PAPER *Special Section on Theoretical Foundations of Computing*

Computation-Universal Models of Two-Dimensional 16-State Reversible Cellular Automata

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SUMMARY A reversible (or injective) cellular automaton (RCA) is a "backward deterministic" CA, i.e., every configuration of it has at most one predecessor. Margolus has been shown that there is a computation-universal two-dimensional 2-state RCA model. Although his model is very interesting, it differs from a standard CA model because of its somewhat spatial and temporal non-uniformity. In this paper, we present two kinds of simple 16-state computation-universal models using the framework of two-dimensional reversible partitioned CA (PCA). Since PCA can be considered as a subclass of standard CA, we can immediately obtain 16-state standard RCA models from them. For each of these models, we designed a configuration which simulates a Fredkin gate. Since Fredkin gate has been known to be a universal logic element, computation-universality of these two models is concluded.

key words: *cellular automata, reversibility, computation-universality*

1. Introduction

A cellular automaton (CA) is a system consisting infinite number of finite automata (celled cells) connected uniformly in a space. By applying a local (transition) function, which determines the next state of each cell depending on the present states of its neighboring cells, to all the cells in parallel, global transition of the entire state (configuration) of the cell space occurs.

A CA is called reversible if every configuration of it has at most one predecessor. Therefore, in a reversible CA (RCA), one can uniquely retrace its movement. Although RCA satisfies such very strong constraint, computation-universality of these systems has been proved. Toffoli[5] showed that every k-dimensional irreversible CA can be simulated by a $(k+1)$-dimensional reversible one, and thus two-dimensional RCA is computation universal. Morita et al.[3] proved that the dimension of computation universal RCA is reduced to 1 by showing any Turing machine can be simulated by a one-dimensional RCA. However, these computation-universal RCAs need many internal states per cell.

Margolus[2], on the other hand, proposed a two-dimensional 2-state model of RCA, and showed its universality. Thus, in the two-dimensional case, very simple universal RCA can be constructed. Though his model is interesting, it differs from the standard CA formulation because it is slightly non-uniform both in time and in space.

In this paper, we investigate how the number of states of universal RCA in standard formulation can be reduced. To do so, we use a partitioned cellular automaton (PCA)[3] as a subsidiary framework. Each cell of PCA is partitioned into the equal number of parts to the neighborhood size, and the information stored in each part is sent to only one of the neighboring cells. In PCA, the reversibility of a local function is equivalent to that of a global function. Therefore, to design a reversible model in the framework of PCA is easier than to design in the framework of standard CA.

Here we shall give two models of computation-universal two-dimensional 16-state RPCA. Since PCA can be regarded as a subclass of standard CA, universal 16-state standard RCAs are immediately obtained. The local function of the first model satisfies the constraints of conservativity, isotropy, and symmetry as well as reversibility, and has some similarity with Margolus model. The second model is conservative and isotropic but not symmetric.

Margolus[2] showed that Billiard Ball Model (BBM)[1] can be embedded in his reversible cellular space. BBM is a kind of computing model in which logical operations are performed by elastic collisions of balls. On the other hand, it has been shown by Fredkin et al.[1] that "Fredkin gate" can be realized in BBM. Fredkin gate is a 3-input 3-output reversible and bit-conserving logic element, and is universal in the sense that any logic circuit can be constructed by using only Fredkin gates and unit delays. By above, universality of Margolus model is concluded.

In this paper, we also go along this approach. We show how signals (corresponding to balls in BBM) propagate, interact, and bounce in each of our RPCA models, then design some building components, and finally give complete configurations which simulate a Fredkin gate.

2. Two-Dimensional Partitioned Cellular Automata

In a standard cellular automaton (CA), a local

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function determines the next state of each cell depending on the present states of its neighboring cells. Figure 1 shows the case of two-dimensional 4-neighbor CA. In such a case, since the local function maps a combination of four cell states into one cell state, it is very difficult to design a CA whose global function is reversible.

Margolus[(2)] proposed a framework of CA having "Margolus meighborhood" to design a reversible one. In his CA, each cell is bordered by alternating solid and dotted lines as shown in Fig. 2(a), and the local function is applied to a block of 4 cells bordered by solid or dotted lines. At even time step, it is applied to the ones with solid border, and at odd time step, to the ones with dotted border. Figure 2(b) is an example of a local function of 2-state CA, to which Margolus gave a proof of universality. In such a CA, the global function becomes reversible if the local function is so. However, it differs from the framework of standard CA by its temporal and spatial non-uniformity, and this difference makes it a little complicated to obtain a reversible CA model with desired property.

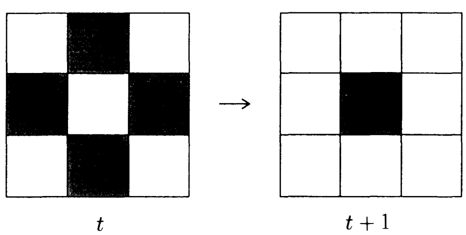


Fig. 1 Domain and range of a local function in a standard 4-neighbor CA.

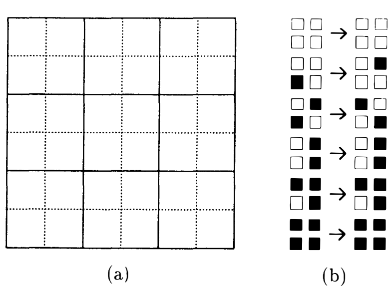


Fig. 2(a) Margolus neighborhood.
(b) The local function of the Margolus model.

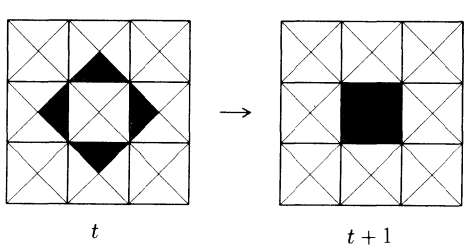


Fig. 3 Domain and range of a local function in a 4-neighbor PCA.

In this paper, to facilitate the clear design of a reversible CA of a standard formulation, we use a framework of partitioned cellular automaton (PCA)[(3)]. In k-neighbor PCA, each cell is partitioned into k parts, and the next state of each cell is determined depending on the present states of k parts taken from k neighboring cells by a specified manner (not depending on the whole states of the neighboring cells). Figure 3 shows a two-dimensional 4-neighbor case.

Definition 2.1: A determinitic two-dimensional 4-neighbor partitioned CA (2PCA(4)) is a system defined by

$$P=(\boldsymbol{Z}^2,\ U,\ R,\ D,\ L,\ f_P)$$

where $\boldsymbol{Z}$ is the set of all integers, U, R, D, and L are non-empty finite sets of up, right, down and left internal states of each cell, and $f_P : D\times L\times U\times R \rightarrow U\times R \times D\times L$ is a mapping called a local function.
A configuration over $U\times R\times D\times L$ (or of P) is a mapping $c : \boldsymbol{Z}^2\rightarrow U\times R\times D\times L$. The set $\{c \mid c : \boldsymbol{Z}^2\rightarrow U\times R\times D\times L\}$ of all configurations of P is denoted by $conf(U\times R\times D\times L)$.

Let "UP" ("RIGHT", "DOWN", "LEFT", respectively) be the projection function which picks out the up (right, down, left) element of a quadruple in $U\times R\times D\times L$. The global function $F_P : conf(U\times R\times D \times L)\rightarrow conf(U\times R\times D\times L)$ of P is defined as follows.

$$\begin{aligned}&F_P(c)(i, j)\\&\quad=f_P(\mathrm{DOWN}(c(i, j+1)),\ \mathrm{LEFT}(c(i+1, j)),\\&\qquad\mathrm{UP}(c(i, j-1)),\ \mathrm{RIGHT}(c(i-1, j)))\\&\qquad\qquad\qquad\qquad(i, j\in\boldsymbol{Z})\end{aligned}$$

Definition 2.2: Let P be a 2PCA(4). P is called locally reversible if f_P is an injection, and is called globally reversible if F_P is an injection.

For 2PCA(4), the following lemmas hold. (Proofs are omitted, since they are essentially the same as in the one-dimensional case[(3)].)

Lemma 2.1: Let P be a 2PCA(4). P is globally reversible iff P is locally reversible.

Thus, in what follows, a globally or locally reversible 2PCA(4) is called simply "reversible" and abbreviated as 2RPCA(4). From this lemma, we can see that in order to obtain a globally reversible 2PCA(4) it is sufficient to design a locally reversible one.

Lemma 2.2: For any 2PCA(4) P, there exists a two-dimensional 4-neighbor (standard) CA whose global transition function is identical with that of P.

This lemma says that 2PCA(4) is a subclass of standard two-dimensional 4-neighbor CA.

3. Fredkin Gate

In this section, we describe several definitions and known results concerning Fredkin gate, which are

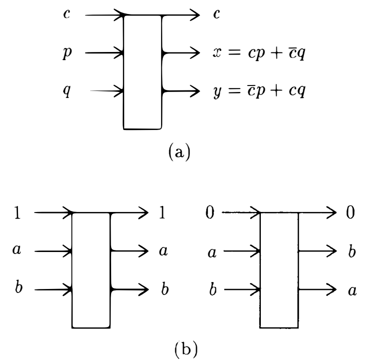


Fig. 4 (a) An F-gate.
(b) The function of an F-gate.

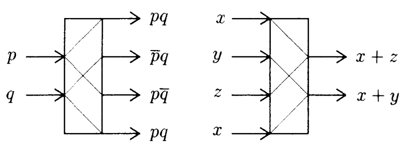


Fig. 5 An I-gate and an inverse I-gate.

needed in the next section (to be more precise see Ref. (1)).

Fredkin gate (F-gate) is a basic element in the theory of Conservative Logic proposed by Fredkin and Toffoli[1]. It is a reversible (i.e., the logical function is injective) and bit-conserving (i.e., the number of 1's is conserved between inputs and outputs) logic gate shown in Fig. 4. It has been known that any combinational logic element (especially AND, OR, NOT, and fan-out element) can be realized only with F-gates. Thus, any sequential circuit can be constructed from F-gates and unit delays.

Fredkin and Toffoli also showed that an F-gate can be embedded in Billiard Ball Model (BBM). It is a reversible and conservative physical model of computation, in which logical operations are performed by elastic collisions of ideal balls. In order to realize an F-gate in BBM, they introduced simpler logic gates called an interaction gate (I-gate) and a switch gate (S-gate) from which an F-gate is constructed.

An I-gate is a 2-input 4-output reversible and bit-conserving gate shown in Fig. 5, and directly realized by a collision of two balls in BBM. The inverse I-gate is a 4-input 2-output gate having the inverse logical function of I-gate. Note that, in the inverse I-gate, the first and the last inputs among four must be the identical value x, and x, y, z must be mutually exclusive (i.e., at most one of them has the logical value 1) in order to keep it reversible.

An S-gate is a 2-input 3-output reversible and bit-conserving gate shown in Fig. 6. An inverse S-gate realizes the inverse function of an S-gate, provided that the inputs satisfy the conditions $cz=0$ and $\bar{c}y=0$.

As in Fig. 7, an S-gate is constructed from an I-gate and an inverse I-gate. An inverse S-gate is realized by the inverse circuit (i.e., the circuit obtained by reversing the directions of arrows) of Fig. 7. Finally, Figure 8 shows how an F-gate is constructed from twos-gates and two inverse S-gates.

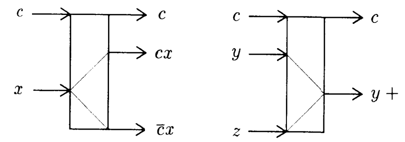


Fig. 6 An S-gate and an inverse S-gate.

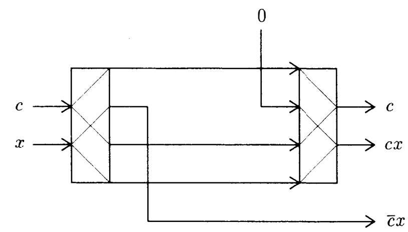


Fig. 7 Construction of an S-gate by an I-gate and an inverse I-gate.

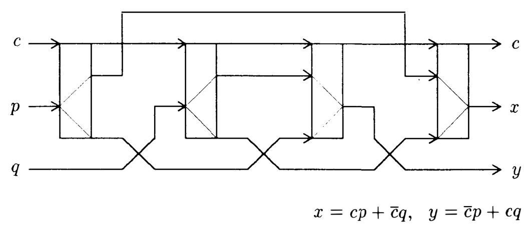


Fig. 8 Construction of an F-gate by S-gates and inverse S-gates.

4. Realization of Fredkin Gates by 16-State 2RPCA (4)

In this section, we give two models (Model 1 and 2) of 16-state 2RPCA (4) which can simulate an F-gate. A 16-state model considered here is a system $P=(\boldsymbol{Z}^2, U, R, D, L, f_P)$ such that $U=R=D=L=\{0, 1\}$. For 16-state 2RPCA(4), we define the notions of conservativity, isotropy, and symmetry as follows. (Note that the notions of isotropy and symmetry can be generalized to k^4-state systems such that $U=R=D=L=\{s_0, \cdots, s_{k-1}\}$.)

Definition 4.1: A 16-state 2PCA(4) P is called conservative if the following condition holds for any (x_0, x_1, x_2, x_3), $(y_0, y_1, y_2, y_3)\in\{0, 1\}^4$ such that $f_P(x_0, x_1, x_2, x_3)=(y_0, y_1, y_2, y_3)$, where "+" is the (ordinary) addition on $\boldsymbol{Z}$.

$$x_0+x_1+x_2+x_3=y_0+y_1+y_2+y_3$$

Definition 4.2: A 16-state 2PCA(4) P is called isotropic if the following condition holds for any (x_0, x_1, x_2, x_3), $(y_0, y_1, y_2, y_3)\in\{0, 1\}^4$ such that $f_P(x_0, x_1, x_2, x_3)=(y_0, y_1, y_2, y_3)$.

$$f_p(x_3, x_0, x_1, x_2) = (y_3, y_0, y_1, y_2)$$

(Intuitively, the above condition says the local function is invariant under the rotation of 90, 180, and 270 degrees.)

Definition 4.3: A 16-state 2PCA(4) P is called symmetric if the following condition holds for any (x_0, x_1, x_2, x_3), $(y_0, y_1, y_2, y_3) \in \{0, 1\}^4$ such that $f_p(x_0, x_1, x_2, x_3) = (y_0, y_1, y_2, y_3)$ and for any $i \in \{0, 1, 2, 3\}$, where "+" is the addition in mod 4.

(1) $x_i = x_{i+2}$ iff $y_i = y_{i+2}$, and

(2) $x_i = x_{i+1}$ and $x_{i+2} = x_{i+3}$ iff $y_i = y_{i+1}$ and $y_{i+2} = y_{i+3}$.

(Intuitively, the above condition says the patterns (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) have the same set of axes of symmetry.)

4.1 Model 1

The local function of Model 1 is shown in Fig. 9, where the states 0 and 1 in each part are represented by white and black, respectively. This model satisfies the conditions of conservativity, isotropy, and symmetry besides reversibility. We now construct a configuration having the function of an F-gate. In this model, a signal (corresponding to a ball in BBM) consists of two black parts as shown in Fig. 10, and goes straight if no obstacle exists. A stable block is a square consisting of eight black parts which remains unchanged (Fig. 11). Two consecutive stable blocks play a role of a mirror which reflects a signal as shown in Fig. 12. An I-gate is realized by colliding two signals at right angles at some place (Fig. 13). Note that the I-gate configuration itself has no black parts except those of signals. An inverse I-gate is obtained by supplying the input signals from the output positions in the I-gate

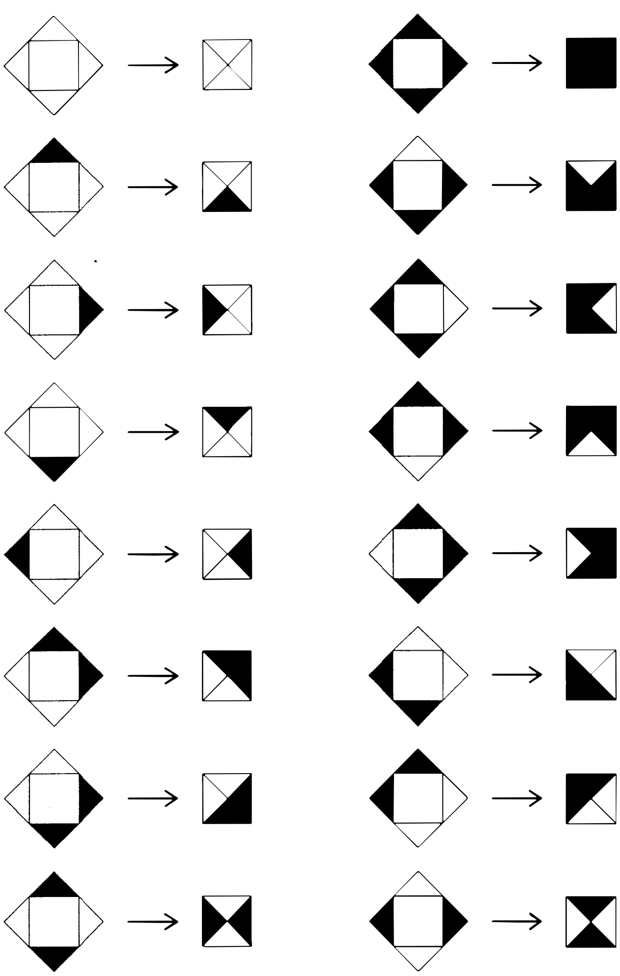


Fig. 9 The local function of 2RPCA(4) Model 1.

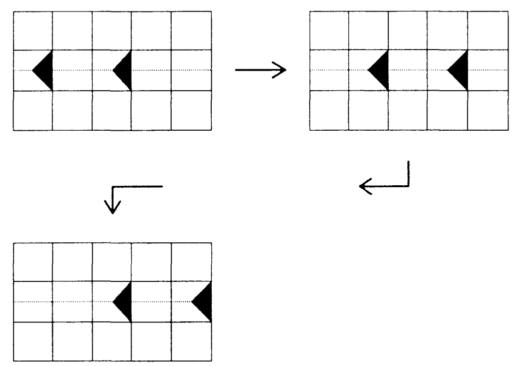


Fig. 10 Propagation of a signal in Model 1.

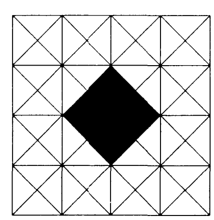


Fig. 11 A stable block in Model 1.

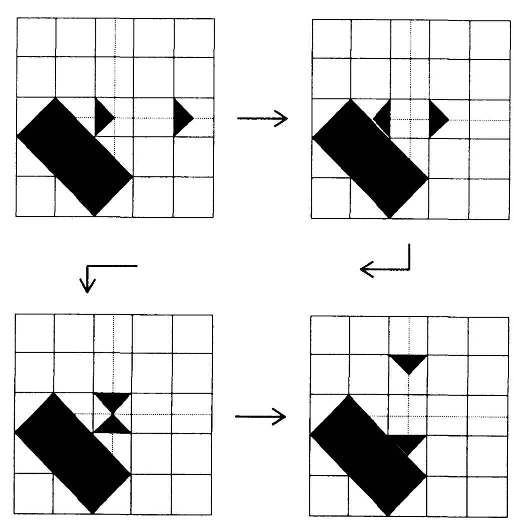


Fig. 12 Reflection of a signal by a mirror in Model 1.

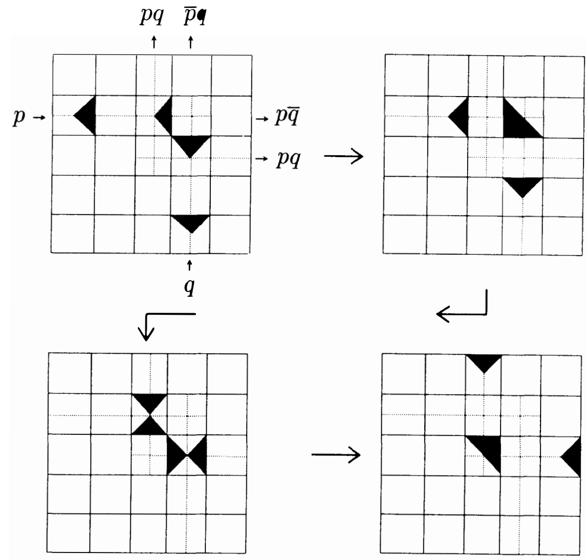


Fig. 13 Realization of an I-gate in Model 1.

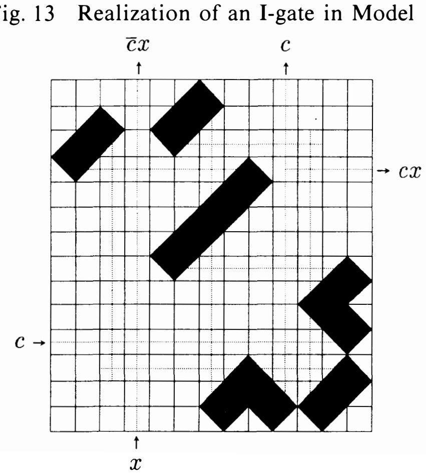


Fig. 14 A configuration of an S-gate in Model 1.

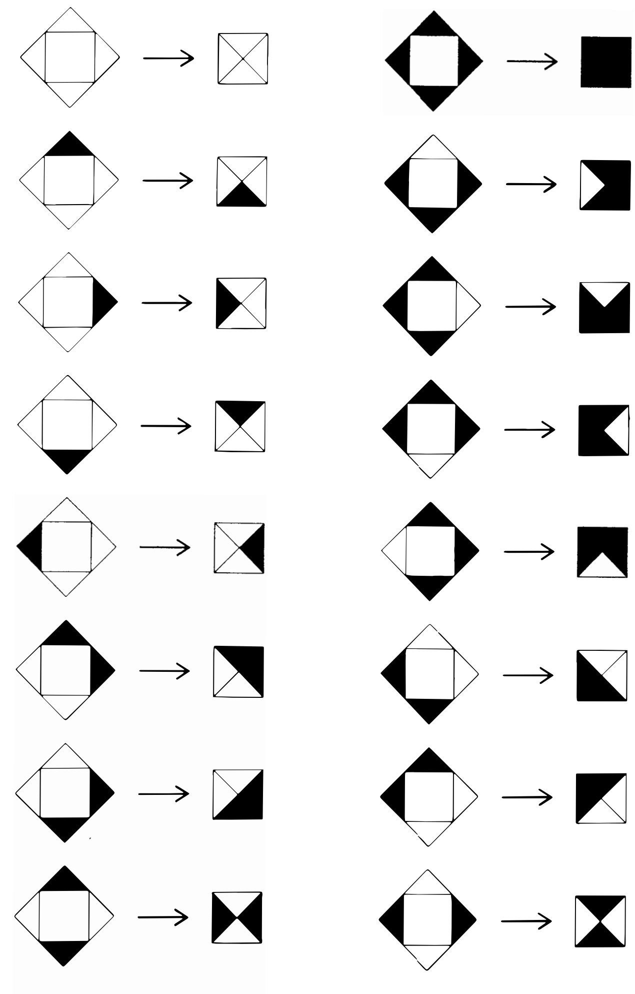
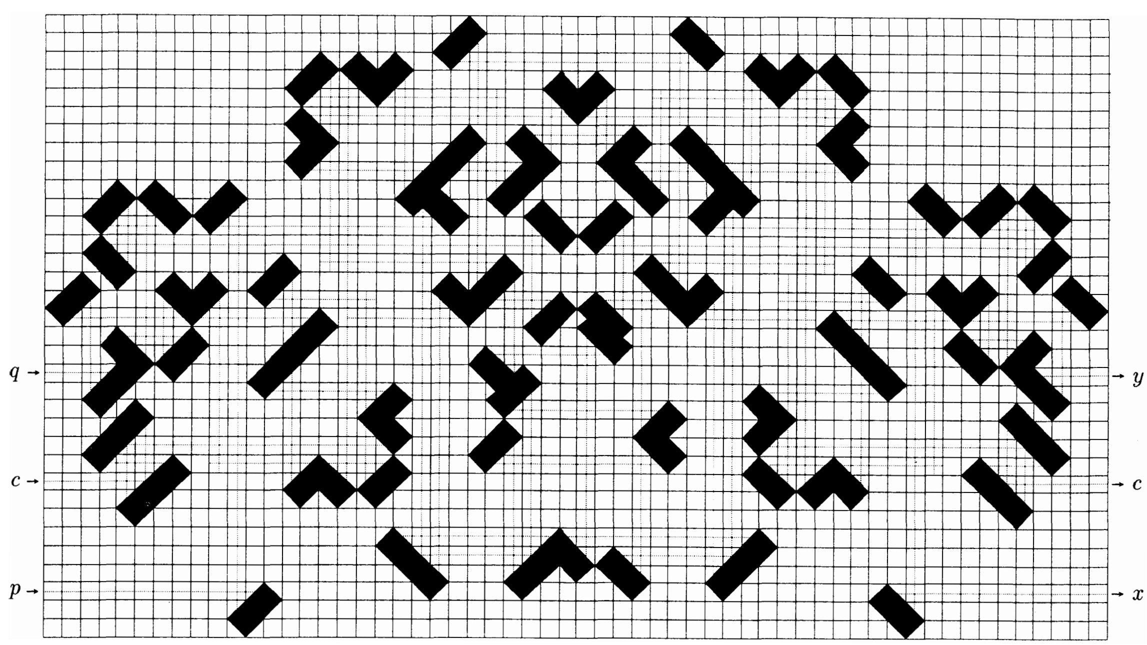


Fig. 16 The local function of 2RPCA(4) Model 2.



$x = cp + \bar{c}q, \quad y = \bar{c}p + cq$

Fig. 15 A configuration of an F-gate in Model 1.

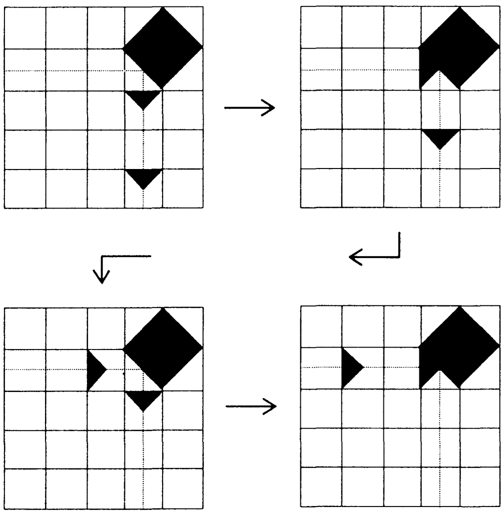


Fig. 17 Reflection of a signal by a mirror in Model 2.

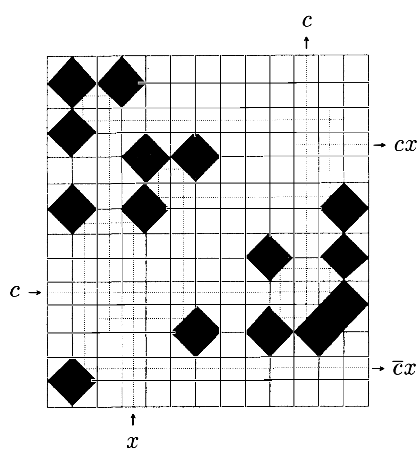


Fig. 18 A configuration of an S-gate in Model 2.

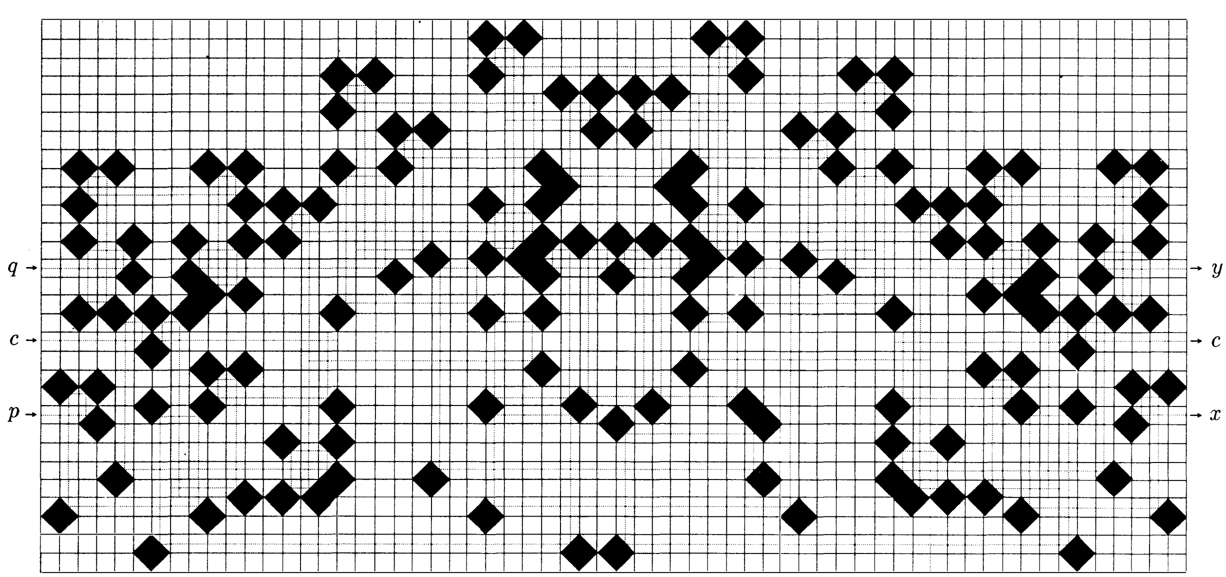
configuration. By combining an I-gate, an inverse I-gate, and mirrors as in Fig. 14 we realize a circuit of Fig. 7 and thus have an S-gate configuration. A configuration of an F-gate is given in Fig. 15, where S-gates, inverse S-gates, and mirrors are appropriately assembled to obtain the circuit shown in Fig. 8. The time delay between the input and the output in this configuration is 164 steps.

4. 2 Model 2

The local function of Model 2 is shown in Fig. 16. It is conservative, isotropic and reversible but not symmetric. A signal, a stable block, and an I-gate are the same as in Model 1. As for a stable block, it is broken by a signal coming from rightward. However, if a signal comes from leftward, it works as a mirror (Fig. 17). Thus, left-turn of a signal is realized by only one block, while right-turn must be realized by three left-turns. Figure 18 is a configuration of an S-gate, and Fig. 19 is that of an F-gate. The time delay in this case is 174 steps.

5. Conclusion

We proposed two simple models of 16-state 2RPCA(4) which can simulate an F-gate. Since logical universality of F-gate is shown in Ref. (1), and more concrete realization method of a universal computing mechanism by F-gates is shown in Ref. (4), universality of the proposed two models is concluded. The configurations of F-gate given here were verified by a computer simulation, where the behavior of them for each combination of input values is displayed as an animation.



$x = cp + \bar{c}q, \quad y = \bar{c}p + cq$

Fig. 19 A configuration of an F-gate in Model 2.

In the case of 2RPCA(4), if the isotropy condition is supposed (in this case $U=R=D=L$ must hold), the above 16-state ($=2^4$-state) universal models are the smallest ones in the number of states, since, apparently, a 1^4-state model is not universal. While we can find many 32-state isotropic universal 2RPCA(5) models, it seems very hard to find other 16-state isotropic universal 2RPCA(4) models except the above two (and the mirror image of Model 2). It is also an open problem whether there exist interesting universal models in the case that the isotropy or the conservativity condition does not hold, or the cases of 2RPCA(3), 2RPCA(2), etc.

The results of this paper suggests the possibility of constructing a computing system by using simple microscopic physical phenomena which obey a reversible and conservative law.

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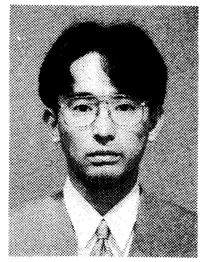

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