広島大学学位請求論文

Singularities of the dual curve of a certain plane curve in positive characteristic

（正標数におけるある平面曲線の双対曲線の特異点）

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主論文

SINGULARITIES OF THE DUAL CURVE OF A CERTAIN PLANE CURVE IN POSITIVE CHARACTERISTIC

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Abstract. It is well known that the Gauss map for a complex plane curve is birational, whereas the Gauss map in positive characteristic is not always birational. Let q be a power of a prime integer. We study a certain plane curve of degree q^2+q+1 for which the Gauss map is inseparable with inseparable degree q. As a special case, we show a relation between the dual curve of the Fermat curve of degree q^2+q+1 and the Ballico-Hefez curve.

1. Introduction

Let p be a prime integer, and q a power of p. We work over an algebraically closed field $\Bbbk$ of charcteristic p. We consider a plane curve C of degree q^2+q+1 defined by a homogeneous polynomial of the form

$$F = \sum_{i,j,k} a_{ijk} x_i x_j^q x_k^{q^2}, \tag{1}$$

where a_{ijk} are coefficients in $\Bbbk$, and $[x_0 : x_1 : x_2]$ is a homogeneous coordinate system in $\mathbb{P}^2$. If a_{ijk} are general, then the plane curve C is smooth. The condition that the defining polynomial of C is of the form (1) is independent of the choice of homogeneous coordinates of $\mathbb{P}^2$ (see Proposition 2.1).

Let $C^\vee$ be the dual curve of the plane curve C. The Gauss map

$$\Gamma : C \to C^\vee ; [x_0 : x_1 : x_2] \mapsto \left[\frac{\partial F}{\partial x_0} : \frac{\partial F}{\partial x_1} : \frac{\partial F}{\partial x_2}\right] \tag{2}$$

is an inseparable morphism. For every i, the partial derivative of F with respect to x_i is

$$\frac{\partial F}{\partial x_i} = \sum_{j,k} a_{ijk} x_j^q x_k^{q^2} = \left(\sum_{j,k} \alpha_{ijk} x_j x_k^q\right)^q, \tag{3}$$

where $\alpha_{ijk} = a_{ijk}^{1/q}$. Thus, if a_{ijk} are general, then the inseparable degree of the Gauss map is q. The purpose of this paper is to study singularities of the dual curve $C^\vee$ of a plane curve C defined by a polynomial of the form (1).

We define $\mathscr{C}$ to be the set of all the projective plane curves defined by homogenious polynomials of the form (1). Note that $\mathscr{C}$ is identified with $\mathbb{P}^{26}$.

Note that all tangent lines of the curve $C \in \mathscr{C}$ intersect C with multiplicity at least q at the tangent point. In our case, a double tangent and a flex are defined as following:

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Definition 1.1. Let m be an integer at least 2. We define an *m-ple tangent* to be a tangent line of C which has distinct m tangent points with multiplicity q, and a *flex* to be a point at which the tangent line intersects C with multiplicity $q+1$. A 2-ple tangent is called a double tangent.

Theorem 1. *Suppose that C is a general member of $\mathscr{C}$. Then*

(i) *the degree of the dual curve $C^\vee$ is $(q^2+q+1)(q+1)$,*
(ii) *the dual curve $C^\vee$ has only ordinary nodes as its singularities,*
(iii) *the number of ordinary nodes of $C^\vee$ i.e. double tangent lines of C, is*

$$\frac{q(q^2+q+1)(q^3+3q^2+3q-1)}{2},$$

and

(iv) *the number of flexes of C is*

$$(q^3+2q^2-q+1)(q^2+q+1).$$

We compare our theorem with the classical situation. Let $\tilde{C}$ be a general *complex* plane curve of degree d. Then the degree of the dual curve $\tilde{C}^\vee$ is $d(d-1)$. Moreover, each flex of $\tilde{C}$ corresponds to a cusp of $\tilde{C}^\vee$, whereas each flex of $C \in \mathscr{C}$ correponds to a smooth point of $C^\vee$. The singularities of $\tilde{C}^\vee$ consist of $\frac{1}{2}d(d-2)(d-3)(d+3)$ ordinary nodes and $3d(d-2)$ cusps.

As a special case, we consider the singularities of the dual curve of the Fermat curve $C_0 \in \mathscr{C}$ of degree q^2+q+1. We will show that the dual curve $C_0^\vee$ is related to the Ballico-Hefez curve.

Let $\gamma_d : \mathbb{P}^2 \to \mathbb{P}^2$ be a morphism defined by $\gamma_d([x_0 : x_1 : x_2]) = [x_0^d : x_1^d : x_2^d]$, and l_0 be a line $x_0+x_1+x_2 = 0$ in $\mathbb{P}^2$.

Definition 1.2. The *Ballico-Hefez curve* is the image of the line l_0 of the morphism γ_{q+1}.

In [5], Hoang and Shimada define the Ballico-Hefez curve to be the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$[s : t] \mapsto [s^{q+1} : t^{q+1} : st^q + s^q t].$$

Note, however, that the image of this morphism is projectively isomorphic to the image of the line l_0 of the morphism γ_{q+1}.

Theorem 2. *Let B be the Ballico-Hefez curve. Let $\gamma_{q^2+q+1} : \mathbb{P}^2 \to \mathbb{P}^2$ be a morphism defined by the above. If $C_0 \in \mathscr{C}$ is the Fermat curve of the degree q^2+q+1, then*

(i) *the dual curve $C_0^\vee$ is $\gamma^{-1}_{q^2+q+1}(B)$, and*
(ii) *the singularities of $C_0^\vee$ consist of $(q^2+q+1)^2(q^2-q)/2$ ordinary nodes, and $3(q^2+q+1)$ singular points with the Milnor number $q^2(q+1)$.*

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2. Preliminaries

From now, let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$.

Proposition 2.1. *Let C be a plane curve. The defining polynomial of C being of the form (1) is a property independent of the choice of homogeneous coordinates.*

Proof. Under the coordinates change

$$x_i = \sum_{l=0}^{2} t_{il} y_l \quad (t_{il} \in k),$$

a homogeneous polynomial F of the form (1) is transformed into

$$\begin{aligned} F &= \sum_{i,j,k} a_{ijk} \left(\sum_{l=0}^{2} t_{il} y_l \right) \left(\sum_{m=0}^{2} t_{jm} y_m \right)^q \left(\sum_{n=0}^{2} t_{kn} y_n \right)^{q^2} \\ &= \sum_{i,j,k} \sum_{l} \sum_{m} \sum_{n} a_{ijk} t_{il} t_{jm}^q t_{kn}^{q^2} y_l y_m^q y_n^{q^2} \\ &= \sum_{l,m,n} b_{lmn} y_l y_m^q y_n^{q^2}, \end{aligned}$$

where $b_{lmn} = \sum_{l,m,n} a_{ijk} t_{il} t_{jm}^q t_{kn}^{q^2}$. □

Lemma 2.1. *If a_{ijk} are general, then the plane curve C is smooth.*

Proof. The Fermat curve of degree $q^2 + q + 1$ is smooth. Being smooth is an open condition. □

3. Proof of the first half of Theorem 1

We define the *reduced Gauss map* $\Gamma_{\mathrm{red}} : C \to (\mathbb{P}^2)^\vee$ of $C \in \mathscr{C}$ by

$$\begin{aligned} &\Gamma_{\mathrm{red}}([x_0 : x_1 : x_2]) \\ &= \left[\left(\frac{\partial F}{\partial x_0}(x_0, x_1, x_2) \right)^{1/q} : \left(\frac{\partial F}{\partial x_1}(x_0, x_1, x_2) \right)^{1/q} : \left(\frac{\partial F}{\partial x_2}(x_0, x_1, x_2) \right)^{1/q} \right]. \end{aligned}$$

Claim 0. *The reduced Gauss map $C \to C^\vee$ is the morphism of separable degree* 1.

Proof. We will prove that the degree of the dual curve of the Fermat curve of degree $q^2 + q + 1$ is $d(d-1)/q$, (see Section 5), and hence the reduced Gauss map of the Fermat curve is the morphism of separable degree 1. Thus the reduced Gauss map $C \to C^\vee$ is also the morphism of separable degree 1. □

We denote the degree of a curve $C \in \mathscr{C}$ by $d = q^2 + q + 1$. If $C \in \mathscr{C}$ is general, then the Gauss map Γ is an inseparable morphism of inseparable degree q by (3). Thus the degree of $C^\vee$ is

$$\frac{d(d-1)}{q} = \frac{(q^2 + q + 1)(q^2 + q)}{q} = (q^2 + q + 1)(q + 1).$$

In order to prove (ii) of Theorem 1, first we prove the following:

Claim 1. *If $C \in \mathscr{C}$ is general, then the curve C has no m-ple tangent line for $m \geq 3$.*

Proof. We define a variety $\mathscr{X}_1$ by

$$\mathscr{X}_1 = \left\{ (Q_0, Q_1, Q_2, l) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \;\middle|\; \begin{array}{l} Q_0 \in l,\ Q_1 \in l,\ Q_2 \in l \\ \text{and } Q_i \neq Q_j \text{ for } i \neq j \end{array} \right\}.$$

Then the action of $\mathrm{PGL}_3(k)$ on $\mathscr{X}_1$ is transitive. Let (P_0, P_1, P_2, l_0) be a point of $\mathscr{X}_1$ and let $[x_0 : x_1 : x_2]$ be a homogeneous coordinate system such that

$$P_0 = [0:0:1],\ P_1 = [0:1:0],\ P_2 = [0:1:1] \text{ and } l_0 = \{x_0 = 0\}.$$

Let C be a plane curve in $\mathscr{C}$. We define an algebraic subset $\mathscr{D}_1$ of $\mathscr{C}$ by

$$\mathscr{D}_1 = \left\{ Y \in \mathscr{C} \;\middle|\; \begin{array}{l} P_0, P_1 \text{ and } P_2 \text{ are smooth points of } Y, \\ \text{and } T_{P_0}Y = T_{P_1}Y = T_{P_2}Y = l_0 \end{array} \right\}.$$

Then $C \in \mathscr{C}$ is in $\mathscr{D}_1$ if and only if

$$a_{222} = 0,\ a_{122} = 0,\ a_{111} = 0,\ a_{211} = 0,\ a_{212} + a_{221} = 0,\ a_{112} + a_{121} = 0,$$
$$a_{022} \neq 0,\ a_{011} \neq 0 \text{ and } a_{011} + a_{012} + a_{021} + a_{022} \neq 0.$$

Therefore $\mathscr{D}_1$ is of codimension 6 in $\mathscr{C}$. Since $\dim \mathscr{X}_1 = 5$, we have

$$\dim \mathscr{X}_1 + \dim \mathscr{D}_1 < \dim \mathscr{C}.$$

Thus if the curve C is general in $\mathscr{C}$, then C does not have any m-ple tangent line for $m \geq 3$. □

Second we prove the following:

Claim 2. *If $C \in \mathscr{C}$ is general, then Γ_{red} is an immersion at every point of C.*

Proof. Let P_0 be the point $[0:0:1]$, and let l_0 be the line $\{x_0 = 0\}$. By linear change of coordinates, we can assume that $P_0 \in C$ and $T_{P_0}C = l_0$. Let (x, y) be affine coordinates such that $[x_0 : x_1 : x_2] = [x : y : 1]$. Then up to multiple constant, the polynomial F can be written as

$$F(x,y,1) = f(x,y) = x + a_{202}x^q + a_{212}y^q + a_{002}x^{q+1} + a_{102}x^q y + a_{012}xy^q$$
$$+ a_{112}y^{q+1} + (\text{terms of degree} \geq q^2).$$

Then we have a local parametrization $x = \phi(t),\ y = t$ of C at P_0 such that the power series $\phi(t)$ is written as

$$\phi(t) = -a_{212}t^q - a_{112}t^{q+1} + a_{012}a_{212}t^{2q} + \cdots.$$

We consider the Gauss map given by (2). Let (η, ζ) be the affine coordinates of $(\mathbb{P}^2)^\vee$ with the origin $l_0 \in (\mathbb{P}^2)^\vee$ such that the point (η, ζ) corresponds to the line $x + \eta y + \zeta = 0$. Then the tangent line of C at $P_t = [\phi(t) : t : 1]$ is

$$\frac{\partial f}{\partial x}(P_t)x + \frac{\partial f}{\partial y}(P_t)y - \frac{\partial f}{\partial x}(P_t)\phi(t) - \frac{\partial f}{\partial y}(P_t)t = 0$$

Therefore the Gauss map locally around P_0 is written as

$$\Gamma(P_t) = \left(\frac{f_y(P_t)}{f_x(P_t)}, -\frac{f_y(P_t)}{f_x(P_t)}t - \phi(t) \right)$$
$$= \left(-\frac{d\phi}{dt}(t), t\frac{d\phi}{dt}(t) - \phi(t) \right).$$

Since

$$-\frac{d\phi}{dt}(t) = a_{112}t^q + (\text{terms of degree} > q)$$

and

$$t\frac{d\phi}{dt}(t) - \phi(t) = a_{212}t^q + (\text{terms of degree} > q),$$

the reduced Gauss map Γ_{red} locally around P_0 is

$$t \mapsto (\alpha_{112}t + (\text{terms of degree} > 1), \alpha_{212}t + (\text{terms of degree} > 1)), \tag{4}$$

where $\alpha_{ijk} = a_{ijk}^{1/q}$. The reduced Gauss map Γ_{red} is not smooth at the point P_0 if and only if $\alpha_{112} = \alpha_{212} = 0$. Since the codimension of the subset

$$\{C \in \mathscr{C} \mid \alpha_{112} = \alpha_{212} = 0\}$$

is 2 in $\mathscr{C}$, the reduced Gauss map Γ_{red} is locally immersion at every point of a general member C of $\mathscr{C}$. □

Suppose that $C \in \mathscr{C}$ is general. We prove that the singular points of the dual curve $C^\vee$ are only ordinaly nodes. Let P_0 and P_1 be the points in the proof of claim 1, and let l_0 be the line $\{x_0 = 0\}$. Suppose that P_0 and P_1 are smooth points of C and $T_{P_0}C = T_{P_1}C = l_0$. Let (x', y') be affine coordinates such that $[x_0 : x_1 : x_2] = [x' : 1 : y']$. Similar to the proof of the claim 2, up to multiple constant, the polynomial F can be written as

$$\begin{aligned} F(x', 1, y') = g(x', y') = x' + a_{101}x'^q + a_{121}y'^q + a_{001}x'^{q+1} + a_{201}x'^q y' + a_{021}x'y'^q \\ + a_{221}y'^{q+1} + (\text{terms of degree} \geq q^2). \end{aligned}$$

Then we have a local parametrization $x' = \psi(t),\ y' = t$, of C at P_0 such that the power series $\psi(t)$ is written as

$$\psi(t) = -a_{121}t^q - a_{221}t^{q+1} + a_{021}a_{121}t^{2q} + \cdots .$$

Let (η, ζ) be the affine coordinates of $(\mathbb{P}^2)^\vee$ with the origin $l_0 \in (\mathbb{P}^2)^\vee$ such that the point (η, ζ) corresponds to the line $x' + \eta y' + \zeta = 0$. The tangent line of C at $P'_t = [\psi(t) : 1 : t]$ is

$$\frac{\partial g}{\partial x'}(P'_t)x' + \frac{\partial g}{\partial y'}(P'_t)y' - \frac{\partial g}{\partial x'}(P'_t)\phi(t) - \frac{\partial g}{\partial y'}(P'_t)t = 0.$$

Therefore the Gauss map Γ locally around P_1 is written as

$$\begin{aligned} \Gamma(P'_t) &= \left(\frac{g_{y'}(P'_t)}{g_{x'}(P'_t)}, -\frac{g_{y'}(P'_t)}{g_{x'}(P'_t)}t - \psi(t)\right) \\ &= \left(-\frac{d\psi}{dt}(t), t\frac{d\psi}{dt}(t) - \psi(t)\right). \end{aligned}$$

Since

$$-\frac{d\psi}{dt}(t) = a_{221}t^q + (\text{terms of degree} > q)$$

and

$$t\frac{d\psi}{dt}(t) - \psi(t) = a_{121}t^q + (\text{terms of degree} > q),$$

we describe the reduced Gauss map

$$t \mapsto (\alpha_{221}t + (\text{terms of degree} > 1),\ \alpha_{121}t + (\text{terms of degree} > 1)) \tag{5}$$

locally around P_1. We define a variety $\mathscr{X}_2$ by

$$\mathscr{X}_2 = \{(Q_0, Q_1, l) \in \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \mid Q_0 \in l, Q_1 \in l \text{ and } Q_0 \neq Q_1\}.$$

Then the action of $\mathrm{PGL}_3(k)$ on $\mathscr{X}_2$ is transitive and $\dim \mathscr{X}_2 = 4$. Let (P_0, P_1, l_0) be the point of $\mathscr{X}_2$ such that $P_0 = [0:0:1]$, $P_1 = [0:1:0]$ and $l_0 = \{x_0 = 0\}$. We define a subset $\mathscr{D}_2$ of $\mathscr{C}$ by

$$\mathscr{D}_2 = \{Y \in \mathscr{C} \mid P_0 \text{ and } P_1 \text{ are smooth points of } Y, \text{ and } T_{P_0}Y = T_{P_1}Y = l_0\}$$

Then $C \in \mathscr{D}_2$ if and only if

$$a_{222} = 0,\ a_{122} = 0,\ a_{111} = 0,\ a_{211} = 0,\ a_{022} \neq 0,\ a_{011} \neq 0.$$

Thus the codimension of $\mathscr{D}_2$ is 4. For $C \in \mathscr{D}_2$, by (4) and (5), the singularities of $C^\vee$ at the point l_0 is not a ordinary node if and only if

$$\begin{vmatrix} \alpha_{112} & \alpha_{212} \\ \alpha_{211} & \alpha_{121} \end{vmatrix} = 0.$$

We define a subset $\mathscr{D}_2'$ of $\mathscr{C}$ by

$$\mathscr{D}_2' = \left\{ Y \in \mathscr{C} \;\middle|\; \begin{array}{l} P_0 \text{ and } P_1 \text{ are smooth points of } Y, \\ T_{P_0}Y = T_{P_1}Y = l_0, \text{ and } Y^\vee \text{ does not have ordinary node at } l_0 \end{array} \right\}.$$

Since the codimension of $\mathscr{D}_2'$ is 5,

$$\dim \mathscr{D}_2' + \dim \mathscr{X}_2 < \dim \mathscr{C}.$$

Therefore, since a_{ijk} are general, the dual curve $C^\vee$ has only ordinary nodes as its singularities.

4. Proof of the second half of Theorem 1

4.1. **Number of the ordinary nodes of $C^\vee$.** Let g and $g^\vee$ be the genera of a general curve $C \in \mathscr{C}$ and its dual curve $C^\vee$, respectively. Let δ be the number of the ordinary nodes of $C^\vee$. Then

$$g = \frac{(d-1)(d-2)}{2} = \frac{\{(q^2+q+1)-1\}\{(q^2+q+1)-2\}}{2}$$

and

$$\begin{aligned} g^\vee =& \frac{(d^\vee-1)(d^\vee-2)}{2} - \delta \\ =& \frac{\{(q^2+q+1)(q+1)-1\}\{(q^2+q+1)(q+1)-2\}}{2} - \delta, \end{aligned}$$

where d and $d^\vee$ are the degree of C and $C^\vee$, respectively, because, by the previous section, $C^\vee$ has only ordinary nodes. By claim 2 of section 3, the reduced Gauss map Γ_{red} is birational onto its image. Thus $g = g^\vee$ and hence we have

$$\begin{aligned} \delta =& \frac{\{(q^2+q+1)(q+1)-1\}\{(q^2+q+1)(q+1)-2\}}{2} \\ &- \frac{\{(q^2+q+1)-1\}\{(q^2+q+1)-2\}}{2} \\ =& \frac{q(q^2+q+1)(q^3+3q^2+3q-1)}{2} \end{aligned}$$

4.2. **Number of the flexes.** We denote by $\mathrm{mult}_P(D_1, D_2)$ the intersection multiplicity of projective plane curves D_1 and D_2 at a point $P \in D_1 \cap D_2$.

Lemma 4.1. *We suppose that $C \in \mathscr{C}$ is a general plane curve in $\mathscr{C}$. If the multiplicity $\mathrm{mult}_u(T_uC, C)$ is more than q at $u \in C$, then the multiplicity $\mathrm{mult}_u(T_uC, C)$ is $q+1$ at $u \in C$ and all other intersection points of T_uC and C are not tangent point.*

Proof. We use the same notation as in Section 3. We define a variety $\mathscr{X}_0$ by

$$\mathscr{X}_0 = \{(Q, l) \in \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \mid Q \in l\}.$$

Then the action of $\mathrm{PGL}_3(k)$ on $\mathscr{X}_0$ is transitive and $\dim \mathscr{X}_0 = 3$. We recall that $[x_0 : x_1 : x_2]$ are homogeneous coordinates, $P_0 = [0 : 0 : 1]$, $P_1 = [0 : 1 : 0]$ and $l_0 = \{x_0 = 0\}$. We define two subsets $\mathscr{D}_0$ and $\widetilde{\mathscr{D}}_0$ of $\mathscr{C}$ by

$$\mathscr{D}_0 = \left\{ Y \in \mathscr{C} \;\middle|\; \begin{array}{l} P_0 \text{ is the smooth point of } Y,\ T_{P_0}Y = l_0 \\ \text{and } \mathrm{mult}_{P_0}(T_{P_0}Y, Y) = q+1 \end{array} \right\}$$

and

$$\widetilde{\mathscr{D}}_0 = \left\{ Y \in \mathscr{C} \;\middle|\; \begin{array}{l} P_0 \text{ is the smooth point of } Y,\ T_{P_0}Y = l_0 \\ \text{and } \mathrm{mult}_{P_0}(T_{P_0}Y, Y) > q+1 \end{array} \right\}.$$

Then the curve $C \in \mathscr{D}_0$ if and only if

$$a_{222} = 0,\ a_{122} = 0,\ a_{212} = 0,\ a_{112} \neq 0 \text{ and } a_{022} \neq 0,$$

and $C \in \widetilde{\mathscr{D}}_0$ if and only if

$$a_{222} = 0,\ a_{122} = 0,\ a_{212} = 0,\ a_{112} = 0 \text{ and } a_{022} \neq 0.$$

Therefore the codimension of $\mathscr{D}_0$ is 3 and that of $\widetilde{\mathscr{D}}_0$ is more than 3 in $\mathscr{C}$. Thus we have

$$\dim \mathscr{X}_0 + \dim \widetilde{\mathscr{D}}_0 < \dim \mathscr{C}.$$

We proved the first half of the lemma. We define a subset $\widetilde{\mathscr{D}}_2$ of $\mathscr{C}$ by

$$\widetilde{\mathscr{D}}_2 = \left\{ Y \in \mathscr{C} \;\middle|\; \begin{array}{l} P_0 \text{ and } P_1 \text{ are the smooth points of } Y,\ T_{P_0}Y = l_0, \\ T_{P_1}Y = l_0 \text{ and } \mathrm{mult}_{P_0}(T_{P_0}Y, Y) = q+1 \end{array} \right\}.$$

Then the curve $C \in \widetilde{\mathscr{D}}_2$ if and only if

$$a_{222} = 0,\ a_{122} = 0,\ a_{111} = 0,\ a_{211} = 0,\ a_{212} = 0,\ a_{112} \neq 0,\ a_{022} \neq 0,\ a_{011} \neq 0.$$

Therefore codimension of $\widetilde{\mathscr{D}}_2$ is 5, and we recall $\dim \mathscr{X}_2 = 4$. Thus, since we have

$$\dim \mathscr{X}_2 + \dim \widetilde{\mathscr{D}}_2 < \dim \mathscr{C},$$

the second half of the lemma is proved. □

Let g be the genus of a general curve $C \in \mathscr{C}$. We use the notion and notation about the correspondence of curves introduced in [3, Chap. 2, Section 5]. Let $T : C \to C$ be correspondence defined by $T(u) = T_uC.C - qu$, $D \subset C \times C$ its curve of correspondence, i.e. $D = \overline{\{(u, v) \mid u \neq v,\ v \in T_uC\}}$. Then the degree of T is

$$\deg T = (q^2 + q + 1) - q = q^2 + 1.$$

Let $\pi_2 : C \times C \to C$ be the projection on second factor. In order to find the degree of T^{-1}, we have to caluculate the number of tangent lines to C, (counted with the intersection multiplicities of D and $\pi_2^{-1}(v)$) other than T_vC passing throght a general point $v \in C$. We consider the projection $\pi_v : C \to \mathbb{P}^1$ from the center $v \in C$

onto a line. Let $\Omega_{C/\mathbb{P}^1}$ be the sheaf of the relative differential of C over $\mathbb{P}^1$. By Hurwitz-formula [4, Chap. IV, Corollary 2.4],

$$2g-2=-2(q^2+q)+\deg R,$$

where the divisor R is the ramification divisor of π_v i.e. $R=\sum_{u\in C}\operatorname{length}(\Omega_{C/\mathbb{P}^1})_u u$. Hence

$$\deg R=q^4+2q^3+2q^2+q-2.$$

Moreover, the length of $(\Omega_{C/\mathbb{P}^1})_v$ is $q-2$. Hence, we have

$$\begin{aligned}\deg T^{-1}&=(q^4+2q^3+2q^2+q-2)-(q-2)\\&=q^4+2q^3+2q^2.\end{aligned}$$

Lemma 4.2. *Let $\pi_1,\ \pi_2 : C\times C\to C$ be the projections on first and second factors, respectively. The divisor D on $C\times C$ is algebraically equivalent to*

$$(q^4+2q^3+2q^2+q)E_u+(q^2+q+1)F_v-q\Delta,$$

where $E_u=\pi_1^{-1}(u),\ F_v=\pi_2^{-1}(v)$ and $\Delta\subset C\times C$ is the diagonal.

Proof. For some $u_0,\ v_0\in C$, we write

$$T(u_0)+qu_0=\sum b_iv_i$$

and

$$T^{-1}(v_0)+qv_0=\sum a_iu_i.$$

Let L be the line bundle

$$L=D-\sum a_iE_{u_i}-\sum b_iF_{v_i}+q\Delta.$$

For any $x\in C$, the restriction of L to E_x is trivial because the divisor $T(x)+qx$ is linearly equivalent to $T(u_0)+qu_0$. By [4, Chap. III, Exercise 12.4], there is a line bundle M on C such that $L\cong\pi_1^*(M)$. Since the restriction of L to F_{v_0} is trivial, the line bundle L is also trivial. Thus D is linearly equivalent to

$$\sum a_iE_{u_i}+\sum b_iF_{v_i}-q\Delta.$$

For any $u,\ v\in C$, the divisors E_{u_i} (resp. F_{v_i}) are algebraically equivalent to E_u (resp. F_v). Note that the degrees of $T(u_0)+qu_0$ and $T^{-1}(v_0)+qv_0$ are

$$\deg(T(u_0)+qu_0)=q^2+q+1$$

and

$$\deg(T^{-1}(v_0)+qv_0)=q^4+2q^3+2q^2+q,$$

and hence the result is proved. □

Lemma 4.3. *If $C\in\mathscr{C}$ is a general plane curve in $\mathscr{C}$, then D and Δ intersect transversally at any point $(u,v)\in D\cap\Delta$.*

Proof. We use the same notations as in Section 3 and Lemma 4.1. We recall that $[x_0:x_1:x_2]$ is homogeneous coordinates, $P_0=[0:0:1]$, $l_0=\{x_0=0\}$ and

$$\mathscr{D}_0=\left\{Y\in\mathscr{C}\;\middle|\;\begin{array}{l}P_0\text{ is the smooth point of }Y,\ T_{P_0}Y=l_0\\ \text{and }\operatorname{mult}_{P_0}(T_{P_0}Y,Y)=q+1\end{array}\right\}.$$

By change of coordinates, we assume that $C \in \mathscr{D}_0$. Let (x, y) be affine coordinates such that $[x_0 : x_1 : x_2] = [x : y : 1]$. Then up to multiple constant, the polynomial F can be written as

$$F(x, y, 1) = x + a_{202}x^q + a_{002}x^{q+1} + a_{102}x^q y + a_{012}xy^q + a_{112}y^{q+1} + a_{220}x^{q^2} + a_{221}y^{q^2} + (\text{terms of degree} > q^2).$$

Then we have a local parametrization $x = \phi_1(t)$, $y = t$ of C at P_0 such that the power series $\phi_1(t)$ is written as

$$\phi_1(t) = -a_{112}t^{q+1} + a_{012}a_{112}t^{2q+1} + \cdots - a_{221}t^{q^2} + (\text{terms of degree} > q^2).$$

Let (P_{t_1}, P_{t_2}) be a point of D in a small neighborhood of (P_0, P_0) such that

$$P_{t_1} = [\phi_1(t_1) : t_1 : 1] \text{ and } P_{t_2} = [\phi_1(t_2) : t_2 : 1].$$

The tangent line of C at P_{t_1} is

$$x = \frac{d\phi_1}{dt}(t_1)y - t_1\frac{d\phi_1}{dt}(t_1) + \phi_1(t_1),$$

and hence

$$x = (-a_{112}t_1^q + a_{012}a_{112}t_1^{2q} + (\text{terms of degree} > 2q))y + (-a_{221}t_1^{q^2} + (\text{terms of degree} > q^2)).$$

Therefore t_2 is the solution of the equation

$$\frac{d\phi_1}{dt}(t_1)y - t_1\frac{d\phi_1}{dt}(t_1) + \phi_1(t_1) - \phi_1(y) = 0 \tag{6}$$

for y that is not t_1 and approaches to 0 when t_1 tends to 0. We can express the left hand side of (6) as

$$\begin{aligned}&(-a_{112}t_1^q + a_{012}a_{112}t_1^{2q} + (\text{terms of degree} > 2q \text{ in } t_1))y\\ &+ (-a_{221}t_1^{q^2} + (\text{terms of degree} > q^2 \text{ in } t_1))\\ &+ a_{112}y^{q+1} - a_{012}a_{112}y^{2q+1} + \cdots + a_{221}y^{q^2} + (\text{terms of degree} > q^2 \text{ in } y)\\ =\ &(y - t_1)^q f_{t_1}(y),\end{aligned}$$

where the power series $f_{t_1}(y)$ is written as

$$f_{t_1}(y) = a_{112}y + a_{221}t_1^q + a_{221}y^q + \cdots.$$

Since $C \in \mathscr{D}_0$, $a_{112} \neq 0$. Thus a solution of $f_{t_1}(y) = 0$ is

$$y = -\frac{a_{221}}{a_{112}}t_1^q + (\text{terms of degree} > q).$$

Therefore we have

$$t_2 = -\frac{a_{221}}{a_{112}}t_1^q + (\text{terms of degree} > q).$$

If (P_{t_1}, P_{t_2}) is a point in Δ, then $t_1 = t_2$. Therefore, if $(P_{t_1}, P_{t_2}) \in D \cap \Delta$, then

$$t_1 = -\frac{a_{221}}{a_{112}}t_1^q + (\text{terms of degree} > q).$$

Thus D and Δ intersect transversally at $(P_0, P_0) \in D \cap \Delta$.

□

By Lemma 4.3, the number of the flexes is equal to the intersection number $(D \cdot \Delta)$ for a general member C of $\mathscr{C}$. Since the self-intersection number of Δ is $2-2g$, the intersection number $(D \cdot \Delta)$ is

$$\begin{aligned}(D \cdot \Delta) &= (\{(q^4+2q^3+2q^2+q)E_u+(q^2+q+1)F_v-q\Delta\} \cdot \Delta)\\ &= q^4+2q^3+3q^2+2q+1-q(2-2g)\\ &= q^5+3q^4+2q^3+2q^2+1\\ &= (q^3+2q^2-q+1)(q^2+q+1).\end{aligned}$$

5. Fermat curve

For any formal power series $f \in \mathbb{k}[[x,y]]$, we define the *Milnor number* $\mu(f)$ by

$$\mu(f) = \dim_{\mathbb{k}} \mathbb{k}[[x,y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

Calculation method of the Milnor number for a formal power series in characteristic zero is well known. (For example, see [6].) However, in positive characteristic, the calculation method and result of the Milnor number differ from the characteristic-zero case in general. In the case of the following lemma, however, the Milnor number is the same as the characteristic-zero case.

Lemma 5.1. *Let a and b be elements in $\mathbb{k} \setminus \{0\}$, and let $f \in \mathbb{k}[[x,y]]$ be a formal power series defined by*

$$f(x,y) = ax^\alpha + by^\beta + \sum_{\alpha\beta < \alpha s + \beta r} c_{r,s} x^r y^s,$$

where α and β satisfy $p \not| \alpha$, $p \not| \beta$ and are relatively prime. Then the Milnor number $\mu(f)$ of f is

$$\mu(f) = (\alpha-1)(\beta-1).$$

Proof. We use notations of [2]. The (β,α)-order of f is

$$\operatorname{ord}_{(\beta,\alpha)}(f) = \alpha\beta.$$

The (β,α)-initial of f is

$$\operatorname{in}_{(\beta,\alpha)}(f) = ax^\alpha + by^\beta.$$

Thus the formal power series f is the semi-quasihomogeneous with respect to (β,α). By the Appendix of [2],

$$\mu(f) = (\alpha-1)(\beta-1).$$

□

Proof of Theorem 2. The morphisms γ_{q^2+q+1} and γ_{q+1} satisfy

$$\gamma_{q^2+q+1} \circ \gamma_{q+1} = \gamma_{q+1} \circ \gamma_{q^2+q+1} = \gamma_{(q^2+q+1)(q+1)}.$$

By the definition of the Ballico-Hefez curve and the line $l = \gamma_{q^2+q+1}(C_0)$, we have

$$B = \gamma_{q+1}(l) = \gamma_{q+1}(\gamma_{q^2+q+1}(C_0)) = \gamma_{q^2+q+1}(\gamma_{q+1}(C_0)) = \gamma_{q^2+q+1}(C_0^\vee),$$

and hence (i) is proved.

We define $X \subset \mathbb{P}^2$ by

$$X = \{x_0 = 0\} \cup \{x_1 = 0\} \cup \{x_2 = 0\}.$$

The Ballico-Hefez curve B has $\frac{q^2-q}{2}$ ordinary nodes on $\mathbb{P}^2 \setminus X$ (see [1, Theorem 2.2]), and no singular points on X. Let H and h be the defining polynomials of $C_0^\vee$ and B, respectively. Using Proposition 1.6 of [5], if $p = 2$, then

$$h = x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_0^q x_2 + x_1^q x_2 + x_0 x_2^q + x_1 x_2^q \\ + \sum_{i=0}^{\nu-1} x_0^{2^i} x_1^{2^i} (x_0 + x_1 + x_2)^{q+1-2^{i+1}}, \tag{7}$$

whereas if p is odd, then

$$h = x_0^{q+1} + x_1^{q+1} + x_2^{q+1} - x_0^q x_1 - x_0^q x_2 - x_0 x_1^q - x_1^q x_2 - x_0 x_2^q - x_1 x_2^q \\ + (x_0^2 + x_1^2 + x_2^2 - 2x_0x_1 - 2x_1x_2 - 2x_2x_0)^{\frac{q+1}{2}}. \tag{8}$$

By (i), the polynomial H satisfies $H(x_0, x_1, x_2) = h(x_0^{q^2+q+1}, x_1^{q^2+q+1}, x_2^{q^2+q+1})$, and two polynomials H and h are symmetric under the permutation of coordinates x_0, x_1 and x_2. First we consider the singularities of $C_0^\vee$ on $\mathbb{P}^2 \setminus X$. The morphism $\gamma_{q^2+q+1} : \mathbb{P}^2 \setminus X \to \mathbb{P}^2 \setminus X$ is étale of degree $(q^2+q+1)^2$. Thus, the ordinary nodes of $C_0^\vee$ on $\mathbb{P}^2 \setminus X$ are $(q^2+q+1)^2(q^2-q)/2$.

Next, we consider the singularities of $C_0^\vee$ on X. $h(0, x_1, x_2) = 0$ if and only if $x_1 = x_2$ by (7) and (8). Moreover, the polynomial H and its partial derivatives $\partial H/\partial x_i = x_i^{q^2+q}(\partial h/\partial x_i)$ vanish at a point in $\{x_0 = 0\}$. Thus all the points on $C_0^\vee \cap \{x_0 = 0\}$ are singular points of $C_0^\vee$. The morphism $\gamma_{q^2+q+1}|_{\{x_0=0\}}$ restricted to $\{x_0 = 0\}$ is degree $q^2 + q + 1$. Thus the number of the singular points of $C_0^\vee$ on $\{x_0 = 0\}$ are q^2+q+1. Therefore, by the polynomial H is symmetric, the number of the singular points of $C_0^\vee$ on X are $3(q^2 + q + 1)$.

Finally, since all Milnor numbers at points in $\gamma_{q^2+q+1}^{-1}([0 : 1 : 1])$ are equal, we should caluculate the Milnor number at the point $[0 : 1 : 1] \in C_0^\vee$. If $p = 2$,

$$h(x_0^{q^2+q+1}, x_1 + 1, 1) = x_0^{q^2+q+1} + x_1^{q+1} + x_0^{q(q^2+q+1)} + x_0^{(q+1)(q^2+q+1)} \\ + \sum_{i=0}^{\nu-1} (x_0^{q^2+q+1})^{2^i} (x_1 + 1)^{2^i} (x_0^{q^2+q+1} + x_1)^{q+1-2^i},$$

whereas if p is odd,

$$h(x_0^{q^2+q+1}, x_1 + 1, 1) = -2x_0^{q^2+q+1} + x_1^{q+1} + x_0^{(q^2+q+1)(q+1)} \\ - x_0^{q(q^2+q+1)} x_1 - 2x_0^{q(q^2+q+1)} - x_0^{q^2+q+1} x_1^q \\ + (x_0^{2(q^2+q+1)} + x_1^2 - 2x_0^{q^2+q+1} x_1 - 4x_0^{q^2+q+1})^{\frac{q+1}{2}}.$$

By Lemma 3.1, the Milnor number of $h(x_0^{q^2+q+1}, x_1 + 1, 1)$ is

$$q(q^2 + q) = q^2(q + 1).$$

□

We confirm that the genus of the Fermat curve agree with the genus of its dual curve. The genus g of the Fermat curve C_0 of the degree $d = q^2 + q + 1$ is

$$g = \frac{(d-1)(d-2)}{2} = \frac{(q^2+q)(q^2+q-1)}{2}.$$

Let μ_P be the Milnor number and let r_P be the number of the branches at a singular point of the dual curve $C_0^\vee$. If a point $P \in C_0^\vee$ is an ordinary node, then $\mu_P = 1$

and $r_P = 2$, whereas if a point P is in $C_0^\vee \cap X$, then $\mu_P = q^2(q+1)$ and $r_P = 1$. Thus the degree $d^\vee$ of $C_0^\vee$ is $(q+1)(q^2+q+1)$, and the genus $g^\vee$ of $C_0^\vee$ is

$$\begin{aligned} g^\vee &= \frac{(d^\vee - 1)(d^\vee - 2)}{2} - \frac{1}{2} \sum_{P \in \mathrm{Sing} C_0^\vee} (\mu_P + r_P - 1) \\ &= \frac{\{(q^2+q+1)(q+1)-1\}\{(q^2+q+1)(q+1)-2\}}{2} \\ &\quad - \frac{1}{2}\{(q^2+q+1)^2(q^2-q) + 3(q^2+q+1)q^2(q+1)\} \\ &= \frac{(q^2+q)(q^2+q-1)}{2}. \end{aligned}$$

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