広島大学学位請求論文

Optimal leading term of solutions to wave equations with strong damping terms

（強摩擦項をもつ波動方程式の
解の最適主要項）

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数学専攻

道久　寛載

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主論文

Optimal leading term of solutions to wave equations with strong damping terms

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Abstract

We analyze the asymptotic behavior of solutions to wave equations with strong damping terms in $\mathbf{R}^n$ $(n \geq 1)$,

$$u_{tt} - \Delta u - \Delta u_t = 0, \qquad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x).$$

If the initial data belong to suitable weighted L^1 spaces, lower bounds for the difference between the solutions and the leading terms in the Fourier space are obtained, which implies the optimality of expanding methods and some estimates proposed in [13] and in this paper.

1 Introduction

In this paper we consider the Cauchy problem of the solution to the linear strongly damped wave equation

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t = 0, & t > 0, \quad x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 1$ and u_0, $u_1 \in L^2(\mathbf{R}^n) \cap L^{1,\gamma}(\mathbf{R}^n)$. The weighted L^1 space $L^{1,\gamma}(\mathbf{R}^n)$ $(\gamma \geq 0)$ is defined by

$$L^{1,\gamma}(\mathbf{R}^n) := \left\{ f \in L^1(\mathbf{R}^n) : \|f\|_{1,\gamma} := \int_{\mathbf{R}^n} (1+|x|)^\gamma |f(x)|\, dx < \infty \right\}.$$

Celebrated mathematical results for equation (1.1) are L^p-L^q decay estimates obtained by Ponce [14] and Shibata [15]. Later seveval mathematicians have studied wave equations with structural damping terms such as $(-\Delta)^\theta u_t$ and exterior domain cases for (1.1). See e.g., [1], [2], [3], [5], [8], [10], [12] and references therein. Especially, [4] has obtained $(L^1 \cap L^2)$-L^2 and L^2-L^2 estimates for the solution to (1.1). They also studied the critical exponent for equation (1.1) with nonlinear terms but asymptotic profiles were not investigated.

Now we focus on reviewing known results on the asymptotic behavior of the solution to (1.1). Recently, Ikehata-Todorova-Yordanov [11] obtained the asymptotic profile in the abstract framework including (1.1). Limited to equation (1.1), they indicated that the solution of (1.1) behaves like

$$e^{-\frac{t|\xi|^2}{2}} \cos(t|\xi|)\widehat{u_0} + e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|)}{|\xi|}\widehat{u_1}, \qquad t \to \infty,$$

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in the Fourier space. However, such diffusion wave properties are not investigated in detail. Quite recently, Michihisa [13] have found a way to obtain higher order asymptotic expansions of the solution to (1.1) in the L^2 framework. He used the explicit solution formula in the Fourier space and defined some suitable functions generated by evolution operators for (1.1) to apply the Taylor theorem efficiently. It is based on a quite natural and simple idea. So the obtained results in [13] seems to be optimal, however, the evidence is not shown anywhere.

When we discuss the optimality of asymptotic expansions, appropriate lower bounds need to be shown. Concerning this optimality, Ikehata [6] proved that the solution $u = u(t,x)$ of (1.1) with $n \geq 3$ satisfies the following inequalities:

$$C_1 \left| \int_{\mathbf{R}^n} u_1(x)\,dx \right| t^{-\frac{n}{4}+\frac{1}{2}} \leq \|u(t)\|_2 \leq C_2 t^{-\frac{n}{4}+\frac{1}{2}}, \qquad t \gg 1,$$

where $C_1 > 0$ is a constant independent of t and the initial data, and $C_2 > 0$ is a constant independent of t. Furthermore, the lower dimensional cases $n = 1, 2$ are also treated by Ikehata-Onodera [9]. They obtained meaningful inequalities corresponding to the above, which says the optimal infinite time blow-up rates are $\sqrt{t}$ $(n = 1)$ and $\sqrt{\log t}$ $(n = 2)$. So in order to obtain L^2 bounded solutions we have to impose the additional condition that the mass of u_1 is zero. However, in this case we face the same problem on the decay rate again. In order to answer this question, in this paper we obtain some asymptotic estimates by taking into account the first moments of the initial value, which were out of interests in [13] but can lead to the optimal lower bound for the L^2 difference between the solution to (1.1) and its leading term.

The rest of this paper is as follows. In Section 2, we confirm the solution formula with some notation. We also define several functions which are components of asymptotic profiles. Main results are stated in Section 3. Theorem 3.4 is the most important result in this paper. We prepare some lemmas in Section 4. Expect for Theorem 3.3, proofs of theorems are in Section 5.

2 Notation

In this paper, $\mathbf{N}$ denotes the set of all natural numbers, and write $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. We also write the surface area of the n-dimensional unit ball as

$$\omega_n = \int_{|\omega|=1} dS = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}.$$

Let us denote the function $\hat{f}$ by the Fourier transform of f,

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

In the Fourier space, the solution $u = u(t,x)$ of (1.1) is formally expressed by

$$\hat{u}(t,\xi) = E_0(t,\xi)\widehat{u_0} + E_1(t,\xi)\left(\frac{|\xi|^2}{2}\widehat{u_0} + \widehat{u_1}\right), \tag{2.1}$$

where E_i $(i = 0, 1)$ are evolution operators given by

$$E_0(t,\xi) := \begin{cases} e^{-\frac{t|\xi|^2}{2}} \cos\left(\frac{t|\xi|\sqrt{4-|\xi|^2}}{2}\right), & |\xi| \leq 2, \\[2ex] e^{-\frac{t|\xi|^2}{2}} \cosh\left(\frac{t|\xi|\sqrt{|\xi|^2-4}}{2}\right), & |\xi| > 2, \end{cases}$$

$$E_1(t,\xi) := \begin{cases} e^{-\frac{t|\xi|^2}{2}} \left[\sin\left(\frac{t|\xi|\sqrt{4-|\xi|^2}}{2}\right) \Big/ \frac{|\xi|\sqrt{4-|\xi|^2}}{2}\right], & |\xi| \le 2, \\ e^{-\frac{t|\xi|^2}{2}} \left[\sinh\left(\frac{t|\xi|\sqrt{|\xi|^2-4}}{2}\right) \Big/ \frac{|\xi|\sqrt{|\xi|^2-4}}{2}\right], & |\xi| > 2. \end{cases}$$

For example, in [11], it is shown that the problem (1.1) has a unique weak solution $u \in C([0,+\infty); H^1(\mathbf{R}^n)) \cap C^1([0,+\infty); L^2(\mathbf{R}^n))$ if $[u_0, u_1] \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. So it is natural to consider $u_0 \in H^1(\mathbf{R}^n)$ but we treat $u_0 \in L^2(\mathbf{R}^n)$ in this paper since we can impose additional regularity on the initial datum u_0 as far as we need.

We define the following functions (see [13]):

$$L_0(a,t,\xi) := \cos\left(t|\xi| - t|\xi|^2 \frac{a}{4+2\sqrt{4-a^2}}\right),$$

$$L_1(a,t,\xi) := \sin\left(t|\xi| - t|\xi|^2 \frac{a}{4+2\sqrt{4-a^2}}\right) \Big/ \frac{|\xi|\sqrt{4-a^2}}{2}.$$

Note that

$$E_i(t,\xi) = e^{-\frac{t|\xi|^2}{2}} L_i(|\xi|, t, \xi), \qquad i = 0, 1.$$

Let $k \in \mathbf{N}_0$. Put

$$e_i^k(t,\xi) := e^{-\frac{t|\xi|^2}{2}} \frac{1}{k!} \frac{\partial^k L_i}{\partial a^k}(0,t,\xi) \cdot |\xi|^k, \qquad i = 0, 1,$$

and

$$m[f]^k(\xi) := \sum_{|\alpha|=k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int_{\mathbf{R}^n} x^\alpha f(x)\, dx\right) (i\xi)^\alpha, \qquad f \in L^{1,k}(\mathbf{R}^n).$$

At the end of this section, we confirm the following remark. Let u be the function defined by (2.1) with u_0, $u_1 \in L^2(\mathbf{R}^n)$. Then there exist constants $\delta \in (0, 1/2)$ and $C_\delta > 0$ such that

$$\|\hat{u}(t)\|_{L^2(|\xi| \ge 1)} \le C_\delta e^{-\delta t} (\|u_0\|_2 + \|u_1\|_2), \qquad t > 0.$$

See [13, Remark 3.1] for details. In this sense the behavior of $\hat{u}$ in the high frequency region is not essential when we consider asymptotic profiles of the solution. So, hereafter, we concentrate on analyzing the behavior of $\hat{u}$ in the low frequency region.

3 Main results

In [13], the author proposed a method for expanding evolution operators E_i $(i = 0, 1)$ but did not provide expanding techniques taking into account the higher moments of the initial data. In order to obtain the precise estimate such as (3.6), we need to prepare more detailed estimates, e.g., (3.3) and (3.5).

Theorem 3.1 *Let $u_1 \in L^{1,\gamma}(\boldsymbol{R}^n)$ with*

$$\gamma \begin{cases} > \frac{1}{2}, & n = 1, \\ > 0, & n = 2, \\ \ge 0, & n \ge 3. \end{cases} \tag{3.1}$$

Then it holds that

$$\left\| E_1(t)\widehat{u_1} - \sum_{k=0}^{[\gamma]} \left(e_1^k(t) \sum_{j=0}^{[\gamma]-k} m[u_1]^j \right) \right\|_{L^2(|\xi|\le 1)} \le C\|u_1\|_{1,\gamma}(1+t)^{-\frac{n}{4}-\frac{\gamma}{2}+\frac{1}{2}}, \qquad t \ge 0. \tag{3.2}$$

Here, $C>0$ is a constant independent of t and u_1. Moreover, it holds that

$$\left\| E_1(t)\widehat{u_1} - \sum_{k=0}^{[\gamma]} \left(e_1^k(t) \sum_{j=0}^{[\gamma]-k} m[u_1]^j \right) \right\|_{L^2(|\xi|\le 1)} = o(t^{-\frac{n}{4}-\frac{\gamma}{2}+\frac{1}{2}}), \qquad t \to \infty. \tag{3.3}$$

Theorem 3.2 *Let $n \ge 1$ and $u_0 \in L^{1,\gamma}(\boldsymbol{R}^n)$ with $\gamma \ge 0$. Then it holds that*

$$\left\| E_0(t)\widehat{u_0} - \sum_{k=0}^{[\gamma]} \left(e_0^k(t) \sum_{j=0}^{[\gamma]-k} m[u_0]^j \right) \right\|_{L^2(|\xi|\le 1)} \le C\|u_0\|_{1,\gamma}(1+t)^{-\frac{n}{4}-\frac{\gamma}{2}}, \qquad t \ge 0. \tag{3.4}$$

Here, $C>0$ is a constant independent of t and u_0. Moreover, it holds that

$$\left\| E_0(t)\widehat{u_0} - \sum_{k=0}^{[\gamma]} \left(e_0^k(t) \sum_{j=0}^{[\gamma]-k} m[u_0]^j \right) \right\|_{L^2(|\xi|\le 1)} = o(t^{-\frac{n}{4}-\frac{\gamma}{2}}), \qquad t \to \infty. \tag{3.5}$$

We refer to the following theorem proved in [6] and [9]. It motivated the derivation of Theorem 3.4 which is the goal of this paper.

Theorem 3.3 [6],[9] *Let $n \ge 1$ and $u = u(t,x)$ be the solution to (1.1) with $u_0 \in L^{1,1}(\boldsymbol{R}^n) \cap H^1(\boldsymbol{R}^n)$, $u_1 \in L^{1,1}(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$. Put*

$$P_1 := \int_{\boldsymbol{R}^n} u_1(x)\,dx.$$

Then it holds that

$$\begin{aligned} C_1^1|P_1|\sqrt{t} &\le \|u(t)\|_2 \le C_2^1\sqrt{t}, && n=1,\\ C_1^2|P_1|\sqrt{\log t} &\le \|u(t)\|_2 \le C_2^2\sqrt{\log t}, && n=2,\\ C_1^n|P_1|t^{-\frac{n}{4}+\frac{1}{2}} &\le \|u(t)\|_2 \le C_2^n t^{-\frac{n}{4}+\frac{1}{2}}, && n\ge 3, \end{aligned}$$

for sufficiently large t. Here, $C_1^n > 0$ $(n \ge 1)$ are constants independent of t and the initial data, and $C_2^n > 0$ $(n \ge 1)$ are constants independent of t.

When the mass of u_1 is zero, we cannot obtain information on lower bounds from Theorem 3.3. However, as we can see Theorem 3.4 below, even if

$$P_1 = \int_{\mathbf{R}^n} u_1(x)\,dx = 0,$$

the right-hand side of (3.6) can be positive as long as one of the quantities

$$\int_{\mathbf{R}^n} x_j u_1(x)\,dx \quad (j=1,\dots,n) \qquad \text{and} \qquad \int_{\mathbf{R}^n} u_0(x)\,dx$$

is not equal to zero. This is the advantage of higher order expansions near zero frequency. Theorems 3.3 and 3.4 indicate that moments of initial data are important quantities for precise asymptotic profiles.

Theorem 3.4 *Let $n \geq 1$ and $\hat{u}$ be the function defined by (2.1) with $u_0 \in L^1(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$, $u_1 \in L^{1,1}(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$. Then it holds that*

$$\begin{aligned}&\left\| \hat{u}(t,\xi) - \left(\int_{\boldsymbol{R}^n} u_1(x)\,dx \right) e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|)}{|\xi|} \right\|_2 \\ &\geq C \sqrt{ \left(\int_{\boldsymbol{R}^n} u_1(x)\,dx \right)^2 + \sum_{j=1}^{n} \left(\int_{\boldsymbol{R}^n} x_j u_1(x)\,dx \right)^2 + \left(\int_{\boldsymbol{R}^n} u_0(x)\,dx \right)^2 } \, t^{-\frac{n}{4}}\end{aligned} \tag{3.6}$$

for sufficiently large t. Here, $C > 0$ is a constant independent of t.

Remark 3.1 *The constant $C > 0$ in (3.6) can be taken not to depend on the initial data if*

$$\int_{\boldsymbol{R}^n} u_1(x)\,dx = 0 \qquad \text{or} \qquad \int_{\boldsymbol{R}^n} u_0(x)\,dx = 0.$$

In general, it depends on the ratio

$$\left| \int_{\boldsymbol{R}^n} u_0(x)\,dx \right| \bigg/ \left| \int_{\boldsymbol{R}^n} u_1(x)\,dx \right|.$$

See the latter part of the proof of Theorem 3.4 for details.

When we consider the case $u_0 \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, $u_1 \in L^{1,1}(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$, it follows from (3.2) with $\gamma = 1$ that

$$\left\| E_1(t)\widehat{u_1} - \sum_{k=0}^{1} \left(e_1^k(t) \sum_{j=0}^{1-k} m[u_1]^j \right) \right\|_{L^2(|\xi|\leq 1)} \leq C\|u_1\|_{1,1}(1+t)^{-\frac{n}{4}}, \qquad t \geq 0.$$

Direct calculation shows

$$\left\| \sum_{k=0}^{1} e_1^k(t) m[u_1]^{1-k} \right\|_{L^2(|\xi|\leq 1)} \leq C\|u_1\|_{1,1}(1+t)^{-\frac{n}{4}}, \qquad t \geq 0.$$

So we obtain

$$\left\| \hat{u}(t) - \left(\int_{\mathbf{R}^n} u_1(x)\,dx \right) e^{-\frac{t|\xi|^2}{2}} \frac{\sin(t|\xi|)}{|\xi|} \right\|_2 \leq C I(u_0,u_1)(1+t)^{-\frac{n}{4}}, \qquad t \geq 0, \tag{3.7}$$

where $I(u_0,u_1) := \|u_0\|_1 + \|u_1\|_{1,1} + \|u_0\|_2 + \|u_1\|_2$.

Inequalities (3.6) and (3.7) imply the optimality of the leading term, i.e., the zero-th order asymptotic expansion and the decay estimate for the difference between the solution to (1.1) and the leading term.

4 Preliminaries

Although proofs of results in [13] were mainly focused on the analysis of E_0, here we deal with E_1 more extensively. The following lemma is a fundamental result for expanding evolution operators.

Lemma 4.1 [13] *Let $n \ge 1$ and $p \in \boldsymbol{N}_0$. Then there exists a constant $C > 0$ such that*

$$\left\| E_1(t) - \sum_{k=0}^{p} e_1^k(t) \right\|_{L^2(|\xi|\le 1)} \le C(1+t)^{-\frac{n}{4}-\frac{p}{2}}, \tag{4.1}$$

$$\left\| E_0(t) - \sum_{k=0}^{p} e_0^k(t) \right\|_{L^2(|\xi|\le 1)} \le C(1+t)^{-\frac{n}{4}-\frac{p+1}{2}}, \tag{4.2}$$

for $t \ge 0$.

Proof. We reconfirm the proof of (4.1). Here we consider $\xi \in \mathbf{R}^n$ with $|\xi| \le 1$. By the Taylor theorem we see that

$$\begin{aligned} E_1(t,\xi) - \sum_{k=0}^{p} e_1^k(t,\xi) &= e^{-\frac{t|\xi|^2}{2}} \left[L_1(|\xi|, t, \xi) - \sum_{k=0}^{p} \frac{1}{k!} \frac{\partial^k L_1}{\partial a^k}(0, t, \xi) \cdot |\xi|^k \right] \\ &= e^{-\frac{t|\xi|^2}{2}} \frac{1}{(p+1)!} \frac{\partial^{p+1} L_1}{\partial a^{p+1}}(\tau|\xi|, t, \xi) \cdot |\xi|^{p+1} \end{aligned}$$

for some $0 \le \tau \le 1$. The function

$$\frac{a}{4 + 2\sqrt{4 - a^2}}$$

and its derivatives are all bounded for $0 \le a \le 1$. Thus, for $k \in \mathbf{N}_0$, there exists a constant $C > 0$ such that

$$\left| \frac{\partial^k L_1}{\partial a^k}(a, t, \xi) \right| \le \frac{C}{|\xi|} \sum_{\ell=0}^{k} (t|\xi|^2)^\ell \tag{4.3}$$

for $0 \le a \le 1$, $t \ge 0$ and $\xi \in \mathbf{R}^n$. So we obtain

$$\left\| E_1(t) - \sum_{k=0}^{p} e_1^k(t) \right\|_{L^2(|\xi|\le 1)} \le C \sum_{\ell=0}^{p+1} \left\| (t|\xi|^2)^\ell |\xi|^p e^{-\frac{t|\xi|^2}{2}} \right\|_{L^2(|\xi|\le 1)} \le C(1+t)^{-\frac{n}{4}-\frac{p}{2}}$$

for $t \ge 0$. One can similarly prove inequality (4.2). □

The following lemma is used to expand the Fourier transform of the initial data. Lemma 4.2 is already proved by [7] but we give a simpler proof than that of [7].

Lemma 4.2 [7] *Let $n \ge 1$ and $\gamma \ge 0$.*

(i) *It holds that*

$$\left| \hat{f}(\xi) - \sum_{k=0}^{[\gamma]} m[f]^k(\xi) \right| \le C|\xi|^\gamma \int_{\boldsymbol{R}^n} |x|^\gamma |f(x)|\, dx, \qquad \xi \in \boldsymbol{R}^n, \tag{4.4}$$

for $f \in L^{1,\gamma}(\boldsymbol{R}^n)$. Here, $C > 0$ is a constant independent of ξ and f;

(ii) For $c > 0$ and $f \in L^{1,\gamma}(\mathbf{R}^n)$, one has

$$\lim_{t\to\infty} t^{\frac{n}{4}+\frac{\gamma}{2}} \left\| e^{-ct|\xi|^2} \hat{f}(\xi) - \sum_{k=0}^{[\gamma]} m[f]^k(\xi) e^{-ct|\xi|^2} \right\|_2 = 0. \tag{4.5}$$

Proof. First, we prove (i). The Taylor theorem yields

$$e^{-ix\cdot\xi} - \sum_{|\alpha|\le[\gamma]} \frac{(-1)^{|\alpha|}}{\alpha!} x^\alpha (i\xi)^\alpha = \frac{1}{[\gamma]!}\int_0^1 (1-\tau)^{[\gamma]} \frac{d^{[\tau]+1}}{d\tau^{[\gamma]+1}} e^{-i\tau x\cdot\xi}\, d\tau.$$

We calculate

$$\begin{aligned}\mathcal{R}_{[\gamma]+1}(x,\xi) &:= \frac{1}{[\gamma]!}\int_0^1 (1-\tau)^{[\gamma]} \frac{d^{[\tau]+1}}{d\tau^{[\gamma]+1}} e^{-i\tau x\cdot\xi}\, d\tau \\ &= \frac{(-i)^{[\gamma]+1}}{[\gamma]!}(x\cdot\xi)^{[\gamma]+1}\int_0^1 (1-\tau)^{[\gamma]} e^{-i\tau x\cdot\xi}\, d\tau.\end{aligned}$$

Since

$$\sup_{|x||\xi|\ge 1} \frac{\left|\mathcal{R}_{[\gamma]+1}(x,\xi)\right|}{|x|^\gamma|\xi|^\gamma} \le \sup_{|x||\xi|\ge 1}\left(\frac{1}{(|x||\xi|)^\gamma} + \sum_{|\alpha|\le[\gamma]}\frac{1}{\alpha!}\frac{1}{(|x||\xi|)^{\gamma-|\alpha|}}\right) < \infty,$$

there exists a constant $C>0$ such that

$$\left|\mathcal{R}_{[\gamma]+1}(x,\xi)\right| \le C|x|^\gamma|\xi|^\gamma$$

for $x\in\mathbf{R}^n$ and $\xi\in\mathbf{R}^n$. Hence we obtain

$$\begin{aligned}&\left|\hat{f}(\xi) - \sum_{k=0}^{[\gamma]} m[f]^k(\xi)\right| = \left|\hat{f}(\xi) - \sum_{|\alpha|\le[\gamma]}\frac{(-1)^{|\alpha|}}{\alpha!}\left(\int_{\mathbf{R}^n} x^\alpha f(x)\,dx\right)(i\xi)^\alpha\right| \\ &= \left|\int_{\mathbf{R}^n}\left(e^{-ix\cdot\xi} - \sum_{|\alpha|\le[\gamma]}\frac{(-1)^{|\alpha|}}{\alpha!}x^\alpha(i\xi)^\alpha\right) f(x)\,dx\right| = \left|\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}(x,\xi)f(x)\,dx\right| \\ &\le C|\xi|^\gamma\int_{\mathbf{R}^n}|x|^\gamma|f(x)|\,dx\end{aligned}$$

for $f\in L^{1,\gamma}(\mathbf{R}^n)$.

Next, we check (ii). First, we calculate

$$\begin{aligned}&\left\|e^{-ct|\xi|^2}\hat{f} - \sum_{k=0}^{[\gamma]} m[f]^k e^{-ct|\xi|^2}\right\|_2^2 = \int_{\mathbf{R}^n} e^{-2ct|\xi|^2}\left|\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}(x,\xi)f(x)\,dx\right|^2 d\xi \\ &= t^{-\frac{n}{2}-\gamma}\int_{\mathbf{R}^n} e^{-2c|\xi|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}\left(x,\frac{\xi}{\sqrt{t}}\right)f(x)\,dx\right|^2 d\xi.\end{aligned}$$

Now we see there exists a constant $C>0$ such that

$$\left|t^{\frac{\gamma}{2}}\mathcal{R}_{[\gamma]+1}\left(x,\frac{\xi}{\sqrt{t}}\right)f(x)\right| \le Ct^{\frac{\gamma}{2}}|x|^\gamma\left|\frac{\xi}{\sqrt{t}}\right|^\gamma|f(x)| = C|\xi|^\gamma|x|^\gamma|f(x)|$$

for $x\in\mathbf{R}^n$, $\xi\in\mathbf{R}^n$ and $t>0$. So, for arbitrary fixed $\xi\in\mathbf{R}^n$, the Lebesgue dominated convergence theorem assures

$$\lim_{t\to\infty}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}\left(x,\frac{\xi}{\sqrt{t}}\right)f(x)\,dx\right|^2 = 0$$

under $f \in L^{1,\gamma}(\mathbf{R}^n)$ since the following estimates holds for each $x \in \mathbf{R}^n$ and $\xi \in \mathbf{R}^n$:

$$t^{\frac{\gamma}{2}}\mathcal{R}_{[\gamma]+1}\left(x, \frac{\xi}{\sqrt{t}}\right) = t^{-\frac{[\gamma]+1-\gamma}{2}}\frac{(-i)^{[\gamma]+1}}{[\gamma]!}(x\cdot\xi)^{[\gamma]+1}\int_0^1 (1-\tau)^{[\gamma]}e^{-i\tau x\cdot\frac{\xi}{\sqrt{t}}}\,d\tau \longrightarrow 0, \qquad t \to \infty.$$

Furthermore, from the above estimate one can also show there exists a constant $C > 0$ such that

$$e^{-2c|\xi|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}\left(x, \frac{\xi}{\sqrt{t}}\right)f(x)\,dx\right|^2 \le C\|f\|_{1,\gamma}^2|\xi|^{2\gamma}e^{-2c|\xi|^2}$$

for $\xi \in \mathbf{R}^n$ and $t > 0$. Again applying the Lebesgue dominanted convergence theorem, we obtain

$$\lim_{t\to\infty}\int_{\mathbf{R}^n}e^{-2c|\xi|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_{[\gamma]+1}\left(x, \frac{\xi}{\sqrt{t}}\right)f(x)\,dx\right|^2 d\xi = 0,$$

which implies the desired estimate. □

Remark 4.1 *Since*

$$\sum_{|\alpha|\le[\gamma]}\frac{(-1)^{|\alpha|}}{\alpha!}x^\alpha(i\xi)^\alpha = \sum_{k=0}^{[\gamma]}\frac{1}{k!}(-ix\cdot\xi)^k,$$

we have just applied the Taylor theorem for a single-variable function in the proof of assertion (i).

The following lemma is a generalized version of the results obtained in [6] whose proof is based on the double angle formulae and the Riemann-Lebesgue lemma.

Lemma 4.3

(i) *Let $n \ge 1$ and let γ satisfy* (3.1). *Then it follows that*

$$\int_{|\xi|\le 1}|\xi|^{2\gamma}e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 d\xi \ge \frac{\omega_n}{4}\left(\int_0^1 x^{2\gamma+n-3}e^{-x^2}\,dx\right)t^{-\frac{n}{2}-\gamma+1} \tag{4.6}$$

for sufficiently large t;

(ii) *Let $n \ge 1$ and $\gamma \ge 0$. Then it follows that*

$$\int_{|\xi|\le 1}|\xi|^{2\gamma}e^{-t|\xi|^2}|\cos(t|\xi|)|^2\,d\xi \ge \frac{\omega_n}{4}\left(\int_0^1 x^{2\gamma+n-1}e^{-x^2}\,dx\right)t^{-\frac{n}{2}-\gamma} \tag{4.7}$$

for sufficiently large t.

Remark 4.2 *Here we also confirm the corresponding upper bounds to show their optimality. In* [I], *condition* (3.1) *is essentially used to recover the integrability of* $\sin(t|\xi|)/|\xi|$ *near* $\xi = 0$.

[I] *If $n \ge 1$ and γ satisfies* (3.1), *then it follows that*

$$\begin{aligned}\int_{|\xi|\le 1}|\xi|^{2\gamma}e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 d\xi &\le \int_{|\xi|\le 1}|\xi|^{2\gamma-2}e^{-t|\xi|^2}\,d\xi \\ &\le e(1+t)^{-\frac{n}{2}-\gamma+1}\int_{|\eta|\le\sqrt{t}}|\eta|^{2\gamma-2}e^{-|\eta|^2}\,d\eta \\ &= e\omega_n(1+t)^{-\frac{n}{2}-\gamma+1}\int_0^{\sqrt{t}}x^{n+2\gamma-3}e^{-x^2}\,dx \\ &\le C(1+t)^{-\frac{n}{2}-\gamma+1}, \qquad t \ge 0;\end{aligned}$$

[II] *If $n \geq 1$ and $\gamma \geq 0$, one has*

$$\begin{aligned}
\int_{|\xi|\leq 1} |\xi|^{2\gamma} e^{-t|\xi|^2} |\cos(t|\xi|)|^2 \, d\xi &\leq \int_{|\xi|\leq 1} |\xi|^{2\gamma} e^{-t|\xi|^2} \, d\xi \\
&\leq e(1+t)^{-\frac{n}{2}-\gamma} \int_{|\eta|\leq \sqrt{t}} |\eta|^{2\gamma} e^{-|\eta|^2} \, d\eta \\
&= e\omega_n (1+t)^{-\frac{n}{2}-\gamma} \int_0^{\sqrt{t}} x^{2\gamma+n-1} e^{-x^2} \, dx \\
&\leq C(1+t)^{-\frac{n}{2}-\gamma}, \qquad t \geq 0.
\end{aligned}$$

Proof of Lemma 4.3. Let $n \geq 1$ and γ satisfy (3.1). Then we have

$$\begin{aligned}
&\int_{|\xi|\leq 1} |\xi|^{2\gamma} e^{-t|\xi|^2} \left| \frac{\sin(t|\xi|)}{|\xi|} \right|^2 d\xi = t^{-\frac{n}{2}-\gamma+1} \int_{|\eta|\leq\sqrt{t}} |\eta|^{2\gamma} e^{-|\eta|^2} \left| \frac{\sin(\sqrt{t}|\eta|)}{|\eta|} \right|^2 d\eta \\
&\geq t^{-\frac{n}{2}-\gamma+1} \int_{|\eta|\leq 1} |\eta|^{2\gamma} e^{-|\eta|^2} \left| \frac{\sin(\sqrt{t}|\eta|)}{|\eta|} \right|^2 d\eta \\
&= \omega_n t^{-\frac{n}{2}-\gamma+1} \int_0^1 x^{2\gamma+n-3} e^{-x^2} \sin^2(\sqrt{t}x) \, dx \\
&= \frac{\omega_n}{2} \left(\int_0^1 x^{2\gamma+n-3} e^{-x^2} \, dx \right) t^{-\frac{n}{2}-\gamma+1} - \frac{\omega_n}{2} t^{-\frac{n}{2}-\gamma+1} \int_0^1 x^{2\gamma+n-3} e^{-x^2} \cos(2\sqrt{t}x) \, dx
\end{aligned}$$

for $t \geq 1$. Assumption (3.1) assures that $x^{2\gamma+n-3} e^{-x^2} \in L^1(0,1)$ (see also [I] in Remark 4.2) and the Riemann-Lebesgue lemma implies

$$\lim_{t\to\infty} \int_0^1 x^{2\gamma+n-3} e^{-x^2} \cos(2\sqrt{t}x) \, dx = 0.$$

So we obtain assertion (i). Assertion (ii) can be proved similarly. □

5 Proofs of main results

Proof of Theorem 3.1. If $u_1 \in L^{1,\gamma}(\mathbf{R}^n)$ with γ satisfying (3.1), we see that

$$\begin{aligned}
&E_1(t,\xi)\widehat{u_1} \\
&= \left[\sum_{k=0}^{[\gamma]} e_1^k(t,\xi) + \left(E_1(t,\xi) - \sum_{k=0}^{[\gamma]} e_1^k(t,\xi) \right) \right] \left[\sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) + \left(\widehat{u_1}(\xi) - \sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) \right) \right] \\
&= \left(\sum_{k=0}^{[\gamma]} e_1^k(t,\xi) \right) \left(\sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) \right) \\
&\quad + \left(\sum_{k=0}^{[\gamma]} e_1^k(t,\xi) \right) \left(\widehat{u_1}(\xi) - \sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) \right) + \left(E_1(t,\xi) - \sum_{k=0}^{[\gamma]} e_1^k(t,\xi) \right) \left(\sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) \right) \\
&\quad + \left(E_1(t,\xi) - \sum_{k=0}^{[\gamma]} e_1^k(t,\xi) \right) \left(\widehat{u_1}(\xi) - \sum_{k=0}^{[\gamma]} m[u_1]^k(\xi) \right).
\end{aligned}$$

From (4.3), there exists a constant $C>0$ such that

$$\left|\sum_{k=0}^{[\gamma]} e_1^k(t,\xi)\right| \le C\frac{e^{-\frac{t|\xi|^2}{2}}}{|\xi|}\sum_{\ell=0}^{[\gamma]}(t|\xi|^2)^\ell$$

for $t\ge 0$ and $\xi\in\mathbf{R}^n$ with $|\xi|\le 1$. Thus we have

$$\begin{aligned}\left\|\left(\sum_{k=0}^{[\gamma]} e_1^k(t)\right)\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)} &\le C\|u_1\|_{1,\gamma}\sum_{\ell=0}^{[\gamma]}\left\|(t|\xi|^2)^\ell|\xi|^{\gamma-1}e^{-\frac{t|\xi|^2}{2}}\right\|_{L^2(|\xi|\le 1)}\\ &\le C\|u_1\|_{1,\gamma}(1+t)^{-\frac{n}{4}-\frac{\gamma}{2}+\frac{1}{2}},\qquad t\ge 0,\end{aligned}$$

with the aid of (4.4) and [I] in Remark 4.2. It follows from (4.1) that

$$\left\|\left(E_1(t)-\sum_{k=0}^{[\gamma]} e_1^k(t)\right)\left(\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)} \le C\|u_1\|_{1,[\gamma]}(1+t)^{-\frac{n}{4}-\frac{[\gamma]}{2}},\qquad t\ge 0.$$

The last term is estimated by

$$\left\|\left(E_1(t)-\sum_{k=0}^{[\gamma]} e_1^k(t)\right)\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)} \le C\|u_1\|_{1,\gamma}(1+t)^{-\frac{n}{4}-\frac{[\gamma]+\gamma}{2}},\qquad t\ge 0.$$

Furthermore, one has

$$\begin{aligned}&\left(\sum_{k=0}^{[\gamma]} e_1^k(t,\xi)\right)\left(\sum_{k=0}^{[\gamma]} m[u_1]^k(\xi)\right)\\ &\quad=\sum_{k=0}^{[\gamma]}\left(e_1^k(t,\xi)\sum_{j=0}^{[\gamma]-k} m[u_1]^j(\xi)\right)+\sum_{k=1}^{[\gamma]}\left(e_1^k(t,\xi)\sum_{j=[\gamma]-k+1}^{[\gamma]} m[u_1]^j(\xi)\right).\end{aligned}$$

If $[\gamma]=0$, then the second term of the right-hand side does not appear. So we consider the case $\gamma\ge 1$. In this case it follows from (4.3) that

$$\begin{aligned}\left\|\sum_{k=1}^{[\gamma]}\left(e_1^k(t)\sum_{j=[\gamma]-k+1}^{[\gamma]} m[u_1]^j\right)\right\|_{L^2(|\xi|\le 1)} &\le C\|u_1\|_{1,[\gamma]}\sum_{k=1}^{[\gamma]}\sum_{\ell=0}^{k}\sum_{j=[\gamma]-k+1}^{[\gamma]}\left\|(t|\xi|^2)^\ell|\xi|^{k+j-1}e^{-\frac{t|\xi|^2}{2}}\right\|_{L^2(|\xi|\le 1)}\\ &\le C\|u_1\|_{1,[\gamma]}(1+t)^{-\frac{n}{4}-\frac{[\gamma]}{2}},\qquad t\ge 0.\end{aligned}$$

Thus we obtain (3.2).

In order to prove (3.3), it suffices to check

$$\left\|\left(\sum_{k=0}^{[\gamma]} e_1^k(t)\right)\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)} = o(t^{-\frac{n}{4}-\frac{\gamma}{2}+\frac{1}{2}}),\qquad t\to\infty.$$

We calculate

$$\left\|\left(\sum_{k=0}^{[\gamma]} e_1^k(t)\right)\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)} \le C\sum_{k=0}^{[\gamma]}\left\|(t|\xi|^2)^k\frac{e^{-\frac{t|\xi|^2}{2}}}{|\xi|}\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)}$$

$$\le C\sum_{k=0}^{[\gamma]}\sup_{a\ge 0}\left(a^k e^{-\frac{a}{4}}\right)\left\|\frac{e^{-\frac{t|\xi|^2}{4}}}{|\xi|}\left(\widehat{u_1}-\sum_{k=0}^{[\gamma]} m[u_1]^k\right)\right\|_{L^2(|\xi|\le 1)}$$

$$\le Ct^{-\frac{n}{4}-\frac{\gamma}{2}+\frac12}\left(\int_{\mathbf{R}^n}\frac{e^{-\frac{|\eta|^2}{2}}}{|\eta|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_1\left(x,\frac{\eta}{\sqrt{t}}\right)u_1(x)\,dx\right|^2 d\eta\right)^{\frac12}.$$

Recalling the proof of Lemma 4.2, there exists a constant $C>0$ such that

$$\frac{e^{-\frac{|\eta|^2}{2}}}{|\eta|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_1\left(x,\frac{\eta}{\sqrt{t}}\right)u_1(x)\,dx\right|^2\le C\|u_1\|_{1,\gamma}^2|\eta|^{2\gamma-2}e^{-\frac{|\eta|^2}{2}}$$

for $\eta\in\mathbf{R}^n$ and $t>0$. By a similar argument to the proof of (ii) in Lemma 4.2 with [I] in Remark 4.2, the Lebesgue dominated convergence theorem yields

$$\lim_{t\to\infty}\int_{\mathbf{R}^n}\frac{e^{-\frac{|\eta|^2}{2}}}{|\eta|^2}\left|t^{\frac{\gamma}{2}}\int_{\mathbf{R}^n}\mathcal{R}_1\left(x,\frac{\eta}{\sqrt{t}}\right)u_1(x)\,dx\right|^2 d\eta=0.$$

So we obtain (3.3) and the proof is now complete. □

The proof of Theorem 3.2 is similar to that of Theorem 3.1, which is simpler. Hence we omit the proofs of (3.4) and (3.5).

Proof of Theorem 3.4. It follows from (3.3) with $\gamma=1$ and (3.5) with $\gamma=0$ that

$$\left\|\hat{u}(t)-\left(\int_{\mathbf{R}^n}u_1(x)\,dx\right)e^{-\frac{t|\xi|^2}{2}}\frac{\sin(t|\xi|)}{|\xi|}\right\|_2=\left\|\hat{u}(t)-e_1^0(t)m[u_1]^0\right\|_2\ge\left\|\hat{u}(t)-e_1^0(t)m[u_1]^0\right\|_{L^2(|\xi|\le1)}$$

$$\ge\left\|\sum_{k=0}^{1}e_1^k(t)m[u_1]^{1-k}+e_0^0(t)m[u_0]^0\right\|_{L^2(|\xi|\le1)}-\left\|E_1(t)\widehat{u_1}-\sum_{k=0}^{1}\left(e_1^k(t)\sum_{j=0}^{1-k}m[u_1]^j\right)\right\|_{L^2(|\xi|\le1)}$$

$$-\left\|E_0(t)\widehat{u_0}-e_0^0(t)m[u_0]^0\right\|_{L^2(|\xi|\le1)}-\left\|\frac{|\xi|^2}{2}E_1(t)\widehat{u_1}\right\|_{L^2(|\xi|\le1)}$$

$$\ge\left\|\sum_{k=0}^{1}e_1^k(t)m[u_1]^{1-k}+e_0^0(t)m[u_0]^0\right\|_{L^2(|\xi|\le1)}-o(t^{-\frac{n}{4}})-o(t^{-\frac{n}{4}})-O(t^{-\frac{n}{4}-\frac12}),\qquad t\to\infty.$$

Here, the following estimate is just used:

$$\left\|\frac{|\xi|^2}{2}E_1(t)\right\|_{L^2(|\xi|\le1)}^2\le C\int_{|\xi|\le1}|\xi|^2e^{-t|\xi|^2}\,d\xi\le C(1+t)^{-\frac{n}{2}-1},\qquad t\ge0.$$

Recall that

$$\sum_{k=0}^{1}e_1^k(t,\xi)m[u_1]^{1-k}(\xi)=e_1^0(t,\xi)m[u_1]^1(\xi)+e_1^1(t,\xi)m[u_1]^0(\xi)$$

$$=-i\sum_{|\alpha|=1}\left(\int_{\mathbf{R}^n}x^\alpha u_1(x)\,dx\right)e^{-\frac{t|\xi|^2}{2}}\frac{\sin(t|\xi|)}{|\xi|}\xi^\alpha-\frac18\left(\int_{\mathbf{R}^n}u_1(x)\,dx\right)t|\xi|^2e^{-\frac{t|\xi|^2}{2}}\cos(t|\xi|),$$

$$e_0^0(t,\xi)m[u_0]^0(\xi)=\left(\int_{\mathbf{R}^n}u_0(x)\,dx\right)e^{-\frac{t|\xi|^2}{2}}\cos(t|\xi|),$$

and so we have

$$\begin{aligned}
&\left\|\sum_{k=0}^{1} e_1^k(t)m[u_1]^{1-k} + e_0^0(t,\xi)m[u_0]^0\right\|_{L^2(|\xi|\le 1)}^2 \\
&= \int_{|\xi|\le 1} e^{-t|\xi|^2} \left| i\sum_{|\alpha|=1}\left(\int_{\mathbf{R}^n} x^\alpha u_1(x)\,dx\right)\frac{\sin(t|\xi|)}{|\xi|}\xi^\alpha \right. \\
&\qquad \left. +\frac{1}{8}\left(\int_{\mathbf{R}^n} u_1(x)\,dx\right)t|\xi|^2\cos(t|\xi|) - \left(\int_{\mathbf{R}^n} u_0(x)\,dx\right)\cos(t|\xi|)\right|^2 d\xi \\
&= \int_{|\xi|\le 1} e^{-t|\xi|^2}\left\{\left[\sum_{|\alpha|=1}\left(\int_{\mathbf{R}^n} x^\alpha u_1(x)\,dx\right)\frac{\sin(t|\xi|)}{|\xi|}\xi^\alpha\right]^2\right. \\
&\qquad \left. +\left[\frac{1}{8}\left(\int_{\mathbf{R}^n} u_1(x)\,dx\right)t|\xi|^2 - \left(\int_{\mathbf{R}^n} u_0(x)\,dx\right)\right]^2 |\cos(t|\xi|)|^2\right\} d\xi \\
&= \sum_{j=1}^{n}\left(\int_{\mathbf{R}^n} x_j u_1(x)\,dx\right)^2 \int_{|\xi|\le 1} e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\xi_j\right|^2 d\xi \\
&\qquad + 2\sum_{1\le j<k\le n}\left(\int_{\mathbf{R}^n} x_j u_1(x)\,dx\right)\left(\int_{\mathbf{R}^n} x_k u_1(x)\,dx\right)\int_{|\xi|\le 1} e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 \xi_j\xi_k\,d\xi \qquad (5.1) \\
&\qquad + \int_{|\xi|\le 1} e^{-t|\xi|^2}\left[\frac{1}{8}\left(\int_{\mathbf{R}^n} u_1(x)\,dx\right)t|\xi|^2 - \left(\int_{\mathbf{R}^n} u_0(x)\,dx\right)\right]^2 |\cos(t|\xi|)|^2\,d\xi.
\end{aligned}$$

Now we calculate the first term. For all $j = 1, \dots, n$, it follows from (4.6) that

$$\begin{aligned}
&\int_{|\xi|\le 1} e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\xi_j\right|^2 d\xi = \int_{|\xi|\le 1} e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 \xi_1^2\,d\xi \\
&= \frac{1}{n}\int_{|\xi|\le 1} |\xi|^2 e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 d\xi \ge \frac{\omega_n}{4n}\left(\int_0^1 x^{n-1}e^{-x^2}\,dx\right)t^{-\frac{n}{2}}
\end{aligned}$$

for sufficiently large t.

We can ignore the second term on the right-hand side of (5.1). This is just because this term never appears if $n = 1$, and it also holds that

$$\int_{|\xi|\le 1} e^{-t|\xi|^2}\left|\frac{\sin(t|\xi|)}{|\xi|}\right|^2 \xi_j\xi_k\,d\xi = 0, \qquad 1 \le j < k \le n,$$

when $n \ge 2$. So let us estimate the integral

$$\begin{aligned}
I &:= \int_{|\xi|\le 1}\left(\frac{P_1}{8}t|\xi|^2 - P_0\right)^2 e^{-t|\xi|^2}|\cos(t|\xi|)|^2\,d\xi \\
&= t^{-\frac{n}{2}}\int_{|\eta|\le\sqrt{t}}\left(\frac{P_1}{8}|\eta|^2 - P_0\right)^2 e^{-|\eta|^2}|\cos(\sqrt{t}|\eta|)|^2\,d\eta,
\end{aligned}$$

where

$$P_j := \int_{\mathbf{R}^n} u_j(x)\,dx, \qquad j = 0, 1.$$

If $P_1 = 0$, then it follows from (4.7) that

$$I = P_0^2 \int_{|\xi|\le 1} e^{-t|\xi|^2} |\cos(t|\xi|)|^2 \, d\xi \ge P_0^2 \frac{\omega_n}{4} \left(\int_0^1 x^{n-1} e^{-x^2} \, dx \right) t^{-\frac{n}{2}}, \qquad t \gg 1.$$

Conversely, in the case $P_0 = 0$, from (4.7), one has

$$I = \frac{P_1^2 t^2}{64} \int_{|\xi|\le 1} |\xi|^4 e^{-t|\xi|^2} |\cos(t|\xi|)|^2 \, d\xi \ge P_1^2 \frac{\omega_n}{256} \left(\int_0^1 x^{n+3} e^{-x^2} \, dx \right) t^{-\frac{n}{2}}, \qquad t \gg 1.$$

Finally, we deal with the case $P_1 \neq 0$ and $P_0 \neq 0$. Let $\delta > 0$ be an arbitrary real number. Then, one has

$$I \ge t^{-\frac{n}{2}} \int_{\delta \le |\eta| \le 2\delta} \left(\frac{P_1}{8} |\eta|^2 - P_0 \right)^2 e^{-|\eta|^2} |\cos(\sqrt{t}|\eta|)|^2 \, d\eta$$

for $t \ge 4\delta^2 > 0$. Now we choose

$$\delta := 4 \sqrt{\frac{|P_0|}{|P_1|}} > 0.$$

If $|\eta| \ge \delta$, then

$$\left| \frac{P_1}{8} |\eta|^2 - P_0 \right| \ge \frac{|P_1|}{8} |\eta|^2 - |P_0| \ge \frac{|P_1|}{16} |\eta|^2 \ge |P_0|.$$

So we can estimate I in two ways:

$$\begin{aligned} I &\ge t^{-\frac{n}{2}} P_0^2 \int_{\delta \le |\eta| \le 2\delta} e^{-|\eta|^2} |\cos(\sqrt{t}|\eta|)|^2 \, d\eta \\ &\ge \frac{\omega_n}{4} \left(\int_\delta^{2\delta} x^{n-1} e^{-x^2} \, dx \right) P_0^2 t^{-\frac{n}{2}}, \end{aligned}$$

$$\begin{aligned} I &\ge t^{-\frac{n}{2}} \left(\frac{P_1}{16} \right)^2 \int_{\delta \le |\eta| \le 2\delta} |\eta|^4 e^{-|\eta|^2} |\cos(\sqrt{t}|\eta|)|^2 \, d\eta \\ &\ge \frac{\omega_n}{1024} \left(\int_\delta^{2\delta} x^{n+3} e^{-x^2} \, dx \right) P_1^2 t^{-\frac{n}{2}}, \end{aligned}$$

for sufficiently large $t \ge 4\delta^2$. Therefore we obtain the desired statement of Theorem 3.4. □

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