33.3% to a lower value (25%; 30%)</td><td>from 60 % to a lower value (25%; 40%)</td><td>from 50% to a higher value (60%; 70%; 80%)</td><td>from 33.3% to a higher value (50%; 40%)</td><td>from 20 % to a higher value (50%; 33.3)</td></tr>
<tr><td>N = 68</td><td>25</td><td>6</td><td>7</td><td>2</td><td>13</td><td>4</td><td>1</td><td>3</td><td>3</td><td>4</td></tr>
<tr><td></td><td colspan="4">40 responses did not changed their initial expectations</td><td colspan="3">18 responses have decreased their initial expectation</td><td colspan="3">10 responses have increased their initial expectation</td></tr>
</table>

Additionally, the students expressed several reasons to keep or change their initial estimations after knowing the given condition. Accordingly, their responses were categorized (*inductive data analysis process*) under the principal categories of **G**eneralizer [**G**], **CON**servative [**CON**], **COR**relational [**COR**], and **R**ational [**R**] thinkers; also, each category included sub-categories (except **R**). Table 34 summarizes the distribution of such manners of reasoning among the students. Moreover, in this study, while both **G** and **CON** defined intrasubjective probability, **COR** and **R** represented intersubjective probability (Chernoff, 2008), as explained in Section ***5.3.3.***

Table 34. PSMTs' manners of reasoning in the context of giving birth after knowing that the woman gave birth to two boys before

Major categories	***Frequency***	***Percentage***
Generalizer [**G** = HOL + A]	40	58.8 %
Conservative [**CON** = SO.C + SU.C]	8	11.8 %
Correlational [**COR** = HOR + V]	16	23.5 %
Rational [**R**]	4	5.9 %
Minor categories	***Frequency***	***Percentage***
HOL	33	48.5 %
A	7	10.3 %
SO.C	3	4.4 %
SU.C	5	7.4 %
HOR	3	4.4 %
V	13	19.1 %
R	4	5.9 %
Total	***68***	***100.0***

In detail, the next discussion explains each type of reasoning. It starts with students who disregarded the condition of giving birth to two boys before from the analysis; these included both **G** and **CON** thinkers, as follows:

- **First: PSMTs who disregarded the given condition from the analysis**

Based on students' responses, two broad categories of ***Generalizer*** and ***CONservative thinkers*** were inferred, whose ways of reasoning are simplified in Tables 35 and 36.

G thinkers kept their initial estimation, in which they agreed that the probability of giving birth to a girl after knowing that the woman gave birth to two boys before remains the same as the probability of giving birth to a girl without any given circumstances. Consequently, their common response was "*there is no difference between our expectations or the way we thought in both situations.*" Although **G** thinkers judged P(G|BB) as equal to P(G), their stated reasons were quite different. While **HOL** thinkers highlighted the process itself (*the random process*), the outcomes of that process were strengthened by **A** thinkers.

Table 35. Generalizer thinkers' typical responses in the context of giving birth

<table>
<tr><th rowspan="2"></th><th colspan="3">Generalizer thinkers [G]</th></tr>
<tr><th colspan="2">Holistic [HOL]</th><th>Atomistic [A]</th></tr>
<tr><td rowspan="2">Students' typical responses</td><td colspan="3">The probability will not change because</td></tr>
<tr><td>we still have the same possibilities of S = {B, G; B, G, twins; or B, G, BB, BG, GG}, as all expected outcomes.</td><td>determining a baby's gender implies a random process, in which we cannot predict the outcome with certainty. Hence, the way of predicting is always the same, no matter first, second, or third born.</td><td>the situation signifies a random process, which means that its events should be independent. Thus, there is no relationship between the previous babies' gender and that of the newborn, and our estimation remains the same as Item A.</td></tr>
<tr><td rowspan="2">N = 40</td><td>26</td><td>6</td><td rowspan="2">7 responses</td></tr>
<tr><td colspan="2">33 responses</td></tr>
</table>

This manner of reasoning can be interpreted in light of the *anchoring and adjustment bias* that demands adjusting the initial value to yield the final answer (Tversky & Kahneman, 1974). From this aspect, **G** thinkers first generated a preliminary judgment called the *anchor*; then, in the second stage, they adjusted that judgment to incorporate the additional given information. Nevertheless, their adjustment was insufficient (Lieder, Griffiths, Huys, & Goodman, 2017). Concretely, **G** thinkers first developed their anchor from experiencing the first situation of Item A; that is, any random experiment yielded a fair distribution wherein P(G) = P(B) = P(twins) and its outcomes were independent (i.e., *equiprobable bias*). Later, in the second stage of handling the new situation of Item B, they perceived that situation through the previously generated anchor. Consequently, they acknowledged that for any random process, all outcomes are independent and equally likely to happen without thinking of how the new information about the previous babies' gender may (or may not) alter the sample space or even the expected outcomes. This analysis is consistent with what Epley and Gilovich (2004) noted regarding *anchoring and adjustment*, which can occur without an externally provided anchor (i.e., Item B did not declare any percentages to hold from it), as some subjects seem to generate their own anchors and adjust from them (*self-generated anchors*).

In this regard, it is evident that the principal reason for the anchoring and adjustment bias, which was shared by **G** thinkers, was the *overgeneralization process*. It reflected a mental heuristic in which individuals search for a general formula that always works. Such general formula of P(G|BB) = P(G) illuminated both **HOL** and **A** thinkers' understanding of *the independence concept*, which appeared when *indicating the random process (whole) and the outcomes of that process (parts)*, respectively. Thus, the reason why **G** thinkers held their initial estimation was a rational cognitive motive, which coincides with Lieder et al.'s (2017) interpretation of the anchoring bias that can be understood as "a signature of resource-rational information processing rather than a sign of human irrationality" (p. 29).

On one hand, **HOL** thinkers agreed that their estimation should be the same as before because (as they stated) "*the process of determining a baby's gender in both situations of A and B reflects a random experiment with various possible outcomes that are expressed by the sample space elements; also, the favorable outcome cannot be predicted with certainty*." Consequently, they supposed that the provided condition (i.e., giving birth to two boys before) would not affect their estimation since the sample space of the experiment still has the same equally likely expected outcomes (e.g., {B, G, twins}). On the other hand, **A** thinkers stressed on the outcomes and judged them to be independent. As they reported, "*because the process of predicting a baby's gender embodied a random experiment, its results had to be independent; then, there was no relationship between the events of first, second, and third births*." In other words, the probability of giving birth to a girl did not depend on previous babies' gender; or the gender of the thirdborn baby would not be affected by either the first or second born babies' genders. This resembled responses stating that "*if the woman delivered two boys before, this does not guarantee that she will give birth to a girl or even a boy later.*"

Based on the above analysis, both **HOL** and **A** thinkers shared the *anchoring bias* inherent in *overgeneralizing the independence concept*; it motived them to anchor and fix their initial estimation of giving birth to a girl and drop the given condition from the context. One possible reason for this is the type of conventional pedagogical probabilistic activities that are mostly practiced in both teacher education and school curricula. Because students were not used to modeling a real phenomenon, their inferences were inspired by the theoretical mathematical formula $P(A|B) = P(A)$ that may not fully satisfy such a realistic context. As described by Díaz et al. (2010), "statistical data will rarely lead to exact equality for independent events, and perfect independence is not found in 'real' applications" (p. 151). Additionally, Kataoka et al. (2010) highlighted that most curriculum activities handle the concept of independence with the context of chronological events. Hence, another likely reason for **G** thinkers (or more obviously **A**) to exhibit drastic reliance on the *independence notion* is the formulation of Item B (chronological).

As stated earlier, besides the category of **G** thinkers, another category emerged to define students who also disregarded the given condition, that is, the conservative thinkers [**CON**] who shared the responses displayed in Table 36.

Table 36. Conservative thinkers' typical responses in the context of giving birth

	Conservative thinkers [CON]	
	Socially conservative [SO.C]	**Subjectively conservative [SU.C]**
Students' typical responses	We have changed the expectation because some women give birth to the same gender always. Therefore, this woman may resemble such a case in which she gives birth to only boys. Then, the probability of giving birth to a girl will be lower than the initial estimation.	Regardless of whether the expectation changes or not, the probability of giving birth to a girl remains a matter of Allah's will.
N = 8	***3 responses***	***5 responses***

Again, the common feature between **G** and **CON** thinkers is that both excluded the given condition when analyzing the second situation. Although the anchoring bias, which was rooted in the overgeneralization heuristic, held the source of this exclusion for **G** thinkers, the same exclusion of the given condition had another root beyond the anchoring bias for the **CON** thinkers.

For more clarification, socially conservative thinkers [**SO.C**] changed their numerical estimation when they were asked to interpret Item B, which means that they were aware of the difference between situations A and B. Nevertheless, they insisted on sharpening the socially shared belief that *some women always give birth to the same gender*. This reflects their inability to overcome such a belief, which prevented them from considering the given condition. Similarly, the subjectively conservative thinkers [**SU.C**] did not clarify clearly whether their expectations would change or not; instead, they maintained the concept of *Allah's will* to explain the new situation in a similar manner of dropping the provided condition.

This description indicates that the reason why **CON** thinkers could not manipulate the given condition in their arguments was not purely cognitive, but rather it was inherent in some held beliefs. This can be explained in terms of Tversky and Kahneman's (1974) psychological analysis of *the illusion of validity*, defined as "the unwarranted confidence which is produced by a good fit between the predicted outcome and the input information" (p. 1126). Accordingly, because some **CON** thinkers strongly believe that some women can give birth to the same gender (**SO.C**) and others have faith in Allah's will (**SU.C**), they both decided to intentionally exclude the given condition from their interpretation. Such exclusion helped them ensure a good fit and consistency between the expected outcome (i.e., a baby girl) and their self-input (i.e., what they believe in) to perform the situation in Item B. Hence, the *illusion of validity* signified the reason why **CON** thinkers (a) retrieved their initial beliefs regarding the first situation in Item A to interpret the new one in B, or (b) exposed similar beliefs in judging Item B that was avoided before for Item A. This illustrates why the number of students who shared such beliefs increased from four (see Tables 24 and 25) to eight (see Table 36). In that sense, the category of **CON** thinkers not only defines students who fixed their reasons, but also exposes several original dogmas that were difficult to overcome. This means that although some respondents tried to ignore these beliefs in their answers to Item A, they were not able to manage that in Item B.

The above-stated analysis implicitly clarifies why **G** thinkers did not have the *illusion of validity* since they generalized their way of thinking and anchored their estimation, without awareness of the factors that may limit such a generalization. On the contrary, **CON** thinkers were conscious of the differences between Items A and B. However, when they tried to state their reasons to explain why the probability in Item B varied from Item A, they, again, relied upon their beliefs. That is consistent with what Tversky and Kahneman (1974) explained regarding the *illusion of validity*: it "persists even when the judge is aware of the factors that limit the accuracy of his predictions" (p. 1126).

In this regard, the commonality between **G** and **CON** thinkers is that both shared the *anchoring bias*. Nonetheless, the notion and source of such a bias was distinct. While it emerged that when **G** thinkers kept the initial estimation as a result of the *overgeneralization heuristic* so that the mathematical interpretation arose in their responses, for **CON** thinkers, it appeared because of the *illusion of validity*, which directed them to retrieve what they believed in and then anchor from it. In other words, although the *anchoring bias* originated in *overgeneralization* (*cognitive source*) and was displayed when the mathematical estimation was sustained for **G** thinkers, for **CON** thinkers, it was inherent in the *illusion of validity* (*belief source*) and appeared when maintaining the non-mathematical reasons. Hence, the numerical estimation and afforded arguments were both anchored because of the *overgeneralization* and *illusion of validity* for **G** and **CON** thinkers, respectively; then, the given condition was dropped, and P(G|BB) was, ultimately, judged as equal to P(G). Although **CON** thinkers' responses (see Table 36) indicate a numerical change, their stated reasons for why such change occurred were the same as in Item A (i.e., the afforded arguments were anchored).

- **Second: PSMTs who incorporated the given condition in the analysis**

Through continuing the analysis process, two other broad categories of correlational [**COR**] and rational thinkers [**R**] were inferred to portray students who emphasized the condition when interpreting the new situation of Item B. Table 37 clarifies their responses.

Although all students who belonged to these categories considered the condition (i.e., giving birth to two boys before) to revise the probabilistic estimation in Item B, there were some differences in how they perceived and utilized such a condition. This is detailed in the following discussion.

Table 37. Correlational and rational thinkers' typical responses in the context of giving birth

<table>
<tr><th rowspan="2"></th><th colspan="2">Correlational thinkers [COR]</th><th colspan="2" rowspan="2">Rational thinkers [R]</th></tr>
<tr><th>Horizontal [HOR]</th><th>Vertical [V]</th></tr>
<tr><td rowspan="2">Students' typical responses</td><td colspan="4">The probability will change because</td></tr>
<tr><td>in the new situation of Item B, S has the three outcomes of {B, B, G}; therefore, P(G) = 1/3.</td><td>if a woman gave birth to two boys before, her probability of giving birth to another boy is higher than that of a girl. Or, if a woman gave birth to two boys, then it is more likely that her third child is going to be a girl.</td><td>the new situation does not look very dissimilar to the first situation. However, if the number of boys continues to increase compared with girls, this could be an indicator of a genetic or biological issue.</td><td>some of the previously considered conditions (in Item A), like when we supposed that the woman might have a miscarriage, do not exist anymore. Rather, in this situation, we already understood that the woman has high probability of giving birth to a child, whether a boy or a girl.</td></tr>
<tr><td rowspan="2">N = 20</td><td rowspan="2">3 responses</td><td rowspan="2">13 responses</td><td>2</td><td>2</td></tr>
<tr><td colspan="2">4 responses</td></tr>
</table>

The first emerged category is of **correlational thinkers [COR]**, which include students who thought of the relationship among the various mentioned outcomes in the new situation (i.e., first, second, and third births of a boy, boy, and girl, respectively). Moreover, this category comprises **horizontal** [**HOR**] and **vertical thinkers** [**V**], whose reasoning has the following characteristics:

HOR thinkers interpreted the new situation of Item B not as a diachronic one that incorporates a sequence of events, in which the first and second outcomes are known, and the third outcome is yet uncertain; rather, they consider it as a static one-stage situation. They assumed that Item B implied an experiment of three equally likely outcomes of two boys and one girl; further, the question was how to determine the probability of one possible event among the three outcomes. **HOR** thinkers tried to overcome the complexity in Item B by converting it into a one-stage experiment wherein it was easier to calculate the probability. Consequently, they modified the conditional probability context to a simple unconditional situation, which they used to judge. Accordingly, they modeled the new problem of Item B as if S = {B, B, G}; then P(G)= 1/3. Such reasoning resembles what Lysoe (2008) reported during the study of prospective lower secondary school teachers' understanding of simple and compound events, in which a common heuristic among them was identified and termed as the *one-step heuristic*. This occurred when the students found the answer by "simply transforming a two-step problem into a one-step problem or simple trial" (p. 2).

Additionally, from a psychological viewpoint, the case of **HOR** thinkers can also be interpreted considering the *availability heuristic*, which indicates that the individual estimates the likelihood of an event based on the ease with which the relevant mental operations of retrieval, construction, or association can be performed (Tversky & Kahneman, 1974). In this regard, **HOR** thinkers were judged to share the *availability heuristic*, which appeared when they reduced the diachronic conditional probabilistic situation into a simple one. Furthermore, this simple situation is faster to calculate, or presumably more available based on their past experiences; both issues designate a couple of plausible causes for such a *reduction process*: the *retrievability of instances* or *imaginability bias* (Tversky & Kahneman, 1974). While the former may reflect the recent activities or tasks that **HOR** thinkers performed or most of the examples they encountered, the latter signifies a more cognitive construction, in which the mind attempts to reduce the load of the complicated computational rules of P(G|BB) and instead develop a simpler formula such as P(G).

The other explored sub-category from **COR** is the **vertical thinkers** [**V**], which symbolizes students who associated the previous babies' gender with the thirdborn's gender. Thinking of such association oriented some **V** thinkers to decrease their initial estimation by clarifying that if the woman gave birth to two boys before, then the probability of giving birth to a boy, as a third child, will be higher than giving birth to a girl. Following the same reasoning, other students who were also classified under **V** thought that because the woman delivered two boys previously, she is more likely to give birth to a girl the third time; accordingly, they increased their first estimation, which they stated earlier in Item A.

V thinkers interpreted the conditional probability of P (G|BB) as a *causal relationship* in which the conditioning event of BB (giving birth to two boys before), which already occurred, signifies the *cause*, and the conditioned event of G (having a girl as a third child) is the *consequence*. This resembles what was reported earlier regarding **o*** thinkers who could not differentiate between the two concepts of *causality* and *conditionality*. Notably, the *causal conception* has been discussed widely in different studies. Such conception considers a cognitive more than being induced by teaching; besides, it hides the reversible character of conditional probability, wherein this notion of reversibility is needed to understand the Bayes theorem and statistical inference (Batanero & Sanchez, 2005; Díaz & de la Fuente, 2007; Savard, 2014; Tversky & Kahneman, 1982). Perhaps one reason why the *causal conception* was widespread among students who employed various manners of reasoning and was emerged in different probabilistic situations (unconditional and conditional) is that we often build our knowledge based on causes and effects. As individuals perceive the idea of causation intuitively, our conceptions about causation are sometimes biased, and at other times, there is a confusion between causality and conditionality (Falk, 1986).

Besides the *causal conception*, some **V** thinkers shared the *gambler fallacy* that describes the belief that after a long run of the same result in a random process, the probability of the same event occurring in the subsequent trial is lower (Batanero & Sanchez, 2005; Lysoe, 2008; Savard, 2014). Although all **V** thinkers considered the previous two births being boys as a *cause* to speculate the probability of giving birth to a girl as a third child, some of them thought more theoretically in manner similar to interpreting conventional probabilistic activities (e.g., tossing a coin). This means that if the first outcome was B and the second also B, then G is more likely to be the next outcome.

As displayed in Table 37, the last inferred category characterizes **rational thinkers** [**R**], which include students who also acknowledged that the new situation of Item B must be modified from Item A. Consequently, they decided to update their initial estimation regarding the probability of giving birth to a girl based on the provided information of delivering two boys before. Although **R** thinkers judged that the probability of giving birth to a girl in Item B remains different from their initial estimation before admitting any conditions, they declared that this does not imply a causal relation wherein giving birth to two boys before caused an increase or decrease in the probability (similar to **V** thinkers). Instead, for **R** thinkers, the given condition must be recognized within a group of other multiple factors (*determinants*), wherein all these factors act together to update the estimation.

Concretely, some **R** thinkers considered the information about the previous babies' gender as *an indicator of a genetic state* in which the continuous process of delivering baby boys could designate a genetic issue. Consequently, for such a case, the probability of giving birth to a girl will be lower than giving birth to a boy. Besides, other **R** thinkers grasped the given condition as a *sign of the woman's ability to deliver her baby.* In that case, although the miscarriage was regarded as one possible outcome in interpreting Item A before knowing any information about the previous babies, such a miscarriage is less likely to occur in the second situation. As some **R** thinkers explained, acknowledging that a woman can deliver a baby, based on the new information, is going to reduce the number of elements in the sample space (after eliminating the miscarriage outcome); then, the probability of giving birth to a girl must be updated (increased). In other words, the provided condition indicated that a woman had high chance of delivering a baby, whether a boy or girl. Such reasoning reflects Tarr and Jones's (1997) argument regarding the ability to recognize *the reduction of the sample space in conditional probability problems*, which determines a typically high level of understanding of conditional probability.

Before proceeding to the next part of the data analysis (conceptual difficulties when calculating conditional probability from a two-way table), it is valuable to regard the relationship between reasoning types in both Items A and B. This was one advantage of utilizing the same context (i.e., gender) to clarify students' manners of reasoning: one when the situation was simple (Item A) and another when it was conditional (Item B). Alternatively stated, this part attempts to explain how students' reasoning in an unconditional probabilistic situation might be related to their reasoning in a conditional one. Accordingly, Table 38 presents a cross-tabulation analysis for students' reasoning in Items A and B, wherein twelve associations of reasoning emerged.

Table 38. PSMTs' reasoning in unconditional vs. conditional probabilistic situation

		PSMTs' reasoning in Item B (Conditional probabilistic situation)				
PSMTs' reasoning in Item A (Unconditional probabilistic situation)		**G**	**CON**	**COR**	**R**	***Total***
	M	15	2	3	0	***20***
	S	22	5	11	3	***41***
	O	3	1	2	1	***7***
	Total	***40***	***8***	***16***	***4***	***68***

According to Table 38, the most prevalent operated associations of reasoning were (**S**, **G**) and (**M**, **G**). It suggested that most of the current study participants who reasoned either **subjectively** or **mathematically** in Item A were more likely to employ type **G** reasoning when this item was conditioned in B. In that sense, and generally speaking, when students are confronted with probabilistic social phenomena, they are more inclined to employ either subjective or theoretical probability to model such a phenomenon; however, most of them think similarly if that phenomenon was conditioned. More precisely, they believe that the condition does not matter and the estimation remains the same, whatever with or without conditions.

In detail, regarding the association of (**S**, **G**), about 32% of the participants, in the beginning, conditioned the probabilistic phenomenon of giving birth through issues like baby sonar results, ultrasound scan, or Allah's will (see Table 24). Nonetheless, when they were given a condition, they thought that such a condition had nothing to do with the probability. In other words, although they admitted the possibility of a probabilistic situation to be conditioned by several factors, they could not clarify how such conditions may influence the estimation. This is an interesting finding wherein students believed that several conditions might exist and restrict the probability. Nevertheless, when they were given a specific condition to scrutinize how it might affect that probability, they could not recognize it. Thus, and because they failed to discuss so, they, alternatively, argued that the estimation remained the same in both conditional and unconditional situations. On the other side, the case of (**M**, **G**) was more reasonable and expected; it reflected a mathematical way of modeling random phenomena, which was performed by about 22% of the participants. According to such a way, students emphasized merely sample space elements to estimate probability whatever that phenomenon was or was not conditioned. In other words, for those students, the probabilistic estimation stayed dependent on the sample space elements. Furthermore and based on the previously defined conceptions and cognitive biases in students' reasoning, the next table allocates these conceptions to both associations of (**S**, **G**) and (**M**, **G**).

Table 38. Characteristics of PSMTs' (**S**, **G**) and (**M**, **G**) associations of reasoning

<table>
<tr><th>Probabilistic reasoning</th><th>Shared biases and conceptions (commonalities)</th><th colspan="2">Specific biases and conceptions</th></tr>
<tr><td rowspan="2">(M, G) reasoning (15 cases)</td><td rowspan="2">– The students shared the equiprobable bias.
– They were insensitive to the prior probabilities of outcomes.
– They interpreted the conditional probability in the same way as simple probability [i.e., $P(A|B) = P(A)$]
– They practiced the anchoring and adjustment bias.</td><td>– They did not understand the required assumptions for applying theoretical probability.
– They tended to overgeneralize the notion of randomness.
(m, HOL) = (9 cases)</td><td>– They did not understand the required assumptions for applying theoretical probability.
– They tended to overgeneralize the notion of independence.
(m, A) = (2 cases)</td></tr>
<tr><td>– They could not perceive the realistic context in such a way of connecting it with the mathematical explanation.
– They tended to overgeneralize the notion of randomness.
(m*, HOL) = (3 cases)</td><td>– They could not perceive the realistic context in such a way of connecting it with the mathematical explanation.
– They tended to overgeneralize the notion of independence.
(m*, A) = (1 case)</td></tr>
</table>

<table>
<tr>
<td rowspan="3">(S, G) reasoning (22 cases)</td>
<td rowspan="3">– The students interpreted the conditional probability in the same way as the simple probability.
– They shared the anchoring and adjustment bias</td>
<td colspan="2">– They practiced the prediction bias.
– They anchored the reason because of the illusion of validity.
– They overgeneralized the notion of randomness.
(s, HOL) = (2 Cases)</td>
</tr>
<tr>
<td>– Their reasons were more cognitive.
– They tended to overgeneralize the notion of randomness.
(s**, HOL) = (16 cases)</td>
<td>– Their reasons were more cognitive.
– They tended to overgeneralize the notion of independence
(s**, A) = (4 cases)</td>
</tr>
<tr>
<td>– Their reasons were more cognitive.
– They were unable to overcome some socially shared beliefs.
(s**, SO.C) = (1case)</td>
<td>– Their explanation was more cognitive.
– They were unable to overcome the concept of Allah's will.
(s**, SU.C) = (1 case)</td>
</tr>
</table>

5.3.2 PSMTs reasoning in a two-way contingency table

Although students' argumentations on how a given condition (new information) can be incorporated to alter the probability were characterized previously, their conceptual difficulties in calculating the conditional probability from a two-way table are analyzed in this section. Both results work together to capture students' reasoning in a conditional probabilistic situation; hence, the *notion of subjective probability as a general classifier* can be clarified.

As reported in Appendix 8, students were asked to respond to the two equivalent Items of **E1** and **E2**. Accordingly, their answers to all questions were first analyzed (see Figure 19). Moreover, their solutions to Q3 and Q4 in both Items **E1** and **E2**, which required conditional probability calculations, were categorized under the two major categories of those who (a) dropped and (b) operated the condition when computing the conditional probability (see Table 20). Before going into detail, Tables 40 and 41 summarize the frequencies of students' correct answers to all questions, considering that each question was answered by 34 students[20]; besides, Figure 19 displays the presented data in both tables.

As shown in Figure 19, almost all students were able to calculate simple probabilities. However, for both joint and conditional probabilities, multiple wrong answers were given. Surprisingly, the highest percentage of students' wrong answers emerged in calculating joint

[20] As reported earlier in Chapter 2, 34 students responded to each item of E1 and E2. While the former was answered by 16, 12, and 6 students in the second, third, and fourth years, respectively, 16, 11, and 7 students responded to the latter (see Table 8). Thus, the presented percentages in Tables 40 and 41 were calculated with respect to 34 students.

probabilities, while nearly half of the students (74 correct responses among 136) determined the conditional probabilities correctly. Perhaps one reason for students' efficiency in computing the conditional probabilities is that the data were presented in the obvious form of frequencies, which are easier to calculate (Gigerenzer & Hoffrage, 1995) compared with using probabilities or percentages.

Table 40. PSMTs' calculations of simple, joint, and conditional probabilities in Item E1

Item E1 questions	*Corresponding mathematical form*	*Correct answer is*	*Number of correct answers*
Q1 [The probability that a student prefers ElAhly]	Simple probability [P (ElAhly)]	[500/800]	33 (97.1 %)
Q2 [The probability that a student is in school B and prefers ElZamalek at the same time]	Joint probability [P (School B ∩ ElZamalek)]	[110/800]	6 (17.6 %)
Q3 [If you knew that the selected student prefers ElAhly, what is the probability that this student is in school A?]	Conditional probability [P (School A| ElAhly)]	[195/500]	22 (64.7 %)
Q4 [If you knew that the selected student belongs to school A, what is the probability that this student prefers ElAhly?]	Conditional probability [P (ElAhly|School A)]	[195/385]	20 58.8 %

Table 41. PSMTs' calculations of simple, joint, and conditional probabilities in Item E2

Item E2 questions	*Corresponding mathematical form*	*Correct answer is*	*Number of correct answers*
Q1 [The probability that a student has enrolled to teach the secondary level]	Simple probability [P (Secondary level)]	[200/460]	(15, 9, 6) = 30 88.2 %
Q2 [The probability that a student has enrolled in the science class at the elementary level]	Joint probability [P (Science class ∩ Elementary level)]	[70/460]	(5, 3, 2) = 10 29.4 %
Q3 [If you knew that the selected student has enrolled in the mathematics class, what is the probability that this student teaches the secondary level?]	Conditional probability [P (Secondary level| Math class)]	[110/300]	(9, 5, 2) = 17 50 %
Q4 [If you knew that the selected student taught the secondary level, what is the probability that this student has enrolled in the mathematics class]	Conditional probability [P (Math class| secondary level)]	[110/200]	(7, 4, 4) = 15 44.1 %

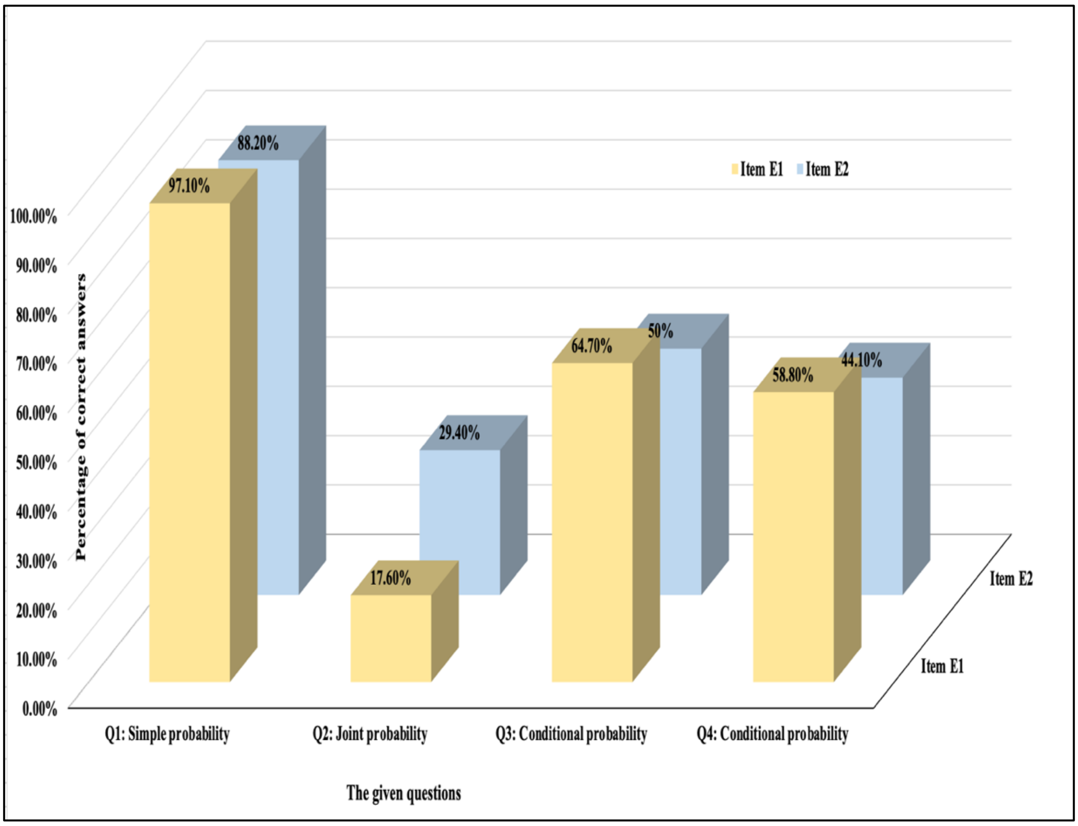


Figure 19. Percentages of PSMTs' correct answers to Items E1 and E2's questions

While the previous data outlined percentages of students' correct answers to all of **E1** and **E2**'s questions, the next discussion focuses on their conceptual difficulties when calculating the conditional probabilities. In other words, the following analysis explains students' solutions to Q3 and Q4 for both Items **E1** and **E2**. Table 42 summarizes students' numerical answers to Q3 and Q4—both the correct and wrong answers.

Table 42. PSMTs' numerical answers to Q3 and Q4 of Items E1 and E2

	Item E1											
	Q3						***Q4***					
	Correct answer	***Wrong answers***					***Correct answer***	***Wrong answers***				
	195/500	195/385	385/500	195/800			195/385	385/500	195/500	190/385		
N	***22 Correct answer***	8	3	1			***20 Correct answer***	7	4	3		
		12 wrong answers						***14 wrong answers***				
	34 answers						***34 answers***					
	Item E2											
	Q3						***Q4***					
	110/300	110/200	200/300	300/110	110/460	300/460	110/200	110/300	110/460	200/300	160/200	90/200
N	***17 Correct answer***	9	2	2	2	2	***15 Correct answer***	8	7	2	1	1
		17 wrong answers						***19 wrong answers***				
	34 answers						***34 answers***					

- **First: PSMTs who disregarded the given condition from the analysis**

Generally, if a condition was not employed to calculate the conditional probability of P(A|B), then it will be judged to be the same as P(A), which would happen if and only if A and B were independent events, as discussed before in the context of giving birth. Furthermore, mathematically, $P(A|B) = P(A \cap B)/P(B)$ refers to the probability of event A's occurrence given that an event B has already occurred. In such a situation, the numerator defines the number of favorable outcome occurrences in the reduced sample space, and the denominator is the total number of outcomes in that reduced sample space, which means that the *sample space's reduction* defines a crucial idea to calculate the conditional probability (Batanero et al., 2015; Reaburn, 2013; Watson & Kelly, 2007). Accordingly, if students did not consider the condition when calculating the conditional probability, they would think that the sample space of the new experiment must remain as before without conditions. Because of this, as reported in Table 20, it was assumed that if the condition was disregarded from the analysis, two conceptions might appear, wherein the common feature between them is that students still think of the sample space as if the situation were not conditioned.

For the first conception, the students might think that P(A|B) = P(A). In such a case, they would give the numerical answers of 385/800 and 500/800 to Q3 and Q4 in Item **E1**, respectively. Similarly, the fractions of 200/460 and 300/460 would be provided as answers to Q3 and Q4 in Item **E2**, respectively (see Table 20). Accordingly, and considering the reported results in Table 42, interestingly, no students shared this conception, which was often caused by *overgeneralizing the independence concept*, as interpreted earlier in the context of giving birth. In other words, there were no answers indicating that students judged the conditional probability of P(A|B) to be equal to P(A) in either **E1** or **E2**.

Additionally, the other expected conception was the *confusion between joint and conditional probability*. Accordingly, students might think that $P(A|B) = P(A \cap B)$; then, they would provide 195/800 as an answer to both Q3 and Q4 of Item **E1** and 110/460 to both Q3 and Q4 of Item **E2** (see Table 20). This already happened, wherein 10 students (7.4 %) shared this conception, as detailed in Table 42. The *confusion between joint and conditional probability* has been discussed by multiple studies; nevertheless, the number of students who shared such confusion was more than the current study participants. For example, Díaz and de la Fuente (2007) reported that 31% of the (university) students confused conditional with joint probability. Moreover, 13.7% of participants (prospective primary school teachers) were confused between conditional and compound probability in Batanero et al.'s (2015) study compared with 17% of future teachers in Estrada and Díaz's (2006) research. This is likely because the current study's participants were still studying courses of statistics and probability during the period of data collection; thus, they more easily read the two-way table and interpreted its cells, which may also explain why fourth-year students asked the researcher to remind them of the formula (see the ethical considerations section).

- **Second: PSMTs who incorporated the given condition in the analysis**

As reported in Table 20, if the students incorrectly incorporated the conditions to calculate conditional probabilities, the fallacy of the *transposed conditional* would emerge. In such a case, they were expected to give the numerical answers of 195/385 and 195/500 to Q3 and Q4 of **E1**, respectively. Moreover, the fractions of 110/200 and 110/300 would be the answers to Q3 and Q4 of **E2**, respectively. On the contrary, if they correctly implemented the given condition, the correct answer would result. Nonetheless, the focus here is not on students who successfully answered the questions but rather on those who had some conceptual difficulties.

According to Table 42, 29 students (21.3%) shared the *transposed conditional* wherein they calculated P(B|A) instead of P(A|B). This fallacy was first addressed by Falk (1986) and described as the lack of distinction between the two directions of the conditional probability of P(A|B) and P(B|A). Moreover, Falk (1986) asserted that such confusion is widespread among students and professionals at all levels. Although the fallacy of the *transposed conditional* was prevalent among the current study's participants, Batanero et al. (2015) reported that only around 6% (12 out of 197 cases) of their sample of prospective primary school teachers experienced it. Nevertheless, for university students, previous research conclusions have varied. For example, the *transposed conditional* was shared by 59% of the subjects in Díaz and de la Fuente's (2007) investigation. As they explained, it stemmed from students' confusion between conditioning and conditioned events and the role of both when computing the conditional probability. This mismatches with Reaburn's (2013) findings in which only 13% of her study participants were unable to determine the difference between P(A|B) and P(B|A); she also stated that this high proportion of correct responses was not surprising because the participants were students entering introductory applied statistics units.

- **Third: Other cases of PSMTs who provided unexpected numerical answers**

Considering the above analysis, 39 among 62 wrong responses were explained: 10 indicated a *confusion between joint and conditional probability* compared with 29 that revealed *the transposed conditional*. Yet, 23 wrong answers (see Table 42) have not been clarified; these are the focus of this section.

Following the same analysis that focused on whether the sample space was reduced or not, we start with the students who answered **300/460** to Q3 of Item E2, wherein the denominator 460 describes the entire population of the E2 experiment. As shown in Table 42, 12 students kept the sample spaces of 800 and 460 for E1 and E2, respectively: 10 of them *confused the joint with the conditional probability* as reported earlier, while 2 others provided the answer of **300/460** to Q3 of Item E2, in which the sample space was the same as without conditions (no reduction).

To judge the answer of **300/460**, the students' followed steps to reach that answer were clarified. In detail, for Q3, students were asked to determine the probability that someone teaches the secondary level knowing that the person has enrolled in mathematics class [P (secondary level | mathematics class)]. Accordingly, they answered **300/460,** which merely defines the probability of someone enrolling in mathematics class (see Appendix 8); in other words, they reduced P(A|B) to P(B), so that the conditioning event took much attention. In this regard, students first confused the *conditioning event* with the *conditioned event* and supposed that "the enrollment in the mathematics class" was the conditioned event that we needed to calculate its probability; then, they dropped the conditioning event of "teaching the secondary level." Mathematically, two steps were employed: *first, the students unconsciously modified P(A|B) to P(B|A), and second, they reduced P(B|A) to P(B).* However, the first step did not indicate the confusion of the transposed conditional but rather a *confusion between the conditioning and conditioned events*, since the former is relevant to calculate the probability and the latter addresses ambiguity between the events. Furthermore, the second step involved *overgeneralizing the independence conception,* wherein the students assumed independence in the data (Estrada & Díaz, 2006).

Still, 21 responses have to be judged, and the commonality among them is that the sample spaces of both E1 and E2 experiments were altered, which implicitly indicates that students tried to manipulate the condition in their analysis. However, that manipulation was not done appropriately because the students could not reach the correct answer. The next discussion attempts to categorize such responses by giving more attention to the numerator (favorable outcome) and denominator (sample space) of students' answers. Perhaps there are other ways to classify such wrong answers, but the focus on numerators and denominators is favored because it matches the process of analyzing Item B. Concretely, all responses that exhibited a change in the sample space were first categorized, as they symbolized students who considered the condition in their analysis (as in Item B). Moreover, sharpening the numerator and denominator helped clarify how students understood the relationships among events embedded in a conditional probabilistic situation. In that sense, both sections can work together to reach a clear conclusion.

– *First*: 10 students defined the reduced sample space correctly, but they could not determine the favorable outcomes. Accordingly, responses of 385/500 and 190/385 were given to Q3 and Q4 of E1, respectively; similarly, a response of 200/300 was given to Q3 of E2 and 160/200 and 90/200 were given to Q4 of E2.

Both answers of **385/500** and **200/300** (5 responses) indicate that students computed the conditional probability of P(A|B) not through the formula P(A∩B)/ P(B) *but using P(A)/P(B).* In this case, *students did not admit that the condition has to reduce the number of both the sample space and favorable outcome elements* since the conditional probability P(A|B) denotes that the only events of interest are those in subset A that can be found in subset B (Reaburn, 2013). Accordingly, they wrongly judged that (a) P(School A | ElAhly) = P(School A) / P (ElAhly) = (385/800) / (500/800) = **385/500** and P(Secondary level | Math class) = P(Secondary level) / P(Mathematics class) = (200/460) / (300/460) = **200/300** for Q3 in E1 and E2, respectively. In these answers, the numerator of 385 determined who preferred both ElAhly and ElZamalek for Q3 of E1, while 200 represented who belonged to both the mathematics and science classes for Q3 of E2 (see Appendix 8). This indicates that the students were *reluctant to believe that the condition restricts not only the sample space but also the favorable outcome*.

Additionally, on one hand, 4 students gave answers of **190/385** and **90/200** to Q4 in both E1 and E2, respectively, which indicates that they were *confused about the conditioned event.* The students mistakenly calculated P(ElZamalek | School A) and P(Science class | Secondary level) instead of the required P(ElAhly | School A) and P(Mathematics class | Secondary level), respectively. Such type of response was considered a *simple confusion more than a conceptual difficulty* because it reflected that students perceive how the conditional probability works and correctly identified the proper cells in the two-way table to compute it. However, they mistakenly *confused the conditioned event with another event in the experiment.* On the other hand, the answer of **160/200**, which was provided by 1 student, to Q4 of E2 reveals a combination of two previously identified errors. That student (a) *confused the required conditioned event of* mathematics class with the event of the science class; accordingly, P(Mathematics class | Secondary level) was replaced by P(Science class | Secondary level). Then, the student (b) calculated P (Science class | Secondary level) through P (Science class) / P (Secondary level) that equals (160/460) / (200/460) = **160/200**, which explains that students' inability to think that *the condition limits the favorable outcome (the numerator).*

– *Second*: *a conceptual error in which P(A|B) equals P(B)/P(A)* was practiced by 9 students. Accordingly, both answers of **385/500** and **200/300** were provided to Q4 of E1 and E2, respectively (see Table 42).

Although the sample space seemed to be reduced as in the previously addressed cases, this did not designate a correct identification of either the sample space or favorable outcomes. In detail, 7 students calculated P (ElAhly | School A) by dividing P(School A) by P(ElAhly) to answer Q4 of E1: (385/800) / (500/800) = **385/500**. Moreover, the corresponding formula of Q4 in E2 [i.e., P (Mathematics class | Secondary level)] was computed by two students as P (Secondary level) / P (Mathematics class) = (200/460) / (300/460) = **200/300**.

This error, which resembled 6.6% of all responses and 14.5% among the wrong ones, designate two fallacies that students practiced when determining the conditional probability P(A|B); both fallacies also appeared in other cases (see the above analysis). In the beginning, the students *confused the conditioned event of A with the conditioning event of B*, which oriented them to change P(A|B) to P(B|A). Then, they were *reluctant to believe that the condition has to restrict the favorable outcome* to (A∩B) and not all B elements; thus, students wrongly judged P(B|A) as equal to P(B)/P(A) instead of P(A∩B)/ P(A).

– *Lastly*: as presented in Table 42, two students gave a wrong response of **300/110** instead of 110/300 to P (Secondary level| Mathematics class), which was the required probability for Q3 of E2; mathematically stated, both students calculated P(A|B) through P(B)/ P(A∩B) instead of the correct opposite formula.

This error expresses students' unawareness of two essential axioms in probability theory: (a) the probability of any event cannot be higher than 1 and (b) the joint probability of two events must be lower than each of a single event; both are closely connected. This is because if we are aware that the joint probability is lower than the probability of a single event [P(A∩B) < P(B)], we can recognize that the fraction of P(B) / P(A∩B) will be greater than 1 since the dominator of P(A∩B) is smaller than the numerator of P(B). A similar error in which students were unconscious of probability axioms and got some probabilities higher than 1 was reported by Batanero et al. (2015).

Based on the previous analysis, Table 43 summarizes students' conceptual difficulties when calculating the conditional probability from a two-way table. It shows that the fallacy of the *transposed conditional* was the most frequent error among the participants (21.3%), followed by the *confusion between joint and conditional probability* (7.4%).

Table 43. PSMTs' conceptual difficulties in calculating the conditional probability

Major categories	*Conceptual difficulty*		*Frequency*	*Percentage*
PSMTs who have disregarded the given condition from the analysis	Confusion between joint and conditional probability	P (A|B) = P (A∩B)	10	7.4%
	Combination of the confusion between conditioning and the conditioned event; and the independence conception	P (A|B) = P (B)	2	1.5%
PSMTs who have incorporated the given condition in the analysis	The fallacy of transposed conditional	P (A|B) = P (B|A)	29	21.3%
	Reluctance to believe that the condition restricts not only the sample space but also the favorable outcome	P (A|B) = P(A)/P(B)	5	3.7%
	Confusion between the conditioned event and another event in the experiment		4	2.9%
	Combination of the confusion between the conditioned event and another event and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome		1	0.7%
	Combination of the confusion between conditioning and the conditioned event; and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome	P (A|B) = P(B)/P(A)	9	6.6%
	Unawareness of basic probability axioms	P (A|B) = P(B)/P(A∩B)	2	1.5%
Total			***62[21]***	***45.6%***

5.3.3 Characteristics of PSMTs' reasoning in a conditional probabilistic situation

This section endeavors to integrate the previous two sections' results that explained how the students reasoned in conditional probabilistic situations in which the *subjective probability* can be characterized. In doing so, these results were consolidated while considering Chernoff's (2008) distinction between intrasubjective (*personal belief-type probability*) and intersubjective probability (*interpersonal belief-type probability*), as well as Stanovich et al.'s (2008) model on normative reasoning and thinking errors (Chiesi & Primi, 2014).

[21] There were 62 wrong responses among 136 (students' answers to both Q3 and Q4 in Items E1 and E2); accordingly, the provided percentages in Table 43 were computed with respect to 136 responses.

Acknowledging that intrasubjective probability is less objective than intersubjective probability (Chernoff, 2008; Hacking, 2001), the researcher interpreted that *intrasubjective thinkers designate cases of students who disregarded the condition from the analysis*. The reason is that, mathematically, the only way to get the normative (i.e., objective or correct) answer when reasoning in a conditional probabilistic situation is to incorporate the condition in the analysis, which restricts the outcomes of interest to those that exist in the reduced sample space. From this aspect, students who dropped the condition from the analysis were considered to possess *less of an element of objectivity* compared with those who thought to employ that condition (whatever mistakenly or correctly).

According to the detailed analysis, students who neglected the condition when interpreting uncertain conditional situations shared several conceptions and cognitive biases. As listed in Table 43, these biases designate the *anchoring and adjustment bias*, which was rooted in either the overgeneralization heuristic or illusion of validity; *confusion between joint and conditional probability*; and a *combination of the confusion between conditioning and the conditioned event and the independence conception*. Such cases were classified under intrasubjective probability; in other words, students who experienced these conceptions and biases were judged to be *intrasubjective thinkers*.

On the contrary, for *intersubjective probability,* a sense of objectivity was intended. This objectivity reflects fulfilling the probability axioms under the idea that the probability must be updated in light of newly available information (Borovcnik, 2012; Chernoff, 2008). Thus, *intersubjective thinkers* were students who decided to involve the condition to revise their prior probabilistic estimation. Although some of them performed this successfully (i.e., rational thinkers and students who solved the conditional probability questions correctly), others shared several conceptions and biases during the process of involvement. These are cases of students who exhibited the *availability heuristic*, which originated from either the retrievability of instances or the imaginability bias (one-step heuristic), the *causal conception*, and *the gambler fallacy*; besides, those who shared the (a) fallacy of transposed conditional, (b) reluctance to believe that the condition restricts not only the sample space but also the favorable outcome, (c) confusion between the conditioned event and another event, (d) combination of the confusion between the conditioned event and another event and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome, (e) combination of the confusion between conditioning and the conditioned event and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome, and (f) unawareness of basic probability axioms (see Table 43).

Additionally, to describe how both *belief-type probabilities* were operated to fulfil the normative answer, Stanovich et al.'s (2008) model of normative reasoning (see Figure 20) is valuable. According to this model, if individuals do not possess the necessary knowledge to produce a normatively correct response, the *mindware gap* signifies the cause of that error. On the other hand, if they own such knowledge, two issues are predicted to happen: they still provide an incorrect response or get a correct solution. While the former often results from an *override failure that* reflects that individuals hold the rule but do not base the answer on it, the latter occurs when they have the *cognitive ability* to utilize their mindware and solve the problem.

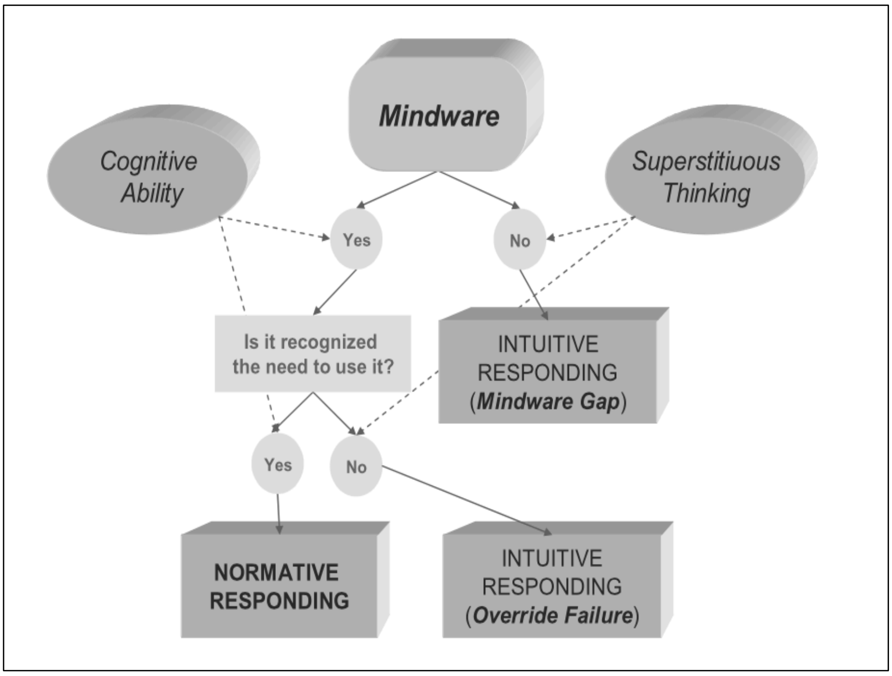


Figure 20. Stanovich et al.'s (2008) model on normative reasoning and thinking errors. Retrieved from Chiesi and Primi (2014, p. 179)

Based on Stanovich et al.'s (2008) model, the previously identified conceptions and biases, which were shared by both intrasubjective and intersubjective thinkers, were occurred because of either the *mindware gap* or *override failure* (*contaminated mindware*).

As detailed before, the intrasubjective thinkers practiced the anchoring and adjustment bias in the context of giving birth. This bias describes that the individual's final judgment is biased toward the initial probabilistic value (Lecoutre et al., 1990); furthermore, it was inherited in either overgeneralization heuristic (**G** thinkers) or the illusion of validity (**CON** thinkers). **G** thinkers kept their initial mathematical estimation and supposed that the process of determining the baby's gender is always the same, whether with or without conditions, which was called by overgeneralization heuristic. On the contrary, **CON** thinkers admitted the difference between both situations (with and without conditions), but they maintained their non-mathematical reasons (i.e., the illusion of validity).

In this regard, *the overgeneralization heuristic,* which was manipulated by **G** thinkers, represents *a mindware gap* wherein students did not have enough knowledge of how the interpretation of real situations may differ from the traditional activities of probability in terms of the ideal assumed notion of independence, particularly, for the conditional phenomena. This reflects why Mises (1928, as cited in Díaz & de la Fuente, 2007) criticized the formal mathematical definition of independence because it is not intuitive at all. On the other hand, the *illusion of validity* stemmed from a *contaminated mindware*; it provoked the override failure that made **CON** thinkers unable to overcome some of their beliefs (see Table 36). Thus, they retrieved the initial estimation despite their awareness of the differences between the unconditional and conditional situations. Although such beliefs were regarded previously as a specific type of probabilistic reasoning (the case of **s*** thinkers), they describe contaminated knowledge for **CON** thinkers because they hinder them from fulfilling the conditional probabilistic situation. This matches what Toplak, Liu, Macpherson, Toneatto, and Stanovich (2007) argued about the *contaminated mindware that reflects superstitious thinking in the case of probabilistic reasoning*.

Similarly, when students were asked to reason in a two-way table to determine some conditional probabilities, 12 of them interpreted P(A|B) as equal to either **P(A∩B)** or **P(B)** (see Table 43). Both thought of the sample space of the experiment as same as that without conditions; accordingly, they were judged to be *intrasubjective thinkers*, as stated before. Moreover, students' errors occurred because of the *mindware gap*. The reason is that such errors either designate a lack of knowledge regarding the conditional probability formula or a failure to read the two-way table.

According to the detailed analysis, the overgeneralization heuristic, confusion between joint and conditional probability, combination of the confusion between conditioning and the conditioned event and independence conception, and illusion of validity defined ***intrasubjective probability,* as conceptualized by the students**.

Although the first three conceptions indicated a *mindware gap,* the last one reflected *a contaminated mindware.* Both lack of knowledge (*mindware gap*) and inhibited beliefs (*contaminated mindware)* were described in some previous studies. For example, Begg and Edward (1999) revealed that very few in-service and pre-service primary school teachers understood the concept of independence. Additionally, Kataoka et al. (2010) stated that several misconceptions of independence and conditional probability persist even for the students who had formally studied it. They also reported that most respondents (master students and PSMTs in Brazil, and high school students in Mexico) applied the original sample space size to compute the conditional probability. Similar results were obtained by different studies with multiple groups of participants (e.g., Díaz et al., 2010; Gusmão et al., 2010). Furthermore, the *contaminated mindware* that describes the *illusion of validity*, wherein some students insisted on several shared beliefs, were also demonstrated by some researchers. For example, Amir and Williams (1999) reported that the "common culture influences the informal ideas of chance and probability the individual acquires: the 'ethnomathematics'' (D'Ambrosio, 1985) of probability" (p. 85). Also, Larose et al. (2010) discussed similar ideas under the term of social representations that defines the socially shared construct or the knowledge of common sense; it directs the predictability of behaviors among individuals of a social group.

Intersubjective thinkers include students who attempted to revise their prior estimation and incorporate the condition in the analysis; they also shared some conceptions and biases. In the context of giving birth, the *availability heuristic* that resulted from either retrievability of instances or the imaginability bias (**HOR** thinkers), and the *causal conception* and the *gambler fallacy* (**V** thinkers) emerged. As summarized in Table 44, while the *availability heuristic* was assigned to the *mindware gap*, both the gambler fallacy and the causal conception were classified under the *override failure*; the former was originated from students' lack of knowledge, and the latter indicated contaminated mindware.

Although all students who performed such fallacies admitted that the probability in a conditional situation (Item B) differed from that in an unconditional situation (Item A), those who shared the *availability heuristic* (**HOR** thinkers) tried to reduce the load of the complex computational rules of P(G|BB) and instead solve the formula of P(G). This means that, practically, they thought of the conditional situation as similar to a simple one with the three outcomes of B, B, and G. As explained before, students operated the *availability heuristic* because of either retrievability of instances or imaginability bias, which means that they lacked the required knowledge about the conditional probability. Consequently, their minds tried to recover straightforward examples of a simple probability situation, which occupied their mindware because it was easy to operate.

On the other side, **V** thinkers practiced both the *gambler fallacy* and the *causal conception*; accordingly, their initial estimation after knowing that the woman delivered two boys before was altered, and they were thus committed to the *override failure* (see Table 44). This reflected that students who shared such fallacies possessed *contaminated* knowledge of the conditional probability, which prevented them from operating the given condition successfully to interpret the conditional situation. Their *contaminated mindware* reflects the dominance of conventional probabilistic activities in which students interpreted the real phenomena through the lens of such activities. Consequently, using the *gambler fallacy*, students supposed that after having two boys, the probability of having another boy is lower. In other words, students' mindware was full of simple traditional probabilistic tasks that negatively affected their interpretation of real phenomena. Yet, holding the *gambler fallacy* not only reflected students' *contaminated knowledge* regarding real situations, but it also further indicated their inability to overcome the belief about the *law of small numbers* (Stohl, 2005).

Moreover, the emergence of the causal conception in which students judged the conditioning event of delivering two boys to be the cause and the conditioned event of delivering a girl to be the consequence symbolized *contaminated knowledge* about the concept of conditionality. Although the students intended to interpret the conditional phenomena, they mixed performing that conditionality with causality. This is reported by Batanero et al. (2016) who found that in several real-life situations, causal and probabilistic approaches are intermingled, and how to separate the random influence from the causal represents a challenge; in this regard, understanding the notion of conditionality helps to overcome such a challenge.

Similarly, when students utilized the two-way table to compute conditional probabilities, six types of errors emerged (see Table 43) and were judged to represent the *intersubjective belief type probability*. This is because these students modified the sample space of the experiment, which indicated their struggles to involve the condition in the calculation. Still, to determine whether such errors originated from the *mindware gap* or o*verride failure*, the students' employed formulas to calculate the conditional probability were sharpened.

In detail, students who shared the fallacy of the transposed conditional and were confused between the conditioned event and another event understood how the conditional probability formula works and correctly applied its procedures. Nonetheless, the former could not differentiate that formula from its opposite, and the latter mistakenly (maybe just a simple mistake) computed the conditional probability of another event, not the required one in the question. Hence, both cases were regarded as o*verride failure.* Although students grasped the conditional probability, some of them maintained *contaminated knowledge* because of which they suppose that P(A|B) = P(B|A), while the others failed to determine which event was asked to estimate its probability. On the contrary, the other four errors were more critical, wherein students exposed a lack of knowledge of the conditional probability formula. As presented in Table 43, 14 students thought that the conditional probability of P(A|B) defines the ratio between P(A) and P(B). These students were reluctant to believe that the condition limits the favorable outcome (not merely the sample space); consequently, they could not determine the correct numerator to calculate the conditional probability. Besides, two other students exhibited unawareness of the basic probability axioms. All these cases were assigned to the *mindware gap.* Although the students changed the values of sample spaces (denominators) (intersubjective thinkers), such a change did not indicate their knowledge about either the mechanism of how the conditional probability formula works or the probability axioms.

Finally, students who (a) interpreted the given condition of delivering two boys before as a sign of the woman's ability to have a new baby and consequently recognized the idea of *sample space reduction* in the giving birth context, and (b) correctly calculated conditional probabilities in the two-way table were exceptional cases. They did not share the previously described conceptions and errors; therefore, they were assessed (by the researcher) to possess the *cognitive ability* to reason in conditional probabilistic situations. Hence, the whole results are summarized in the following table:

Table 44. A model of PSMTs' R(in) P that is related to a conditional probabilistic situation

Origin of belief	**Belief-type probabilities**	
	Intrasubjective probability (personal belief type) [Intrasubjective thinkers]	***Intersubjective probability (interpersonal belief type) [Intersubjective thinkers]***
Mindware gap	– Overgeneralization heuristic (cognitive source) – Confusion between joint and conditional probability – Combination of the confusion between conditioning and the conditioned event and the independence conception	– Availability heuristic – Reluctance to believe that the condition restricts not only the sample space but also the favorable outcome – Combination of the confusion between the conditioned event and another event; and reluctance to believe that the condition restricts not only the sample space but also the favorable outcome – Combination of the confusion between conditioning and the conditioned event; and reluctance to believe that the condition restricts not only the sample space but also the favorable outcome – Unawareness of basic probability axioms
Override failure	– Illusion of validity (belief source)	– Gambler fallacy – Causal conception – Fallacy of transposed conditional – Confusion between the conditioned event and another event in the experiment
Cognitive ability		– Recognition of the idea of sample space reduction – Calculation of the conditional probability from a two-way table correctly

5.4 Issues of trustworthiness and ethical considerations

- **Issues of trustworthiness**

For validating qualitative research findings, issues of trustworthiness must be discussed; this would confirm how such findings can be trusted in terms of several criteria such as credibility, transferability, dependability, and confirmability (Korstjens & Moser, 2018). Accordingly, to validate the above-reported findings (categories of PSMTs' reasoning) of the inductive and deductive data analysis processes, two criteria were considered: *credibility* and *dependability.*

On one hand, *credibility* signifies the confidence in the truth of research findings; it "establishes whether the research findings represent plausible information drawn from the participants' original data and is a correct interpretation of the participants' original views" (Korstjens & Moser, 2018, p. 121). Moreover, *triangulation* defines one possible strategy to ensure such *credibility* (Ibid.). It is also considered a powerful technique that facilitates the validation of data through cross verification from two or more sources, in which weaknesses in the inferred data from one source can be strengthened by another source, thereby increasing the validity and reliability of the results (Honorene, 2016; Joint United Nations Programme on HIV/AIDS, 2010). Therefore, to verify this study results, two types of triangulation were utilized: data and theory triangulation (Turner & Turner, 2019). While the former was employed through involving multiple groups of participants of second-, third-, and fourth-year university students during their preparation program, as detailed earlier in Table 4, the latter was embedded in the process of analyzing the consistency between empirical results and existing theories (the step of "*incorporate the emerged categories into a model*").

In detail, the process of defining the categories of PSMTs' reasoning has relied on three different contexts (i.e., giving birth, throwing a die, and weather predictability). This process involved utilizing two approaches in which such category can be characterized: (a) psychological studies that highlight individuals' cognitive biases in reasoning under uncertainty (e.g., the work of Tversky and Kahneman, 1974) and (b) educational investigations that determine learners' difficulties and misconceptions regarding probability (e.g., Konold, 1989; Batanero et al. 2010; see Appendix 5). Furthermore, a detailed explanation of similarities and differences between the resulted types of reasoning in each context, including how both are related to the relevant theories, was provided (see the summary in Figure 17). This also represented a theoretical triangulation (Miles et al., 2013).

On the other hand, the *dependability*, which reflects the stability of findings over time (Korstjens & Moser, 2018), was confirmed by checking the *intracoder reliability*. Intracoder reliability expresses the coder's consistency across time; that verifies the ability of coding protocols (the previously summarized in Figures 15, 16, and 17) to result in the consistent categorization of content (Lacy, Watson, Riffe, & Lovejoy, 2015). From this aspect, Cohen's Kappa coefficient was calculated to check the consistency between two rounds of coding processes, in which all PSMTs' responses to Items A, B, C, and D were analyzed twice using the detailed procedures in Section **5.1** (Elo et al., 2014; Schreier, 2012). The first data analysis was conducted in October and November 2019, while the second time was conducted in April and May 2020. The results are presented in Tables 45–48[22], which indicate the moderate (Items A and C) and strong (Items B and D) levels of agreement and consistency between the two rounds of coding (McHugh, 2012).

Table 45. Cohen's Kappa value for Item A's analysis

Symmetric Measures					
		Value	Asymptotic Standard Error[a]	Approximate T[b]	Approximate Significance
Measure of Agreement	Kappa	.788	.069	8.393	.000
N of Valid Cases		68			
a. Not assuming the null hypothesis. b. Using the asymptotic standard error assuming the null hypothesis.					

Table 46. Cohen's Kappa value for Item B's analysis

Symmetric Measures					
		Value	Asymptotic Standard Error[a]	Approximate T[b]	Approximate Significance
Measure of Agreement	Kappa	.832	.059	10.368	.000
N of Valid Cases		68			
a. Not assuming the null hypothesis. b. Using the asymptotic standard error assuming the null hypothesis.					

Table 47. Cohen's Kappa value for Item C's analysis

Symmetric Measures					
		Value	Asymptotic Standard Error[a]	Approximate T[b]	Approximate Significance
Measure of Agreement	Kappa	.768	.064	11.865	.000
N of Valid Cases		68			
a. Not assuming the null hypothesis. b. Using the asymptotic standard error assuming the null hypothesis.					

[22] The process of analyzing Items A, B, and C involved 68 responses compared with 48 for Item D.

Table 48. Cohen's Kappa value for Item D's analysis

Symmetric Measures					
		Value	Asymptotic Standard Error[a]	Approximate T[b]	Approximate Significance
Measure of Agreement	Kappa	.887	.054	11.317	.000
N of Valid Cases		48			

a. Not assuming the null hypothesis.
b. Using the asymptotic standard error assuming the null hypothesis.

- **Ethical considerations**

It is also worthy to state the *ethical considerations* that may affect the validity and generalizability of the current study's findings. These considerations include issues related to the (a) study participants, principally PSMTs, (b) process of collecting and analyzing the data, and (c) the questionnaire items.

First, all participants agreed to be engaged in this study; as reported before, PSMTs were freely asked about their availability and willingness to cooperate. This satisfies the ethical considerations of educational research in Egypt in which participants have the right to accept or reject to cooperate; they also have the freedom to answer (or not) some questions (Elzewiney, 2014). Therefore, some of the PSMTs (third-year students) were fully appreciated when they told the researcher that they had a schedule in conflict with the time of implementing the study questionnaire, and they would not be able to respond to the task of weather predictability at that time (see Table 8). Furthermore, to protect the study participants from harm in any way (Elzewiney, 2014), they were (a) informed that their responses to the study questionnaire would not be related to their academic assessment and (b) their identities would be secured through anonymity.

Additionally, most of the participating PSMTs were previously taught a micro-teaching course by the researcher. On one hand, this supported the process of analyzing the collected data in which it was easy for the researcher to interpret PSMTs' responses by merely relying on what they wrote in the questionnaire. On the other hand, because of the cordial relationship between the researcher and PSMTs, they were probably a little careless in either writing their reasoning or further explaining their answers; they believed that the researcher would be able to interpret whatever they wrote. This may have affected the validity of the researcher's interpretation since the analysis processes depended entirely on what PSMTs explained in the questionnaire. Moreover, this analysis was conducted by only the researcher because it was a little challenging to engage another researcher to interpret the collected data, especially in light of linguistics sensitivity and contextual concerns. Nevertheless, as reported before, to reduce such possible bias, the analysis process was handled twice with a long time difference of about six months.

Second, regarding procedures of data collection and analysis, as the data were obtained from both PSMTs and some pupils, it is reliable to report that data from the pupils were not collected by the researcher; instead, two in-service teachers did so. However, an interview was conducted with those teachers before and after the data collection (see Section **5.1**). Also, all pupils' responses to the probability context survey were analyzed by the researcher. In contrast, the procedures for collecting (and analyzing) the data obtained from PSMTs were entirely conducted by the researcher during the micro-teaching session, which was, according to the faculty's regulations, the appropriate time to collect such data (Elzewiney, 2014). PSMTs who did not engage in this study were taught the micro-teaching course as usual. Yet, in this regard, several concerns have to be mentioned, as follows:

– During the initial stage, when PSMTs asked the researcher about the validity of identifying issues such as women's bodily appearance and Allah's will as possible factors influencing the probability of giving birth (during the interview), they were told to mention whatever they think. They also were informed that these reasons and criteria would be more valuable than the mathematical given percentages. Such feedback from the researcher might be a source of some bias, wherein PSMTs thought more to identify any condition that may affect the probability of giving birth to a girl, whatever came to their mind. It may also be one probable reason why the subjective manner of reasoning dominated PSMTs' responses to that problem.
– In the stage of employing the study questionnaire, and again related to the problem of giving birth, some PSMTs suggested that the event of the birth of twins was a possible expected outcome. This may also be why many participants considered twins in their answers. Moreover, regarding questions that required conditional probability calculations from a two-way table, fourth-year PSMTs requested a similar example to remember the formula. Provision of such an example, and not their understanding of the concept, maybe why they did not exhibit conceptual difficulties in determining the conditional probability.
– Regarding the analysis process, as detailed earlier, primary data were obtained from the study questionnaire and were analyzed twice by the researcher. Thus, during the second cycle of the analysis, some issues were modified to ensure consistency between both cycles' results; accordingly, a clear conclusion can be reached.

For instance: about PSMTs' responses to Item **B**, during the first round of data analysis, their responses were categorized in terms of their given numerical answers, whether they kept or changed their initial estimations. Nevertheless, after repeating such analysis, it was observed that there was a large difference between the number of assigned codes to each category (not like in the case of Items **A**, **C**, and **D**). The reason was that some respondents (conservative thinkers) had changed their numerical estimations, but they stated the same reasons that they gave before to Item **A**, which confused the researcher regarding in which category to include these types of responses. Furthermore, it was realized that such manner of categorization, which depends merely on PSMTs' numerical estimations, could not provide an understanding of how they perceived the given condition based on which the subjective probability, as a general classifier, could be characterized. Accordingly, during the second round of analysis, PSMTs' responses were restructured based on whether they emphasized or disregarded the given condition instead of addressing their numerical answers. This supported the interpretation of the conservative thinkers' category, which included those who retrieved a particular explanation after admitting the differences between Items **A** and **B** (e.g., emphasizing God's attribution to a phenomenon), regardless of whether they did or did not change their numerical estimations.

Finally, concerning the questionnaire items, one crucial issue to mention here is that all the items (except E1 and E2) incorporated open questions. On one hand, this was beneficial to the essential purpose of the current exploratory study, which may be regarded as a guide on probability education research in the Egyptian context. Although some studies were conducted to investigate PSMTs' statistical thinking (e.g., Osman, 2010), no previous investigations related to probability have been conducted in the past 10 years. Accordingly, national research demands some studies that examine the current situation; this study may afford such a detailed description, specifically for PSMTs.

On the other hand, some of PSMTs' conceptions and biases probably emerged because of the form of the questions themselves; these questions did not intend to determine the existence of specific pre-defined conceptions (like in some other studies), but rather to investigate the current state of PSMTs' knowledge about teaching probability. More precisely, regarding Items **A** and **B** that were related to the context of giving birth, as reported before, the subjective manner of reasoning appeared obviously in PSMTs' responses to **A** compared with those to Items **C** and **D** related to throwing a die and weather predictability, respectively. Moreover, during the analysis of PSMTs' responses to Item **B** in which they were asked to determine the probability of giving birth to a girl knowing that the woman had delivered two boys before, it was difficult to judge their ideas scientifically, that is, whether the babies' genders are correlated or independent from each other. Accordingly, the data interpretation focused on how PSMTs understood the condition of "*two boys before*," regardless of the scientific knowledge embedded in that situation.

CHAPTER 6: CONCLUSIONS AND RECOMMENDATIONS

This chapter summarizes the basic ideas of the current study for which PSMTs' knowledge for teaching probability in Egypt was conceptualized from the PoPR. It is presented in three sections. First, the answer to each research question is outlined. The study implications and recommendations are described in the second section. Lastly, the limitations of this study and some directions for future research are discussed.

Before we begin, it is important to first provide a brief overview of the basic ideas and questions that the current study addresses.

As probability involves a substantial knowledge that is often manipulated in our daily lives, needed in everyday settings for all citizens in decision-making situations, and required for the training of specialists, educational authorities in many countries recognized and included it in the official curricula from the primary level to teacher education. It helps to form a specific type of reasoning—probabilistic reasoning—from which we can formally structure our uncertain thinking about random phenomena, and, accordingly, overcome our deterministic thinking and accept the existence of chance in nature. Despite that, various issues exist regarding deficiencies of probability education. One such problem that is of concern to the current study is the potentially inadequate preparation of mathematics teachers to teach probability, notably, in the Egyptian context. From this aspect, and because of the influence of teachers' knowledge on students' learning, this study approached PSMTs' knowledge for teaching probability from the PoPR, which emphasizes their thinking processes and cognitive biases. More precisely, it responded to three questions:

- **RQ1**. What is the current status of "statistics and probability" education in Egypt?
- **RQ2**. What is the definition of mathematics teachers' professional knowledge for teaching probability from the PoPR?
- **RQ3.** What are the characteristics of PSMTs' knowledge for teaching probability in Egypt from the PoPR?

Accordingly, the next sections were organized.

6.1. Answers to the research questions

6.1.1 The answer to the first research question

As reported, the first research question aimed at *defining the current status of statistics and probability education in Egypt,* which was mainly discussed in Chapter 3. It was presented from (a) a local perspective that was expressed in the governmental policy, standards of the Faculty of Education, and the state of pupils’ achievement; and (b) an international viewpoint with much focus on the Egyptian school curriculum of probability compared to the NZ curriculum.

From a local perspective, the government has advocated the need to enhance PSMTs’ professional competence to meet pupils’ needs, under the broad goal of improving the quality of education practice to fit the global systems. In that sense, PSMTs’ preparation has been given consideration. This is particularly the case with teaching the content of statistics and probability, because learning such content is less emphasized in PSMTs’ preparation not only as a discipline (9.2% of the studied hours during a four-year preparation program) but also as a content that should be deliberated pedagogically to promote their PCK. Furthermore, Egyptian pupils’ achievement in the content area of *Data and chance* remains the lowest among all mathematics areas, as TIMSS reports for 2003 and 2007 revealed. Thus, studying how PSMTs can be prepared to teach statistics and probability signifies one plausible way to consider pupils’ insufficiencies, especially in terms of (a) the international consensus regarding the influence of teachers’ knowledge on pupils’ achievement and (b) the current status of the Egyptian employment system in which merely completing a university degree is sufficient to practice the teaching profession.

Within the domain of statistics and probability, **probability** denotes a core concept for which most of the textbooks' activities, in Egypt, are intended to promote pupils' probabilistic understanding. This was revealed by analyzing the declared activities within statistics units for both primary and lower secondary school textbooks. Moreover, that analysis relied on exploring the correspondence between the objectives of such activities and Burrill and Biehler's (2011) list of fundamental statistical ideas.

Additionally, the intended and implemented Egyptian school curricula of probability was analyzed from an international viewpoint. On one hand, the intended curriculum was compared with the NZ curriculum, which serves as a working model for pedagogical reforms in other countries. The comparison involved a summative content analysis that aimed at quantifying the usage of probabilistic words in both the intended curricula documents, which were provided by the official websites of Egypt and NZ. Such a quantification was conducted through (a) operating the OSA, which helped in determining the declared probabilistic situations, propositions, procedures, and terms in both the curricula; and (b) assigning the resultant entities to the corresponding fundamental probabilistic ideas in Batanero et al.’s (2016) list. Accordingly, and after ensuring trustworthiness of this comparative analysis, several issues were defined. One essential finding that reflects the deficiency of the intended curriculum of probability in Egypt is as follows: although it meets the NZ curriculum in strengthening the probabilistic concepts that are relevant to randomness, events and sample space, and modeling and simulation, ideas of independence and conditional probability, probability distribution and expectation, and convergence and law of large numbers have not yet been considered in Egypt.

On the other hand, the implemented curriculum of probability, which was defined by the national school textbooks' activities, was analyzed through the OSA. Hence, the probabilistic situations, propositions, procedures, and terms within the textbooks' discourses were defined and assigned to Batanero et al.'s (2016) classification of various probability interpretations. As a result, the analysis revealed another aspect of deficiency in the Egyptian curriculum of probability: the lack of addressing the experimental probability interpretation. Moreover, the subjective probability approach was neglected until grade 9.

Generally speaking, these findings highlighted (a) the limitedness of PSMTs' preparation to teach statistics and probability and (b) deficiencies in the content of probability taught in the schools in Egypt. Accordingly, this study approached PSMTs' knowledge for teaching probability as one plausible way to overcome such constraints that affect the quality of probability education and, ultimately, pupils' probabilistic reasoning. For instance, when teaching probability predominantly utilizes a theoretic approach rather than a frequentist one, pupils tend to develop conceptions based on deterministic reasoning (Konold, 1995). With regard to that view, investigating the current state of PSMTs' knowledge implies one essential step toward designing fruitful pedagogical preparation: it should be grounded in their reasoning processes.

6.1.2 *The answer to the second research question*

The second research question aimed at *defining mathematics teachers' professional knowledge for teaching probability from the PoPR*; two essential steps were involved in answering this question. The first step outlined *knowledge for practice*, which indicated mathematics teachers' professional knowledge for teaching probability as defined in the literature. The second step manifested the psychological facet of teachers' knowledge; it was exemplified by their conceptions (*knowledge in practice*) and reasoning processes. These ideas were discussed in Chapter 4, where the study framework was represented.

About the first step, an extensive literature review was conducted in light of the MKT model to define mathematics teachers' professional knowledge for teaching probability; alternatively stated, the essential ideas that mathematics teachers need to understand to teach probability effectively were outlined based on previous scientifically approved research in the field (*knowledge for practice).* Thus, the aspects of **KoP**, **KoTP**, **KoSPK**, and **KoPL** were explored; they expressed mathematics teachers' Knowledge of Probability, Knowledge of Teaching Probability, Knowledge of Students Probability Knowledge, and Knowledge of Probability Language, respectively. While the first three aspects corresponded to SMK, KCT, and KCS in the MKT model, respectively, the fourth aspect of **KoPL** defined a distinct component for the probability instruction, which has not been displayed explicitly in the MKT.

The **KoP** reveals the heart of mathematics teachers' professional knowledge for teaching probability. It indicates epistemological reflection on the meaning of probability, which requires an understanding of its (a) objective facet that emphasizes the mathematical rules that govern random processes, and (b) subjective facet that sharpens the information available to the person assigning that probability. While the former can be informed through the theoretical and experimental interpretations, conditional probability determines an essential concept to understand the latter.

Within the objective view, theoretical probability indicates a fraction whose numerator is the number of favorable cases and denominator is the number of all equally likely cases; yet, because of the equiprobability, the theoretical interpretation is difficult to apply outside games of chance. Moreover, the experimental probability signifies a hypothetical number toward which the relative frequency tends during the stabilization process when random sequences are considered (Sharma, 2016). It also has the practical limitation of only obtaining an estimation, which alters from one series of repetitions to another, and it cannot be applied when it is not possible to replicate an experiment under the same conditions. Thus, neither interpretation is suitable to address every situation; instead, the appropriate approach should be applied depending upon the context.

Within the subjective view, the concept of conditional probability specifies a prerequisite for understanding subjective probability (Jones et al., 2007). It describes an update of the predictor's knowledge of a particular event when new information is available; thus, it keeps the dual object-subject character of probability (Kapadia & Borovcnik, 2010). Accordingly, this study relied on PSMTs' reasoning in a conditional probabilistic situation to characterize the notion of subjective probability. This was grounded in Chernoff's (2008) classification of subjective probability into intrasubjective probability (*personal belief-type*) and intersubjective probability (*interpersonal belief-type*), and in Borovcnik's (2012) review of the educational perspective on conditional probability. As the latter declared, conditional probability (a) fulfils probability axioms for the objectivists, and (b) reflects that any probability is conditional to available information and is related to the idea of updating it in light of new evidence.

In addition to **KoP**, which was sharpened in the current study, several issues that are related to the three other components of **KoTP**, **KoSPK**, and **KoPL** were found. The **KoTP** included concerns on how to teach probability, which was defined in the following terms: (a) Warming up the probability lesson, it should focus on developing students' intuitive understanding of uncertainty to capture the formal concept of probability. (b) Accessing the probability activities, teachers should define simple, compound, and conditional probability; understand concepts of variability, expectation, randomness, and independence; distinguish between mutually exclusive, joint, and independent events; and draw inferences about a population from random samples—they should also separate a mathematical problem from the statistical one and recognize interrelationships among mathematics, statistics, and probability. (c) Teachers should be able to connect and differentiate among various probability interpretations wherein the law of large numbers plays an essential role in connecting theoretical and experimental probability. In this regard, teachers have to choose between two methodological directions to highlight such a connection: the mapping direction that starts with the theoretical interpretation or the inference direction that begins with doing an experiment (Nilsson, 2013). This choice depends on whether the sample space is known by or hidden from students. (d) Teachers should utilize various representations wherein employ tables; area models; Venn, pipe, or tree diagrams; and computerized simulators to facilitate students' understanding of probability concepts.

For the **KoSPK**, teachers should build their instruction on students' existing knowledge of probability; thus, they have to perceive students' prior knowledge (e.g., ratios, proportions, percentages, fractions, and rational number), misconceptions, difficulties, and levels of cognitive development that are related to probability. Lastly, **KoPL** considers the probability language as a fundamental aspect of teachers' knowledge. Resultantly, how teachers use that language to connect students' daily intuitions of chance, which are manifested in their natural conversation, with the academic language of probability is important. Besides, teachers' capacity to distinguish both the languages is crucial, because sometimes the usage of the terms related to probability during the formal instruction differs from how these words function in everyday situations (e.g., the concept of fairness).

The issues discussed above described the first step in defining the study framework. This step recognized mathematics teachers' professional knowledge for teaching probability in terms of **KoP**, which indicates their understanding of the probability concepts and also crosses with knowledge of the language, knowledge of teaching, and knowledge of students to construct **KoPL**, **KoTP**, and **KoSPK**, respectively; furthermore, those four components are interrelated.

Although some studies have been conducted to investigate teachers' knowledge for teaching probability by utilizing the MKT framework (e.g., Birel, 2017; Brijlall, 2014; Chick & Baker, 2005; Contreras et al., 2011; Danişman & Tanişli, 2017), the focus was more on assessing those teachers' practical knowledge, which was often described as being insufficient or inadequate. Moreover, most of these studies neither regarded teachers' reasoning processes nor the cognitive biases that may cause such insufficient practices (e.g., Torres et al., 2016). Thus, this study framework not only relied on what was raised in the previous research (i.e., the first employed step to conceptualize the study framework) but also attempted to define a new angle that may help exhibit the psychological facet of teachers' knowledge for teaching probability, which is represented by their reasoning processes and conceptions. These ideas were outlined in the second step to characterize the study framework.

In the second step, the study premises were defined as follows: (a) conceptions represent knowledge in evolution; (b) reasoning indicates an individual cognitive process to interpret the acquired knowledge; and (c) there is a reciprocal relationship between conceptions and reasoning.

Conception is knowledge created through the interaction between individuals and their milieu (Brousseau, 1998); it can be valid in certain circumstances but cannot be generalized across all. Moreover, it is not reasonable to pretend that a specific type of conception might explain a certain level of understanding because classifying such conceptions into levels of conceptual understanding does not recognize the value of individuals' reasoning to make sense of phenomena (Savard, 2007). This is highly valuable for probability education since individuals' worlds are full of diverging conceptions connected to probability (Kapadia & Borovcnik, 2010), and these conceptions signify a critical component for the process of knowledge construction (Smith et al., 1993).

Additionally, the probabilistic reasoning was selected to be the study perspective in light of Kapadia and Borovcnik's (2010) recommendation to replace Heitele's (1975) ideas with an approach that looks at concepts from a non-mathematical perspective, that is, has a cognitive psychological nature and focuses on how the mind works. Also, the PoPR appreciates the dual character of probability, wherein subjectivity itself is one plausible approach to interpret a probabilistic situation. Moreover, probabilistic reasoning considers two main concepts of variability and randomness; variability indicates that the outcome is not determined, and randomness signifies uncertainty and independence.

Based on the definition of conceptions and reasoning, the relationship between them was identified as follows: depending upon how we do reason in an uncertain situation that contains probability knowledge (theoretical constructs), our conceptions can be clarified.

In light of this, the study framework, which defines mathematics teachers' professional knowledge for teaching probability from the PoPR, was represented (see Figure 11). It embodied interrelationships among professional knowledge, conceptions, and reasoning. According to the framework, mathematics teachers' professional knowledge for teaching probability (*knowledge for practice*) consolidates knowledge of probability (**KoP**) that crosses with knowledge of the language, teaching, and students to assemble knowledge of probability language (**KoPL**), knowledge of teaching probability (**KoTP**), and knowledge of students' probability knowledge (**KoSPK**), respectively. Nonetheless, practically, during the actual teaching, each teacher transmits this knowledge through a lens that is probability conceptions, which indicate teacher's practical knowledge (*knowledge in practice*). These conceptions may match the scientific knowledge (theoretical static constructs); however, in some cases, there is a gap between how a teacher perceives probability knowledge and professional knowledge for teaching probability if his/her conceptions do not fully fulfil probability theory. The existence of such a gap reflects teachers' various ways of reasoning under uncertainty; placing the focus on reasoning processes (the study perspective) could help characterize this gap.

Accordingly, through the lens of probabilistic reasoning, mathematics teachers' professional knowledge for teaching probability includes these redefined aspects: R(in)P, R(in)PL, R(in)TP, and R(in)SPK, which determine their reasoning in situations that involve knowledge of probability, probability language, teaching probability, and students probability knowledge, respectively.

6.1.3 *The answer to the third research question*

The third research question endeavored to characterize PSMTs' knowledge for teaching probability in Egypt from the PoPR; it sharpened the aspect of PSMTs' R(in)P that is related to (a) simple unconditional and (b) conditional probabilistic situations. While the former helped to incorporate different probability conceptions together in one schema, the notion of subjective probability was clarified based on the latter. Furthermore, both issues were handled through a field study in which a sample of sixty-eight PSMTs, who were enrolled in the four-year mathematics teachers' preparation program during the academic year 2018–2019 at the

Faculty of Education, Tanta University, Egypt, was engaged in this study. Additionally, the data were collected using a questionnaire; it included six items that were developed in terms of (a) the value of adopting a social problem, (b) the school curriculum viewpoint, (c) the pupils' perspective, and (d) issues of previous research.

First, to define PSMTs' reasoning in a simple unconditional probabilistic situation, they were asked to interpret three probabilistic tasks of giving birth, throwing a die, and weather predictability. These tasks were constructed in light of the above-stated criteria to cover different probability contexts. After the data were obtained, the analysis processes involved two types of coding: (a) inductive coding that was performed in light of Thomas's (2006) steps through NVivo software to analyze PSMTs' responses to the first problem of how to judge the probability of giving birth to a girl, and (b) deductive coding to analyze PSMTs' answers to the other tasks of how to determine the probability of obtaining number 5 in an experiment of throwing a die and interpret a 60% probability of rain. Both processes were repeated once again after around six months, and accordingly, intracoder reliability was calculated through Cohen's Kappa coefficient to verify the stability of findings over time and ensure the trustworthiness of the results. It gives reasonable values of .788, .768, and .887 for the three tasks, respectively. As a result, four manners of reasoning emerged that were designated by **M**, **S**, **O,** and **I** to describe PSMTs, whose thinking was mathematical, subjective, empirical, and intuitive, respectively.

In the beginning, the analysis process for PSMTs' responses to the problem of determining the probability of giving birth to a girl revealed three categories of thinkers: **M**athematically (29.4%), **S**ubjectively (60.3%), and **O**utcome-oriented (10.3%). These categories also included multiple subcategories of **m**, **m***; **s**, **s***, **s****; and **o** and **o***. Furthermore, similar manners of reasoning emerged among PSMTs' interpretations of tasks of throwing a die and weather predictability, but not consistently. Concretely, for the activity of throwing a die, the major categories of **M** (73.5%) and **O** (26.5%) resulted, compared to **M** (14.6%), **S** (10.4%), **O** (50%), and the distinct category of **I** (25%) for the task of weather predictability. Besides, new sub-categories of **m**** and **o**** were inferred, with the former only emerging in PSMTs' answers to the activity of throwing a die, while the latter appeared in both the contexts of throwing a die and weather predictability. Additionally, some PSMTs' conceptions and cognitive biases were deduced and assigned to the three factors of *variability*, *randomness*, and *contextual recognition*, which describe the process of probabilistic reasoning.

Type **M** reasoning represented the most used manner of thinking that appeared in the three contexts; it handled the *theoretical probability* to model the uncertain phenomena.

The *variability* for **M** thinkers indicated that the outcome varies depending upon the possible events in the sample space. Moreover, PSMTs exhibited the *equiprobability* bias in understanding *randomness*, which, for them, meant that the random character of the experiment stayed sufficient evidence of equiprobable outcomes (Lecoutre, 1992). Nevertheless, they altered in terms of *contextual recognition*. On one hand, **m** thinkers were able to connect the mathematical model of the given situations with realistic conditions; but, they overestimated the replicability of the experimental results. On the other hand, **m*** thinkers showed an abstract mindset that declined realistic circumstances to interpret the situation theoretically. While **m** thinkers exhibited *equiprobability* as a result of employing the *representativeness heuristic*, **m*** thinkers shared the *equiprobability* because of the *overgeneralization heuristic.* Hence, *contextual recognition* appeared in utilizing both the *representativeness and overgeneralization* heuristics; more precisely, the former defined the realistic circumstances as equal in occurrence, and the latter dropped the practical obstacles and sought a general formula instead.

Also, as reported earlier, type **m**** reasoning appeared in the context of throwing a die; it exhibited an adequate understanding of the *theoretical probability* in terms of the three factors of *variability, randomness,* and *contextual recognition.* For this, **m**** thinkers could (a) differentiate between randomness and fairness, and (b) identify the required assumptions to handle the proper probability interpretation according to the circumstances of the phenomena.

The second prevalent model of thinking was **O**; it was also observed in the three contexts. Although such type of reasoning was intended to employ the experimental probability to model an uncertain situation, it did not reflect an adequate recognition of that interpretation since **O** thinkers manipulated the experimentations not to define the probability but rather to describe the favorable outcome. Apart from that was **o**** thinkers who interpreted the *experimental probability* as the relative frequency of occurrence of an event in a large number of trials.

All **O** thinkers admitted the *variability* of outcomes; however, how they perceived it was a little different. For **o** and **o*,** the *variability* did not depend on frequencies but on *one single trial* at which the favorable outcome could be expected. On the contrary, **o**** thinkers understood that the estimation alters depending upon the ratio between frequencies to the total number of trials. Moreover, several conceptions that are relevant to *randomness* were exposed. For **o** thinkers, both the *prediction conception* and *prediction bias* emerged; while the former signified a misunderstanding of the expectation's intention, the latter judged that expectation precisely. **o*** thinkers also shared the *causal conception* wherein they confused *causality and* *conditionality;* they considered the conditioning event to be a cause for the favorable outcome that represented the result. This denies the *randomness* that demands *independence*. On the contrary, **o**** thinkers displayed an adequate recognition of the *randomness* through admitting *the law of large numbers* at which although the favorable outcome cannot be certainly anticipated, the number of trials has to be increased to get an accurate prediction. Thus, *randomness* for **o**** thinkers exhibited a fair distribution in the long term of many trials. Regarding the *contextual recognition*, all **O** thinkers identified the *data-context* that includes the real conditions from which the problem arose (e.g., *Backgammon games)*. However, the *task-context* that includes defining the sequence of the task and its motivating story (Hershkowitz et al., 2001) was recognized only by **o**** thinkers; this recognition helped them think of the limitations of utilizing the experimental probability.

The third manner of reasoning that PSMTs performed was **S**; it defined probability as a degree of credibility in the occurrence of an event, and it was judged based on various types of information wherein the probability could be revised and updated. **S** reasoning appeared in both the contexts of giving birth and weather predictability, but not in throwing a die, which reflects the value of daily situations in exposing the subjective facet of the probability (e.g., Chassapis & Chatzivasileiou, 2008; Konold, 1995; Larose et al., 2010; Musch & Ehrenberg, 2010; Savard, 2008).

All **S** thinkers considered *variability* of outcomes through which the expected outcome altered upon the information regarding the phenomenon under study becoming available. Nevertheless, the nature of this information that explained the *variability* for **s*** stayed different from both **s** and **s****. Although **s** and **s**** attributed the *variability* to multiple cognitive criteria, **s*** thinkers stressed the religious *conception of Allah's will* that was judged in light of acknowledging the influence of socio-cultural factors on learners' conceptions to be a particular type of probabilistic reasoning; it was neither a *personalist interpretation* nor *superstitious reasoning.* Furthermore, regarding *randomness,* **s** thinkers shared the *prediction bias* that appeared previously for **o** thinkers, and some of them maintained the *dependence conception*, interpreting two dependent events as if one is the cause for the other. While **s** thinkers neglected the *randomness* that demands *independence*, both **s*** and **s**** recognized such randomness when they explicitly reported that expected outcomes could not be predicted precisely. Additionally, all **S** thinkers referred to a *data context* wherein they relied on several real conditions to explain the presented tasks. They also recognized the *task context* within which these tasks might be approached through the subjective probability.

Lastly, the fourth manner of reasoning that emerged only in PSMTs' responses to the weather predictability task was **I**; it used qualitative expressions to explain a probabilistic estimation. **I** thinkers recognized the *variability* that involved merely two options of the favorable outcome or any other event, as in the case of **o** and **o*** thinkers. Besides, **I** thinkers' understanding of *randomness* was expressed qualitatively in light of a continuous decision line that ranges from 0% to 100%, to quantify their probabilistic judgment. Based on Lysoe's (2008) categories of uncertain words' usage, **I** thinkers were judged to have novice recognition of *task-context* because they were in the second level wherein their interpretation of probability includes qualitative idioms.

Similar to the previously determined procedures, PSMTs had to interpret two different conditional probabilistic contexts through which their conditional probabilistic reasoning could be characterized. In the first context, they were asked to determine the probability of giving birth to a girl after adding the condition that the woman had delivered two boys before. Accordingly, their responses were analyzed inductively (Linneberg & Korsgaard, 2019). Moreover, after around six months, the data analysis processes were repeated to verify Cohen's Kappa coefficient, which gave a reasonable value of .832. Additionally, in the second context, PSMTs calculated conditional probabilities from a two-way table and their answers were analyzed deductively in light of what the previous studies reported about the expected errors in computing the conditional probability.

The analysis of PSMTs' responses to the problem of giving birth to a girl after adding the new condition revealed four manners of reasoning: **G**eneralizer (58.8%), **Con**servative (11.8%), **Cor**relational (23.5%), and **R**ational thinkers (5.9%). While **G** and **CON** defined *intrasubjective probability* (personal belief-type probability) through which PSMTs dropped the condition from their analysis, **COR** and **R** reflected *intersubjective probability* (interpersonal belief-type), involving the same condition in the interpretation. Furthermore, several conceptions and cognitive biases emerged. On one hand, **G** thinkers exhibited the *anchoring bias* that was invoked by *overgeneralizing* the *independence concept,* and **CON** thinkers shared the *illusion of validity*. On the other hand, **COR** thinkers practiced *the availability heuristic, causal conception*, and, in some cases, the gambler fallacy. On the contrary, **R** thinkers exhibited an adequate knowledge of the conditional probability whereby employing a condition might affect the sample space of the experiment was expressed.

Additionally, PSMTs' conceptual difficulties in calculating the conditional probability were also defined. Those who disregarded the given condition from the analysis shared a (a) confusion between joint and conditional probability [$P(A|B) = P(A \cap B)$] (7.4%), or (b) a combination of the confusion between conditioning and conditioned event, and independence conception [P(A|B) = P(B)] (1.5%). On the other side, PSMTs who incorporated the given condition in the analysis (except those who correctly solved the problems) exposed the following: (a) the fallacy of transposed conditional [P(A|B) = P(B|A)] (21.3%), (b) the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome [P(A|B) = P(A)/P(B)] (3.7%), (c) confusion between the conditioned event and another event in the experiment (2.9%), (d) combination of the confusion between the conditioned event and another event and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome (0.7%), (e) combination of the confusion between conditioning and conditioned event, and the reluctance to believe that the condition restricts not only the sample space but also the favorable outcome [P(A|B) = P(B)/P(A)] (6.6%), and (f) unawareness of basic probability axioms [P (A|B) = P(B)/P(A∩B)] (1.5%). After all, tables 32 and 44 exhibited how PSMTs reasoning in simple and conditional probabilistic situations was conceptualized, respectively.

6.2. The study implications and recommendations

The results of the current study provide a wide range of implications not only in the field of teacher education but also for pre-university education in terms of both curriculum and teaching approaches, specifically in the Egyptian context.

About the curriculum and as discussed in Chapter 3 regarding the weaknesses of the current school curriculum of probability for both primary and lower secondary levels, it is recommended to increase the activities that approach the epistemic subjective side of probability to be in balance with the existing objective ones. This way, pupils would perceive the dual character of the probability concept, which, on one hand, defines the mathematical rules that govern random phenomena, and on the other hand, reflects a degree of certainty in the occurrence of an event. While the first facet is more objective and intends to measure the magnitude of a certain phenomenon, the latter is subjective and depends on the information available to the person at the time of the investigation. Furthermore, understanding both the facets of probability would help pupils handle the variety of uncertain situations in their daily lives, as each interpretation has some limitations and weaknesses that may be overcome by the other one.

Although the above paragraph highlights the need to modify the current school curriculum of probability, it also gives some suggestions for the initial preparation of mathematics teachers. In this regard, the study recommends statistics educators who are responsible for teaching statistics and probability courses (the discipline itself) differentiate between the objective and subjective sides of probability during their instruction; they should also strengthen both the circumstances and limitations for utilizing each interpretation (SMK). Along with that, and pedagogically, mathematics educators have to support PSMTs in establishing a connection between their understanding of probability theory and how to teach it, in light of the school curriculum and pupils' needs (PCK). Moreover, it is also recommended to discuss with PSMTs what activities can be designed to expose pupils' subjective probabilistic reasoning; notably, such a manner of reasoning is prevalent in handling daily situations and affected by their social practices.

As revealed above, it is somewhat difficult to discuss issues of pre-university education without referring to teacher education that was the essential focus of the current study. In this context, the following question emerges: what other recommendations can be transmitted to the educators who prepare PSMTs to teach probability effectively in light of the study findings and results? The investigation has sharpened characteristics of PSMTs' R(in)P that defines how they reason in an uncertain situation (a situation that includes probability knowledge); consequently, several conceptions and cognitive biases were identified. Some of these conceptions (a) were a mismatch with the professional knowledge for teaching probability (the knowledge that the scientific community recommends mathematics teachers to understand to teach probability efficiently) and (b) would affect their teaching (implementation) and, consequently, future pupils' probabilistic reasoning. Accordingly, to overcome such concerns, especially in light of the international agreement regarding the influence of teachers' knowledge on pupils' understanding and achievement, the study recommends the following:

At first, and based on the study findings, it is recommended to shift the focus of the probability discussion from listing the three primary interpretations of probability (i.e., theoretical, experimental, and subjective) to describing the factors that underpin the process of probabilistic reasoning (i.e., variability, randomness, and contextual recognition). This designates one possible step of moving from growing the content knowledge to strengthening the process knowledge. Alternatively stated, it recommends a shift from "what probability knowledge do PSMTs have?" to "how do PSMTs perceive the probability knowledge?" This mirrors the question, "how do PSMTs reason probabilistically?" On one hand, it responds to several recommendations in the field, and on the other hand, it signifies a way to approach PSMTs' different manners of reasoning, which is highly significant because it reflects a possible reason for why they share different conceptions of probability. This way, and as stated earlier in the significance of the study (see Chapter 1), pedagogical courses can be built constructively to be in line with PSMTs' reasoning processes.

Shifting the focus to the process will alter the way of teaching itself. For example, to approach the subjective probability through learning conditional probability, independence, and Bayes theorem, it is advised not only to strengthen the formula and techniques that govern most teacher education courses but also to emphasize how such procedures work intuitively and in real circumstances. From this aspect, it is valuable to implement real examples, particularly in teaching conditional probability, to expose how acquiring new information helps to change our prior estimation (e.g., Díaz & Fuente, 2007). This is reported in Chapter 4; the probability is often introduced to learners using easy examples such as tossing coins or rolling dice that have quantifiable sample spaces. These examples do not provide an adequate basis for understanding subjective probabilities and lead to a belief that probabilities are empirical properties of the scenario rather than a measure of our knowledge of the outcomes.

It is also helpful to cover a wide range of probabilistic problems that depend on different contexts, especially, the uncertain social problems that enable PSMTs to deal with high levels of ambiguity (i.e., probabilistic reasoning) (Lord, 1994). Besides, the activities that could be translated into multiple forms are important to introduce, since learners sometimes exhibit fallacious reasoning or even conceptual difficulty merely because of the task form. As Gigerenzer and Hoffrage (1995) reported, some students had much success with Bayesian problems when the information was presented in frequency formats. This would promote PSMTs' procedural and conceptual knowledge of probability (e.g., Reaburn, 2013), which in turn affects pupils' understanding of probability.

More specifically, concerning issues of subjective probability, where probability does not have measurable characteristics but rather reflects the individuals' beliefs, it is recommended to raise PSMTs' awareness of their personal experiences and beliefs and of how probability theory can be beneficial in calibrating these beliefs objectively to be in line with these theories. This way, PSMTs will be able to handle such concerns that may appear during actual interaction with their pupils, especially in light of the weakness of the school curriculum that seems to ignore the subjective side of probability. It is what we often call KCS, which describes the interplay between knowledge of the content and students. Therefore, such attention supports strengthening PSMTs' KCS that is related to probability.

Additionally, the usage of technology and computerized simulators is also recommended to be implemented in both school and teacher education (e.g., Batanero & Sanchez, 2005; Batanero et al., 2016). As discussed in Chapter 4, such simulators provide valuable representations that could facilitate learning the probabilistic concepts and, at the same time, eliminate some possible misconceptions. In other words, these simulators can quickly generate random experiments and exhibit the effect of sample size (Batanero et al., 2005); consequently, individuals can evaluate their prior judgments and correct their preconceptions or beliefs that may not be consistent with the probability theory (e.g., Contreras et al., 2010). This means that one way to convince learners that their solutions to probability problems are wrong is to challenge these solutions with experiments (Díaz & Batanero, 2009). Furthermore, and related to what was discussed earlier regarding the need to change the teaching approaches of PSMTs, these technologies can support such change; it enables PSMTs to work with real data where the difference between empirical results and theoretical probabilistic models can be visualized (Batanero et al., 2016). Such an approach matches the effective constructive teaching that should be based on knowledge of pupils' preconceptions, since they construct the meaning by connecting the new knowledge to what they already believe to be true (Garfield, 1995). Because of this, several researchers recommend mathematics teachers start by teaching their pupils experimental probability (e.g., Andrew, 2009).

As a part of the didactical preparation, it is recommended that mathematics educators to pay more attention to probability language. Concretely, this study has manifested probability language as an essential aspect of mathematics teachers' professional knowledge for teaching probability. It signifies one unique feature of the study framework, which has not explicitly been considered in other models of teachers' knowledge (e.g., the MKT). Such an idea can indicate what should be considered during the preparation of PSMTs, especially for the probability domain that has not been provided with much evidence previously. As reported in the TEDS-M, which focused on the standard repertoire of mathematics education represented by the three domains of Numbers, Algebra, and Geometry, the area of *Data and chance* was unequally implemented, and its content reduced to basic concepts of probability and data handling (Döhrmann, Kaiser, & Blömeke, 2012; Li & Wisenbaker, 2008). In this view, PSMTs need to be made conscious of the possible confusion between the informal daily language of probability and the formal one, and stand to use such informal intuitive language to approach formal concepts.

6.3. The study limitations and directions for future research

Depending upon the study delimitations (see Chapter 1) and the declared issues of trustworthiness (see Chapter 5), several concerns should be taken into consideration when interpreting the current study's results and findings.

First, the theoretical analysis that conceptualized the study framework was limited to ICOTS' papers in 2010, 2014, and 2018, and several other studies accessible through search engines (e.g., Google, Google Scholar). As a result, the conceptualization may disregard some other aspects that should be considered to address mathematics teachers' professional knowledge for teaching probability; possibly, such aspects have been highlighted in other research papers that were not reviewed. The study premises are also closely related to this. They define the researcher's interpretation of possible interrelationships among professional knowledge (theoretical constructs), practical knowledge (conceptions), and reasoning processes as one plausible way to exhibit the psychological aspect of probability, which was described during the whole discussion by the expression "the perspective of probabilistic reasoning (PoPR)". These premises were utilized to complement the literature review and finalize the conceptual framework (see Chapter 4). Although the construction of the framework was revised several times and then modified in light of the supervisor's and colleagues' comments and critiques, this may not be adequate to guarantee its validity. In other words, if other researchers follow the same procedures of both the literature review and study premises, they might get another conceptualization. This concern may also weaken the findings of the second research question.

Second, and as reported in Chapter 1, the implementation of the study questionnaire was limited to a non-random sample of PSMTs who belong to Tanta University, the only governmental university in Gharbia governorate in Egypt. Consequently, the drawn conclusions may not be valid for all the characteristics of the whole population of PSMTs across universities. Furthermore, these conclusions and results may alter if the same survey were carried out with another sample of PSMTs in other countries. This is particularly true in the case of issues of subjective probability reasoning that is highly influenced by socio-cultural factors.

Finally, another two interrelated constraints of the current study are (a) the proposed items of the questionnaire through which the data were collected and (b) the process of analyzing that data, both were detailed in Chapter 5. In brief, the questionnaire items were formulated loosely to match the nature of the current exploratory investigation, which could be the basis for similar future national studies. Nonetheless, perhaps such open questions, especially the problem of giving birth to a girl, affected PSMTs' responses and oriented them to reason superficially out of the educational context and probability theory. Also, the collected data were interpreted by the researcher only depending merely upon the study questionnaire, which means that there was neither opportunity to engage other researchers during the data interpretation process nor to conduct posterior interviews with the PSMTs. This may limit the validity of the analysis and, consequently, the obtained conclusions. However, as reported in Chapter 5, PSMTs' responses to each item of the study questionnaire were analyzed twice through the described protocols, and the calculated Cohen Kappa coefficients were reasonable.

Based on the recommendations and limitations of the study, the following directions for future research are proposed:

- More systematic approaches can be adopted to clarify what aspects of mathematics teachers' professional knowledge are relevant for teaching probability, either through a different theoretical analysis of the current research or by examining the implementation of lessons by the in-service teachers.
- Similar studies might be conducted over different groups (e.g., PSMTs in other universities, in-service teachers) to get a broad and profound understanding of the current state of mathematics teachers' knowledge for teaching probability in Egypt.
- Other future investigations might be carried out using other specific questions that can validate the current study results; for example, one could use the Conditional Probability Reasoning test (CPR) proposed by Díaz and de la Fuente (2007) to assess PSMTs' conceptions in conditional probability. Moreover, and specifically about the conditional probability, since, as stated earlier (see the study limitations), no afterward interviews were conducted with the PSMTs, it is important to consider this concern in future research. More concretely, although the current study supposed that the PSMTs were thinking of the condition when calculating the conditional probability from the two-way table as same as the problem of giving birth, this may not be true. PSMTs probably picked the numbers randomly from the tables without thinking of any relationship between both contexts. That issue was not fully clear in the current study since there were no interviews with the participants. In this sense, the study advocate for more research regarding PSMTs' interpretation of the two-way table, which paves the way toward a trend of research that focuses on how conditional probability can be taught in teacher education,

further to pupils, in a way that makes them realize the connection between different conditional probabilistic contexts (e.g., social phenomena and mathematical tasks).

- A logical subsequent step for this study would be to design a teaching experiment that may help eliminate PSMTs conceptions and cognitive biases. Thus, another direction for future studies could be proposing a course of study or a teaching strategy and investigating its effectiveness to improve PSMTs' knowledge for teaching probability. In this view, it is crucial to consider how to overcome PSMTs' shared conceptions and biases (e.g., equiprobable bias, causal conception, independence conception, overgeneralization heuristics, and the illusion of validity). In particular, the equiprobable bias. Acknowledging the importance of the equally likely principle, which considers an essential premise to apply the theoretical probability wherein most curriculum activities can be handled, much more research is needed to clarify when (when not) that premise is valid. In other words, since the equally likely principle is relevant to the theoretical probability, it is important to confirm students' understandability of that principle before modeling the probabilistic situation. They have to think about whether the equally likely principle valid to the situation they are going to model or not; accordingly, if yes, they may use theoretical probability and if not, they have to think of another probability interpretation. Additionally, and under the view of PSMTs' pedagogical preparation, it is also possible to do further research on the characteristics of activities that may, on one hand, enhance PSMTs' understanding of objective probability and, on the other hand, expose their subjective probabilistic reasoning.
- More specifically, regarding the emergence of randomness, variability, and contextual recognition in PSMTs' reasoning under uncertainty, more research is needed to argue whether we have to change the prevalent way of teaching and focus on these factors; furthermore, how to move toward such a challenging change.
- Finally, the dilemma about the relationship between conditional and unconditional probabilistic reasoning remains a notable area of further studies.

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APPENDICES

Appendix 1:

Classification of the subjects that PSMTs study during the whole duration of the preparation program based on Grossman's model[23]:

General Pedagogical Subjects (GPK)		
Code	***Subject name***	***Number of hours***
Edu111	Introduction to Education	3
Curr112	Environmental Education	2
Edu112	Philosophical and scientific thinking	2
Curr113	Health Education	2
EDU 121	The teacher and the teaching profession	3
MH 121	Developmental Psychology	2
Curr412	Curriculum	2
Psy211	Educational Psychology 1	2
MH221	Social Psychology	1
Psy221	Cognitive Psychology	1
Curr222	Education Technology	3
COMP 211	School and classroom administration	2
Psy222	Psychology of learning 2	2
Edu311	Adult education and its applications	2
MH321	Psychology with special needs	3
Psy421	Individual Differences and Psychological Measurement	3
Curr321	Methods of teaching people with special needs	2
Edu421	Educational Thought and its applications	2
Total number of hours		***39***

Teaching Methods and Practicum Training (PCK)		
Code	***Subject name***	***Number of hours***
MAT 122	History of mathematics and philosophy	3
curr211	Microteaching	3
Curr312	IT education specialization	2
Curr221	Microteaching2	2
Curr311	Teaching Methods 1	3
Curr313	Computer in specialization	2

[23] The academic program for the Faculty of Education, Tanta university has been retrieved from http://tdb2.tanta.edu.eg/acad_catalog/under.aspx?AS_FACULTY_INFO_ID=3

140	Field Training	4
140	Field Training	4
Curr411	Teaching Methods 2	3
140	Field Training	4
140	Field Training	4
Total number of hours		***34***

General Cultural Subjects (K of Context)		
Code	***Subject name***	***Number of hours***
Curr111	Scientific Culture	2
Ara111	Arabic	2
Eng121	English	4
666	Human rights	2
Comp221	International Education	1
EDU 221	School and community	1
MH411	Mental health and psychological counseling	3
Edu321	Education and the issues of the day	2
Comp321	Parenting supervision	1
Comp421	Education system in Egypt and contemporary trends	2
Total number of hours		***20***

Advanced Mathematics, Statistics, and Sciences (SMK)		
Code	**Subject name**	**Number of hours**
PHS 111	physics 1 properties of the material electrical and magnetics	5
MAT 111	Basics of mathematics	5
MAT 112	High algebra	4
MAT 113	Calculus 1	4
MAT 114	Statics 1	4
PHS121	Physics 2 heat and geometrical optics	5
MAT 123	Engineering analytical level	5
MAT 124	Differentiation and integration 2	4
MAT 125	Dynamics 1	5
MAT211	Linear algebra	4
MAT212	Differentiation and integration 3 - Mathematics applications	6
MAT 214	Statics 2	4
MAT 215	Introduction to Computer Programming	4
PHS 221	Physics 4 Physical Optics and AC	5
MAT 221	Analytical Engineering in vacuum	3
MAT 423-MAT 424	Applied Mathematics Fluid Dynamics 2 Applied Mathematics quantum mechanics	6

MAT 222	Differential Equations	3
MAT 223	Dynamics 2	3
MAT 224	Introduction to Statistics and probabilities	3
MAT 225	Real analysis	3
PHS 211	Physics 3 Thermodynamics and Modern Physics	5
MAT 312	Theory Composition	3
MAT 421	General Topology	3
MAT 314	Applied Mathematics rigid body dynamics	5
MAT 315	Electrostatic and magnetic	5
MAT 316	Chaos Theory	4
MAT321	Numerical Analysis	4
MAT322	Special Functions and Partial Differential Equations	4
MAT323	Statistical analysis	4
MAT 324 - MAT425	Applied Mathematics (analytical dynamics) - Applied Mathematics (Mechanics multiple circles)	6
MAT326	Mechanics Astronomy and Space	3
MAT413	Functional Analysis	4
MAT414-MAT415	Applied Mathematics General Theory of Relativity Applied Mathematics dynamics inhibitions 1	6
MAT416	Advanced statistical methods	5
MAT 411	Mathematics vital	3
MAT412	Rings and fields	3
MAT 422	Sports programming and software packages	4
MAT 421	Complex Analysis	4
MAT 425	Differential Geometry	4
MAT 426	Statistical Mechanics	4
Total number of hours		***168***

Appendix 2:

Results of analyzing school textbooks' activities that are relevant to the domain of statistics for both primary and lower secondary levels; from grade 1 to 9

This analysis was conducted in light of the fundamental statistical ideas that are declared by Burrill and Biehler (2011). Thus, these ideas are first summarized; then, accordingly, the results of the analysis process are reported, as follows:

	The characteristics of the related activity
Data	It aims at presenting data as numbers with a context or reflects the processes of obtaining such data (types of data and ways of collecting it).
Variation	It intends to recognize or measure the variability to predict, explain, or control a phenomenon (measures of variability).
Distribution	It includes concepts of tendencies from empirical distributions, random variables from theoretical distributions, and summaries in the sampling distribution.
Representation	It combines the graphical or other data representations (e.g., tables, graphs).
Association and modelling relations between two variables	It displays the relationships among statistical variables for categorical and numerical data (e.g., regression models).
Probability models for data generating processes	It represents the hypothetical relationships generated from theory (theoretical probability); simulations, or large data set (experimental probability); and, quantifying the variability in data including long-term stability (i.e., probability theories).
Sampling and inference	It aims at clarifying the relation between samples and population, and how to conclude with some degree of certainty.

A: Results of analyzing textbooks' activities for grade 1, 2, 3, and 4[24]

Core statistical ideas	**School textbooks' activities for grades 1, 2, 3 and 4**				**Codes**
	1st Grade, 2nd Term, Unit4: Statistics, Lesson 1: Statistics (6 activities)	*2nd G, 2nd T, U5: Statistics, L1: Collecting and Representing Data (5 As)*	*3rd G, 2nd T, U5: Statistics and Probability, L1: Represent Data (6 As), L2: Probability (9 As), L3: Revision (3 As)*	*4th G, 2nd T, U4: Statistics and Probability, L1: Collecting, Displaying and Representing Data (11 As). L2: The Probability (8 As). L3: Revision (3 As)*	

[24] The terms U, T, L, and A refer to the Unit, Term (semester), Lesson, and Activity, respectively; too, the reported activities are arranged by the numbers 1,2, 3, ..., Etc.

Data (Organize data)	1,3,5,6 (tables)	1,2,4,5 (tables)	L1: 1,2 (tables). L3: 1 (identify appropriate way to collect data)	L1: 1,2,3 (noticing), 4 (experimenting), 5 (field study), 6,9 (arrange data in table with sets). L3: 2 (table)	**19**
Representation (Represent data)	2,3,6 (small blocks) 4,5 (number line horizontally or vertically)	1,2,3 (marks and bar graph)	L1: 3,4,5,6 (bar and line graph). L3: 2 (bar graph)	L1: 7,11 (bar and line graph), 8,10 (double bar graph). L3: 1 (appropriate way)	**18**
Probability models for data generating processes (Understand the meaning of probability)			L2: 1,2 (certain, possible, impossible event), 3 (expectation to great/ less extent), 4 (extent of expectation: great, moderate, weak, none), 5 (the probability of certain, impossible, and possible event), 6,7,8,9 (procedures of calculating probability). L3: 3 (procedures for calculating the probability in relation to certain, possible, impossible event)	L2:1 (certain, possible, impossible event), 2 (meaning of probability). L2: 3,4,5,6,7,8 (calculate the probability of occurrence of a random event). L3: 3 (calculate probability).	**19**
Codes	**9**	**7**	**18**	**22**	**56**

B: Results of analyzing the textbooks' activities for grade 5 and 6

	School textbooks' activities for grades 5 and 6				Codes
Core statistical ideas	*5th G, 1st T, U4: Probability, L1: Experimental probability (7 As). L2: Theoretical probability (10 As). L3: Exercises (9 Ac)*	*5th G, 2nd T, U5: Statistics, L1: Collecting data (3 As), L2: Organizing and Displaying Data (3 As), L3: Reading Tables and Line Graphs (3 As), L4: Representing Data by Histogram and Frequency Polygon (1 A), L5: Representing Data Using Pie Graphs (1 A), L6: Revision (4 As)*	*6th G, 1st T, U4: Statistics L1: Kinds of statistics data (5 As). L2: collecting descriptive statistic data (3 As). L3: collecting statistics quantitative data (4 As). L4: Representing quantitative statistics data by the frequency curve (5 As). L5: Exercises (4 As)*	*6th G, 2nd T U4: Statistics and Probability, L1: Representing statistical data using the circular sector (12 As). L2: Random experiment (4 As). L3: Probability (7 As). L4: Revision (9 As)*	
Data (identify appropriate method to collect data)		L1: 1 (observing, recording, tables), 2 (survey, table), 3 (measuring, table). L2: 1,2,3 (form frequency tables and cumulative frequency tables with sets)	L1:1, 2,3,4,5. L5:1 (descriptive vs quantitative data). L2:1, 2,3. L3: 1,2,3,4. L4:1, 2,3,4,5. L5:2,3,4 (understand the collected data from the frequency table)	L1: 6 (tables). L4: 9 (survey)	**29**
Representation (Data representation)		L2: 1,3 (represent simple frequency table/ with intervals using bar graph). L3: 1 (pictograph),	L2: 1, 2 (forming the frequency table). L3:1, 2,3,4. L5:2 (forming the frequency table with sets). L4: 1 (represent data using frequency	L1: 1 (circular sector concept)2, 3,4,5,7,8,9,10, 11,12 (circular sector). L2: 2,3 (tree	**41**

		2,3 (triple bar graph). L4: 1 (Histogram and frequency polygon). L5: 1 (pie graph). L6: 1,2 (histogram and frequency polygon), 3,4 (pie graph).	polygon), 1,2,3,4,5. L5:2,3,4(represent data using frequency curve),	structure). L4: 1 (circular sector)	
Probability models for data generating processes (Understand meaning of probability through**)**	L1: 1,2,3,4, L3: 9 (experimental probability concept), 5,6,7, L3:8 (calculate the expected times of occurrence for an event). L2:1,2,3,4 (sample space), 5 (event), 6,7,8,9, 10, L3: 1,3,4,5,6,7 (determine the events and calculate theoretical probability). L3: 2 (calculate complementary event)			L2: 1 (random experiment concept), 2,3,4 (sample space concept). L3: 1,2,3 (sample space), 4 (event), 5,6,7 (probability of possible, impossible, certain event). L4: 2,3,4 (sample space), 5,6,7,8 (probability).	**44**
Codes	**26**	**17**	**36**	**35**	**114**

C: Results of analyzing textbooks' activities for grade 7

Core statistical ideas	School textbooks' activities for grade 7		Codes
	7th G, 1st T, Unit 3: Statistics (Measures of Central Tendency), L1: Arithmetic Mean (3 As). L2: Median (2 As). L3: Mode (2 As)	*7th G, 2nd T, U2: Probability and Statistics, L1: Samples (7 As). L2: Probability (9 As). L3: Revision (3 As)*	
Data		L3: 3 (design a survey to collect data)	**1**
Distribution	L1:1,3 (Identify the concept of arithmetic mean), 2 (calculate the value of mean). L2: 1,2 (concept of median, its order and value). L3:1,2 (concept of mode).		**7**
Probability models for data generating processes		L2: 1,2,3,4,5 (concept of experimental probability and how to calculate it), 6,7,8,9 (calculate the probability of an event as subset of the sample space).	**9**
Sampling and inference		L1: 1 (concept of sample and its relation to population, 2 (how to choose random sample systematically), 3 (concept of random sample), 4,5,6,7, L3: 1, 3 (use the calculator to generate random numbers). L3:2 (use Excel program to generate random numbers).	**10**
Codes	**7**	**20**	**27**

D: Results of analyzing textbooks' activities for grade 8

Core statistical ideas	School textbooks' activities for grades 8		Codes
	8th G, 1st Term, U3: Statistics, L1: Collecting and organizing data (6 As), L2: Ascending and descending cumulative frequency table and their graphical representation (6 As). L3: Arithmetic mean, median, and mode (6 As). L4: Revision (6 As)	*8th G, 2nd T. U3: Probability, L1: Probability (7 As). L2: Revision (6 As)*	
Data	L1: 1,2, L4: 1,2,4,5 (Collect, analyse, interpret), L1: 3,4,5,6, L2:1,2 (organize data using frequency table with sets). L2: 1,2, L3: 4,5 (forming ascending/descending cumulative frequency table).		**16**
Distribution	L1: 2 (determine the mode). L3: 1,3, L4: 1,2 (finding mean for frequency table with sets), L3: 2 (concept of mean). L3 4,5, L4:3,4 (finding median for frequency table with sets). L3: 6, L4: 5,6 (finding mode for frequency table with sets).		**13**

Representation	L2: 1,2,3,4,5,6, L3: 4,5, L4: 3,4 (represent ascending/descending cumulative frequency curve)). L3: 6, L4: 5,6 (represent data using histogram).		**13**
Probability models for data generating processes		L1: 1,2,3,4,5,6,7 (identify sample space, possible, impossible, certain event, calculate the probability of an event). L2: 1, 3,4,5,6 (calculate the probability of an event).	**12**
Sampling and inference		L1: 4,6, L2: 2,3 (drawing conclusion with some degree of certainty).	**4**
Codes	**42**	**16**	**58**

E: Results of analyzing textbooks' activities for grade 9

Core statistical ideas	**School textbooks' activities for grade 9**		**Codes**
	9th G, 1st Term, U3: Statistics, L1: Collecting Data (6 As). L2: Dispersion (8 As). L3: Revision (10 As)	*9th G, 2nd T, U3: Probability, L1: Operation on events (14 As). L2: Complementary Event and Difference between two events (6 As). L3: Revision (7 As)*	
Variation	L2: 2 (identify range as a simplest way to represent variation), L2:3, L3: 1,2,7 (standard deviation for raw data), L2:4,5, L3:3,8 (SD for frequency table), L2:6,8, L3:4,9 (SD for frequency table with sets), L2:7 (SD using calculator), L3: 10 (SD using EXCEL program).		**15**
Distribution	L2:1,8, L3: 3,4,7 (calculate the arithmetic mean for raw data and frequency table).		**5**
Representation	L2:8 (represent frequency table with sets using frequency polygon).	L1: 5,8 (represent events using Venn diagram), 14 (find the intersection and union of two events from Venn diagram).	**4**
Association and modelling relations			

between two variables			
Probability models for data generating processes		L1:1,2, L3:4,5,7 (calculate the probability of an event), L1: 3,6,9 (identify sample space), L1: 4,7,10 (identify an event), L1:5,8, L2:4, L3:1 (probability of occurring two events together), L1: 11, L3: 2 (probability of the union of two events), L1: 12,13 (identify the relationship between the intersection and union of two events). L2: 1,2 (identify the relationship between an event and its complementary), L2:3, L3:6 (relationship between probability of an event and probability of its complementary), 4 (calculate the probability of complementary event), 5 (identify difference between two events), 6 (calculate the probability of two events). L3: 3 (exclusive events).	**27**
Sampling and inference	L1: 1, L3: 5 (identify primary and secondary resources to collect data), 2 (mass population and sample as methods to collect data), L1:3,6, L3: 6 (sample: biased and random: simple and layer), 4(compare sample with population), 5(choose random numbers using calculator).		**8**
Codes	**29**	**30**	**59**

Appendix 3:

Results of analyzing the implemented curriculum of probability in Egypt: how was the implemented curriculum characterized? Retrieved from Elbehary, 2019.

	Situations	*Propositions*	*Procedures*	*Term and embedded concepts*
Intuitive meaning	- Use students' daily life context to grasp certain, possible, and impossible events (3^{rd} **G**) - Discuss the meaning of great and moderate probability (3^{rd} **G**) - Handle students' personal judgments to determine the degree of probability (4^{th} **G**)	- Relationship between possible, impossible, certain events and personal expectations (3^{rd} **G**) - Relationship between types of events and its probability (3^{rd}, 4^{th} **G**)		- Guess, expect, and predict (3^{rd}: 6^{th} **G**) - Great, moderate, less, weak and none (3^{rd}: 6^{th} **G**) - Certain, possible, and impossible event (3^{rd}, 6^{th} **G**) - Defective Vs functional (5^{th} **G**) - Success Vs failure (5^{th} **G**)
Classical meaning	- Discuss the theoretical meaning of probability through some random experiments (e.g., tossing a coin, rolling a dice, gender of a newborn, results of a football match, spin a spinner) (4^{th}: 6^{th} **G**) - Define the sample space not only in a simple random experiment but also in two steps trial and assigned probability of some events (5^{th} **G**, 6^{th} **G**). - Explain the meaning of the random experiment (6^{th} **G**)	- Tossing two coins once is equivalent to tossing one coin two consecutive times (6^{th} **G**) - Relationship between the sample space and events (6^{th} **G**) - Relationship between the type of an event and its probability (e.g., if $A = \phi$, then $P(A) = 0/n(S) = 0$) (6^{th} **G**) - The probability can be written as a fractional, decimal, or in the form of a percentage (6^{th} **G**)	- Apply the law of theoretical probability (3^{rd}: 6^{th} **G**) - Determine the probability of impossible, possible, and certain events (3^{rd} **G**) - Compare among decimals, fractions, and percentages (4^{th}, 6^{th} **G**) - Determine the elements of sample space (4^{th}: 6^{th} **G**) - Represent the sample space using a tree diagram (6^{th} **G**)	- Fair coin, H, T, HH, TT (3rd: 6th G) - Equally likely, same, symmetric, and identical (4th: 6th G) - Ratios, decimals, and percentages % (4th G) - All possible outcomes (4th: 6th G) - Odd, even, prime, divisible by, greater or smaller than or between, and 2-digit number (3rd: 6th G) - Sample space (S) (5th, 6th G) - Theoretical probability (5th, 6th G) - Random experiment (6th G) - Tree diagram (6th G) - A, n (A), n (S), P (A), φ, (6th G) - inequality ≥ (6th G)
Frequentist meaning	- Doing a simple random experiment of tossing a coin 10, 20, 50, 100 times (5^{th} **G**) - Propose a survey to ask students about preferred sport and language (5^{th} **G**) - interpret the favorable cases by knowing the probability of a small sample (5^{th} **G**) - Inference into the probability of a small sample trough reflecting the population's characteristics (5^{th} **G**)	- Relationship between the experimental and theoretical probability (5^{th} **G**)	- Apply the law of experimental probability (5^{th} **G**) - Calculate the number of expected times by knowing the probability of previous trials (5^{th} **G**)	- Regular coin (5^{th} **G**) - Survey (5^{th} **G**) - Sample (5^{th} **G**) - Experimental probability (5^{th} **G**) - Favorable, preferred, and favorite (5^{th} **G**)
Axiomatic meaning	- Discuss the relationships among all possibilities of some random experiments (4^{th}: 6^{th} **G**)	- For $A \subset S$, $0 \leq p(A) \leq 1$) (4^{th}: 6^{th} **G**) - the sum of probabilities for all possible events = 1 (4^{th} **G**) - Relationship between the probability of an event and its complementary (e.g., success vs failure, defective vs functional) (4^{th}, 5^{th} **G**)	- Calculate the probability of a complementary event (4^{th} **G**) - Calculate the probability of event A union B (4^{th}, 5^{th} **G**)	- Subset (5^{th} **G**) - A or B ($A \cup B$) (5^{th} **G**)

Appendix 4:

List of the contributed papers on probability education in ICOTS8, 9, and 10

I. In ICOTS8 (2010), 15 (out of 127) papers on probability were found, as follows:

Title	*The quoted sections to define paper's purpose*		*The assigned category*
	Section 1	*Section 2*	
Paradoxical games as a didactic tool to train teachers in probability (Batanero et al., 2010)	"we suggest the interest of classical paradoxes in the history of probability to organise some didactic activities directed to train teachers in probability." (p.2)	"Batanero et al. (2004) proposed an activity based on the Bertrand's box paradox that serves to compare the frequentist and Laplace's conceptions of probability, and to reflect on the concepts of dependent experiments and conditional probability" (p.2)	B (Effectiveness of paradoxical games on changing teachers' conceptions of probability)
Students' opinion on the subjects of statistics and probability in secondary schools of Lisbon, Portugal (Caldeira & Mouriño, 2010)	"This work aims at analysing students' opinion about these subjects." (p.1)	"In this study, we analysed the opinion of students from secondary school about the subjects of Statistics and Probability." (p.4)	Not relevant (Non-cognitive aspect)
Linking probability to real-world situations: how do teachers make use of the mathematical potential of simulation programs? (Theis & Savard, 2010)	"we conducted a one-year design experiment involving 4 high school teachers. We trained the participants in various concepts of probability and accompanied them to prepare classroom situations, which they used in their classrooms. In this paper, we analyze how the participating teachers used a simulation software we provided them." (p.1)	"In this paper, we present the preliminary results of this analysis. We will discuss the following issues: a) the teachers used the simulation programs mainly to show their pupils that gambling activities are not in favour of the gambler in the long term, b) the teachers had difficulties making the most of other probabilistic concepts that could potentially have been taught through the simulation software." (p.2)	D (The effectiveness of simulation software to explain probability concepts)
The impact of using pupils' daily social practices as well as computerized simulators as a teaching medium on motivation and knowledge construction regarding probabilities among high school pupils (Grenon et al., 2010)	"Our research data show that learning while playing, by using computerized simulators as a teaching medium, is effective in motivating pupils and in building knowledge." (p.1)	"In this study we propose to investigate the recourse to active methods of teaching, using realistic contexts based on the knowledge that pupils have of gambling games. And to do this, we shall construct learning situations integrating computerized simulators that will sustain their motivation while learning probabilities." (p.1)	D (Effectiveness of computerized simulators as a teaching medium on probability knowledge construction)

The effect of contextualising probability education on differentiating the concepts of luck, chance, and probabilities among middle and high school pupils in Quebec (Larose et al., 2010)	"Within the scope of this study, carried out among over 1,600 pupils in middle and high schools, we have collected their implicit definitions of gambling, luck, and probabilities." (p.1)	"What are the social representations that correspond to preconceptions regarding the definition of such concepts as luck, chance, and probability among pupils in middle and high school? This is what we have explored by asking two samples, distinct but complementary, of Quebec pupils (N= 1,882) to define what each concept meant." (p.1)	A (Investigate students' social representations and preconceptions or probability)
Implications of educational reform in Cyprus on the teaching of probability and statistics at the secondary school level (Papaieronymou, 2010)	"This paper examines the educational reform currently taking place in Cyprus and its implications on the teaching of statistics and probability at the secondary school level." (p.1)	"the implications of reform discussed in this paper should be considered as educators in Cyprus prepare for instruction on probability and statistics and as the various committees appointed by the Cyprus Ministry of Education and Culture prepare the revised curriculum materials." (p.4)	C (General description of the current status of statistics and probability education)
A semiotic analysis of "Mônica's random walk": activity to teach basic concepts of probability (Gusmão et al., 2010)	"We analyzed the activity "Mônica's random walk1 ", in the learning environment paper-and-pencil, which presents the basic concepts of probability" (p.1)	"In this work we apply the technique of OSA to analyze how interact the two meanings (institutional and personal) of probability during the activity "Monica's random walk", and if this helps to teach probability, in an attempt to evaluate the outline of the teaching sequence aiming to adapt it in the future to the virtual environment of AVALE." (p.2)	A (Analyzing teachers' implementation of Mônica's random walk activity to teach the probability concepts)
Simulating the risk without gambling: can student conceptions generate critical thinking about probability? (Savard, 2010)	"lesson plans about probability were designed and implemented in a grade four classroom. In this teaching experiment, students were asked to simulate the spinning of the wheel using a spinner." (p.1)	"A teaching experiment was conducted in a grade four classroom in a Quebec City suburb. The aims of this teaching experiment were to study the probabilistic thinking of the students and to see how this thinking was developed within fake gambling situations." (p.2)	B (Studying students' probabilistic thinking in fake gambling situations)
Motivation and self-efficacy related to probability and statistics: task-specific motivation	"We concentrate on the data of 350 prospective teachers, who were asked about content domain-specific and	"What interest and self-efficacy dispositions do learners have related to the areas of mathematics (general), statistics, and	Not relevant (Non-cognitive aspect)

and proficiency (Gundlach et al., 2010)	taskspecific motivation and self-efficacy and about solutions to given tasks which were parallelised with the motivation questionnaire." (p.1)	when being confronted with particular tasks? • What interdependencies among these variables and with the proficiency of solving tasks can be observed?" (p.2)	
The teaching of statistics and probability in mathematics undergraduate courses (Viali, 2010)	"This study analyzed the curriculums of Mathematics undergraduates programs in Brazil." (p.1)	"The main aim of the study was to verify the instruction hours of the probability and statistics courses and what they represent out of the total number of hours of the course."	C (Curriculum analysis of statistics and probability courses)
Pre-service teachers' understanding of probability distributions: a multilevel statistical analysis (Chadjipadelis & Anastasiadou, 2010)	"this study posed a fundamental question: Does a project improve Greek pre-service teachers' understanding of probability distributions?" (p.1)	"In this paper the problem of pre-service teachers' approaches in solving tasks in probability distributions is discussed. Two groups of students took part in the study. The control group participates in teacher-centred teaching environment. The experimental group participates in student-centred teaching environment. Experimental group students were allowed to become involved to creation of their own task along with academic demands. Those students got a more meaningful learning and achieved higher performance." (p. 4)	D (Effectiveness of students-centered approach on pre-service teachers' understanding of probability distributions)
Changing the understanding of probability in talented children (Estrella & Olfos, 2010)	"This paper refers to the effectiveness of an instructional sequence of lessons related to Probability, which were implemented to 11 to 13 years old talented children." (p.1)	"This summary of responses highlights the high number of erroneous or partially correct responses for items four and six. Items four and six correspond to contexts about fair coins and dice" (p.3)	D (Effectiveness of instructional sequence on probability understanding)
Reviewing and promoting research in probability education electronically (Kapadia & Borovcnik, 2010)	"This paper provides an incisive and reflective summary on which researchers can build, while the latter enables developments relevant for other areas of research too. Hyperlinks are included throughout." (p.1)	"The articles show that the community regains interest in probability education." (p.5)	Not relevant (Literature review on current research efforts about probability education)

The future of interactive, electronic research: an exemplar from probability education (Borovcnik & Kapadia, 2010)	"This one goes on to discuss the influence of new technology in how research is presented and how this may change even the nature of research." (p.1)	"Finally, we discuss how electronic publishing affects the research and paves the way for future research, especially for younger researchers." (p.5)	Not relevant (Analyzing the influence of electronic publications)
Principles and strategies in teaching probability (Leviatan, 2010)	"We propose to teach tertiary probability focusing on general probabilistic principles that lead to general probabilistic problem-solving strategies." (p.1)	In section 1 we present some of the theoretical principles, in section 2 we describe some resulting strategies, and finally we offer directions for future research." (p.1)	C (Proposed principles to teach probability focusing on probability axioms)

II. In ICOTS9 (2014), 8 (out of 127) papers on probability were found, as follows:

Title	***The quoted sections to define paper's purpose***		***The assigned category***
	Section 1	*Section 2*	
The potential of a grounded theory approach to study teaching probability (Eckert, 2014)	"I propose a research methodology founded on the theoretical assumptions of symbolic interactionism combined with a grounded theory approach. The purpose of this paper is to outline such a research methodology that focuses on teaching as classroom interaction between teachers and students." (p. 1)	"The discussion aims to emphasize the possibilities by this way of studying teachers' knowledge for teaching probability and refine the methodological construct." (p.1)	B (Effectiveness of grounded theory to study teachers' knowledge with a classroom interaction)
Measuring the basics of probabilistic reasoning: the IRT-based construction of the probabilistic reasoning questionnaire (Primi et al., 2014)	"The aim of the present study was to develop a scale to measure basic probabilistic reasoning skills, which are deemed necessary to successfully complete introductory statistics courses. Specifically, our aim was to accurately measure low levels of ability in order to identify students with difficulties." (p.1)	"(for examples: "A ball was drawn from a bag containing 10 red, 30 white, 20 blue, and 15 yellow balls. What is the probability that it is neither red nor blue? a. 30/75; b. 10/75; c. 45/75; and "60% of the population in a city are men and 40% are women. 50% of the men and 30% of the women smoke. We select a person from the city at random. What is the probability that this person is a smoker? a. 42%, b. 50%, c. 85%)" (p.2)	C (Measuring students' probabilistic reasoning levels)
A review of probability and statistics apps for mobile devices (Edwards, 2014)	"This paper reviews some the mobile apps currently available which enable a user to either learn Statistics or to carry out the sorts of summaries and analyses encountered in an	"This paper has identified a host of faults and errors in the apps considered above. However users have no way of knowing about these until they have obtained the app" (p.4)	Not relevant (literature review on currently available mobile apps)

	undergraduate Statistics course." (p.1)		
Meanings of probability in Spanish curriculum for primary school (Torres & Contreras, 2014)	"The aim of this paper is to analyze the probability content in the Spanish curricular guidelines for primary school" (p.1)	"We identify the main probabilistic objects and the probability meanings suggested in these guidelines." (p.1)	A (Curriculum analysis with a consideration of subjective probability)
Training prospective teachers for teaching of probability at secondary school in Colombia (Torres, 2014)	"The aim of this paper is to analyze the training about probability and how to teach probability offered by some of the Colombian universities which have undergraduate programs for prospective secondary school teachers." (p.1)	"the Colombian curriculum explicitly considers classical and frequentist approaches to probability while the subjective approach is only implicit." (p.2)	A (Curriculum analysis with a consideration of subjective probability)
Probability and statistics in access exams to Spanish universities (Díaz et al., 2014)	"We perform a crossed analysis between curricular guidelines and items appearing in the exams, in four Spanish regions, so as to detect prevalent units, similarities and differences. Thus, we check what competencies and curricular units appear in the official curriculum related to Statistics and Probability, and how they are assessed in the exams." (p.1)	"essential and original points in the official curriculum for Applied Mathematics such as the interpretation of issues related to social sciences from a mathematical point of view, critical assessment of the obtained results or the importance of inferential statistic to study commercial, economical and political situations are poorly represented in PAU." (p.3) "in general, the questions about real life problems are underrated in PAU." (p.4)	C (Curriculum analysis with a consideration of the relationship between statistics and probability)
What's missing in teaching probability and statistics: building cognitive schema for understanding random phenomena (Kuzmak, 2014)	"An analysis of verbal protocols of 24 college students, who interact with and describe random phenomena involving the mixture of colored marbles, is presented, using cognitive schema to represent the subjects' expressed understanding." (p.1)	"In this paper, I apply the construct of schema to "random phenomena," as a means to formally describe a mature understanding of random phenomena; to illustrate the relative complexity and abstractness of the schema; to support analyzing students' understanding; to clarify teaching objectives regarding probability and statistics, and to identify directions for instructional improvement." (p.2)	A (Analyzing college students understanding of random phenomena using chance games) .

		"Gambler's fallacy, predicting mixedup sequence works" (p.4)	
Overview of prospective mathematics teachers' probabilistic thinking (Moreno & Cardeñoso, 2014)	"This paper presents an overview of the models of probabilistic thinking constructed by 583 prospective mathematics teachers in Mendoza, Argentina. The goal was to gain insight into the personal meanings that these future teachers attribute to random phenomena and to the estimation of their probability." (p.1)	"the purpose of the present work was to discover what beliefs and conceptions the pupils have regarding randomness and probability." (p.1) "Four clusters emerged and, in accordance with their characteristics and in hierarchical order, were labelled: deterministic, personalistic, uncertainty, and contingency." (p.3)	A (Investigate the current state of PSMTs probabilistic thinking

III. In ICOTS10 (2018), 2 (out of 79) papers on probability were found, as follows:

Title	***The quoted sections to define paper's purpose***		***The assigned category***
	Section 1	*Section 2*	
Comparing the efficiency of mathematical V. intuitive explanations in conditional probability (Levy & Stukalin, 2018)	"The students were presented with the Monty Hall problem and then received one of 3 explanations to the counter-intuitive solution to the paradox: Group 1 served as a control group and did not receive any explanation. Group 2 students were shown a mathematical solution using a tree diagram. Group 3 students were presented with the following intuitive solution: "Imagine that after selecting a door at random you are given the choice of either holding on to your initial choice or opening the two remaining doors. Obviously the second option is better"." (p. 1)	"Results show that the group that was shown an intuitive solution to the Monty Hall problem performed better than the other groups in the test question." (p.2) "These initial findings stress the importance of exposing statistics students to counter intuitive problems, and specifically to the underlying intuition behind the solutions of such problems." (p.2)	B (Effectiveness of providing the intuitive explanation of probability paradoxes on pre-service teachers' performance in conditional probability)
Teaching probability and statistics to pre-service elementary school teachers (Takagi, 2018)	"Here, I introduce a syllabus of the class about the statistical charts such as bar and line graphs, pie charts and histogram." (p.1)	"We obtain some conclusions through the class as follows: Many students don't have some fundamental acknowledgments about statistical charts." (p.2)	D (Experiment a syllabus to teach statistical charts for pre-service elementary school teachers)

Appendix 5:

Definitions of primary probability interpretations and learners' cognitive biases in reasoning under uncertainty

I. The primary probability interpretations

Probability interpretation	***Underlined conditions, circumstances, and limitations to considers***
Theoretical Probability is a fraction whose numerator is the number of favorable cases and whose denominator is the number of all possible cases in the set of sample space	– The possible outcomes should be equally likely to occur. – It is only applicable in a finite set of possible outcomes (sample space) of a random process. – It enables one to calculate probabilities before any trials are performed. – It is difficult to be operated for complex daily situations (e.g., weather events, accident risks. Too, in some classroom situations (e.g., the case of rolling an unfair die). – It mirrors the idea of fairness; a decision is fair if it is made by an ideal chance device.
Experimental Probability can be determined by dividing the number of times an event occurs by the total number of performed trials.	– It cannot present the probability of an event when it is impossible to repeat the experiment a very large number of times. – No number can be fixed to ensure an optimal estimation for the probability. – It is required to experiment to obtain the relative frequencies concerning the outcomes and consequently estimate the probability . – As the number of trials increases, the experimental probability approaches the theoretical probability.
Subjective probability Probability indicates a degree of belief or preferences of a person based on personal judgment and information about the situation.	– The given judgment depends on several factors; for example, the knowledge of the subject, the conditions of the observation, the kind of event whose uncertainty is reflected on, and available data about the random phenomena. – It is closely connected with the Bayesian formula that allowed for revising a prior estimation of probability by processing new information, for estimating a new posterior probability.

II. Learners' cognitive biases in reasoning under uncertainty

Conception, bias, or heuristic	***Characteristics of the individual's reasoning***
Availability	– Estimate the probability by the ease with which instances come to mind instead of the complete data.
Dependence	– Think of past events influence future events, or as if the outcomes are associated with each other. Accordingly, randomness is neither admitted nor recognized.
Personalist interpretation	– Animism attribution of phenomena to God; the lucky or skilled person; conditions and rules of the game; or the mechanism of the object manipulation (e.g., the flipping technique).
Prediction	– Judge the prediction exactly (the prediction has the meaning of exact prediction).

Representativeness	– Estimate the likelihood of an event based on how well it represents some aspects of the apparent population (i.e., the degree of similarity between sample and population). Hence, the more representative an event is judged, the higher the probability it takes. – Believe that even small samples should reflect the population distribution or the process by which random outcomes are generated.
Gambler fallacy (Negative recency effect)	– Believe that after a long run of the same result in a random process, the probability of the same event occurring in the next trial is lower. – Try to balance the outcomes of a probability sequence without considering the independence among trials.
Unpredictability	– Think that outcomes cannot be predicted. Thus, the individual is not able to evaluate or predict the probability of such outcomes (its matter of randomness or chance).
Anchoring bias	– Estimate the probability based on some initial values (developed in light of given words in the problem, or partial computations) that were adjusted to yield the final answer.
The conjunction fallacy	– Think that the compound probability could be higher than the probability of each a single event (overestimate).
Effect on a sample size	– Estimate the probability without considering the sample size (law of small numbers).
Base-rate fallacy	– Fail to take base rates into account when judging probabilities. Accordingly, the individual tends to ignore the population base rate; particularly, in Bayes' problems, since both statistics of population and selective part of this population have to be considered together to solve a task correctly.
The time axis fallacy (The chronological conception, or the Falk phenomena)	– Interpret the conditional probability of P(A│B) as a temporal relationship, which is the conditioning event B should always precede the occurrence of event A. – Reluctant to believe that an event could condition another event that occurs before it.
Causal conception	– Interpret the conditional probability of P (A|B) as if the conditioning event B signifies a cause of A; then, A is the consequence (cause vs. effect). Yet, if A is perceived as a possible cause of B, then P(A/B) represents a diagnostic relationship.
Transposed conditional	– Confuse between (P(A|B) and P(B|A).
Cardinal conception	– Interpret the conditional probability of P(A|B) to be the ratio CARD ($A \cap B$)/ Card (B). That is correct in the case of finite equiprobable sample spaces. However, in case of a continuous sample space or the probabilities for the simple events are not equal, this conception leads to an error.
Equiprobability bias	– Think of random events as being equiprobable by their very nature (even when they are not). Thus, the outcomes are judged to be equally likely when their probabilities are not equal.
Outcome approach	– Interpret the goal of the question as a request to predict the outcome of a single trial. Consequently, instead of reflecting the distribution of

	occurrences in a series of events (the big picture), the individual focuses on the result of a single trial. – Evaluate the predictions as being right or wrong after one trial. Thus, the individual's judgment takes the form yes-no on whether an outcome will occur on a particular trial, which signifies a deterministic model of the situation. – Base the predictions on causal analysis, rather than the distributional information. Hence, the assigned numbers that reflect the probability are used occasionally to measure the strength of the causal factors. – Evaluate the probabilities in terms of their closeness to the values of 0%, 50%, and 100% (i.e., impossible, possible, certain).
Sample space	– Fail to recognize that all outcomes can occur.
Visual effect (Visual appearance)	– Think that the position of an object may change its probability to occur (e.g., the parts of a spinner make different probabilities depending upon their place).
Illusory correlation	– Maintain personal expectations and beliefs about the relationship between the variables, regardless of the empirical data or the evidence that indicates that such variables are independent.

Appendix 6:

The probability contexts survey that was distributed to pupils

Those are seven different settings at which the probability is manifested. Choose and prioritize three of them based on its importance and frequent usage in our daily life situations:

The situation	***Example***		***Arrange it***
To predict the weather circumstances	It is most probable to rain tomorrow		
To predict the result of a handball match for your school team	It is a weak possibility to win the handball competition		
To predict the gender of the newborn baby	The probability of giving birth to a girl equals 50%		
To express the status of a patient person	The probability of living to 90 years old equals 40%		
To express what we prefer	Your friend probably prefer science compared to mathematics		
To predict the quality of some products	The probability that the lamp produced by a factory is defective equals 3%.		
To predict the winner for some chance games	The probability of getting number 4 when throwing a die equals 3%.		

Appendix 7:

The probability contexts survey that was distributed to PSMTs

The following table summarizes seven various contexts at which the probability can be operated: they all express primary and lower-secondary textbooks' viewpoint. Based on your understanding of the principal concepts of theoretical, experimental, and conditional probability that you had studied before, could you determine the appropriateness of each situation to approach each probability interpretation? (i.e., Which setting could be suitable to teach each probability concept for your prospective pupils). Please note that some contexts can be adapted to approach more than one concept (i.e., you may select multiple interpretations for each setting).

The situation	***An example***	***The probability interpretation***		
		Theoretical	***Experimental***	***Conditional***
To predict the weather circumstances	It is probable to rain tomorrow 			
To predict the result of a handball match for your school team	it is a weak possibility to win the handball competition 			
To predict the gender of the newborn baby	The probability of giving birth to a girl equals 50% 			
To express the status of a patient person	The probability of living to 90 equals 40% 			
To express what we prefer	Your friend probably prefer science compared to mathematics 			
To predict the quality of some products	The probability that the lamp produced by a factory is defective equals 3%. 			

Appendix 8:

A questionnaire on reasoning in probability [R(in)P]

– ***First part (sub-questionnaire 1)***

- ***Item A:*** Knowing that there is a pregnant woman

Q1. What is the probability of giving birth to a girl?

Q2. Are there any conditions to determine that probability? "In other words, explain the reasons because of which you have decided the proposed probabilistic ratio (try to reflect and state the criteria that helped you to judge, or any conditions that you may think may change your estimation)"

- ***Item B:*** If you knew that a woman had gave birth to two boys before, and she will give birth to her third child

Q1. What is the probability of giving birth to a girl in that new case (i.e., after incorporating the given condition)?

Q2. Explain how have you determined such probability? In other words, why do you think that your expectation in the first situation (i.e., firstborn) is the same or different than in the second one (i.e., third born)?

– ***Second part (sub-questionnaire 2)***

- ***Item C:*** How can you explain to your prospective students the various strategies that could be employed to determine the probability of getting number 5 in a random experiment of rolling a die one time?
- ***Item D:*** Explain the meaning of this statement from your own perspective: the probability of raining tomorrow equals 60%.

- ***Item E1*:** These are the results of a questionnaire that was applied to a sample of 800 students in two schools of A and B to know which football team they prefer more, whether ElAhly or ElZamalek:

	ElAhly	**ElZamalek**	***Total***
School A students	195	190	385
School B students	305	110	415
Total	***500***	***300***	***800***

Suppose we select a student randomly;

Q1. What is the probability that a student prefers ElAhly?

Q2. What is the probability that a student in school B and prefers ElZamalek at the same time?

Q3. If you knew that the selected student prefers ElAhly, what is the probability that this student in the school A?

Q4. If you knew that the selected student belongs to school A, what is the probability that this student prefers ElAhly?

- ***Item E2***: These are the data for the students teachers who are enrolled in both Mathematics and Science classes for elementary and secondary education:

	Mathematics class	**Science class**	***Total***
Elementary level	190	70	260
Secondary level	110	90	200
Total	***300***	***160***	***460***

Suppose we select a student randomly;

Q1. What is the probability that a student has enrolled to teach the Secondary level?

Q2. What is the probability that a student has enrolled in the Science class for the elementary level?

Q3. If you knew that the selected student has enrolled in the Mathematics class, what is the probability that this student teaches the secondary level?

Q4. If you knew that the selected student taught the secondary level, what is the probability that this student has enrolled in the Mathematics class?