広島大学学位請求論文

Generalized Cousin-I condition and intermediate pseudoconvexity in a Stein manifold

（Stein 多様体での一般化された Cousin-I 条件と中間的擬凸性）

２０２０年

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主論文

Generalized Cousin-I condition and intermediate pseudoconvexity in a Stein manifold

Shun Sugiyama

Abstract

Let D be an open subset of an n-dimensional Stein manifold, where $n \geq 2$. Assume that the canonical map $H^{n-1}(D, \mathcal{O}) \to H^{n-1}(D, \mathcal{M})$ is injective. Then, we prove that D is pseudoconvex of order 1, which generalizes the well-known theorem of Cartan-Behnke-Stein. Moreover we introduce a new proof of theorem of Eastwood–Vigna Suria.

1 Introduction

According to the well-known theorem of Cartan-Behnke-Stein [4, 6], every Cousin-I open subset of $\mathbb{C}^2$ is Stein. Here, an open set D in an n-dimensional Stein manifold X is said to be *Cousin-I* if any additive Cousin problem has a solution. This condition is equivalent to the injectivity of the canonical map $H^1(D, \mathcal{O}) \to H^1(D, \mathcal{M})$, where $\mathcal{M}$ denotes the sheaf of all germs of meromorphic functions on D (see Grauert–Remmert [11, p. 137]).

On the other hand, there is an intermediate geometric notion which generalizes pseudoconvexity. An open set D in an n-dimensional complex manifold X is said to be pseudoconvex of order $n-q$, where $1 \leq q \leq n$, if its complement $X \setminus D$ has the same continuity as an analytic set of pure dimension $n-q$.

The object of this paper is to generalize Cousin-I condition and describe its relation to pseudoconvexity of order $n-q$. Precisely, we prove that an open set D in an n-dimensional Stein manifold X is pseudoconvex of order 1 if the canonical map $H^{n-1}(D, \mathcal{O}) \to H^{n-1}(D, \mathcal{M})$ is injective (Theorem 5.1). In the case where $n = 2$, this result is nothing but the theorem of Cartan-Behnke-Stein for an open set D in a Stein manifold X of dimension two (see Kajiwara–Kazama [13, Corollary 3] and Berg [5, Corollary]). Moreover we introduce a new proof of theorem of Eastwood–Vigna Suria.

2 Preliminaries

We denote by $\|\cdot\|$ the Euclidian norm on $\mathbb{C}^n$ and by $|\cdot|$ the maximum norm on $\mathbb{C}^n$. Let $\mathsf{B}_n(c, r) = \{z \in \mathbb{C}^n \ ; \ \|z - c\| < r\}$ and $\mathsf{P}_n(c, r) = \{z \in \mathbb{C}^n \ ; \ |z - c| < r\}$ for every $c \in \mathbb{C}^n$ and $r \in (0, \infty]$. We call the set $\mathsf{B}_n(c, r)$ the *ball* of radius r with center c in $\mathbb{C}^n$ and the set $\mathsf{P}_n(c, r)$ the *polydisk* of radius r with center c in $\mathbb{C}^n$. Throughout this paper, X always stands for an n-dimensional complex manifold. An upper semicontinuous function u is said to be *subpluriharmonic* on X if for every open set $D \Subset X$ and for every pluriharmonic function h which is defined on a neighborhood of $\overline{D}$ and satisfies the inequality $u \leq h$ on ∂D, we have the inequality $u \leq h$ on $\overline{D}$ (see Fujita [9]). An upper semicontinuous function u is *q-plurisubharmonic* on X, where $1 \leq q \leq n$, if for every domain D in $\mathbb{C}^q$ and for every holomorphic function f on D to X, the function $u \circ f$ is subpluriharmonic on D. We obtain the following proposition which generalizes Lemma 1 in Yasuoka [21].

Proposition 2.1. *Let D be an open subset of $\mathbb{C}^n$ and u an upper semicontinuous function. If u is not subpluriharmonic on D, then there exist $c \in D$, $\rho > 0$, a function $h : \overline{\mathsf{B}_n(c,\rho)} \to \mathbb{R}$ which is real-analytic near $\overline{\mathsf{B}_n(c,\rho)}$ and a constant $K > 0$ such that $\mathsf{B}_n(c,\rho)$ is relatively compact in D, $u(c) = h(c)$, $u \leq h$ on $\overline{\mathsf{B}_n(c,\rho)}$ and*

$$\mathrm{i}\partial\bar{\partial}h = -\mathrm{i}K\sum_{\nu=1}^{n} dz_\nu \wedge d\bar{z}_\nu$$

on $\mathsf{B}_n(c,\rho)$.

Proof. By Proposition 3 in Fujita [9], there exist a relatively compact open ball $Q = \mathsf{B}_n(a,R)$, a function $g : \overline{Q} \to \mathbb{R}$ which is pluriharmonic near $\overline{Q}$ and $b \in Q$ such that $u \leq g$ on ∂Q and $u(b) > g(b)$. Replacing g, we can assume that $u < g$ on ∂Q and $u(b) > g(b)$. Since the function $u - g$ is upper semicontinuous on $\overline{Q}$, we can put $M = \max_{z\in\partial Q}\{u(z) - g(z)\} < 0$. Take an arbitrary $K \in (0, -M/R^2)$. Because the function $u - g + K\|z-a\|^2$ is upper semicontinuous on $\overline{Q}$, there exists $c \in \overline{Q}$ such that

$$N = \max_{z\in\overline{Q}}\{u(z) - g(z) + K\|z-a\|^2\} = u(c) - g(c) + K\|c-a\|^2.$$

Since $b \in Q$ and $u(b) - g(b) + K\|b-a\|^2 > 0$, we have $N > 0$. Moreover, $u(z) - g(z) + K\|z-a\|^2 \leq M + KR^2 < 0$ for every $z \in \partial Q$. Therefore, we obtain $c \in Q$. Take an arbitrary $\rho > 0$ such that $\mathsf{B}_n(c,\rho)$ is relatively compact in Q. The function $h(z) = g(z) - K\|z-a\|^2 + N$ is real-analytic on Q. We see that $u(c) = h(c), u \leq h$ on Q and

$$\mathrm{i}\partial\bar{\partial}h = -\mathrm{i}K\sum_{\nu=1}^{n} dz_\nu \wedge d\bar{z}_\nu$$

on Q. □

Proposition 2.2. *Let $c \in \mathbb{C}^n$, $r > 0$ and $f \in \mathcal{O}(\mathsf{B}_n(c,r))$ with $\Im(f(c)) = 0$. Set*

$$P(z) = \sum_{|\nu|\leq 2}\frac{1}{\nu!}\frac{\partial^{|\nu|}e^f}{\partial z^\nu}(c)(z-c)^\nu.$$

Then, for every $\varepsilon \in (0, e^{-\Re(f(c))})$, there exist $\rho \in (0, r)$, $\delta > 0$ and $M > 0$ such that

$$\log |P(z) - t| \leq \Re(f(z)) - \varepsilon t + M \|z - c\|^3$$

on $\overline{\mathsf{B}_n(c, \rho)} \times [0, \delta]$.

Proof. We may assume that $c = 0$. As the function e^f is holomorphic on $\mathsf{B}_n(0, r)$, we obtain the Tayler series expansion

$$e^{f(z)} = \sum_{\nu} \frac{1}{\nu!} \frac{\partial^{|\nu|} e^f}{\partial z^\nu}(0) z^\nu$$

of e^f which converges on $\mathsf{B}_n(0, r)$. Put

$$R(z) = \sum_{|\nu| \geq 3} \frac{1}{\nu!} \frac{\partial^{|\nu|} e^f}{\partial z^\nu}(0) z^\nu.$$

We have $e^f = P + R$ on $\mathsf{B}_n(0, r)$. Take an arbitrary $\rho_1 \in (0, r)$. Consider the expression of the form $R(z) = \sum_{|\nu|=3} g_\nu(z) z^\nu$ by holomorphic functions $g_\nu \in \mathcal{O}(\mathsf{B}_n(0, r))$. Then there exists $M_1 > 0$ such that

$$|R(z)| \leq \sum_{|\nu|=3} |g_\nu(z)| \, |z^\nu| \leq M_1 \|z\|^3$$

on $\overline{\mathsf{B}_n(0, \rho_1)}$. Let $h_1 = \Re(f)$, $h_2 = \Im(f)$ and $\varepsilon \in (0, e^{-h_1(0)})$. We define the function $F(z, t)$ on $\mathsf{B}_n(0, r) \times \mathbb{R}$ by

$$F(z, t) = \left(e^{h_1(z) - \varepsilon t}\right)^2 - \left|e^{h_1(z) + \mathrm{i} h_2(z)} - t\right|^2.$$

By a simple calculation, we obtain the inequality

$$\frac{\partial F}{\partial t}(0, 0) = 2e^{2h_1(0)} \left(-\varepsilon + e^{-h_1(0)}\right) > 0.$$

It follows that there exist $\rho_2 > 0$ and $\delta > 0$ such that $\partial F(z, t)/\partial t > 0$ on $\overline{\mathsf{B}_n(0, \rho_2)} \times [-\delta, \delta]$. Thus, $F(z, t) \geq F(z, 0) = 0$ on $\overline{\mathsf{B}_n(0, \rho_2)} \times [0, \delta]$. It means that

$$\left|e^{h_1(z) + \mathrm{i} h_2(z)} - t\right| \leq e^{h_1(z) - \varepsilon t}$$

on $\overline{\mathsf{B}_n(0, \rho_2)} \times [0, \delta]$. Let $\rho = \min\{\rho_1, \rho_2\}$. Then, for every $(z, t) \in \overline{\mathsf{B}_n(0, \rho)} \times [0, \delta]$, we have

$$\begin{aligned} |P(z) - t| &\leq \left|e^{h_1(z) + \mathrm{i} h_2(z)} - t\right| + |R(z)| \\ &\leq e^{h_1(z) - \varepsilon t}(1 + M \|z\|^3), \end{aligned}$$

where $M = M_1 \max_{\|z\| \leq \rho_2} e^{-h_1(z) + \varepsilon \delta}$, and consequently

$$\log |P(z) - t| \leq h_1(z) - \varepsilon t + \log(1 + M \|z\|^3) \leq h_1(z) - \varepsilon t + M \|z\|^3.$$

□

3 A characterization of pseudoconvexity of general order

In this chapter, we introduce the definition of intermediate pseudoconvexity and give a characterization of intermediate pseudoconvexity by Hartogs figures. This characterization is useful in the calculation of the cohomology groups.

Definition 3.1 (see Tadokoro [19], Fujita [9] and Matsumoto [14])**.** *Let $1 \leq q \leq n-1$. An open set D in X is called pseudoconvex of order $n-q$ if it satisfies the condition:*

> *Let $\xi \in E = X \setminus D$, $(U; z_1, \dots, z_n)$ a coordinate neighborhood containing ξ and $z_1(\xi) = \xi_1, \dots, z_n(\xi) = \xi_n$. Suppose that there exists $r > 0$ such that*
>
> $$\left\{ x \in U \; ; \; z_i(x) = \xi_i \ (1 \leq i \leq n-q), \, 0 < \sum_{i=n-q+1}^{n} |z_i(x) - \xi_i|^2 < r \right\}$$
>
> *has no point of E. Then there exists $s > 0$ such that for every $(\eta_1, \dots, \eta_{n-q})$ with $|\eta_i - \xi_i| < s \, (1 \leq i \leq n-q)$, the set*
>
> $$\left\{ x \in U \; ; \; z_i(x) = \eta_i \ (1 \leq i \leq n-q), \, \sum_{i=n-q+1}^{n} |z_i(x) - \xi_i|^2 < r \right\}$$
>
> *contains at least one point of E.*

Moreover, we say that every open set in X is pseudoconvex of order 0.

An open set D in X is pseudoconvex in the original sense if and only if it is pseudoconvex of order $n-1$. Note that pseudoconvexity of general order is a boundary local condition, namely, if for each $\xi \in \partial D$ there exists a neighborhood U of ξ such that $D \cap U$ is pseudoconvex of order $n-q$ in U, then D is pseudoconvex of order $n-q$.

Proposition 3.1 (Sugiyama [17, Propostion 3.1])**.** *Let D be an open subset of $\mathbb{C}^n$, q an integer such that $1 \leq q \leq n-1$ and $b, c \in (0,1)$. Put $\mathsf{H}_e = \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q} \; ; \; |\zeta_1| < 1, \, |\zeta_2| < b\} \cup \{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q} \; ; \; c < |\zeta_1| < 1, \, |\zeta_2| < 1\}$. The condition $(\star)$ implies that $-\log d_D$ is q-plurisubharmonic on D, where d_D is the boundary distance function with respect to the Euclidian norm.*

$(\star)$ *Let $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{C}^n \to \mathbb{C}^n, (z_1, \dots, z_n) \mapsto (w_1, \dots, w_n)$, be a biholomorphic map which satisfies the following two conditions:*

- *$\varphi(\mathsf{H}_e) \subset D$.*
- *There exist polynomials $P_j(z_1, \dots, z_n)$, $Q_j(w_1, \dots, w_n)$ of degree at most two such that $\varphi_j(z_1, \dots, z_n) = P_j(z_1, \dots, z_n)$ and $(\varphi^{-1})_j(w_1, \dots, w_n) = Q_j(w_1, \dots, w_n)$ for every $j = 1, \dots, n$.*

Then we have that $\varphi(\mathsf{P}_n(0,1)) \subset D$.

Proof. We improve the argument in Yasuoka [21] and Sugiyama [17]. Seeking a contradiction, suppose that $-\log d_D$ is not q-plurisubharmonic on D. Because of Proposition 3.10 in Pawlaschyk–Zeron [16], there exists $w \in \partial\mathsf{B}_1(0,1)$ such that $-\log d_D(z : w)$ is not q-plurisubharmonic on D, where $d_D(z : w)$ is distance to the boundary in direction w. According to Theorem 2 in Fujita [10], there exists a q-dimensional complex affine subspace L of $\mathbb{C}^n$ such that $-\log d_D(z : w)$ is not subpluriharmonic on $L \cap D$. Write $0_k = (0, \ldots, 0) \in \mathbb{C}^k$ for every $k \in \mathbb{N}$. Using a unitary transformation, we can suppose that $0_n \in L \cap D$ and $L = \mathbb{C}^q \times \{0_{n-q}\}$. Since the function $-\log d_D(z : w)$ is not subpluriharmonic on $L \cap D$, it follows that $w \notin L$. By a unitary transformation again, we may assume $w = e_{q+1}$, where e_{q+1} is the unit vector whose $q+1$-th component is 1. Let $d(\zeta) = d_D((\zeta, 0_{n-q}) : e_{q+1})$ for any $\zeta \in \mathbb{C}^q$. There exist $(a, 0_{n-q}) \in L \cap D$, $r > 0$, a function $g : \overline{\mathsf{B}_q(a,r)} \to \mathbb{R}$ which is real-analytic near $\overline{\mathsf{B}_q(a,r)}$ and a constant $K > 0$ such that $-\log d(a) = g(a)$, $-\log d \leq g$ on $\overline{\mathsf{B}_q(a,r)}$ and

$$\mathrm{i}\partial\bar{\partial}g = -\mathrm{i}K \sum_{\nu=1}^{q} d\zeta_\nu \wedge d\bar{\zeta}_\nu$$

on $\mathsf{B}_q(a,r)$ by Proposition 2.1. The function $h_1 = -g - K\sum_{\nu=1}^{q} |\zeta_\nu|^2$ is pluriharmonic on $\mathsf{B}_q(a,r)$. Therefore there exists $f \in \mathcal{O}(\mathsf{B}_q(a,r))$ such that $h_1 = \Re(f)$ and $\Im(f(a)) = 0$ (see Fritzsche–Grauert [8, p. 318]). Without loss of generality we can assume $a = 0_q$. From Proposition 2.2, there exist $\rho_1 \in (0,r)$, $\delta > 0$ and $M > 0$ such that

$$\log|P(\zeta) - t| \leq h_1(\zeta) - \varepsilon t + M\|\zeta\|^3$$

on $\overline{\mathsf{B}_q(0,\rho_1)} \times [0,\delta]$, where

$$P(\zeta) = P(\zeta_1, \ldots, \zeta_q) = \sum_{|\nu| \leq 2} \frac{1}{\nu!} \frac{\partial^{|\nu|} e^{f(0)}}{\partial \zeta^\nu} \zeta^\nu, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_q).$$

Take an arbitrary $\rho \in (0, \min\{\rho_1, K/M\})$. Put $\mathsf{B} = \mathsf{B}_q(0,\rho)$. If $\|\zeta\| \leq \rho$ and $0 < t \leq \delta$ then,

$$\begin{aligned}\log|P(\zeta) - t| &\leq h_1(\zeta) - \varepsilon t + M\|\zeta\|\,\|\zeta\|^2 \\ &\leq h_1(\zeta) - \varepsilon t + K\|\zeta\|^2 = -g(\zeta) - \varepsilon t < -g(\zeta).\end{aligned}$$

If $0 < \|\zeta\| \leq \rho$ and $0 \leq t \leq \delta$ then,

$$\begin{aligned}\log|P(\zeta) - t| &\leq h_1(\zeta) - \varepsilon t + M\|\zeta\|\,\|\zeta\|^2 \\ &< h_1(\zeta) - \varepsilon t + K\|\zeta\|^2 = -g(\zeta) - \varepsilon t \leq -g(\zeta).\end{aligned}$$

It follows that

$$|P(\zeta) - t| < e^{-g(\zeta)} \leq e^{\log d(\zeta)} = d(\zeta) = d_D((\zeta, 0_{n-q}) : e_{q+1}), \tag{1}$$

on $\overline{\mathsf{B}} \times [0, \delta] \setminus \{(0_q, 0)\}$. On the other hand, we have

$$|P(0_q)| = \left|e^{f(0_q)}\right| = e^{h_1(0_q)} = d(0_q) = d_D(0_n : e_{q+1}). \tag{2}$$

By the definition of the function $d_D(z : e_{q+1})$, there exists $s \in \partial\mathsf{B}_1(0, 1)$ such that $sP(0_q)e_{q+1} \in \partial D$. We define the holomorphic map $\psi : \mathbb{C}^{q+1} \times \mathbb{C}^{n-(q+1)} \to \mathbb{C}^n$ by

$$\psi(z_1, \dots, z_n) = \begin{cases} z_j & 1 \leq j \leq n,\ j \neq q + 1, \\ s(P(z_1, \dots, z_q) - z_{q+1}) & j = q + 1. \end{cases}$$

Take an arbitrary polydisk $\mathsf{P} = \mathsf{P}_q(0, \rho_2)$ such that $\overline{\mathsf{P}} \subset \overline{\mathsf{B}}$. By inequalities (1) and (2), we obtain $\psi(\partial\mathsf{P}\times[0, \delta]\times\{0_{n-(q+1)}\}) \subset D$ and $\psi(\overline{\mathsf{P}}\times\{t\}\times\{0_{n-(q+1)}\}) \subset D$ for any $t \in (0, \delta]$. We can choose $\varepsilon_0 > 0$ such that $\psi(\partial\mathsf{P} \times \overline{\mathsf{B}_1(0, \varepsilon_0)} \times \{0_{n-(q+1)}\}) \subset D$. Take an arbitrary $\delta_0 \in (0, \varepsilon_0/2)$, the set $\mathsf{B}_1(\delta_0, \varepsilon_0 - \delta_0)$ satisfies $\overline{\mathsf{B}_1(\delta_0, \varepsilon_0 - \delta_0)} \subset \overline{\mathsf{B}_1(0, \varepsilon_0)}$ and $0 \in \mathsf{B}_1(\delta_0, \varepsilon_0 - \delta_0)$. Set $\phi(z_1, \dots, z_q, z_{q+1}, \dots, z_n) = \psi(z_1, \dots, z_q, \delta_0 - z_{q+1}, \dots, z_n)$. This holomorphic map ϕ is biholomorphic. In fact, we can get the map $\phi^{-1} : \mathbb{C}^n \to \mathbb{C}^n$, $(w_1, \dots, w_n) \mapsto (z_1, \dots, z_n)$,

$$\phi^{-1}(w_1, \dots, w_n) = \begin{cases} w_j & 1 \leq j \leq n,\ j \neq q + 1, \\ w_{q+1}/s - P(w_1, \dots, w_q) + \delta_0 & j = q + 1. \end{cases}$$

There exists $\varepsilon > 0$ such that $\phi(\partial\mathsf{P}\times\overline{\mathsf{B}_1(0, \varepsilon_0 - \delta_0)}\times\overline{\mathsf{P}_{n-(q+1)}(0, \varepsilon)}) \subset D$, because $\partial\mathsf{P}\times\overline{\mathsf{B}_1(0, \varepsilon_0 - \delta_0)}\times \{0_{n-(q+1)}\}$ is a compact set. Moreover, we see that $\phi(\overline{\mathsf{P}} \times \{0_{n-q}\}) \subset D$ and $\phi(\delta_0 \cdot e_{q+1}) \notin D$. Since $\partial\mathsf{P} \times \overline{\mathsf{B}_1(0, \varepsilon_0 - \delta_0)} \times \overline{\mathsf{P}_{n-(q+1)}(0, \varepsilon)}$ and $\overline{\mathsf{P}} \times \{0_{n-q}\}$ are compact sets in $\mathbb{C}^n$, we can define a biholomorphic map φ which satisfies the condition (2) of the statement of lemma such that $\varphi(\mathsf{H}_e) \subset D$ and $\varphi(\mathsf{P}_n(0, 1)) \not\subset D$. This is a contradiction. □

The following theorem is a generalization of Lemmata 1 and 2 in Kajiwara–Kazama [13] (see also Lemma 2.1 in Abe [1]).

Theorem 3.1 (Sugiyama [17, Theorem 3.1])**.** *Let D be an open subset of $\mathbb{C}^n$ and q an integer such that $1 \leq q \leq n$. Then the following two conditions are equivalent.*

(1) *D is pseudoconvex of order $n - q$ in $\mathbb{C}^n$.*

(2) *D satisfies the condition $(\star)$.*

Proof. In the case where $n = q$, the assertion is trivial. So we can assume that $1 \leq q \leq n - 1$. (1) $\to$ (2). This is a direct result of Theorem 2 in Fujita [9]. (2) $\to$ (1). According to Theorem 2 in Fujita [9] and Theorem 3.1, this is trivial. □

4 Existence of meromorphically tirivial cocycles

The goal of this section is to organize the Kajiwara–Kazama's method [13, p. 8]. In particular, we will make an open covering that satisfies good conditions and a holomorphic cocycle that is meromorphically trivial.

Proposition 4.1 (cf. Kajiwara–Kazama [13, p. 8]). *Let X be an n-dimensional complex manifold and D an open set in X. Assume that there exist a holomorphic map $F : X \to \mathbb{C}^n, x \mapsto (w_1, \ldots, w_n)$, an open set $U \subset X$ and a point $a = (a_1, \ldots, a_n) \in \mathsf{P}_n(0, 1 + 2\varepsilon)$ such that $F(U)$ is biholomorphic to a polydisk $\mathsf{P}_n(0, 1 + 2\varepsilon)$, $U \cap D \neq \emptyset$ and $a \notin F(U \cap D)$. Put $T_1 = \{x \in X \ ; \ |w_1(x)| < 1 + 2\varepsilon\}$, $T_2 = \{x \in X \ ; \ |w_j(x)| < 1+2\varepsilon \ (j = 2, \ldots, n)\}$, $T_3 = T_1 \cap T_2 \cap U$, $T_4 = \{x \in T_2 \ ; \ |w_1(x)| > 1 + \varepsilon\} \cup \{x \in T_2 \backslash T_3 \ ; \ |w_1(x)| < 1+2\varepsilon\}$, $D_1 = \{x \in D \cap T_3 \ ; \ w_1 \neq a_1\} \cup \{D \cap T_4\}$ and $D_j = \{x \in D \ ; \ w_j \neq a_j\}$ for $j = 2, \ldots, n$. Then $\mathcal{D} = \{D_j\}_{j=1}^n$ is an open covering of D.*

Proof. Take an arbitrary point $x \in D$. If $w_j(x) \neq a_j$ for some $j = 2, \ldots, n$, then $x \in D_j$. So we may assume that $w_j(x) = a_j$ for every $j = 2, \ldots, n$. Then $x \in T_2$. If $|w_1(x)| > 1 + \varepsilon$, then we can get $x \in D_1$ because $x \in T_4$. In the case where $|w_1(x)| \leq 1 + \varepsilon$ and $x \notin T_3$, then we have that $x \in D_1$. In the case where $|w_1(x)| \leq 1 + \varepsilon$ and $x \in T_3$. If $w_1(x) \neq a_1$, we obtain $x \in D_1$. If $w_1(x) = a_1$, this contradicts $a \notin F(U \cap D)$. Thus we can get $D = \bigcup_{j=1}^n D_j$. □

Proposition 4.2 (cf. Kajiwara–Kazama [13, p. 8]). *Let T_1, T_2, T_3, T_4, D_j $(j = 1, \ldots, n)$ and $\mathcal{D}$ be the same as in Proposition 4.1. Assume that X is Stein. Then there exist $\rho \in \mathcal{M}(T_2)$ such that*

(1) $f = \dfrac{\rho}{(w_2 - a_2) \cdots (w_n - a_n)} \in Z^{n-1}(\mathcal{D}, \mathcal{O}) \cap \delta\left(C^{n-2}(\mathcal{D}, \mathcal{M})\right)$,

(2) $\rho = \dfrac{1}{w_1 - a_1} + \rho_3$ *on* T_3, *where* $\rho_3 \in \mathcal{O}(T_3)$.

Proof. Notice that $1/(w_1 - a_1) \in \mathcal{O}(T_3 \cap T_4)$. Since X is Stein, T_2 is Stein. The set $\{T_3, T_4\}$ is an open covering of T_2. So we can find holomorphic functions $\rho_j \in \mathcal{O}(T_j)$ for $j = 3, 4$ which satisfies $1/(w_1 - a_1) = \rho_4 - \rho_3$ on $T_3 \cap T_4$. We define

$$\rho = \begin{cases} \rho_4 & \text{on } T_4, \\ \rho_3 + \dfrac{1}{w_1 - a_1} & \text{on } T_3. \end{cases}$$

This function ρ is a meromorphic function on T_2. Since $f \in \mathcal{O}(D_1 \cap \cdots \cap D_n)$, we can define $f \in Z^{n-1}(\mathcal{D}, \mathcal{O})$. Moreover $D_1 \subset T_2$, so we have that $f \in \mathcal{M}(D_1 \cap \cdots \cap D_{n-1})$. Thus $f \in \delta\left(C^{n-2}(\mathcal{D}, \mathcal{M})\right)$. □

5 Generalized Cartan-Behnke-Stein's theorem

We introduce a generalized Cousin-I condition. An open set D in X is called q-*Cousin-I*, where $1 \leq q \leq n-1$, if the canonical map $H^q(D,\mathcal{O}) \to H^q(D,\mathcal{M})$ is injective. Note that D is 1-Cousin-I if and only if D is Cousin-I (see Grauert–Remmert [11, p. 137]). Let $b \in (0,1)$. We put $\mathsf{T}_{n-1} = \{z \in \mathbb{C}^n \ ; \ b < |z| < 1\} = \bigcup_{j=1}^n U_j$. Here, $U_j = \{z \in \mathsf{P}_n(0,1) \ ; \ b < |z_j| < 1\}$ $(j = 1,\ldots,n)$. It follows from $0 \notin \mathsf{T}_{n-1}$ that we can define $\frac{1}{z_1\cdots z_n} \in H^{n-1}(\mathsf{T}_{n-1},\mathcal{O})$.

Lemma 5.1. *Let $n \geq 2$, then T_{n-1} is not $(n-1)$-Cousin-I. Moreover $\frac{1}{z_1\cdots z_n} \neq 0$ in $H^{n-1}(\mathsf{T}_{n-1},\mathcal{O})$ but $\frac{1}{z_1\cdots z_n} = 0$ in $H^{n-1}(\mathsf{T}_{n-1},\mathcal{M})$.*

Proof. We obtain $H^k(\mathsf{T}_{n-1},\mathcal{F}) \cong H^k(\mathcal{U},\mathcal{F})$ for any $k \geq 0$ and for any analytic coherent sheaf $\mathcal{F}$ because $\mathcal{U} = \{U_j\}$ is a Stein open covering of T_{n-1}. Assume that $g = \frac{1}{z_1\cdots z_n} = 0$ in $H^{n-1}(\mathsf{T}_{n-1},\mathcal{O}) \cong H^{n-1}(\mathcal{U},\mathcal{O})$. There exist $g_j \in \mathcal{O}(V_j)$ $(j = 1,\ldots,n)$ such that $\delta(\{g_j\}) = g$, where $V_j = U_1 \cap \cdots \cap \hat{U}_j \cap \cdots \cap U_n$ and δ is the coboundary operator. The set V_j is a Reinhardt domain with a center origin. Therefore the function g_j can be expanded into the Laurent series with a center origin. It follows from the uniqueness of the representation of the Laurent series that there exists a $j \in \{1,\ldots,n\}$ such that g_j has the term of $\frac{1}{z_1\cdots z_n}$. It is a contradiction from $g_j \in \mathcal{O}(V_j)$. Moreover, we define $f_j \in C^{n-2}(\mathcal{U},\mathcal{M})$ by

$$f_j = \begin{cases} \dfrac{1}{z_1\cdots z_n} & \text{on } V_1, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain $\delta(f_j) = g$. Thus $g = 0$ in $H^{n-1}(\mathsf{T}_{n-1},\mathcal{M})$. □

Lemma 5.2. (cf. Watanabe [20, Lemma 4]) *Let $n \geq 2$, $b, c \in (0,1)$ and $b < |d| < 1$. Put $\mathsf{H}_e = \{(z_1,\ldots,z_{n-1},z_n) \in \mathbb{C}^n \ ; \ |(z_1,\ldots,z_{n-1})| < 1, |z_n - d| < b\} \cup \{(z_1,\ldots,z_{n-1},z_n) \in \mathbb{C}^n \ ; \ c < |(z_1,\ldots,z_{n-1})| < 1, |z_n - d| < 1\}$. Then the set H_e is not $(n-1)$-Cousin-I. Moreover $\frac{1}{z_1\cdots z_n} \neq 0$ in $H^{n-1}(\mathsf{H}_e,\mathcal{O})$ but $\frac{1}{z_1\cdots z_n} = 0$ in $H^{n-1}(\mathsf{H}_e,\mathcal{M})$.*

Proof. Let $U_j = \{(z_1,\ldots,z_n) \in \mathsf{P}_n((0,\ldots,0,d),1) \ ; \ c < |z_j| < 1\}$ $(j = 1,\ldots,n-1)$ and $U_n = \{(z_1,\ldots,z_n) \in \mathbb{C}^n \ ; \ |(z_1,\ldots,z_{n-1})| < 1, |z_n - d| < b\}$. The set $\mathcal{U} = \{U_j\}$ is a Stein covering of H_e. To obtain a contradiction, we assume that $g = \frac{1}{z_1\cdots z_n} = 0$ in $H^{n-1}(\mathsf{H}_e,\mathcal{O}) \cong H^{n-1}(\mathcal{U},\mathcal{O})$. In the case where $n = 2$, there exist $f_j \in \mathcal{O}(U_j)$ $(j = 1,2)$ such that $g = f_2 - f_1$ on $U_1 \cap U_2$. We notice that $f_2 = g + f_1$ is holomorphic on $(U_1 \setminus \{z_2 = 0\}) \cup U_2$. Moreover the function $f_1 = f_2 - g$ is holomorphic on $(U_2 \setminus \{z_1 = 0\}) \cup U_1$. Thus function f_1 can be extended to $\mathsf{P}_2((0,d),1) \setminus \{z_1 = 0\}$ and also the function f_2 can be extended to $\mathsf{P}_2((0,d),1) \setminus \{z_2 = 0\}$ (see Jarnicki–Pflug [12, p. 182]). $\mathsf{P}_2((0,d),1)$ is an open neighborhood of $(0,0)$ and put $G = \mathsf{P}_2((0,d),1) \setminus \{(0,0)\}$. So we can choose $\varepsilon > 0$ such that $T = \left\{z \in \mathbb{C}^2 ; 0 < |z_1| < \varepsilon, |z_2| < \varepsilon\right\} \cup \left\{z \in \mathbb{C}^2 ; |z_1| < \varepsilon, 0 < |z_2| < \varepsilon\right\} \subset G$. Thus $\{g\} = 0 \in H^1(T,\mathcal{O})$. This contradicts Lemma 5.1.

In the case where $n \geq 3$, there exist $g_j \in \mathcal{O}(V_j)$ $(j = 1 \ldots, n)$ such that $\delta(\{g_j\}) = g$, where $V_j = U_1 \cap \cdots \cap \hat{U}_j \cap \cdots \cap U_n$ and δ is the coboundary operator. $G^{(n)}_{\nu_1 \cdots \nu_{n-1}} \in C^{n-2}(\mathcal{U}, \mathcal{O})$ defined by

$$G^{(n)}_{1 \cdots n-1} = -z_n g_n - (-1)^{2+n} \frac{1}{z_1 \cdots z_{n-1}},$$
$$G^{(n)}_{\nu_1 \cdots \nu_{n-1}} = -z_n g_l,$$

where $\nu_1 \cdots \nu_{n-1} = 1 \cdots \hat{l} \cdots n$ and $l \neq n$. By a simple calculation, we have $\delta(G^{(n)}_{\nu_1 \cdots \nu_{n-1}}) = 0$. There exists an element $G \in C^{n-3}(\mathcal{U}, \mathcal{O})$ such that $\delta(G) = G^{(n)}_{\nu_1 \cdots \nu_{n-1}}$ according to the lemma of Andreotti–Grauert [3, p. 218]. In detail,

$$\sum_{k=1}^{n-1} (-1)^{k-1} G_{1 \cdots \hat{k} \cdots n-1}(z) = G^{(n)}_{1 \cdots n-1}(z) = -z_n g_n - (-1)^{2+n} \frac{1}{z_1 \cdots z_{n-1}}$$

for any $z \in V_n = U_1 \cap \cdots \cap U_{n-1}$. By restricting the above equation to $\{z_n = 0\}$, we get

$$\sum_{k=1}^{n-1} (-1)^{k-1} G_{1 \cdots \hat{k} \cdots n-1}(z_1, \ldots, z_{n-1}, 0) = (-1)^{n-1} \frac{1}{z_1 \cdots z_{n-1}}.$$

On the other hand, the set $\{z_n = 0\} \cap \mathsf{H}_e = \{(z_1, \ldots, z_{n-1}, 0) \in \mathbb{C}^n \ ; \ c < |(z_1, \ldots, z_{n-1})| < 1\}$ is identified with the set T_{n-2}. This contradicts Lemma 5.1. In particular we see that $g = 0$ in $H^{n-1}(\mathsf{H}_e, \mathcal{M})$ because of the proof of Lemma 5.1. Therefore H_e is not $(n-1)$-Cousin-I. □

Theorem 5.1 (Sugiyama [17, Theorem 5.1])**.** *Let $n \geq 2$, X an n-dimensional Stein manifold and D an open subset of X. If D is $(n-1)$-Cousin-I, then D is pseudoconvex of order* 1.

Proof. We use the argument in Kajiwara–Kazama [13, pp. 7–9] and Mori [15, pp. 186–191]. To obtain a contradiction, suppose that D is not pseudoconvex of order 1. There exists a point $x_0 \in \partial D$ such that for any neighborhood U of x_0, then $D \cap U$ is not pseudoconvex of order 1 in U. Since X is Stein, we can take holomorphic functions $\psi_j \in \mathcal{O}(X)$ $(j = 1, \ldots, n)$ which satisfies the following two conditions:

- $\psi_j(x_0) = 0$ $(j = 1, \ldots, n)$.
- The family $\{\psi_1, \ldots, \psi_n\}$ forms a coordinate system in the connected component U of $\{x \in X \ ; \ |\psi_j(x)| < K \ (j = 1, \ldots, n)\}$ containing x_0 for some $K > 0$.

Define a holomorphic mapping $\psi : X \to \mathbb{C}^n, \ x \mapsto (\psi_1(x), \ldots, \psi_n(x)) = (z_1(x), \ldots, z_n(x))$. Then we have that $\psi(U) = \{z \in \mathbb{C}^n \ ; \ |z_j| < K \ (j = 1, \ldots, n)\}$. By Theorem 3.1, there exist a biholomorphic map $\varphi : \mathbb{C}^n \to \mathbb{C}^n, (w_1, \ldots, w_n) \mapsto (z_1, \ldots, z_n)$ and $\varepsilon > 0$ such that

- Put $\mathsf{H} = \{w \in \mathbb{C}^n \ ; \ |w_i| < 1 \ (i = 1, \dots, n)\}$
  $\cup \{w \in \mathbb{C}^n \ ; \ 1 - 2\varepsilon < |w_j| < 1 + 2\varepsilon \ (j = 1, \dots, n-1), \ |w_n| < 1 + 2\varepsilon\}$.
  Then we heve $\varphi(\mathsf{H}) \subset \psi(D \cap U)$.
- $\varphi(E) \subset \psi(U)$, where $E = \{w \in \mathbb{C}^n \ ; \ |w_j| < 1 + 2\varepsilon \ (j = 1, \dots, n)\}$.
- There is a point $a = (a_1, \dots, a_{n-1}, a_n) \in \mathbb{C}^n$ such that $a_j = 0 \ (j = 1, \dots, n-1)$, $1 < |a_n| < 1 + 2\varepsilon$ and $\varphi(a) \notin \psi(D \cap U)$.

Put $\varphi^{-1} \circ \psi : X \to \mathbb{C}^n \ x \mapsto (w_1, \dots, w_n)$. The family $\{w_1, \dots, w_n\}$ forms a coordinate system in U. By Proposition 4.1 and Proposition 4.2, we can take open sets T_1, T_2, T_3, T_4, an open covering $\mathcal{D}$ of D and $\rho \in \mathcal{M}(T_2)$. Let f, g and h be the functions given by

$$f = \frac{\rho}{w_2 \cdot w_3 \cdots w_{n-1}(w_n - a_n)},$$
$$g = \frac{1}{w_1 \cdot w_2 \cdots w_{n-1}(w_n - a_n)},$$
$$h = \frac{\rho_3}{w_2 \cdot w_3 \cdots w_{n-1}(w_n - a_n)}.$$

By Proposition 4.2, we obtain $f \in Z^{n-1}(\mathcal{D}, \mathcal{O}) \cap \delta\left(C^{n-2}(\mathcal{D}, \mathcal{M})\right)$. As D is $(n-1)$-Cousin-I, it follows that $\{f\} = 0$ in $H^{n-1}(D, \mathcal{O})$. We can take a refinement $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of $\mathcal{D}$ which holds the following two properties:

- $A = A_1 \cup \cdots \cup A_n$, $A_i \cap A_j$ is empty if $i \neq j$.
- $\{U_{\alpha_j}\}_{\alpha_j \in A_j}$ is a Stein covering of D_j and $\mathcal{U}$ is a Stein covering of D.

From $f = 0 \in H^{n-1}(D, \mathcal{O}) \cong H^{n-1}(\mathcal{U}, \mathcal{O})$, it is concluded that there exists $f^{(0)} \in C^{n-2}(\mathcal{U}, \mathcal{O})$ such that $f = \delta\left(f^{(0)}\right)$. The function $h(x)$ is holomorphic on $T_3 \cap D_2 \cap \cdots \cap D_n$. By setting

$$h^{(0)} = \begin{cases} -h & \text{on } T_3 \cap U_{\nu_2} \cap \cdots \cap U_{\nu_n},\ \nu_j \in A_j \ (j = 2, \dots, n), \\ 0 & \text{otherwise,} \end{cases}$$

we can define $h^{(0)} \in C^{n-2}(T_3 \cap \mathcal{U}, \mathcal{O})$. Here $T_3 \cap \mathcal{U} = \{T_3 \cap U_\alpha\}_{\alpha \in A}$ is an open covering of $T_3 \cap D$. The cochain $h^{(0)} + f^{(0)} \in C^{n-2}(T_3 \cap \mathcal{U}, \mathcal{O})$ satisfies that

$$g = \delta\left(h^{(0)} + f^{(0)}\right) \ \text{on } T_3 \cap U_{\nu_1} \cap \cdots \cap U_{\nu_n},$$

where $\nu_j \in A_j \ (j = 1, \dots, n)$. Thus $g = 0$ in $H^{n-1}(T_3 \cap \mathcal{U}, \mathcal{O})$. Since $\mathsf{H}' = \psi|_U^{-1} \circ \varphi(\mathsf{H}) \subset D \cap U$, it follows that $g|_{\mathsf{H}'} = 0$. This contradicts Lemma 5.2. □

Corollary 5.1 (Kajiwara–Kazama [13, Corollary 3] and Berg [5, Corollary])**.** *Let X be an 2-dimensional Stein manifold and D an open set in X. Then D is Stein if and only if D is Cousin-I.*

6 A new proof of theorem of Eastwood–Vigna Suria

In this section, we shall extend Theorem 5.1. Moreover we give a new proof of theorem of Eastwood–Vigna Suria. Firstly, we state this theorem.

Theorem (Eastwood–Vigna Suria [7, Theorem 3.8])**.** *Let D be an open set in an n-dimensional Stein manifold and q an integer with $1 \leq q \leq n$. If D satisfies $H^k(D, \mathcal{O}) = 0$ for every $k = q, \ldots, n-1$, then D is pseudoconvex of order $n-q$.*

For our purposes, we introduce two lemmata. The original two lemmata were proved by Abe [2]. Here we shall prove in an intermediate case. Let D be an open set in X. Let D_1 and D_2 be open sets in D. If $n \geq 3$, then we take $w_3, \ldots, w_n \in \mathcal{O}(D)$ and put $D_\nu = \{w_\nu \neq 0\}$ for $3 \leq \nu \leq n$. In addition, we assume that $D = \bigcup_{\nu=1}^{n} D_\nu$. Let $h \in \mathcal{O}(D_1 \cap D_2)$. Then we can define

$$\eta = \frac{h}{w_3 \cdots w_n} \in Z^{n-1}(\{D_\nu\}_{\nu=1}^{n}, \mathcal{O})$$

For any $2 \leq s \leq n-1$ and $3 \leq k_1 < \cdots < k_{s-1} \leq n$, let $\eta^{(k_1 \cdots k_{s-1})}_{\nu_1 \cdots \nu_{n-s+1}} \in C^{n-s}(\{D_\nu\}_{\nu=1}^{n}, \mathcal{O})$ be the cochain defined by

$$\eta^{(k_1 \cdots k_{s-1})}_{\nu_1 \cdots \nu_{n-s+1}} = \begin{cases} (-1)^{(s-1)+k_1+\cdots+k_{s-1}} \frac{h}{w_{\nu_3} \cdots w_{\nu_{n-s+1}}} & \text{if } \{\nu_1, \ldots, \nu_{n-s+1}, k_1, \ldots, k_{s-1}\} = \{1, \ldots, n\}, \\ 0 & \text{otherwise,} \end{cases}$$

on $D_{\nu_1} \cap \cdots \cap D_{\nu_{n-s+1}}$, where $1 \leq \nu_1 < \cdots < \nu_{n-s+1} \leq n$.

Lemma 6.1 (Abe [2, Lemma 5.1])**.** *For every $2 \leq s \leq n-1$ and $3 \leq k_1 < \cdots < k_{s-1} \leq n$, then we have that*

$$\delta\eta^{(k_1 \cdots k_{s-1})} = \sum_{j=1}^{s-1} (-1)^{j-1} w_{k_j} \eta^{(k_1 \cdots \hat{k}_j \cdots k_{s-1})}.$$

Proof. This lemma was proved by Abe [2]. For reader's convenience, we present a proof of this lemma. Take arbitrary numbers $1 \leq \nu_1 < \cdots < \nu_{n-s+2} \leq n$. Firstly we consider the case where $\{\nu_1, \ldots, \hat{\nu}_i, \ldots, \nu_{n-s+2}, k_1, \ldots, k_{s-1}\} \subsetneq \{1, \ldots, n\}$ for every $i \in \{1, \ldots, n-s+2\}$. Since $\eta^{(k_1 \cdots k_{s-1})}_{\nu_1 \cdots \hat{\nu}_i \cdots \nu_{n-s+2}} = 0$ for every $i \in \{1, \ldots, n-s+2\}$, we have that

$$\left(\delta\eta^{(k_1 \cdots k_{s-1})}\right)_{\nu_1 \cdots \nu_{n-s+2}} = \sum_{i=1}^{n-s+2} (-1)^i \eta^{(k_1 \cdots k_{s-1})}_{\nu_1 \cdots \hat{\nu}_i \cdots \nu_{n-s+2}} = 0$$

on $D_{\nu_1} \cap \cdots \cap D_{\nu_{n-s+2}}$.

If there exists $q \in \{1, \ldots, s-1\}$ such that $\{\nu_1 \ldots, \nu_{n-s+2}, k_1, \ldots, \hat{k}_q, \ldots, k_{s-1}\} = \{1, \ldots, n\}$, then there exists $p \in \{1, \ldots, n-s+2\}$ such that $\nu_p = k_q$ and we have $\{\nu_1, \ldots, \hat{\nu}_p, \ldots, \nu_{n-s+2}, k_1, \ldots, k_{s-1}\}$

$= \{1,\ldots,n\}$. This is a contradiction. It follows that $\{\nu_1,\ldots,\nu_{n-s+2},k_1,\ldots,\hat{k}_j,\ldots,k_{s-1}\} \subsetneq \{1,\ldots,n\}$ for every $j \in \{1,\ldots,s-1\}$ and therefore we have that

$$\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}_{\nu_1\cdots\nu_{n-s+2}} = 0 = \left(\delta\eta^{(k_1\cdots k_{s-1})}\right)_{\nu_1\cdots\nu_{n-s+2}}$$

on $D_{\nu_1}\cap\cdots\cap D_{\nu_{n-s+2}}$.

Next we consider the case where there exists $p \in \{1,\ldots,n-s+2\}$ such that $\{1,\ldots,n\} = \{\nu_1,\ldots,\hat{\nu}_p,\ldots,\nu_{n-s+2},k_1,\ldots,k_{s-1}\}$. Then there exists $q \in \{1,\ldots,s-1\}$ such that $\nu_p = k_q$. If $i \neq p$, then $\{\nu_1,\ldots,\hat{\nu}_i,\ldots,\nu_{n-s+2},k_1,\ldots,k_{s-1}\} \subsetneq \{1,\ldots,n\}$ and therefore $\eta^{(k_1\cdots k_{s-1})}_{\nu_1\cdots\hat{\nu}_i\cdots\nu_{n-s+2}} = 0$. It follows that

$$\begin{aligned}\left(\delta\eta^{(k_1\cdots k_{s-1})}\right)_{\nu_1\cdots\nu_{n-s+2}} &= \sum_{i=1}^{n-s+2}(-1)^{i-1}\eta^{(k_1\cdots k_{s-1})}_{\nu_1\cdots\hat{\nu}_i\cdots\nu_{n-s+2}} = (-1)^{p-1}\eta^{(k_1\cdots k_{s-1})}_{\nu_1\cdots\hat{\nu}_p\cdots\nu_{n-s+2}}\\ &= (-1)^{p-1}\frac{(-1)^{(s-1)+k_1+\cdots+k_{s-1}}h}{w_{\nu_3}\cdots\hat{w}_{\nu_p}\cdots w_{\nu_{n-s+2}}}\\ &= (-1)^{(p+s-2)+k_1+\cdots+k_{s-1}}\frac{w_{\nu_p}h}{w_{\nu_3}\cdots w_{\nu_{n-s+2}}}\end{aligned}$$

If $j \neq q$, then $\{\nu_1,\ldots,\nu_{n-s+2},k_1,\ldots,\hat{k}_j,\ldots,k_{s-1}\} \subsetneq \{1,\ldots,n\}$ and therefore $\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}_{\nu_1\cdots\nu_{n-s+2}} = 0$. We can get

$$\begin{aligned}\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}_{\nu_1\cdots\nu_{n-s+2}} &= (-1)^{q-1}w_{k_q}\eta^{(k_1\cdots\hat{k}_q\cdots k_{s-1})}_{\nu_1\cdots\nu_{n-s+2}}\\ &= (-1)^{q-1}w_{\nu_p}\frac{(-1)^{(s-2)+k_1+\cdots+\hat{k}_q+\cdots+k_{s-1}}h}{w_{\nu_3}\cdots w_{\nu_{n-s+2}}}\\ &= (-1)^{(q+s-3)+k_1+\cdots+\hat{k}_q+\cdots+k_{s-1}}\frac{w_{\nu_p}h}{w_{\nu_3}\cdots w_{\nu_{n-s+2}}}\end{aligned}$$

on $D_{\nu_1}\cap\cdots\cap D_{\nu_{n-s+2}}$. Since $k_q = (p-1)+(q-1)+1 = p+q-1$, we have that $\{(p+s-2)+k_1+\ldots+k_{s-1}\} - \{(q+s-3)+k_1+\cdots+\hat{k}_q+\cdots+k_{s-1}\} = p-q+1+k_q = p-q+1+(p+q)-1 = 2p$. So we can obtain

$$\left(\delta\eta^{(k_1\cdots k_{s-1})}\right)_{\nu_1\cdots\nu_{n-s+2}} = \sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}_{\nu_1\cdots\nu_{n-s+2}}$$

on $D_{\nu_1}\cap\cdots\cap D_{\nu_{n-s+2}}$. □

Let q be an integer with $1 \leq q \leq n-3$ and $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ a Stein open covering of D which is a refinement of $\{D_\nu\}_{\nu=1}^n$. Let $\alpha : \Lambda \to \{1,\dots,n\}$ be a map such that $U_\lambda \subset D_{\alpha(\lambda)}$ for any $\lambda \in \Lambda$. Then α indues the canonical homomorphisms

$$\alpha^* : C^k(\{D_\nu\}_{\nu=1}^n, \mathcal{O}) \to C^k(\mathcal{U}, \mathcal{O})$$

for any $k \geq 0$.

Next, we assume that $H^k(D, \mathcal{O}) = 0$ for every $k = q+1, \dots, n-2$. The following lemma is an intermediate version of lemma of Abe [2].

Lemma 6.2 (cf. Abe [2, Lemma 5.2])**.** *Assume that $F^{(k_0)} = \alpha^*(\eta)$ is trivial in $H^{n-1}(\mathcal{U}, \mathcal{O})$. Then there exist cochains $f^{(k_1\cdots k_{s-1})} \in C^{n-s-1}(\mathcal{U}, \mathcal{O}), 1 \leq s \leq n-q-1, q+2 \leq k_1 < \cdots < k_{s-1} \leq n$, and cocycles $F^{(k_1\cdots k_{s-1})} \in Z^{n-s}(\mathcal{U}, \mathcal{O}), 1 \leq s \leq n-q$, which satisfy the following two conditions:*

- *For every $1 \leq s \leq n-q-1$ and $q+2 \leq k_1 < \cdots < k_{s-1} \leq n$, we have that*

$$\delta f^{(k_1\cdots k_{s-1})} = F^{(k_1\cdots k_{s-1})}.$$

- *For every $2 \leq s \leq n-q$ and $q+2 \leq k_1 < \cdots < k_{s-1} \leq n$, we have that*

$$F^{(k_1\cdots k_{s-1})} = -\sum_{j=1}^{s-1}(-1)^{j-1} w_{k_j} f^{(k_1\cdots \hat{k}_j \cdots k_{s-1})} + \alpha^*\left(\eta^{(k_1\cdots k_{s-1})}\right).$$

Proof. Since $F^{(k_0)} = 0$ in $H^{n-1}(\mathcal{U}, \mathcal{O})$, there exists $f = f^{(k_0)} \in C^{n-2}(\mathcal{U}, \mathcal{O})$ such that $\delta f = F^{(k_0)}$. Next we consider the case where $s = 2 \leq n-q$. We put $F^{(k)} = -w_k f + \alpha^*(\eta^{(k)}) \in C^{n-2}(\mathcal{U}, \mathcal{O})$ for every $k \in \{q+2, \dots, n\}$. By Lemma 6.1, we have that $\delta\eta^{(k)} = w_k \eta$ on $D_1 \cap \cdots \cap D_n$. Therefore,

$$\begin{aligned}\delta\left(F^{(k)}\right) &= \delta\left(-w_k f + \alpha^*(\eta^{(k)})\right) = -w_k\delta(f) + \alpha^*\left(\delta\eta^{(k)}\right)\\ &= -w_k F^{(k_0)} + \alpha^*(w_k\eta) = w_k(-F^{(k_0)} + \alpha^*(\eta)) = 0\end{aligned}$$

It follows that $F^{(k)} \in Z^{n-2}(\mathcal{U}, \mathcal{O})$. Now $H^{n-2}(\mathcal{U}, \mathcal{O}) = 0$ therefore there exists $f^{(k)} \in C^{n-3}(\mathcal{U}, \mathcal{O})$ such that $\delta f^{(k)} = F^{(k)}$ for every $k \in \{q+2, \dots, n\}$. Finally we consider the case where $3 \leq s \leq n-q$. By induction hypothesis, we already have $f^{(k_1\cdots k_{t-1})} \in C^{n-t-1}(\mathcal{U}, \mathcal{O})$ and $F^{(k_1\cdots k_{t-1})} \in Z^{n-t}(\mathcal{U}, \mathcal{O})$ for $1 \leq t \leq s-1$ and $q+2 \leq k_1 < \cdots < k_{t-1} \leq n$. Let

$$F^{(k_1\cdots k_{s-1})} = \sum_{j=1}^{s-1}(-1)^{j-1} w_{k_j} f^{(k_1\cdots \hat{k}_j \cdots k_{s-1})} + \alpha^*\left(\eta^{(k_1\cdots k_{s-1})}\right) \in C^{n-s}(\mathcal{U}, \mathcal{O})$$

for $q+2 \leq k_1 < \cdots < k_{s-1} \leq n$.

We have that

$$
\begin{aligned}
\delta F^{(k_1\cdots k_{s-1})} &= -\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\delta f^{(k_1\cdots\hat{k}_j\cdots k_{s-1})} + \alpha^*\left(\delta\eta^{(k_1\cdots k_{s-1})}\right)\\
&= -\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\Big\{-\sum_{i=1}^{j-1}(-1)^{i-1}w_{k_i}f^{(k_1\cdots\hat{k}_i\cdots\hat{k}_j\cdots k_{s-1})}\\
&\quad -\sum_{i=j+1}^{s-1}(-1)^{i-2}w_{k_i}f^{(k_1\cdots\hat{k}_j\cdots\hat{k}_i\cdots k_{s-1})} + \alpha^*\left(\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}\right)\Big\} + \alpha^*\left(\delta\eta^{(k_1\cdots k_{s-1})}\right)\\
&= \sum_{i<j}(-1)^{i+j}w_{k_i}w_{k_j}f^{(k_1\cdots\hat{k}_i\cdots\hat{k}_j\cdots k_{s-1})} - \sum_{j<i}(-1)^{j+i}w_{k_j}w_{k_i}f^{(k_1\cdots\hat{k}_j\cdots\hat{k}_i\cdots k_{s-1})}\\
&\quad -\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\alpha^*\left(\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}\right) + \alpha^*\left(\delta\eta^{(k_1\cdots k_{s-1})}\right)\\
&= \alpha^*\left(-\sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})} + \delta\eta^{(k_1\cdots k_{s-1})}\right)
\end{aligned}
$$

Since $\delta\eta^{(k_1\cdots k_{s-1})} = \sum_{j=1}^{s-1}(-1)^{j-1}w_{k_j}\eta^{(k_1\cdots\hat{k}_j\cdots k_{s-1})}$ by Lemma 6.1, we have that $\delta F^{(k_1\cdots k_{s-1})} = 0$. It follows that $F^{(k_1\cdots k_{s-1})} \in Z^{n-s}(\mathcal{U},\mathcal{O})$. If $3 \le s \le n-q-1$, then we have that $H^{n-s}(\mathcal{U},\mathcal{O}) = 0$ and therefore there exists $f^{(k_1\cdots k_{s-1})} \in C^{n-s-1}(\mathcal{U},\mathcal{O})$ such that $\delta f^{(k_1\cdots k_{s-1})} = F^{(k_1\cdots k_{s-1})}$. □

Theorem 6.1. *Let X be an n-dimensional Stein manifold, q an integer such that $1 \le q \le n$ and D an open subset of X. If D satisfies the following two conditions:*

- *D is $(n-1)$-Cousin-I.*
- *$H^k(D,\mathcal{O}) = 0$ for every $k = q,\dots,n-2$.*

Then D is pseudoconvex of order $n-q$.

Proof. In the case where $n = q$, the assertion is trivial. So we can assume that $1 \le q \le n-1$. To obtain a contradiction, suppose that D is not pseudoconvex of order $n-q$. There exists a point $x_0 \in \partial D$ such that for any neighborhood U of x_0, then $D \cap U$ is not pseudoconvex of order $n-q$ in U. Since X is Stein, we can take holomorphic functions $\psi_j \in \mathcal{O}(X)$ $(j = 1,\dots,n)$ which satisfies the following two conditions:

- $\psi_j(x_0) = 0$ $(j = 1,\dots,n)$
- The family $\{\psi_1,\dots,\psi_n\}$ forms a coordinate system in the connected component U of $\{x \in X\ ;\ |\psi_j(x)| < K\ (j = 1,\dots,n)\}$ containing x_0 for some $K > 0$.

We define a holomorphic mapping $\psi : X \to \mathbb{C}^n, x \mapsto (\psi_1(x), \dots, \psi_n(x))$, then ψ maps U biholomorphically onto $\{z \in \mathbb{C}^n\,;\, |z_j| < K\ (j = 1, \dots, n)\}$. By Theorem 3.1, there exist a biholomorphic map $\varphi : \mathbb{C}^n \to \mathbb{C}^n, (w_1, \dots, w_n) \mapsto (z_1, \dots, z_n)$, $\varepsilon > 0$, and $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ such that $\varphi(\mathsf{H}_q(2\varepsilon)) \subset \psi(U \cap D)$, $\varphi(\mathsf{P}_n(0, 2\varepsilon)) \subset \psi(U)$, $a_1 = a_2 = \cdots = a_q = a_{q+2} = \cdots = a_n = 0$, $1 \le |a_{q+1}| \le 1 + 2\varepsilon$ and $\varphi(a) \notin \psi(U \cap D)$. Here

$$\begin{aligned}\mathsf{H}_q(2\varepsilon) = &\big\{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q}\,;\, 1 - 2\varepsilon < |\zeta_1| < 1 + 2\varepsilon\ , |\zeta_2| < 1 + 2\varepsilon\big\}\\ &\cup \big\{(\zeta_1, \zeta_2) \in \mathbb{C}^q \times \mathbb{C}^{n-q}\,;\, |\zeta_1| < 1,\ |\zeta_2| < 1\big\}\,.\end{aligned}$$

We can put $\phi = \varphi^{-1} \circ \psi : X \to \mathbb{C}^n, x \mapsto (w_1(x), w_2(x), \dots, w_n(x))$. The family $\{w_1, \cdots, w_n\}$ forms a coordinate system in U. Moreover we have $\mathsf{H}_q(2\varepsilon) \subset \phi(U \cap D)$, $\mathsf{P}_n(0, 2\varepsilon) \subset \phi(U)$ and $a \notin \phi(U \cap D)$. By Proposition 4.1 and Proposition 4.2, we can take open sets T_1, T_2, T_3, T_4, an open covering $\mathcal{D}$ of D and $\rho \in \mathcal{M}(T_2)$. Let f, g and h be functions defied by

$$\begin{aligned} f &= \frac{\rho}{w_2 \cdot w_3 \cdots w_q \cdot (w_{q+1} - a_{q+1}) \cdot w_{q+2} \cdots w_n},\\ g &= \frac{1}{w_1 \cdot w_2 \cdots w_q \cdot (w_{q+1} - a_{q+1})},\\ h &= \frac{\rho_3}{w_2 \cdot w_3 \cdots w_q \cdot (w_{q+1} - a_{q+1})}.\end{aligned}$$

We can define $f \in Z^{n-1}(\mathcal{D}, \mathcal{O})$ and get $f = 0 \in H^{n-1}(D, \mathcal{M})$. Since $H^{n-1}(D, \mathcal{O}) \to H^{n-1}(D, \mathcal{M})$ is injective, so we have that f is trivial in $H^{n-1}(D, \mathcal{O})$. We can take a refinement $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{D}$ which holds the following two properties:

- $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$, $\Lambda_i \cap \Lambda_j$ is empty if $i \ne j$.
- $\{U_{\lambda_j}\}_{\lambda_j \in \Lambda_j}$ is a Stein covering of D_j and $\mathcal{U}$ is a Stein covering of D.

By applying Lemma 6.2, we can take $F^{(q+2\cdots n)}_{\nu_1 \cdots \nu_{q+1}} \in Z^q(\mathcal{U}, \mathcal{O})$. Since $H^q(\mathcal{U}, \mathcal{O}) = 0$, there exists $f_{\nu_1 \cdots \nu_q} \in C^{q-1}(\mathcal{U}, \mathcal{O})$ such that $F^{(q+2\cdots n)}_{\nu_1 \cdots \nu_{q+1}} = \delta(f_{\nu_1 \cdots \nu_q})$. By a simple calculation,

$$\begin{aligned} F^{(q+2\cdots n)}_{\nu_1 \cdots \nu_{q+1}} &= -\sum_{j=q+2}^{n} (-1)^{j-q-2} w_j f^{(q+2\cdots \hat{j} \cdots n)}_{\nu_1 \cdots \nu_{q+1}} + (-1)^{(n-q-1)+(q+2)+\cdots+n} \frac{\rho}{w_2 \cdots w_q (w_{q+1} - a_{q+1})}\\ &= \sum_{j=1}^{q+1} (-1)^{j-1} f_{\nu_1 \cdots \hat{\nu}_j \cdots \nu_{q+1}}, \end{aligned}$$

on $U_{\nu_1} \cap \cdots \cap U_{\nu_{q+1}}$. By putting $L_q = \{w_{q+2} = \cdots = w_n = 0\} \subset X$, we obtain

$$\begin{aligned} \sum_{j=1}^{q+1} (-1)^{j-1} f_{\nu_1 \cdots \hat{\nu}_j \cdots \nu_{q+1}} &= (-1)^{n-q+\cdots+n+1} \frac{\rho}{w_2 \cdots w_q (w_{q+1} - a_{q+1})}\\ &= (-1)^{n-q+\cdots+n+1} (h + g) \end{aligned}$$

on $L_q \cap T_3 \cap U_{\nu_1} \cap \cdots \cap U_{\nu_{q+1}}$, where $\nu_j \in \Lambda_j$ $(j = 1, \ldots, q+1)$. The set $\{D_j \cap L_q\}_{j=1}^{q+1}$ is an open covering of $D \cap T_3 \cap L_q$ and $\mathcal{U} \cap L_q \cap T_3 = \{U_{\nu_j} \cap L_q \cap T_3\,;\, \nu_j \in \Lambda_j,\ j = 1, \ldots, q+1\}$ is a refinement of it. By setting

$$h^{(0)}(x) = \begin{cases} -h(x) & \text{if } x \in T_3 \cap U_{\nu_2} \cap \cdots \cap U_{\nu_{q+1}},\ \nu_j \in \Lambda_j\ (j = 2, \ldots, q+1) \\ 0 & \text{otherwise,} \end{cases}$$

we can define $h^{(0)} \in C^{q-1}(\mathcal{U} \cap L_q \cap T_3, \mathcal{O})$, since $h \in \mathcal{O}(T_3 \cap D_2 \cap \cdots \cap D_{q+1})$. And also we can define $g \in Z^q(\mathcal{U} \cap L_q \cap T_3, \mathcal{O})$. In addition, we can get

$$\sum_{j=1}^{q+1} (-1)^{j-1} f_{\nu_1 \cdots \hat{\nu}_j \cdots \nu_{q+1}} + h^{(0)} = g$$

on $T_3 \cap U_{\nu_1} \cap \cdots \cap U_{\nu_{q+1}}$, where $\nu_j \in \Lambda_j$ $(j = 1, \ldots, q+1)$. It follows that $g = 0$ in $H^q(T_3 \cap \mathcal{U} \cap L_q, \mathcal{O})$. Recall that $\mathsf{H}_q(2\varepsilon) \subset \phi(U \cap D)$, so we can get $g = 0$ in $H^q(\phi|_U^{-1}(\mathsf{H}_q(2\varepsilon)) \cap \mathcal{U} \cap L_q, \mathcal{O})$. This contradicts Lemma 5.2. Thus D is pseudoconvex of order $n - q$. □

Corollary 6.1. *Let X be an n-dimensional Stein manifold and D an open set in X. Then D is Stein if and only if D satisfies the following two conditions:*

- *D is $(n-1)$-Cousin-I.*
- *$H^k(D, \mathcal{O}) = 0$ for every $k = 1, \ldots, n-2$.*

Corollary 6.2 (Eastwood–Vigna Suria [7, Theorem 3.8])**.** *Let X be an n-dimensional Stein manifold, q an integer with $1 \leq q \leq n$ and D an open set in X. If D satisfies $H^k(D, \mathcal{O}) = 0$ for every $k = q, \ldots, n-1$, then D is pseudoconvex of order $n - q$.*

Acknowledgments

Firstly, I would like to express the deepest appreciation to Professor Makoto Abe, who introduced me to the theory of analytic function of several complex variables, gave me helpful advices and carefully read my papers. I would like to thank Professor Masafumi Yoshino for giving me helpful and valuable comments. I also would like to thank Professor Tetsu Shimomura, who provided a good chance to talk about my results at the seminar on potential theory at Hiroshima University, and gave me helpful suggestions. I would like to show my appreciation to Professor Kentaro Hirata, who taught me potential theory and carefully read my papers. I am grateful to Professor Tadashi Shima for reading the manuscript of my papers and giving me valuable suggestions. Finally, I would like to thank Professor Yoshihiro Mizuta for giving me valuable comments at the seminar on potential theory at Hiroshima University.

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