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Relation	

A Characterization of Subpluriharmonicity for a Function of Several Complex Variables

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Abstract

We give a characterization of a subpluriharmonic function of several complex variables in the sense of Fujita (J. Math. Kyoto Univ., 30:637–649, 1990) by using polynomial functions of degree at most two.

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1. Introduction

Let D be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. According to Fujita [2], we say that u is subpluriharmonic if for every relatively compact domain G in D and for every real-valued pluriharmonic function h defined near $\overline{G}$, the inequality $u \leq h$ on ∂G implies the inequality $u \leq h$ on $\overline{G}$. If $n = 1$, then an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where D is an open set of $\mathbb{C} = \mathbb{R}^2$, is subpluriharmonic if and only if u is subharmonic.

By Yasuoka [9, Theorem 1], an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where D is an open set of $\mathbb{C}$, is subharmonic if and only if for every open disk B relatively compact in D and for every polynomial $P(z)$ of a complex variable z of degree at most two, the inequality $u(z) \leq \Re(P(z))$ on ∂B implies the inequality $u(z) \leq \Re(P(z))$ on $\overline{B}$.

In this paper, we generalize this fact to several complex variables. That is to say, we prove that an upper semicontinuous function $u : D \to [-\infty, +\infty)$, where D is an open set of $\mathbb{C}^n$, is subpluriharmonic if and only if for every open ball B relatively compact in D and for every polynomial $P(z_1, z_2, \ldots, z_n)$ of n complex variables $z_1, z_2, \ldots, z_n$ of degree at most two, the inequality $u(z) \leq \Re(P(z))$ on ∂B implies the inequality $u(z) \leq \Re(P(z))$ on $\overline{B}$, where $z = (z_1, z_2, \ldots, z_n)$ (see Theorem 3.2).

2. Preliminaries

Let $N \in \mathbb{N}$ and let $\mathbf{F}$ denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbf{F}^N$, that is,

$$\|x\| := \left(\sum_{\nu=1}^{N} |x_\nu|^2\right)^{1/2}$$

for every $x = (x_1, x_2, \ldots, x_N) \in \mathbf{F}^N$. For every $c \in \mathbf{F}^N$ and for every $r \in (0, +\infty]$, the set

$$\mathbf{B}(c, r) := \{x \in \mathbf{F}^N \mid \|x - c\| < r\}$$

is said to be the *open ball* of *radius* r with *center* c in $\mathbf{F}^N$. For every point $c \in \mathbf{F}^N$ and for every subset E of $\mathbf{F}^N$, the number

$$\operatorname{dist}(c, E) := \inf\{\, \|x - c\| \mid x \in E \,\}$$

is said to be the *distance* from c to E.

We denote by $z_1, z_2, \dots, z_n$ the complex coordinates of $\mathbb{C}^n$. Let $N \in \mathbb{N}$ and let D be an open set of $\mathbb{C}^n$. A $\mathcal{C}^2$ function $u : D \to \mathbb{R}$ is said to be *pluriharmonic* if

$$\frac{\partial^2 u}{\partial z_\mu \partial \overline{z_\nu}} = 0$$

on D for every $\mu, \nu = 1, 2, \dots, n$ (see, for instance, Fritzsche-Grauert [1, p. 318]). An upper semicontinuous function $u : D \to [-\infty, +\infty)$ is said to be *subpluriharmonic* if for every relatively compact open set G of D and for every pluriharmonic function h defined near $\overline{G}$, the inequality $u \leq h$ on ∂G implies the inequality $u \leq h$ on $\overline{G}$ (cf. Fujita [2, 3]). As is noted in Fujita [2, p. 638] (see also Fujita [3, Proposition 2]), the subpluriharmonic functions on D exactly coincide with the $(n-1)$-plurisubharmonic functions on D in the sense of Hunt-Murray [5, Definition 2.3].

By the second statement of Słodkowski [6, Lemma 4.4], we have the following proposition (see also Sugiyama [8, Proposition 2.1]).

Proposition 2.1 (Słodkowski**).** Let D be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. If u is not subpluriharmonic on D, then there exist $c \in D$, $r \in \big(0, \operatorname{dist}(c, \partial D)\big)$, $K > 0$, and a function f holomorphic near $\overline{\mathbf{B}(c, r)}$ such that $u(c) = \Re(f(c))$ and

$$u \leq \Re(f) - K\|z - c\|^2$$

on $\overline{\mathbf{B}(c, r)}$.

3. Results on Subpluriharmonic Functions

We denote by $\mathbb{C}[z_1, z_2, \dots, z_n]$ the algebra of polynomial functions of n complex variables $z_1, z_2, \dots, z_n$ with coefficients in $\mathbb{C}$. We have the following lemma which refines Proposition 2.1.

Lemma 3.1. Let D be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. If u is not subpluriharmonic on D, then there exist $c \in D$, $r \in (0, \operatorname{dist}(c, \partial D))$, $K > 0$, and $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$ with $\deg P \leq 2$ such that $u(c) = \Re(P(c))$ and

$$u \leq \Re(P) - K\|z - c\|^2$$

on $\overline{\mathbf{B}(c, r)}$.

Proof. By Proposition 2.1, there exist $c \in D$, $R \in \big(0, \operatorname{dist}(c, \partial D)\big)$, $L > 0$, and a function f holomorphic near $\overline{\mathbf{B}(c, R)}$ such that $u(c) = \Re\big(f(c)\big)$ and

$$u \leq \Re(f) - L\|z - c\|^2$$

on $\overline{\mathbf{B}(c, R)}$. Let

$$f(z) = \sum_{\alpha} a_\alpha (z - c)^\alpha$$

be the Taylor expansion of $f(z)$ near $\overline{\mathbf{B}(c, R)}$, where $z = (z_1, z_2, \dots, z_n)$ (see, for instance, Fritzsche-Grauert [1, p. 24]). Let

$$P(z) := \sum_{|\alpha| \leq 2} a_\alpha (z - c)^\alpha$$

for every $z \in \mathbb{C}^n$ and let

$$Q(z) := \sum_{|\alpha| \geq 3} a_\alpha (z - c)^\alpha$$

for every $z \in \overline{\mathbf{B}(c, R)}$. Then, we have $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$, $\deg P \leq 2$, $f = P + Q$ on $\overline{\mathbf{B}(c, R)}$ and there exists $M > 0$ such that

$$|Q(z)| \leq M\|z - c\|^3$$

on $\overline{\mathbf{B}(c, R)}$. Take an arbitrary $K \in (0, L)$. For any $r \in (0,\ \min\{R, (L - K)/M\}]$, we have

$$\begin{aligned}
&\Re(P) - K\|z - c\|^2 \\
&= \Re(f) - \Re(Q) - K\|z - c\|^2 \\
&\geq u + L\|z - c\|^2 - M\|z - c\|^3 - K\|z - c\|^2 \\
&= u + (L - K - M\|z - c\|)\,\|z - c\|^2 \\
&\geq u
\end{aligned}$$

on $\overline{\mathbf{B}(c, r)}$. On the other hand, we have that $\Re\big(P(c)\big) = \Re\big(f(c)\big) = u(c)$. ■

We have the following theorem, which generalizes Yasuoka [9, Theorem 1] to several complex variables and also refines the first statement of Słodkowski [6, Lemma 4.4].

Theorem 3.2. Let D be an open set of $\mathbb{C}^n$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following two conditions are equivalent.

(1) u is subpluriharmonic.

(2) For every $c \in D$, there exists $R \in (0, \operatorname{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$ with $\deg P \le 2$, the inequality $u \le \Re(P)$ on $\partial \mathbf{B}(c, r)$ implies the inequality $u \le \Re(P)$ on $\overline{\mathbf{B}(c, r)}$.

Proof. (1) → (2). Since the real part of a holomorphic function is pluriharmonic, the assertion follows.

(2) → (1). Suppose that u is not subpluriharmonic. Take an arbitrary $R > 0$. Then, by Lemma 3.1, there exist $c \in D$, $r \in (0, \min\{R, \operatorname{dist}(c, \partial D)\})$, $K > 0$, and $Q \in \mathbb{C}[z_1, z_2, \dots, z_n]$ with $\deg Q \le 2$ such that $u(c) = \Re(Q(c))$ and

$$u \le \Re(Q) - K\|z - c\|^2$$

on $\overline{\mathbf{B}(c, r)}$. Then,

$$P := Q - Kr^2 \in \mathbb{C}[z_1, z_2, \dots, z_n],$$

$\deg P \le 2$, and $u \le \Re(Q) - Kr^2 = \Re(P)$ on $\partial \mathbf{B}(c, r)$ while $u(c) > \Re(P(c))$, which is a contradiction. ■

Corollary 3.3 (cf. Yasuoka [9, Theorem 1]**).** Let D be an open set of $\mathbb{C}$ and let $u : D \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following two conditions are equivalent.

(1) u is subharmonic.

(2) For every $c \in D$, there exists $R \in (0, \operatorname{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{C}[z]$ with $\deg P \le 2$, the inequality $u \le \Re(P)$ on $\partial \mathbf{B}(c, r)$ implies the inequality $u \le \Re(P)$ on $\overline{\mathbf{B}(c, r)}$.

Remark 3.4. For a related characterization of pseudoconvexity of a domain in $\mathbb{C}^n$ or over $\mathbb{C}^n$ by using polynomial functions of degree at most two, see Sugiyama [7, Theorem 3.1] as well as Yasuoka [9, Theorem 2].

Remark 3.5. As Example 3.6 below shows, we cannot replace condition (2) in Theorem 3.2 by the following condition (2)'.

(2)' For every $c \in D$, there exists $R \in (0, \operatorname{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{C}[z_1, z_2, \dots, z_n]$ with $\deg P \le 1$, the inequality $u \le \Re(P)$ on $\partial \mathbf{B}(c, r)$ implies the inequality $u \le \Re(P)$ on $\overline{\mathbf{B}(c, r)}$.

Example 3.6. Let $n \in \mathbb{N}$ and let

$$u(z) := {x_1}^2 - 2{y_1}^2 - \sum_{\nu=2}^{n} |z_\nu|^2$$

for every $z \in D = \mathbb{C}^n$, where $z_1 = x_1 + \mathrm{i} y_1$. Then, u is not subpluriharmonic while u satisfies condition (2)' in Remark 3.5.

Proof. By Fujita [2, Proposition 5], a C^2 function is subpluriharmonic if and only if its complex Hessian matrix has at least one nonnegative eigenvalue at any point. Since

$$\left(\frac{\partial^2 u}{\partial z_\mu \partial \overline{z_\nu}}\right) = \begin{pmatrix} -\frac{1}{2} & & & O \\ & -1 & & \\ & & \ddots & \\ O & & & -1 \end{pmatrix},$$

the function u is not subpluriharmonic on $\mathbb{C}^n$. Let G be an arbitrary relatively compact open set of $\mathbb{C}^n$. Let

$$P(z) := \sum_{\nu=1}^{n} c_\nu z_\nu + d,$$

where $c_1, c_2, \dots, c_n, d \in \mathbb{C}$, and assume that $u \le \Re(P)$ on ∂G. Let

$$\mu : \mathbb{C}^n \to \mathbb{C}^n, \quad z \mapsto \left(x_1 + \mathrm{i} \cdot \frac{y_1}{\sqrt{2}}, z_2, \dots, z_n\right),$$

which is an $\mathbb{R}$-linear isomorphism of $\mathbb{C}^n$. Since

$$\left(\frac{\partial^2(u\circ\mu)}{\partial z_\mu \partial \overline{z_\nu}}\right) = \begin{pmatrix} 0 & & & O \\ & -1 & & \\ & & \ddots & \\ O & & & -1 \end{pmatrix},$$

the function $u\circ\mu$ is subpluriharmonic on $\mathbb{C}^n$. On the other hand, the function

$$(\Re(P)\circ\mu)(z) = \Re\left\{\left(a_1 + \mathrm{i}\cdot\frac{b_1}{\sqrt{2}}\right)\cdot z_1 + \sum_{\nu=2}^{n} c_\nu z_\nu + d\right\},$$

where $c_1 = a_1 + \mathrm{i}b_1$, is pluriharmonic on $\mathbb{C}^n$. Since $u\circ\mu \le \Re(P)\circ\mu$ on $\mu^{-1}(\partial G) = \partial(\mu^{-1}(G))$, we have that $u\circ\mu \le \Re(P)\circ\mu$ on $\overline{\mu^{-1}(G)} = \mu^{-1}(\overline{G})$. Thus, we obtain the inequality $u \le \Re(P)$ on $\overline{G}$ and, therefore, u satisfies condition (2)' in Remark 4.3. ■

4. Corresponding Facts for Subharmonic Functions

We denote by $x_1, x_2, \dots, x_N$ the real coordinates of $\mathbb{R}^N$. Let D be an open set of $\mathbb{R}^N$. A $\mathcal{C}^2$ function $u: D \to \mathbb{R}$ is said to be *harmonic* if $\Delta h = 0$ on D. An upper semicontinuous function $u: D \to [-\infty, +\infty)$ is said to be *subharmonic* if for every relatively compact open set G of D and for every harmonic function h defined near $\overline{G}$, the inequality $u \le h$ on ∂G implies the inequality $u \le h$ on $\overline{G}$ (cf. Hörmander [4, p. 141]). Our definition of subharmonic functions does not exclude the function $u \equiv -\infty$.

We denote by $\mathbb{R}[x_1, x_2, \dots, x_n]$ the algebra of polynomial functions of n real variables $x_1, x_2, \dots, x_n$ with coefficients in $\mathbb{R}$. By Hörmander [4, p. 147], we have the following lemma.

Lemma 4.1 (Hörmander**).** Let D be an open set of $\mathbb{R}^N$ and let $u: D \to [-\infty, +\infty)$ be an upper semicontinuous function. If u is not subharmonic, then there exist $c \in D$, $R \in (0, \mathrm{dist}(c, \partial D))$, and $P \in \mathbb{R}[x_1, x_2, \dots, x_N]$ with $\deg P \le 2$ such that $u(c) = P(c)$, $\Delta P < 0$, and $u \le P$ on $\overline{\mathbf{B}(c, R)}$.

We have the following characterization of the subharmonic functions of several real variables, which resembles Theorem 3.2.

Theorem 4.2. Let D be an open set of $\mathbb{R}^N$ and let $u: D \to [-\infty, +\infty)$ be an upper semicontinuous function. Then, the following two conditions are equivalent.

(1) u is subharmonic.

(2) For every $c \in D$, there exists $R \in (0, \mathrm{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{R}[x_1, x_2, \dots, x_N]$ with $\deg P \le 2$ which is harmonic on $\mathbb{R}^N$, the inequality $u \le P$ on $\partial\mathbf{B}(c, r)$ implies the inequality $u \le P$ on $\overline{\mathbf{B}(c, r)}$.

Proof. (1) → (2). The assertion is clear.

(2) → (1). Suppose that u is not subharmonic. Take an arbitrary $R > 0$. Then, by Lemma 4.1, there exist $c \in D$,

$$r \in (0, \min\{R, \mathrm{dist}(c, \partial D)\}),$$

and $Q \in \mathbb{R}[x_1, x_2, \dots, x_N]$ with $\deg Q \le 2$ such that $u(c) = Q(c)$, $\Delta Q < 0$, and $u \le Q$ on $\overline{\mathbf{B}(c, r)}$. Then, $\Delta Q = -2NK$ on $\mathbb{R}^N$ for some constant $K > 0$. Let

$$P := Q + K(\|x - c\|^2 - r^2).$$

Then, $P \in \mathbb{R}[x_1, x_2, \dots, x_N]$, $\deg P \le 2$, $\Delta P = 0$ on $\mathbb{R}^n$, and $P = Q \ge u$ on $\partial\mathbf{B}(c, r)$ although

$$P(c) = Q(c) - Kr^2 = u(c) - Kr^2 < u(c),$$

which is a contradiction. ■

Remark 4.3. As Example 4.4 below shows, if $N \ge 2$, then we cannot replace condition (2) by the following condition (2)' in Theorem 4.2.

(2)' For every $c \in D$, there exists $R \in (0, \mathrm{dist}(c, \partial D))$ such that for every $r \in (0, R]$ and for every $P \in \mathbb{R}[x_1, x_2, \dots, x_N]$ with $\deg P \le 1$, the inequality $u \le P$ on $\partial\mathbf{B}(c, r)$ implies the inequality $u \le P$ on $\mathbf{B}(c, r)$.

Example 4.4. Let $N \ge 2$ and let

$$u(x) := {x_1}^2 - 2{x_2}^2$$

for every $x = (x_1, x_2, \dots, x_N) \in D = \mathbb{R}^N$. Then, u is not subharmonic while u satisfies condition (2)' in Remark 4.3.

Proof. Since $\Delta u = -2$ on $\mathbb{R}^N$, the function u is not subharmonic on $\mathbb{R}^N$ (see, for instance, Hörmander [4, p. 146]). Take an arbitrary relatively compact open set G of $\mathbb{R}^N$ and arbitrary $a_1, a_2, \dots, a_N, b \in \mathbb{R}$. Let

$$P(x) \coloneqq \sum_{k=1}^{N} a_k x_k + b$$

and assume that $u \leq P$ on ∂G. Let

$$\mu : \mathbb{R}^N \to \mathbb{R}^N, \quad x \mapsto \left(x_1, \frac{x_2}{\sqrt{2}}, x_3, \dots, x_N\right),$$

which is an $\mathbb{R}$-linear isomorphism of $\mathbb{R}^N$. Since $(u \circ \mu)(x) = {x_1}^2 - {x_2}^2$, the function $u \circ \mu$ is harmonic on $\mathbb{R}^N$ and therefore is subharmonic on $\mathbb{R}^N$. Since the function

$$(P \circ \mu)(x) = a_1 x_1 + \frac{a_2}{\sqrt{2}} \cdot x_2 + \sum_{k=3}^{N} a_k x_k + b$$

is harmonic on $\mathbb{R}^N$ and satisfies $u \circ \mu \leq P \circ \mu$ on $\mu^{-1}(\partial G) = \partial(\mu^{-1}(G))$, we have that $u \circ \mu \leq P \circ \mu$ on $\overline{\mu^{-1}(G)} = \mu^{-1}(\bar{G})$. It follows that $u \leq P$ on $\bar{G}$ and therefore condition (2)' in Remark 4.3 is satisfied. ■

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