Essays on the Whittaker–Henderson Method of Graduation

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Contents

Chapter 1

Introduction

1.1 Introductory survey

The Whittaker–Henderson(WH) method of graduation is a frequently used smoothing tool in econometric time series analysis. The main purpose of this thesis is to analyze the special cases of the Whittaker–Henderson method of graduation. There are two popular cases of Whittaker–Henderson method of graduation, first one is called exponential smoothing (ES) filter (King and Rebelo, 1993) or WH method of order 1 and the second one is popularly known as Hodrick-Prescott (HP) filter (Hodrick and Prescott, 1997) or WH method of order 2. Though we called it the Whittaker–Henderson method of graduation but the method was first introduced by a German scholar George Bohlman (1899). Later, the method was well developed by Whittaker (1923) and Henderson (1924) separately and now it is popularly known as the Whittaker–Henderson method of graduation.

Bohlman suggested a method for graduating data where he used the first-order differences for graduation. His proposed method is described as the minimization problem of the following function with respect to $x_1, \ldots, x_T$ is:

$$\sum_{t=1}^{T}(y_t - x_t)^2 + \lambda^2 \sum_{t=1}^{T-1}(\nabla x_t)^2, \tag{1.1}$$

where $\boldsymbol{y} = [y_1, \ldots, y_T]'$ denotes univariate time series of T observations, $\boldsymbol{x} = [x_1, \ldots, x_T]'$, $\lambda^2 \geq 0$ is a parameter. Here, $\nabla x_t = x_{t+1} - x_t$ is called the first-order difference and the operator ∇ represents the forward difference operator. The first term, the square of the deviations and the second term represents the smoothing term. The parameter λ^2 is used to control the trade-off between the smoothness of the graduated data and the size of the deviation. If the value of λ^2 increases, the solution becomes smoother.

Whittaker (1923), without knowing about Bohlman's work, published a paper named as *"On a New Method of Graduation"*, where he suggested a method for data smoothing using third-order differences that is $\nabla^3 x_t = (x_{t+3} - 3x_{t+2} + 3x_{t+1} - x_t)$. He considered the following penalized least squares problem:

$$\min_{x_1,\dots,x_T \in \mathbb{R}} f(x_1,\dots,x_T) = \sum_{t=1}^{T} h_t^2 (y_t - x_t)^2 + \lambda^2 \sum_{t=1}^{T-3} (\nabla^3 x_t)^2, \tag{1.2}$$

where $\nabla x_t = x_{t+1} - x_t$ and derived the following optimality condition:

$$h_1^2 y_1 = h_1^2 \widehat{x}_1 + \lambda^2(-1)\nabla^3 \widehat{x}_1, \tag{1.3}$$

$$h_2^2 y_2 = h_2^2 \widehat{x}_2 + \lambda^2 3\nabla^3 \widehat{x}_1 + \lambda^2(-1)\nabla^3 \widehat{x}_2, \tag{1.4}$$

$$h_3^2 y_3 = h_3^2 \widehat{x}_3 + \lambda^2(-3)\nabla^3 \widehat{x}_1 + \lambda^2 3\nabla^3 \widehat{x}_2 + \lambda^2(-1)\nabla^3 \widehat{x}_3, \tag{1.5}$$

$$h_4^2 y_4 = h_4^2 \widehat{x}_4 + \lambda^2 \nabla^3 \widehat{x}_1 + \lambda^2(-3)\nabla^3 \widehat{x}_2 + \lambda^2 3\nabla^3 \widehat{x}_3 + \lambda^2(-1)\nabla^3 \widehat{x}_4, \tag{1.6}$$

$$\vdots$$

For proof of the above condition see section 1.2.1 of this chapter.

On the other hand, Henderson(1924) published an article about the data smoothing method named as *"A New Method of Graduation"*, where he discovered a factorization formula to calculate Whittaker's method in a more simpler way. Later, the method is known as the Whittaker–Henderson's method of graduation.

An important contribution was made by Greville (1957) about Whittaker–Henderson's smoothing process is to express objective function in matrix notation and to solve the system. Greville minimized expression of the form

$$(\boldsymbol{x} - \boldsymbol{y})' \boldsymbol{W} (\boldsymbol{x} - \boldsymbol{y}) + \boldsymbol{x}' \boldsymbol{G} \boldsymbol{x}. \tag{1.7}$$

Here $\boldsymbol{W} = \boldsymbol{I}$ is positive definite matrix and $\boldsymbol{G}$ is a positive semi-definite matrix. Here a small value of the first term is taken to indicate a close approach to the original data, while a small value of the second is considered to reflect a high degree of smoothness. This is the first attempt of minimizing the expression (1.7) for a given $\boldsymbol{y}$ leads to the equation

$$\hat{\boldsymbol{x}} = (\boldsymbol{I}+\boldsymbol{W}^{-1}\boldsymbol{G})^{-1}\boldsymbol{y}. \tag{1.8}$$

For proof of equation (1.8), see the appendix of this chapter.

Hodrick and Prescott (1997) introduced a method, which is known as Hodrick-Prescott (HP) filtering in econometrics and it is also regarded as the Whittaker–Henderson method of graduation of order 2. Hodrick and Prescott (1997) popularized the Whittaker–Henderson method of graduation in modern economics. According to HP (1997), a given time series y_t is the sum of a growth component x_t and a cyclical component c_t ,i.e.,

$$y_t = x_t + c_t, \text{ for } t = 1,2,...,T. \tag{1.9}$$

For determining the growth components, HP filter is

$$\underset{x_1,\ldots,x_T\in\mathbb{R}}{\arg\min}\left[\sum_{t=1}^{T}(y_t-x_t)^2+\lambda\sum_{t=3}^{T}[(x_{t+1}-x_t)-(x_t-x_{t-1})]^2\right], \tag{1.10}$$

where, the parameter $\lambda > 0$ is a tuning/smoothing parameter. Here, the first term measures the fidelity to the datum and the second term measures the smoothness.

Weinert (2006) introduced some algorithms to compute the estimates and the GCV (generalized cross-validation) score for the Whittaker–Henderson smoothing problem. According to him, a known sequence $\{y_t\}$ for T measurements and positive real number λ and a positive integer $p < T$, and the solution sequence is $\{\boldsymbol{x}_t\}$ which minimizes

$$\lambda\sum_{t=1}^{T}(y_t-x_t)^2+\sum_{t=1}^{T-p}(\nabla^p x_t)^2, \tag{1.11}$$

where, ∇ is a forward difference operator. The minimizer of (1.11) is

$$\boldsymbol{A}\hat{\boldsymbol{x}} = \lambda\boldsymbol{y}, \tag{1.12}$$

where

$$\boldsymbol{A} = \lambda\boldsymbol{I}+\boldsymbol{M}'\boldsymbol{M}. \tag{1.13}$$

Here, $\boldsymbol{M}$ is a $(T-p)\times T$ difference matrix. To solve (1.12) he factorized the coefficient matrix. This algorithm is time worthy.

Kim et al. (2009) proposed a filtering method popularly known as "l_1 trend filtering" which is a variation of Hodrick-Prescott(HP)filtering. HP filtering is commonly used for trend estimation of time series data where l_1 trend filtering gives continuous piece wise linear trends. The modification is done by replacing a sum of absolute values $l_1$norm (l_1 norm of a vector $\boldsymbol{x} = [x_1, \dots, x_T]'$ is $\|\boldsymbol{x}\|_1 = \sum_{t=1}^{T} |x_t|$) for the sum of squares used in the second term of HP filtering.

For observed time series $\boldsymbol{y} = [y_1, ..., y_T]'$, l_1 trend filtering is defined as follows,

$$\begin{aligned} \widehat{\boldsymbol{x}} &= \underset{x_1,\dots,x_T \in \mathbb{R}^T}{\arg\min} \left[\sum_{t=1}^{T} (y_t - x_t)^2 + \lambda \sum_{t=3}^{T} |\Delta^2 x_t| \right] \\ &= \underset{\boldsymbol{x} \in \mathbb{R}^T}{\arg\min} \left(\|\boldsymbol{y} - \boldsymbol{x}\|_2^2 + \lambda \|\boldsymbol{D}\boldsymbol{x}\|_1 \right). \end{aligned} \tag{1.14}$$

Yamada (2017) proposed a useful modification of the HP filter. The proposed modified HP filter is:

$$\begin{aligned} & [\widehat{x}_{1-h}, ..., \widehat{x}_0, \widehat{x}_1, ..., \widehat{x}_T, \widehat{x}_{T+1}, ..., \widehat{x}_{T+h}]' \\ = & \underset{x_{1-h},\dots,x_{T+h} \in \mathbb{R}}{\arg\min} \left[\sum_{t=1}^{T} (y_t - x_t)^2 + \lambda \sum_{t=3}^{T+h} (\Delta^2 x_t)^2 \right], \end{aligned} \tag{1.15}$$

and

$$\begin{cases} \widehat{\boldsymbol{x}}_t = \widetilde{\boldsymbol{x}}_t, & t = 1, \dots, T, \\ \widehat{\boldsymbol{x}}_{T+j} = \widetilde{\boldsymbol{x}}_T + j(\widetilde{\boldsymbol{x}}_T - \widetilde{\boldsymbol{x}}_{T-1}), & j = 1, \dots, h. \end{cases} \tag{1.16}$$

Here, $(\widetilde{x}_1, \dots, \widetilde{x}_T)$ is the solution of HP filter. This modified filter provides extrapolation of the trends beyond the sample limit.

Yamada (2017) suggested a new method, which is closely related to l_1 trend filtering and named as "Pure l_1 trend filtering" method. Pure l_1 trend filtering is defined as follows:

$$\widehat{\boldsymbol{z}} = \boldsymbol{D}'(\boldsymbol{D}\boldsymbol{D}')^{-1} \widehat{\boldsymbol{\psi}}, \tag{1.17}$$

where

$$\widehat{\boldsymbol{\psi}} = \underset{\boldsymbol{\psi} \in \mathbb{R}^{(T-2)}}{\arg\min} \left(\|\boldsymbol{y} - \boldsymbol{D}'(\boldsymbol{D}\boldsymbol{D}')^{-1} \boldsymbol{\psi}\|_2^2 + \lambda \|\boldsymbol{\psi}\|_1 \right). \tag{1.18}$$

Note that, $\boldsymbol{D}$ is a tridiagonal Toeplitz matrix of which the first row is $[1,-2,1,0,\ldots,0]$ and $\boldsymbol{D}\boldsymbol{D}'$ is a banded Toeplitz matrix. Since $\boldsymbol{D}$ is of full row rank, $\boldsymbol{D}\boldsymbol{D}'$ is positive definite, which indicates $\boldsymbol{D}\boldsymbol{D}'$ is non-singular.

De Jong and Sakarya (2016) recently derived an explicit formula for the smoother weights of the WH graduation of order 2 which is popularly known as the Hodrick–Prescott filter. It's worth to mention that the $T \times T$ matrix $(\boldsymbol{I}_T + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1}$ is referred to as the smoother matrix of the HP filter. More recently, by applying the SMW formula and the spectral decomposition of a symmetric tridiagonal Toeplitz matrix, Cornea-Madeira (2017) provided a simpler formula of it. We note here that to derive the explicit formula, we apply a different approach to that of Cornea-Madeira (2017). In this thesis, we derived explicit formulas for the Whittaker–Henderson graduation of orders 1 and 2 and also prove that these two smoother matrices are bisymmetric.

1.2 Some preliminary definitions and basic properties:

Suppose $\boldsymbol{B}$ is a $K \times K$ symmetric matrix having K distinct characteristic vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$ and the corresponding characteristic roots are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The characteristic vectors of a symmetric matrix are orthogonal. Now, the matrix $\boldsymbol{V}$ is the eigenvector matrix i.e. $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k]$ and $\boldsymbol{\Lambda} = \mathsf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a eigenvalue matrix. Here we discuss some basic properties of linear algebra which are particularly related to this thesis.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* The trace of a square symmetric matrix $\boldsymbol{B}$ is the sum of its diagonal elements:

$$\mathsf{trace}(\boldsymbol{B}) = \sum_{i=1}^{K} b_{ii},$$

where b_{ii} are the diagonal elements of $\boldsymbol{B}$.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* The transpose of a matrix is an operator which flips a matrix over its diagonal. Transpose of a matrix $\boldsymbol{B}$ is often denoted by $\boldsymbol{B}'$ or $\boldsymbol{B}^T$. If $\boldsymbol{B}$ is an $m \times n$ matrix, then $\boldsymbol{B}'$ is an $n \times m$ matrix, that is, the i-th row, j-th column element of $\boldsymbol{B}'$ is the j-th row, i-th column element of $\boldsymbol{B}$: $[\boldsymbol{B}']_{ij} = [\boldsymbol{B}]_{ji}$.
If $\boldsymbol{A}$ is another matrix then $(\boldsymbol{AB})' = \boldsymbol{B}'\boldsymbol{A}'$.
An example of the transpose of a matrix is:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 3 & 6 \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* The diagonalization of a symmetric matrix $\boldsymbol{B}$ is

$$\boldsymbol{V}'\boldsymbol{B}\boldsymbol{V} = \boldsymbol{\Lambda}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* The spectral decomposition of a symmetric matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = \boldsymbol{V\Lambda V}' = \sum_{k=1}^{K} \lambda_k \boldsymbol{v}_k \boldsymbol{v}_k'.$$

This is also called the eigenvalue decomposition of matrix $\boldsymbol{B}$.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A matrix is called idempotent if it is equal to it's square, that is, $\boldsymbol{B}^2 = \boldsymbol{BB} = \boldsymbol{B}$. If $\boldsymbol{B}$ is symmetric then $\boldsymbol{B}'\boldsymbol{B} = \boldsymbol{B}$.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* The rank of a symmetric matrix is the number of non-zero characteristic roots/eigenvalues it contains.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A square matrix $\boldsymbol{B}$ is called a permutation matrix if every row and every column of $\boldsymbol{B}$ contains only one element 1 and the other elements are 0. For order 2 there exist 2 permutation matrices, they are:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For order 3 there are 6 permutation matrices. In this way, for order n there are $n!$ permutation matrices.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* For any square matrix $\boldsymbol{B}$, the quadratic form can be written as:

$$\boldsymbol{x}'\boldsymbol{Bx}.$$

Now, if $\boldsymbol{x}'\boldsymbol{Bx} > 0$ for all real $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{B}$ is positive definite,
if $\boldsymbol{x}'\boldsymbol{Bx} < 0$ for all real $\boldsymbol{x} \neq \boldsymbol{0}$, then $\boldsymbol{B}$ is negative definite,
if $\boldsymbol{x}'\boldsymbol{Bx} \geq 0$ for all real $\boldsymbol{x}$, then $\boldsymbol{B}$ is positive semi-definite,
if $\boldsymbol{x}'\boldsymbol{Bx} \leq 0$ for all real $\boldsymbol{x}$, then $\boldsymbol{B}$ is negative semi-definite.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* If $\boldsymbol{Bx} = \boldsymbol{x}$ for all $\boldsymbol{x}$ then particularly for the unit vectors $\boldsymbol{x} = \boldsymbol{e}_j$ implies that $\boldsymbol{Be}_j = \boldsymbol{e}_j$ so that $b_{i,j} = \boldsymbol{e}_i'\boldsymbol{Be}_j = \boldsymbol{e}_i'\boldsymbol{e}_j$, which is 0 when $i \neq j$ and 1 when $i = j$, then $\boldsymbol{B} = \boldsymbol{I}$ is called the identity matrix of a specified order. Conversely, if $\boldsymbol{B} = \boldsymbol{I}$, then $\boldsymbol{Bx} = \boldsymbol{x}$ holds for all $\boldsymbol{x}$. An $n \times n$ identity matrix is written

as $\boldsymbol{I}_n$ and $\boldsymbol{I}_n' = \boldsymbol{I}_n^2 = \boldsymbol{I}_n^{-1} = \boldsymbol{I}_n$ holds.
For example,

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A matrix $\boldsymbol{B} = [b_{i,j}]$ is diagonal if all the entries outside the principal diagonal (that is, for $i \neq j$) are zero. The diagonal entries themselves may be or may not be zero. Matrix $\boldsymbol{B}$ can also be written as $\boldsymbol{B} := \mathsf{diag}(b_1, b_2, ..., b_n)$.
An example of a diagonal matrix is:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A real square matrix $\boldsymbol{B}$ is orthogonal if $\boldsymbol{B}'\boldsymbol{B} = \boldsymbol{B}\boldsymbol{B}' = \boldsymbol{I}$. If a matrix is orthogonal, then its rows form an orthonormal set and its columns also form an orthonormal set.
An example of an orthogonal matrix is:

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A tridiagonal Toeplitz matrix is a special (square) Toeplitz matrix where the major and minor diagonal entries are constant and others are zero. Example of a tridiagonal Toeplitz matrix is:

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix},$$

where the empty spaces are filled with zeroes.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* Pentadiagonal Toeplitz matrix is a square matrix of five major and minor diagonal elements are constant. An example of a pentadiagonal Toeplitz matrix is

$$\boldsymbol{D}' = \begin{bmatrix} 6 & -4 & 1 & & & & \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -4 \\ & & & & 1 & -4 & 6 \end{bmatrix},$$

where the empty spaces are filled with zeroes.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* Let, $\boldsymbol{x} = [x_1, ..., x_n]'$ be an $n \times 1$ column vector. A sequence of differences $\Delta x_i = x_i - x_{i-1}$, where $i = 2, ..., n$ is called the linear first order differences, which can be expressed in matrix form as $D_1 x$, where $\boldsymbol{D}_1$ the $(n-1)\times n$ matrix is called a *linear first-order difference matrix* of the following form:

$$\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix},$$

where the empty spaces are filled with zeroes.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A $\boldsymbol{P}^{th}$ order difference matrix is expressed as $\boldsymbol{D}_p$, whose dimension is $(n-p)\times n$ for $n,p\in\mathbb{R}$, is

$$\boldsymbol{D}_p = (-1)^p \begin{bmatrix} 1 & -\binom{p}{1} & \binom{p}{2} & -\binom{p}{3} & \cdots & 1 & & & \\ & 1 & -\binom{p}{1} & \binom{p}{2} & -\binom{p}{3} & \cdots & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & & \ddots & \\ & & & 1 & -\binom{p}{1} & \binom{p}{2} & -\binom{p}{3} & \cdots & 1 \end{bmatrix},$$

where the empty spaces are filled with zeroes.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* l_1 norm of a vector $\boldsymbol{x} = [x_1,\ldots,x_n]'$ is defined as

$$\|\boldsymbol{x}\|_1 = \sum_{t=1}^{n} |x_t|.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* l_2 norm of a vector $\boldsymbol{x} = [x_1,\ldots,x_n]'$ is defined as

$$\|\boldsymbol{x}\|_2 = \left(\sum_{t=1}^{n} |x_t|^2 \right)^{1/2}.$$

Frobenius norm of a matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ is defined as

$$\|\boldsymbol{B}\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^2 \right)^{1/2} = \sqrt{\mathsf{trace}(\boldsymbol{B}'\boldsymbol{B})}.$$

Definition: Suppose, Δ is a backward difference operator, for a vector $\boldsymbol{x} = [x_1, \ldots, x_n]$ the first-order backward difference is $\Delta x_t = x_t - x_{t-1}$. Similarly, the second-order backward difference is

$$\begin{aligned} \Delta^2 x_t &= \Delta(\Delta x_t) = \Delta(x_t - x_{t-1}) = (\Delta x_t - \Delta x_{t-1}) \\ &= (x_t - x_{t-1}) - (x_{t-1} - x_{t-2}) = x_t - 2x_{t-1} + x_{t-2}. \end{aligned}$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A square matrix is called symmetric if the entries of the matrix are equal with respect to the principal diagonal. Suppose, a square matrix $\boldsymbol{B} = [b_{i,j}]$ is symmetric iff $b_{i,j} = b_{j,i}$. Moreover, a symmetric matrix is equal to its transpose, that is, $\boldsymbol{B} = \boldsymbol{B}'$.
For an example, a 3×3 symmetric matrix is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A square matrix that is symmetric with respect to the northeast-to-southwest diagonal or a square matrix such that the values on each line perpendicular to the main diagonal are the same for a given line

is called a persymmetric matrix.
Suppose, a square matrix of order n is $\boldsymbol{B} = [b_{i,j}]$, will be a persymmetric matrix if and only if it satisfies

$$b_{i,j} = b_{n-j+1,n-i+1} \quad \text{for} \quad 1 \leq i,j \leq n.$$

For an exchange matrix, $\boldsymbol{T}$ persymmetric matrix $\boldsymbol{B}$ satisfies the relation, $\boldsymbol{BT} = \boldsymbol{TB'}$.
An example of a 3×3 persymmetric matrix is:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A square matrix is called a centrosymmetric matrix if it is symmetric about its center. A matrix of order n, $\boldsymbol{B} = [b_{i,j}]$ is centrosymmetric if and only if it satisfies

$$\boldsymbol{B}_{i,j} = \boldsymbol{B}_{n-i+1,n-j+1} \quad \text{for} \quad 1 \leq i,j \leq n.$$

A centrosymmetric matrix also satisfies $\boldsymbol{BT} = \boldsymbol{TB}$ relation with the exchange matrix. The general form of a 3×3 centrosymmetric matrix is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{21} \\ b_{13} & b_{12} & b_{11} \end{bmatrix}.$$

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* A matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$

is said to be a skew-symmetric matrix if and only if its entries satisfy the relation

$$\boldsymbol{B}_{i,j} = -\boldsymbol{B}_{n-i+1,n-j+1} \quad \text{for} \quad 1 \le i, j \le n.$$

For any real $\boldsymbol{x}$, a skew-symmetric matrix also satisfies the relation $\boldsymbol{x}'\boldsymbol{B}\boldsymbol{x} = 0$ and $\boldsymbol{B}\boldsymbol{T} = -\boldsymbol{T}\boldsymbol{B}$ for an exchange matrix $\boldsymbol{T}$.

Definition:*[K. M. Abadir and J. R. Magnus (Matrix Algebra)]* Bisymmetric matrices are both symmetric centrosymmetric and symmetric persymmetric matrices. More precisely, a bisymmetric matrix is a square matrix that is symmetric about both of its main diagonals. An $n \times n$ matrix $\boldsymbol{B}$ is bisymmetric if it satisfies both $\boldsymbol{B} = \boldsymbol{B}\boldsymbol{T}$ and $\boldsymbol{B}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{B}$ where $\boldsymbol{T}$ is the $n \times n$ exchange matrix. An example is

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 5 & 4 & 0 \\ 0 & 4 & 5 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}.$$

Discrete cosine transformation (Type-II): The discrete cosine transformation matrix of type 2 is an invertible $n \times n$ square matrix where the n real numbers $x_0, ..., x_{n-1}$ are transformed into the N real numbers $X_0, ..., X_{n-1}$ according to the following formula:

$$X_k = \sum_{t=0}^{n-1} x_t \cos\left[\frac{\pi}{n}\left(t + \frac{1}{2}\right)k\right], \qquad k = 0, \dots, n-1.$$

Woodbury matrix identity: Woodbury matrix identity was originated by Max A. Woodbury. It is also known as matrix inversion lemma. The lemma states that if $\boldsymbol{B}$ and $\boldsymbol{C}$ are respectively $n \times n$ and $k \times k$ square invertible matrices and $\boldsymbol{X}$ and $\boldsymbol{Y}$ are matrices so that and $\boldsymbol{XCY}$ have the same dimensions), then

$$(\boldsymbol{B}+\boldsymbol{XCY})^{-1} = \boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{X}(\boldsymbol{C}^{-1}+\boldsymbol{YB}^{-1}\boldsymbol{X})^{-1}\boldsymbol{YB}^{-1}.$$

The main purpose of this lemma is to perform numerical computations easily where $\boldsymbol{B}^{-1}$ is known and it is desired to compute $(\boldsymbol{B}+\boldsymbol{XCY})^{-1}$. This lemma is a special case of the Binomial inversion theorem.

Sherman-Morrison formula: Sherman-Morrison formula is a special case of the Woodbury matrix identity. Let $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ be a square matrix and it's inverse exist. If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ are two column vectors then,

$$(\boldsymbol{B}+\boldsymbol{uv}')^{-1} = \boldsymbol{B}^{-1} - \frac{\boldsymbol{B}^{-1}\boldsymbol{uv}'\boldsymbol{B}^{-1}}{1+\boldsymbol{v}'\boldsymbol{B}^{-1}\boldsymbol{u}}.$$

Here, $\boldsymbol{uv}'$ is the outer product of two vectors. The above equation is applicable if and only if $1+\boldsymbol{v}'\boldsymbol{B}^{-1}\boldsymbol{u} \neq 0$.

1.2.1 Whittaker (1923)

Let, assume that we are focusing on a number $\boldsymbol{y}_t$ which depends on the parameter t and assume that we have T data $y_1, y_2, \ldots, y_T$ which are affected to uncertainty or irregularity due to, for example, an unexpected observation error. The observations collected from natural circumstances can be affected by irregularities. Statistical data collected from a comparatively small field will be affected by irregularities due to the occasional nature of the field. Now, if we derive two sets of data, which are not similar, from the two fields and construct a table of the differences, we will find that the differences are irregular and can't satisfy its purpose. For this reason, we need to find another sequence of regular differences $x_1, x_2, \ldots, x_T$, whose terms have little difference from the terms of the given sequence $y_1, y_2, \ldots, y_T$. This process is called the graduation of the observation or the smoothing of data, which is gained by creating a balance between fidelity to the data and smoothness of the fitted curve.

Now, the degree of the smoothness of the sequences $x_1, x_2, \ldots, x_T$ may be measured by the smallness of the some of the squares of the third-order differences, that is

$$\begin{aligned} S &= (x_4 - 3x_3 + 3x_2 - x_1)^2 + \cdots + (x_T - 3x_{T-1} + 3x_{T-2} - x_{T-3})^2 \\ &= (\nabla^3 x_4)^2 + (\nabla^3 x_5)^2 + (\nabla^3 x_6)^2 + (\nabla^3 x_7)^2 + \cdots + (\nabla^3 x_T)^2 \\ &= \sum_{t=4}^{T} (\nabla^3 x_t)^2. \end{aligned} \tag{1.19}$$

Let $h_1, h_2, \ldots, h_T$ be some constant then fidelity is measured by the sum of squares of deviations:

$$\begin{aligned} F &= h_1^2(y_1 - x_1)^2 + h_2^2(y_2 - x_2)^2 + \cdots + h_T^2(y_T - x_T)^2 \\ &= \sum_{t=1}^{T} h_t^2 (y_t - x_t)^2. \end{aligned} \tag{1.20}$$

Now, Whittaker's method is to minimize the following function for some value of λ with respect to $x_1, \ldots, x_T$:

$$\min_{x_1,\ldots,x_T \in \mathbb{R}} f(x_1, \ldots, x_T) = \sum_{t=1}^{T} h_t^2 (y_t - x_t)^2 + \lambda^2 \sum_{t=1}^{T-3} (\nabla^3 x_t)^2, \tag{1.21}$$

where $\nabla x_t = x_{t+1} - x_t$. Let $\boldsymbol{y} = [y_1, \dots, y_T]'$, $\boldsymbol{x} = [x_1, \dots, x_T]'$, $\boldsymbol{H} = \mathsf{diag}(h_1^2, \dots, h_T^2)$, and $\boldsymbol{D} \in \mathbb{R}^{(T-3)\times T}$ be a matrix such that

$$\boldsymbol{D}\boldsymbol{x} = [\nabla^3 x_1, \dots, \nabla^3 x_{T-3}]'.$$

Then, (1.21) can be expressed in matrix notation as follows:

$$\min_{\boldsymbol{x}\in\mathbb{R}^T} f(\boldsymbol{x}) = (\boldsymbol{y}-\boldsymbol{x})'\boldsymbol{H}(\boldsymbol{y}-\boldsymbol{x}) + \lambda^2(\boldsymbol{D}\boldsymbol{x})'(\boldsymbol{D}\boldsymbol{x}). \tag{1.22}$$

From

$$\frac{\partial f(\boldsymbol{x})}{\partial \boldsymbol{x}} = -2\boldsymbol{H}(\boldsymbol{y}-\boldsymbol{x}) + 2\lambda^2\boldsymbol{D}'\boldsymbol{D}\boldsymbol{x}, \tag{1.23}$$

letting $\widehat{\boldsymbol{x}}$ denote the solution of (1.21)/(1.22), the optimality condition for (1.21)/(1.22) can be expressed as

$$-\boldsymbol{H}(\boldsymbol{y}-\widehat{\boldsymbol{x}}) + \lambda^2\boldsymbol{D}'\boldsymbol{D}\widehat{\boldsymbol{x}} = \boldsymbol{0}. \tag{1.24}$$

Equivalently,

$$\boldsymbol{H}\boldsymbol{y} = \boldsymbol{H}\widehat{\boldsymbol{x}} + \lambda^2\boldsymbol{D}'\boldsymbol{D}\widehat{\boldsymbol{x}}. \tag{1.25}$$

Given that

$$\boldsymbol{D}\widehat{\boldsymbol{x}} = [\nabla^3 \widehat{x}_1, \dots, \nabla^3 \widehat{x}_{T-3}]' \tag{1.26}$$

and

$$
\boldsymbol{D}' = \begin{bmatrix}
-1 & & & & & & \\
3 & -1 & & & & & \\
-3 & 3 & -1 & & & & \\
1 & -3 & 3 & -1 & & & \\
 & \ddots & \ddots & \ddots & \ddots & & \\
 & & 1 & -3 & 3 & -1 & \\
 & & & 1 & -3 & 3 & -1 \\
 & & & & 1 & -3 & 3 \\
 & & & & & 1 & -3 \\
 & & & & & & 1
\end{bmatrix}, \tag{1.27}
$$

where the empty spaces are filled with zeroes, (1.25) may be expressed as

$$h_1^2 y_1 = h_1^2 \widehat{x}_1 + \lambda^2(-1)\nabla^3 \widehat{x}_1, \tag{1.28}$$

$$h_2^2 y_2 = h_2^2 \widehat{x}_2 + \lambda^2 3\nabla^3 \widehat{x}_1 + \lambda^2(-1)\nabla^3 \widehat{x}_2, \tag{1.29}$$

$$h_3^2 y_3 = h_3^2 \widehat{x}_3 + \lambda^2(-3)\nabla^3 \widehat{x}_1 + \lambda^2 3\nabla^3 \widehat{x}_2 + \lambda^2(-1)\nabla^3 \widehat{x}_3, \tag{1.30}$$

$$h_4^2 y_4 = h_4^2 \widehat{x}_4 + \lambda^2 \nabla^3 \widehat{x}_1 + \lambda^2(-3)\nabla^3 \widehat{x}_2 + \lambda^2 3\nabla^3 \widehat{x}_3 + \lambda^2(-1)\nabla^3 \widehat{x}_4, \tag{1.31}$$

$$\vdots$$

$$h_T^2 y_T = h_T^2 \widehat{x}_T + \lambda^2 \nabla^3 \widehat{x}_{T-3}, \tag{1.32}$$

which are given in Whittaker (1923). Thus, the optimality condition for $t = 4, 5, \ldots, T-3$ can be expressed as

$$h_t^2 y_t = h_t^2 \widehat{x}_t + \lambda^2 \nabla^3 \widehat{x}_{t-3} - 3\lambda^2 \nabla^3 \widehat{x}_{t-2} + 3\lambda^2 \nabla^3 \widehat{x}_{t-1} - \lambda^2 \nabla^3 \widehat{x}_t. \tag{1.33}$$

For simplification we assume that, $h_1 = h_2 = \cdots = h_T$.
Now, if we consider $h_1^2 = h_2^2 = \cdots = h_T^2 = \varepsilon\lambda^2$,where ε is a non-negative constant, then the above system of equations can be written as:

$$
\begin{aligned}
&\varepsilon y_1 = \varepsilon\widehat{x}_1 - \nabla^3\widehat{x}_1,\\
&\varepsilon y_2 = \varepsilon\widehat{x}_2 + 3\nabla^3\widehat{x}_1 - \nabla^3\widehat{x}_2,\\
&\varepsilon y_3 = \varepsilon\widehat{x}_3 - 3\nabla^3\widehat{x}_1 + 3\nabla^3\widehat{x}_2 - \nabla^3\widehat{x}_3,\\
&\varepsilon y_4 = \varepsilon\widehat{x}_4 + \nabla^3\widehat{x}_1 - 3\nabla^3\widehat{x}_2 + 3\nabla^3\widehat{x}_3 - \nabla^3\widehat{x}_4,\\
&\vdots\\
&\varepsilon y_T = \varepsilon\widehat{x}_T + \nabla^3\widehat{x}_{T-3}.
\end{aligned}
$$

So, the above system is equivalent to the following expression

$$\varepsilon y_t = \varepsilon\widehat{x}_t + \nabla^3\widehat{x}_{t-3} - 3\nabla^3\widehat{x}_{t-2} + 3\nabla^3\widehat{x}_{t-1} - \nabla^3\widehat{x}_t. \tag{1.34}$$

Let $F\widehat{x}_t = \widehat{x}_{t+1}$. Then, we obtain

$$
\begin{aligned}
\nabla^3\widehat{x}_{t-3} - 3\nabla^3\widehat{x}_{t-2} + 3\nabla^3\widehat{x}_{t-1} - \nabla^3\widehat{x}_t &= \nabla^3(\widehat{x}_{t-3} - 3\widehat{x}_{t-2} + 3\widehat{x}_{t-1} - \widehat{x}_t)\\
&= \nabla^3(\widehat{x}_{t-3} - 3F\widehat{x}_{t-3} + 3F^2\widehat{x}_{t-3} - F^3\widehat{x}_{t-3})\\
&= \nabla^3(1 - 3F + 3F^2 - F^3)\widehat{x}_{t-3}\\
&= -\nabla^3(F-1)^3\widehat{x}_{t-3}\\
&= -\nabla^6\widehat{x}_{t-3}.
\end{aligned} \tag{1.35}
$$

Substituting (1.35) into (1.34) yields

$$\varepsilon y_t = \varepsilon\widehat{x}_t - \nabla^6\widehat{x}_{t-3}. \tag{1.36}$$

For finding the optimal solution $\widehat{\boldsymbol{x}} = [\widehat{x}_1, ..., \widehat{x}_T]'$ of the above function (1.21), now all the equations except the three first and the three last are of the form

$$\widehat{x}_t - y_t = \frac{1}{\varepsilon}\nabla^6 \widehat{x}_{t-3}, \quad \text{for } t = 4, 5, \dots, T-3. \tag{1.37}$$

Similarly, the first three and the last three equations can also be brought to the same form by introducing new quantities of $\widehat{x}_t$. The quantities are $\nabla^3\widehat{x}_{-1}, \nabla^3\widehat{x}_{-2}, \nabla^3\widehat{x}_0, \nabla^3\widehat{x}_{T+1}$, $\nabla^3\widehat{x}_{T+2}, \nabla^3\widehat{x}_{T+3}$ for which

$$\nabla^3\widehat{x}_{-1} = 0, \nabla^3\widehat{x}_{-2} = 0, \nabla^3\widehat{x}_0 = 0, \nabla^3\widehat{x}_{T+1} = 0, \nabla^3\widehat{x}_{T+2} = 0, \nabla^3\widehat{x}_{T+3} = 0. \tag{1.38}$$

Thus the graduated values $\widehat{x}_t$ satisfy the linear difference equation

$$\varepsilon\widehat{x}_t - \nabla^6\widehat{x}_{t-3} = \varepsilon y_t \quad t = 1, 2, \dots, T, \tag{1.39}$$

and the solution has to satisfy the terminal conditions $\nabla^3\widehat{x}_t = 0$ for $t = -2, -1, 0, T+1, T+2, T+3$. After that Whittaker expanded the equation in powers of ε, which he assumed would be small and solve the linear equations.

1.3 Objectives and Outlines

The main goal of this research is to analyze the special cases of the Whittaker–Henderson method of graduation, which originates in the work of Bohlmann (1899), which is a widely used graduating method. The WH method of graduation has been used for the mortality table construction, in the Actuarial literature. At the beginning of this thesis, some preliminary definitions and relevance methods are discussed. The primary objective of this thesis is to establish alternative methods to gain simpler formulas.
This thesis dissertation consists of four chapters. In chapter 1, the introductory survey of research, some preliminary definitions, examples and relevance methods are discussed.

Chapter 2, we provide an explicit formula for the smoother weights of the Whittaker–Henderson (WH) graduation of order 1 along with some related results, which leads to a richer understanding of the filter. $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$ in is the smoother matrix of the WH graduation and its elements are the smoother weights of the smoothing. The main motivation of this thesis comes from De Jong and Sakarya (2016). They provided an explicit formula for the smoother weights of the WH graduation of order 2, following which Cornea-Madeira (2017) gave a simpler explicit formula. In econometrics, the WH graduation of order 2 is referred to as the Hodrick–Prescott (1997) filter. We note here that to derive the explicit formula, we apply a different approach to that of Cornea-Madeira (2017). This is mainly because the approach given in Chapter 2 leads to a simpler formula. In the last part of this chapter, for comparison, we show another formula based on Cornea-Madeira's (2017) approach. A MATLAB code is included which visualizes the efficiency of the proposed method.

In Chapter 3, we provide an alternative simpler formula for the Hodrick–Prescott (HP) (1997) filter and explains the reason why our approach leads to a simpler formula. The Hodrick–Prescott (HP) filter is a popular method to estimate the trend component of the univariate time series. It is described as a penalized least squares problem and a special case of the Whittaker–Henderson (WH) method of graduation. By implementing the Sherman–Morrison–Woodbury (SMW) formula and a discrete cosine transformation matrix, De Jong and Sakarya (2016) recently derived an explicit formula for the smoother weights of the Hodrick–Prescott filter. In recent times, by applying the SMW formula and the spectral decomposition of a symmetric tridiagonal Toeplitz matrix, Cornea-Madeira (2017) provided a simpler formula. In this chapter, we provide a simpler alternative formula for the smoother weights of the HP filter. A MATLAB code to find the smoother weights of the popular HP filter is included in the last part of the chapter which guaranteed the efficiency of the proposed method.

In Chapter 4, based on the result of Yamada (2019), simple formulas for calculating the smoother matrix of the WH method is provided. In addition, we show some results, which include that two other smoother matrices related to the WH graduation are also bisymmetric.

1.4 Appendix

Proof of Equation (1.8)

Differentiating equation (1.7) we get

$$\boldsymbol{W}(\hat{\boldsymbol{x}} - \boldsymbol{y}) + \boldsymbol{G}\hat{\boldsymbol{x}} = \boldsymbol{0},$$

following which we obtain

$$(\boldsymbol{W} + \boldsymbol{G})\hat{\boldsymbol{x}} = \boldsymbol{W}\boldsymbol{y}.$$

$\boldsymbol{W} + \boldsymbol{G}$ is a positive definite matrix and thus it is invertible. Then, it follows that

$$\hat{\boldsymbol{x}} = (\boldsymbol{I} + \boldsymbol{W}^{-1}\boldsymbol{G})^{-1}\boldsymbol{y}.$$

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Chapter 2

Explicit Formulas for the Smoother Weights of the Whittaker–Henderson Graduation of Order 1

This chapter basically based on a previously published article [Yamada, H. and F. T. Jahra, 2018].

2.1 Introduction

The Whittaker–Henderson (WH) graduation, which originates in the work of Bohlmann (1899), is a widely applied smoothing method. The WH graduation (of order p) is defined as

$$\min_{x_1,\ldots,x_n \in \mathbb{R}} \sum_{t=1}^{n} (y_t - x_t)^2 + \lambda \sum_{t=p+1}^{n} (\Delta^p x_t)^2, \tag{2.1}$$

where $y_1, \ldots, y_n$ represent a sequence of n observations, $\lambda > 0$ is a smoothing parameter, and $\Delta x_t = x_t - x_{t-1}$. For historical remarks on the filter, see Weinert (2007).[1] It may also

[1] See also Phillips (2010) and Nocon and Scott (2012).

be represented in matrix notation as

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} (\boldsymbol{y}-\boldsymbol{x})'(\boldsymbol{y}-\boldsymbol{x})+\lambda(\boldsymbol{D}_p\boldsymbol{x})'(\boldsymbol{D}_p\boldsymbol{x}), \tag{2.2}$$

where $\boldsymbol{y}=[y_1,\dots,y_n]'$, $\boldsymbol{x}=[x_1,\dots,x_n]'$, and $\boldsymbol{D}_p\in\mathbb{R}^{(n-p)\times n}$ is the pth-order difference matrix such that $\boldsymbol{D}_p\boldsymbol{x}=[\Delta^p x_{p+1},\dots,\Delta^p x_n]'$. For example,

$$\boldsymbol{D}_1=\begin{bmatrix}-1 & 1 & 0 & \cdots & 0\\ 0 & -1 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & -1 & 1\end{bmatrix}\in\mathbb{R}^{(n-1)\times n}.$$

Since the objective function of the WH graduation is quadratic and its Hessian matrix is positive definite, it has a unique global minimizer, which is expressed explicitly as

$$\widehat{\boldsymbol{x}}_p=(\boldsymbol{I}_n+\lambda\boldsymbol{D}_p'\boldsymbol{D}_p)^{-1}\boldsymbol{y}, \tag{2.3}$$

where $\boldsymbol{I}_n$ is the identity matrix of order n.

$(\boldsymbol{I}_n+\lambda\boldsymbol{D}_p'\boldsymbol{D}_p)^{-1}$ in (2.3) is the smoother matrix of the WH graduation and its elements are the smoother weights of the smoothing. Recently, De Jong and Sakarya (2016) provided an explicit formula for the smoother weights of the WH graduation of order 2, following which Cornea-Madeira (2017) gave a simpler explicit formula.[2] In this paper, we contribute to the literature by providing an explicit formula for the smoother weights of the graduation of order 1 along with some related results, which enables us to gain a richer understanding of the filter. We note here that to derive the explicit formula, we apply a different approach to that of Cornea-Madeira (2017). This is mainly because, in the case under consideration, the approach given in the present paper leads to a simpler formula. Later, for comparison, we show another formula based on Cornea-Madeira's (2017) approach.

This chapter is organized as follows. In Section 2.2, we provide the main results. In Section 2.3, we show the formula based on Cornea-Madeira's (2017) approach. Section 2.4 concludes the paper.

[2] In econometrics, the WH graduation of order 2 is referred to as the Hodrick–Prescott (1997) filter.

2.2 The main results

Recall that $\boldsymbol{D}_1\boldsymbol{D}_1' = \boldsymbol{Q}_{n-1}$, where

$$\boldsymbol{Q}_r = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \in \mathbb{R}^{r\times r},$$

which is a well-known symmetric tridiagonal Toeplitz matrix (Strang and MacNamara, 2014). From the three-term recurrence relation of multiple angles in sine function, the spectral decomposition of $\boldsymbol{Q}_r$ is:[3] $\boldsymbol{Q}_r = \boldsymbol{V}_r\boldsymbol{\Lambda}_r\boldsymbol{V}_r'$, where $\boldsymbol{\Lambda}_r = \mathsf{diag}(\lambda_{r,1},\ldots,\lambda_{r,r})$ and $\boldsymbol{V}_r = [v_{r,i,j}]_{i,j=1,\ldots,r}$ with

$$\lambda_{r,k} = 2-2\cos\left(\frac{k\pi}{r+1}\right), \quad k=1,\ldots,r, \tag{2.4}$$

$$v_{r,i,j} = \sqrt{\frac{2}{r+1}}\sin\left(\frac{ij\pi}{r+1}\right), \quad i,j=1,\ldots,r. \tag{2.5}$$

Hence the spectral decomposition for $\boldsymbol{D}_1\boldsymbol{D}_1'$ is

$$\boldsymbol{D}_1\boldsymbol{D}_1' = \boldsymbol{V}_{n-1}\boldsymbol{\Lambda}_{n-1}\boldsymbol{V}_{n-1}'. \tag{2.6}$$

We note that (i) $0<\lambda_{n-1,1}<\cdots<\lambda_{n-1,n-1}<4$ and (ii) $\boldsymbol{V}_{n-1}$ is an orthogonal matrix.[4] We apply (2.6) to derive an explicit formula for the smoother weights of the WH graduation of order 1.

Theorem 2.1.

$$\widehat{\boldsymbol{x}}_1 = \boldsymbol{y}-\boldsymbol{D}_1'\boldsymbol{V}_{n-1}(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{\Lambda}_{n-1})^{-1}\boldsymbol{V}_{n-1}'\boldsymbol{D}_1\boldsymbol{y}. \tag{2.7}$$

[3] Pesaran (1973) used the spectral decomposition of a more general matrix.

[4] Also, from $\mathsf{trace}(\boldsymbol{D}_1\boldsymbol{D}_1') = 2(n-1)$ and $|\boldsymbol{D}_1\boldsymbol{D}_1'| = n$, it follows that $\sum_{k=1}^{n-1}\lambda_{n-1,k} = 2(n-1)$ and $\prod_{k=1}^{n-1}\lambda_{n-1,k} = n$.

Proof. Applying the Woodbury matrix identity to $(\boldsymbol{I}_n+\lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1}$, it follows that

$$(\boldsymbol{I}_n+\lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1}=\boldsymbol{I}_n-\boldsymbol{D}_1'(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{D}_1\boldsymbol{D}_1')^{-1}\boldsymbol{D}_1. \tag{2.8}$$

Moreover, from (2.6), the right-hand side of (2.8) may be rewritten as

$$\begin{aligned}\boldsymbol{I}_n-\boldsymbol{D}_1'(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{D}_1\boldsymbol{D}_1')^{-1}\boldsymbol{D}_1&=\boldsymbol{I}_n-\boldsymbol{D}_1'(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{V}_{n-1}\boldsymbol{\Lambda}_{n-1}\boldsymbol{V}_{n-1}')^{-1}\boldsymbol{D}_1\\&=\boldsymbol{I}_n-\boldsymbol{D}_1'\boldsymbol{V}_{n-1}(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{\Lambda}_{n-1})^{-1}\boldsymbol{V}_{n-1}'\boldsymbol{D}_1, \end{aligned}\tag{2.9}$$

which proves (2.7). □

Denote the (i,j) entry of $(\boldsymbol{I}_n+\lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1}$ by $q_{i,j}$ and the Kronecker delta by $\delta_{i,j}$. Let $\widehat{\boldsymbol{x}}_1=[\widehat{x}_{1,1},\dots,\widehat{x}_{1,n}]'$.

Corollary 2.1. *Denote the* (i,j) *entry of* $\boldsymbol{D}_1'\boldsymbol{V}_{n-1}(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{\Lambda}_{n-1})^{-1}\boldsymbol{V}_{n-1}'\boldsymbol{D}_1$ *by* $p_{i,j}$.

$$q_{i,j}=\delta_{i,j}-p_{i,j}=\delta_{i,j}-\sum_{k=1}^{n-1}\frac{(v_{n-1,i,k}-v_{n-1,i-1,k})(v_{n-1,j,k}-v_{n-1,j-1,k})}{\lambda^{-1}+\lambda_{n-1,k}} \tag{2.10}$$

$$=\delta_{i,j}-\frac{2}{n}\sum_{k=1}^{n-1}\frac{\left\{\sin\left(\frac{i\pi k}{n}\right)-\sin\left(\frac{(i-1)\pi k}{n}\right)\right\}\left\{\sin\left(\frac{j\pi k}{n}\right)-\sin\left(\frac{(j-1)\pi k}{n}\right)\right\}}{\lambda^{-1}+2-2\cos\left(\frac{\pi k}{n}\right)} \tag{2.11}$$

for $i,j=1,\dots,n$.

Proof. Firstly, $(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{\Lambda}_{n-1})^{-1}$ in (2.7) may be represented as

$$\begin{aligned}&(\lambda^{-1}\boldsymbol{I}_{n-1}+\boldsymbol{\Lambda}_{n-1})^{-1}\\&\quad=\mathsf{diag}\left\{(\lambda^{-1}+\lambda_{n-1,1})^{-1},\dots,(\lambda^{-1}+\lambda_{n-1,n-1})^{-1}\right\}\\&\quad=\mathsf{diag}\left[\left\{\lambda^{-1}+2-2\cos\left(\frac{1\pi}{n}\right)\right\}^{-1},\dots,\left\{\lambda^{-1}+2-2\cos\left(\frac{(n-1)\pi}{n}\right)\right\}^{-1}\right].\end{aligned}$$

Secondly, because

$$v_{n-1,0,j}=\sqrt{\frac{2}{n}}\sin\left(\frac{0\cdot j\pi}{n}\right)=0,\quad v_{n-1,n,j}=\sqrt{\frac{2}{n}}\sin\left(\frac{n\cdot j\pi}{n}\right)=0,$$

for $j = 1, \ldots, n-1$, $\boldsymbol{D}_1'\boldsymbol{V}_{n-1}$ in (2.7) is expressed by

$$\begin{aligned}
&\boldsymbol{D}_1'\boldsymbol{V}_{n-1} \\
&= \begin{bmatrix} v_{n-1,0,1} - v_{n-1,1,1} & \cdots & v_{n-1,0,n-1} - v_{n-1,1,n-1} \\ \vdots & & \vdots \\ v_{n-1,n-1,1} - v_{n-1,n,1} & \cdots & v_{n-1,n-1,n-1} - v_{n-1,n,n-1} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\frac{2}{n}}\left\{\sin\left(\frac{0\cdot 1\pi}{n}\right) - \sin\left(\frac{1\cdot 1\pi}{n}\right)\right\} & \cdots & \sqrt{\frac{2}{n}}\left\{\sin\left(\frac{0\cdot(n-1)\pi}{n}\right) - \sin\left(\frac{1\cdot(n-1)\pi}{n}\right)\right\} \\ \vdots & & \vdots \\ \sqrt{\frac{2}{n}}\left\{\sin\left(\frac{(n-1)\cdot 1\pi}{n}\right) - \sin\left(\frac{n\cdot 1\pi}{n}\right)\right\} & \cdots & \sqrt{\frac{2}{n}}\left\{\sin\left(\frac{(n-1)\cdot(n-1)\pi}{n}\right) - \sin\left(\frac{n\cdot(n-1)\pi}{n}\right)\right\} \end{bmatrix}.
\end{aligned}$$

Using this result, (2.10) and (2.11) immediately follow from Theorem 2.1. □

Let $\boldsymbol{\iota} = [1, \ldots, 1]' \in \mathbb{R}^n$.

Corollary 2.2. *For given $i, j = 1, 2, \ldots$,*

$$q_{i,j} \to \delta_{i,j} - 2\lambda \int_0^1 \frac{\{\sin(i\pi r) - \sin((i-1)\pi r)\}\{\sin(j\pi r) - \sin((j-1)\pi r)\}}{1 + 4\lambda \sin^2(0.5\pi r)} dr, \quad (n \to \infty). \tag{2.12}$$

For $i, j = 1, \ldots, n$,

$$q_{i,j} \to \frac{1}{n}, \quad (\lambda \to \infty), \tag{2.13}$$

and

$$q_{i,j} \to \delta_{i,j}, \quad (\lambda \to 0). \tag{2.14}$$

Proof. (2.12) immediately follows from (2.11). Recalling that $\boldsymbol{I}_n - \boldsymbol{D}_1'(\boldsymbol{D}_1\boldsymbol{D}_1')^{-1}\boldsymbol{D}_1 = \frac{1}{n}\boldsymbol{\iota}\boldsymbol{\iota}'$, (2.13) follows from (2.7). Finally, (2.14) is evident from (2.10). □

Proposition 2.1. $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1'\boldsymbol{D}_1)^{-1}$ *is a symmetric centrosymmetric matrix.*

Proof. We only prove that $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)^{-1}$ is centrosymmetric. Let $\boldsymbol{J}_n \in \mathbb{R}^{n \times n}$ be the exchange/permutation matrix:

$$\boldsymbol{J}_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Because

$$\boldsymbol{D}_1' \boldsymbol{D}_1 = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{n \times n} \tag{2.15}$$

is a centrosymmetric matrix, $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)$ is also a centrosymmetric matrix. Accordingly, it follows that $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1) = \boldsymbol{J}_n (\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1) \boldsymbol{J}_n$. Recalling that $\boldsymbol{J}_n^{-1} = \boldsymbol{J}_n$, we obtain

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)^{-1} = \boldsymbol{J}_n (\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)^{-1} \boldsymbol{J}_n,$$

which proves that $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)^{-1}$ is a centrosymmetric matrix. $\square$

Remarks. From Proposition 2.1, we immediately obtain $q_{n-i+1,n-j+1} = q_{i,j}$ for $i, j = 1, \ldots, n$. For example, $q_{n,n-2} = q_{1,3}$, $q_{n,n-1} = q_{1,2}$ and $q_{n,n} = q_{1,1}$. Thus, for large n, $q_{n-i+1,n-j+1}$ approximately equals the right-hand side of (2.12).

We also remark that it follows immediately from (2.8) and $\boldsymbol{D}_1 \boldsymbol{\iota} = \boldsymbol{0}$ that

$$\sum_{j=1}^{n} q_{i,j} = 1, \quad i = 1, \ldots, n, \tag{2.16}$$

and $\frac{1}{n}\sum_{i=1}^{n}\widehat{x}_{1,i} = \frac{1}{n}\sum_{i=1}^{n} y_i$.[5]

2.3 Formulas based on Cornea-Madeira's (2017) approach

From (2.15), it follows that $\boldsymbol{D}_1'\boldsymbol{D}_1 = \boldsymbol{Q}_n - \boldsymbol{e}_1\boldsymbol{e}_1' - \boldsymbol{e}_n\boldsymbol{e}_n'$, where $\boldsymbol{e}_1 = [1,0,\dots,0]' \in \mathbb{R}^n$ and $\boldsymbol{e}_n = [0,\dots,0,1]' \in \mathbb{R}^n$. Accordingly, we obtain

$$(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = (\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1' - \lambda\boldsymbol{e}_n\boldsymbol{e}_n')^{-1}, \tag{2.17}$$

where $\boldsymbol{A} = \boldsymbol{I}_n + \lambda\boldsymbol{Q}_n$. In the same way as Cornea-Madeira (2017), we apply the Sherman–Morrison formula twice to (2.17), we obtain the following results:

$$(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = (\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1')^{-1} + \lambda\frac{(\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1')^{-1}\boldsymbol{e}_n\boldsymbol{e}_n'(\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1')^{-1}}{1 - \lambda\boldsymbol{e}_n'(\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1')^{-1}\boldsymbol{e}_n},$$

where

$$(\boldsymbol{A} - \lambda\boldsymbol{e}_1\boldsymbol{e}_1')^{-1} = \boldsymbol{A}^{-1} + \lambda\frac{\boldsymbol{A}^{-1}\boldsymbol{e}_1\boldsymbol{e}_1'\boldsymbol{A}^{-1}}{1 - \lambda\boldsymbol{e}_1'\boldsymbol{A}^{-1}\boldsymbol{e}_1}.$$

Let $\boldsymbol{A}^{-1} = [b_{i,j}]_{i,j=1,\dots,n}$ and denote the first and last column of $\boldsymbol{A}^{-1}$ by $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_n$, respectively. We obtain the following results:

Theorem 2.2.

$$(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = \boldsymbol{A}^{-1} + \frac{2\lambda(\kappa_{11}\boldsymbol{\beta}_1\boldsymbol{\beta}_1' + \kappa_{1n}\boldsymbol{\beta}_1\boldsymbol{\beta}_n' + \kappa_{n1}\boldsymbol{\beta}_n\boldsymbol{\beta}_1' + \kappa_{nn}\boldsymbol{\beta}_n\boldsymbol{\beta}_n')}{\kappa_{11}^2 - \kappa_{1n}^2 - \kappa_{n1}^2 + \kappa_{nn}^2}, \tag{2.18}$$

where $\kappa_{11} = 1 - \lambda b_{1,1}$, $\kappa_{1n} = \lambda b_{1,n}$, $\kappa_{n1} = \lambda b_{n,1}$, *and* $\kappa_{nn} = 1 - \lambda b_{n,n}$.

Proof. See the Appendix. □

[5] Some related discussion may be found in Yamada (2015, 2018).

Since $\boldsymbol{A}^{-1} = \left\{\boldsymbol{V}_n(\boldsymbol{I}_n + \lambda\boldsymbol{\Lambda}_n)\boldsymbol{V}_n'\right\}^{-1} = \boldsymbol{V}_n(\boldsymbol{I}_n + \lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{V}_n'$, we obtain

$$b_{i,j} = \sum_{k=1}^{n}\left(\frac{v_{n,i,k}v_{n,j,k}}{1+\lambda\lambda_{n,k}}\right) = \frac{2}{n+1}\sum_{k=1}^{n}\frac{\sin\left(\frac{i\pi k}{n+1}\right)\sin\left(\frac{j\pi k}{n+1}\right)}{1+\lambda\left\{2-2\cos\left(\frac{\pi k}{n+1}\right)\right\}}, \quad i,j=1,\ldots,n. \tag{2.19}$$

Corollary 2.3. *Let $(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = [q_{i,j}]_{i,j=1,\ldots,n}$. We have*

$$q_{i,j} = b_{i,j} + \frac{2\lambda\left(\kappa_{11}b_{i,1}b_{j,1} + \kappa_{1n}b_{i,1}b_{j,n} + \kappa_{n1}b_{i,n}b_{j,1} + \kappa_{nn}b_{i,n}b_{j,n}\right)}{\kappa_{11}^2 - \kappa_{1n}^2 - \kappa_{n1}^2 + \kappa_{nn}^2}, \tag{2.20}$$

where $\kappa_{11} = 1-\lambda b_{1,1}$, $\kappa_{1n} = \lambda b_{1,n}$, $\kappa_{n1} = \lambda b_{n,1}$, and $\kappa_{nn} = 1-\lambda b_{n,n}$.

Let $\boldsymbol{n}_1 = \{1,3,\ldots,n\}$ if n is odd, $\boldsymbol{n}_1 = \{1,3,\ldots,n-1\}$ if n is even, $\boldsymbol{n}_2 = \{2,4,\ldots,n-1\}$ if n is odd, and $\boldsymbol{n}_2 = \{2,4,\ldots,n\}$ if n is even. The following result corresponds to Cornea-Madeira's (2017) Theorem 1:

Corollary 2.4. *Let $\phi_i = 2\lambda\left(1-2\lambda\sum_{k\in\boldsymbol{n}_i}v_{n,1,k}^2\gamma_k^{-1}\right)^{-1}$ for $i=1,2$, where $\gamma_k = 1+\lambda\lambda_{n,k}$ for $k=1,\ldots,n$. Then, it follows that*

$$(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = \boldsymbol{V}_n(\boldsymbol{I}_n + \lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{V}_n' + \phi_1\boldsymbol{V}_n\boldsymbol{K}_1\boldsymbol{V}_n' + \phi_2\boldsymbol{V}_n\boldsymbol{K}_2\boldsymbol{V}_n', \tag{2.21}$$

where $\boldsymbol{K}_r = [k_{r,i,j}]_{i,j=1,\ldots,n}$ for $r=1,2$ is such that

$$k_{1,i,j} = \begin{cases}\frac{v_{n,i,1}v_{n,j,1}}{\gamma_i\gamma_j}, & i+j: even,\ j: odd\\ 0, & otherwise\end{cases}; \quad k_{2,i,j} = \begin{cases}\frac{v_{n,i,1}v_{n,j,1}}{\gamma_i\gamma_j}, & i+j: even,\ j: even\\ 0, & otherwise\end{cases}.$$

Proof. See the Appendix. □

Finally, concerning $b_{i,j}$ in Corollary 2.3, it follows from (2.19) that, for given $i,j = 1,2,\ldots$,

$$b_{i,j} \to 2\int_0^1 \frac{\sin(i\pi r)\sin(j\pi r)}{1+4\lambda\sin^2(0.5\pi r)}dr, \quad (n\to\infty). \tag{2.22}$$

2.4 Concluding remarks

In this chapter, we provided explicit formulas for the smoother weights of the WH graduation of order 1 and some related results. The results obtained in the chapter are summarized in Theorems 2.1 and 2.2, in Corollaries 2.1, 2.2, 2.3, and 2.4, and in Proposition 2.1.

We note that although we may consider algorithms based on the explicit formulas derived in the paper, they are not necessarily recommended for practical use when n is large, because they may not be numerically efficient even though the smoother matrix is a symmetric centrosymmetric matrix. An efficient algorithm that reduces execution time and memory use is obtainable by performing a Cholesky decomposition of $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)$ and then solving the resulting triangular systems. See Weinert (2007) for further details.

2.5 Numerical example

As an example, we are going to find the inverse of $(\boldsymbol{I}_5 + \lambda \boldsymbol{D}_1' \boldsymbol{D}_1)^{-1} \in \mathbb{R}^{5\times 5}$, where $\lambda = 1$. We know that, the first-order difference matrix

$$\boldsymbol{D}_1 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 5},$$

and

$$\boldsymbol{D}_1' \boldsymbol{D}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{5\times 5}. \tag{2.23}$$

A 5×5 tridiagonal Toeplitz matrix $\boldsymbol{Q}_5$ is as follows

$$\boldsymbol{Q}_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}. \tag{2.24}$$

Now, from equations (2.23) and (2.24) we get the following relation:

$$\boldsymbol{D}_1' \boldsymbol{D}_1 = \boldsymbol{Q}_5 - \boldsymbol{e}_1 \boldsymbol{e}_1' - \boldsymbol{e}_5 \boldsymbol{e}_5', \tag{2.25}$$

where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \boldsymbol{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

are unit vectors.
Note that $\boldsymbol{Q}_5$ has distinct eigenvalues

$$\gamma_j = 2 - 2\cos\left(\frac{\pi j}{6}\right), \quad j = 1, ..., 5,$$

and corresponding eigenvector $\boldsymbol{x}_j = [x_{1,j}, ..., x_{5,j}]'$ with

$$x_{i,j} = \left(\frac{1}{3}\right)^{1/2} \sin\left(\frac{\pi i j}{6}\right), \quad i, j = 1, ..., 5.$$

Let, $\boldsymbol{A} = \boldsymbol{I}_5 + \lambda \boldsymbol{Q}_5$. Now we consider the eigenvalue matrix of $(\boldsymbol{I}_5 + \lambda \boldsymbol{Q}_5)$ is

$$\boldsymbol{\Lambda} = \mathsf{diag}(\lambda_1, ..., \lambda_5),$$

with

$$\lambda_j = 1 + \lambda \gamma_j,$$

We have

$$A^{-1} = (I_5 + \lambda Q_5)^{-1} = \begin{bmatrix} 0.3819444 & 0.1458333 & 0.0555556 & 0.0208333 & 0.0069444 \\ 0.1458333 & 0.4375000 & 0.1666667 & 0.0625000 & 0.0208333 \\ 0.0555556 & 0.1666667 & 0.4444444 & 0.1666667 & 0.0555556 \\ 0.0208333 & 0.0625000 & 0.1666667 & 0.4375000 & 0.1458333 \\ 0.0069444 & 0.0208333 & 0.0555556 & 0.1458333 & 0.3819444 \end{bmatrix},$$

and let $A^{-1} = [b_{i,j}]_{i,j=1,\ldots,5}$ and denote the first and last column of A^{-1} by β_1 and β_5 respectively.

Now, according to the Theorem 2.2, the explicit inverse is:

$$\begin{aligned}(I_5 + \lambda D_1' D_1)^{-1} &= (I_5 + \lambda Q_5 - \lambda e_1 e_1' - \lambda e_5 e_5')^{-1} \\ &= (A - \lambda e_1 e_1' - \lambda e_5 e_5')^{-1} \\ &= A^{-1} + \frac{2\lambda(\kappa_{1,1}\beta_1\beta_1' + \kappa_{1,5}\beta_1\beta_5' + \kappa_{5,1}\beta_5\beta_1' + \kappa_{5,5}\beta_5\beta_5')}{\kappa_{1,1}^2 - \kappa_{1,5}^2 - \kappa_{5,1}^2 + \kappa_{5,5}^2}. \end{aligned} \tag{2.26}$$

Here, $\kappa_{1,1} = 1 - \lambda b_{1,1}$, $\kappa_{1,5} = \lambda b_{1,5}, \kappa_{5,1} = \lambda b_{5,1}$, and $\kappa_{5,5} = 1 - \lambda b_{5,5}$.
So, now we get

$$(I_5 + \lambda D_1' D_1)^{-1} = \begin{bmatrix} 0.618182 & 0.236364 & 0.090909 & 0.036364 & 0.018182 \\ 0.236364 & 0.472727 & 0.181818 & 0.072727 & 0.036364 \\ 0.090909 & 0.181818 & 0.454545 & 0.181818 & 0.090909 \\ 0.036364 & 0.072727 & 0.181818 & 0.472727 & 0.236364 \\ 0.018182 & 0.036364 & 0.090909 & 0.236364 & 0.618182 \end{bmatrix}.$$

2.6 Appendix

Proof of Theorem 2.2

Since $\boldsymbol{A}^{-1} = [b_{i,j}]_{i,j=1,\ldots,n}$ is a centrosymmetric matrix, $b_{n,n} = b_{1,1}$ and $b_{n,1} = b_{1,n}$ and accordingly $\kappa_{11} = \kappa_{nn}$ and $\kappa_{1n} = \kappa_{n1}$. From $(\boldsymbol{A} - \lambda \boldsymbol{e}_1\boldsymbol{e}_1')^{-1} = \boldsymbol{A}^{-1} + \lambda \frac{\boldsymbol{A}^{-1}\boldsymbol{e}_1\boldsymbol{e}_1'\boldsymbol{A}^{-1}}{1-\lambda\boldsymbol{e}_1'\boldsymbol{A}^{-1}\boldsymbol{e}_1} = \boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1'$, we obtain

$$
\begin{aligned}
(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} &= (\boldsymbol{A} - \lambda \boldsymbol{e}_1\boldsymbol{e}_1')^{-1} + \lambda \frac{(\boldsymbol{A} - \lambda \boldsymbol{e}_1\boldsymbol{e}_1')^{-1}\boldsymbol{e}_n\boldsymbol{e}_n'(\boldsymbol{A} - \lambda \boldsymbol{e}_1\boldsymbol{e}_1')^{-1}}{1-\lambda\boldsymbol{e}_n'(\boldsymbol{A} - \lambda \boldsymbol{e}_1\boldsymbol{e}_1')^{-1}\boldsymbol{e}_n} \\
&= \boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1' + \lambda \frac{(\boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1')\boldsymbol{e}_n\boldsymbol{e}_n'(\boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1')}{1-\lambda\boldsymbol{e}_n'(\boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1')\boldsymbol{e}_n} \\
&= \boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1' + \lambda \frac{(\boldsymbol{\beta}_n + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1)(\boldsymbol{\beta}_n' + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1')}{1-\lambda(b_{n,n} + \lambda\kappa_{11}^{-1}b_{n,1}^2)} \\
&= \boldsymbol{A}^{-1} + \lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_1' \\
&\qquad + \lambda \frac{\boldsymbol{\beta}_n\boldsymbol{\beta}_n' + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_n' + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_n\boldsymbol{\beta}_1' + b_{n,1}^2\lambda^2\kappa_{11}^{-2}\boldsymbol{\beta}_1\boldsymbol{\beta}_1'}{1-\lambda(b_{n,n} + \lambda\kappa_{11}^{-1}b_{n,1}^2)}.
\end{aligned}
$$

Since $b_{1,1} = b_{n,n}$, it follows that $\lambda\kappa_{11}^{-1}\{1-\lambda(b_{n,n} + \lambda\kappa_{11}^{-1}b_{n,1}^2)\} + \lambda b_{n,1}^2\lambda^2\kappa_{11}^{-2} = \lambda(1-\lambda b_{1,1})^{-1}(1-\lambda b_{n,n}) = \lambda$. Hence, we obtain

$$
\begin{aligned}
(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} &= \boldsymbol{A}^{-1} + \lambda \frac{\boldsymbol{\beta}_1\boldsymbol{\beta}_1' + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_1\boldsymbol{\beta}_n' + b_{n,1}\lambda\kappa_{11}^{-1}\boldsymbol{\beta}_n\boldsymbol{\beta}_1' + \boldsymbol{\beta}_n\boldsymbol{\beta}_n'}{1-\lambda(b_{n,n} + \lambda\kappa_{11}^{-1}b_{n,1}^2)} \\
&= \boldsymbol{A}^{-1} + \lambda \frac{\kappa_{11}\boldsymbol{\beta}_1\boldsymbol{\beta}_1' + \kappa_{1n}\boldsymbol{\beta}_1\boldsymbol{\beta}_n' + \kappa_{n1}\boldsymbol{\beta}_n\boldsymbol{\beta}_1' + \kappa_{nn}\boldsymbol{\beta}_n\boldsymbol{\beta}_n'}{\kappa_{11}^2 - \kappa_{n1}^2}, \qquad (2.27)
\end{aligned}
$$

which leads to (2.18).

Proof of Corollary 2.4

Since $\boldsymbol{A}^{-1}$ is a centrosymmetric matrix, $b_{n,n} = b_{1,1}$ and $b_{n,1} = b_{1,n}$. Accordingly, (2.27) may be rewritten as

$$
(\boldsymbol{I}_n + \lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\{w_1(\boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_n\boldsymbol{e}_n') + w_2(\boldsymbol{e}_1\boldsymbol{e}_n' + \boldsymbol{e}_n\boldsymbol{e}_1')\}\boldsymbol{A}^{-1},
$$

where

$$w_1 = \frac{\lambda(1-\lambda b_{1,1})}{(1-\lambda b_{1,1})^2-(\lambda b_{1,n})^2}, \quad w_2 = \frac{\lambda^2 b_{1,n}}{(1-\lambda b_{1,1})^2-(\lambda b_{1,n})^2}.$$

Since $\boldsymbol{A}^{-1} = \boldsymbol{V}_n(\boldsymbol{I}_n+\lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{V}_n'$, we obtain

$$(\boldsymbol{I}_n+\lambda\boldsymbol{D}_1'\boldsymbol{D}_1)^{-1} = \boldsymbol{V}_n(\boldsymbol{I}_n+\lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{V}_n' + \boldsymbol{V}_n(\boldsymbol{I}_n+\lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{H}(\boldsymbol{I}_n+\lambda\boldsymbol{\Lambda}_n)^{-1}\boldsymbol{V}_n',$$

where, letting $\boldsymbol{V}_n = [\boldsymbol{v}_1,\ldots,\boldsymbol{v}_n]$,

$$\boldsymbol{H} = \boldsymbol{V}_n\{w_1(\boldsymbol{e}_1\boldsymbol{e}_1'+\boldsymbol{e}_n\boldsymbol{e}_n')+w_2(\boldsymbol{e}_1\boldsymbol{e}_n'+\boldsymbol{e}_n\boldsymbol{e}_1')\}\boldsymbol{V}_n' = w_1(\boldsymbol{v}_1\boldsymbol{v}_1'+\boldsymbol{v}_n\boldsymbol{v}_n')+w_2(\boldsymbol{v}_1\boldsymbol{v}_n'+\boldsymbol{v}_n\boldsymbol{v}_1').$$

Since the (i,j) entry of $\boldsymbol{v}_1\boldsymbol{v}_1'$, $\boldsymbol{v}_n\boldsymbol{v}_n'$, $\boldsymbol{v}_1\boldsymbol{v}_n'$, and $\boldsymbol{v}_n\boldsymbol{v}_1'$ is respectively $v_{n,i,1}v_{n,j,1}$, $v_{n,i,n}v_{n,j,n}$, $v_{n,i,1}v_{n,j,n}$, and $v_{n,i,n}v_{n,j,1}$, it follows that

$$h_{i,j} = w_1v_{n,i,1}v_{n,j,1}+w_1v_{n,i,n}v_{n,j,n}+w_2v_{n,i,1}v_{n,j,n}+w_2v_{n,i,n}v_{n,j,1}, \quad i,j=1,\ldots,n,$$

where $h_{i,j}$ denotes (i,j) entry of $\boldsymbol{H}$. In addition, we have

$$\begin{aligned} v_{n,k,n} &= \sqrt{\frac{2}{n+1}}\sin\left(\frac{nk\pi}{n+1}\right) = \sqrt{\frac{2}{n+1}}\sin\left\{\frac{(n+1)k\pi}{n+1}-\frac{k\pi}{n+1}\right\} \\ &= -\sqrt{\frac{2}{n+1}}\left\{\cos\frac{(n+1)k\pi}{n+1}\sin\frac{k\pi}{n+1}\right\} = -\cos(k\pi)\sqrt{\frac{2}{n+1}}\sin\left(\frac{k\pi}{n+1}\right) \\ &= -\cos(k\pi)v_{n,k,1} = -(-1)^k v_{n,k,1} = (-1)^{k+1}v_{n,k,1}. \end{aligned} \tag{2.28}$$

Accordingly, we obtain

$$\begin{aligned} h_{i,j} &= w_1v_{n,i,1}v_{n,j,1}+w_1v_{n,i,n}v_{n,j,n}+w_2v_{n,i,1}v_{n,j,n}+w_2v_{n,i,n}v_{n,j,1} \\ &= w_1v_{n,i,1}v_{n,j,1}+w_1(-1)^{i+1}v_{n,i,1}(-1)^{j+1}v_{n,j,1}+w_2v_{n,i,1}(-1)^{j+1}v_{n,j,1}+w_2(-1)^{i+1}v_{n,i,1}v_{n,j,1} \\ &= \{w_1+(-1)^{i+j+2}w_1+(-1)^{j+1}w_2+(-1)^{i+1}w_2\}v_{n,i,1}v_{n,j,1} \\ &= \{1+(-1)^{i+j}\}\{w_1+(-1)^{j+1}w_2\}v_{n,i,1}v_{n,j,1}, \quad i,j=1,\ldots,n. \end{aligned}$$

Thus, we have

$$h_{i,j} = \begin{cases} 0 & (i+j : \text{odd}) \\ 2(w_1+w_2)v_{n,i,1}v_{n,j,1} & (i+j : \text{even},\ j : \text{odd}) \\ 2(w_1-w_2)v_{n,i,1}v_{n,j,1} & (i+j : \text{even},\ j : \text{even}) \end{cases}$$

for $i, j = 1, \ldots, n$. From (2.5), it follows that $v_{n,i,j} = v_{n,j,i}$ for $i, j = 1, \ldots, n$. Then, from (2.28), we obtain $v_{n,n,k} = (-1)^{k+1}v_{n,1,k}$. Thus, we have

$$b_{1,n} = \sum_{k=1}^{n} \left(\frac{v_{n,1,k}v_{n,n,k}}{\gamma_k} \right) = \sum_{k=1}^{n} (-1)^{k+1} v_{n,1,k}^2 \gamma_k^{-1}.$$

In addition, $b_{1,1} = \sum_{k=1}^{n} v_{n,1,k}^2 \gamma_k^{-1}$. Then, using these results yields

$$\begin{aligned} w_1 + w_2 &= \frac{\lambda(1-\lambda b_{1,1}) + \lambda^2 b_{1,n}}{(1-\lambda b_{1,1})^2 - (\lambda b_{1,n})^2} = \frac{\lambda(1-\lambda b_{1,1}+\lambda b_{1,n})}{(1-\lambda b_{1,1}-\lambda b_{1,n})(1-\lambda b_{1,1}+\lambda b_{1,n})} \\ &= \frac{\lambda}{1-\lambda b_{1,1}-\lambda b_{1,n}} = \lambda \left(1 - 2\lambda \sum_{k \in \boldsymbol{n}_1} v_{n,1,k}^2 \gamma_k^{-1} \right)^{-1}. \end{aligned}$$

Likewise, we obtain

$$\begin{aligned} w_1 - w_2 = &= \frac{\lambda(1-\lambda b_{1,1}) - \lambda^2 b_{1,n}}{(1-\lambda b_{1,1})^2 - (\lambda b_{1,n})^2} = \frac{\lambda(1-\lambda b_{1,1}-\lambda b_{1,n})}{(1-\lambda b_{1,1}-\lambda b_{1,n})(1-\lambda b_{1,1}+\lambda b_{1,n})} \\ &= \frac{\lambda}{1-\lambda b_{1,1}+\lambda b_{1,n}} = \lambda \left(1 - 2\lambda \sum_{k \in \boldsymbol{n}_2} v_{n,1,k}^2 \gamma_k^{-1} \right)^{-1}. \end{aligned}$$

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Chapter 3

An Explicit Formula for the Smoother Weights of the Hodrick–Prescott Filter

This chapter basically based on a previously published article [Yamada, H. and F. T. Jahra, 2019].

3.1 Introduction

The Hodrick–Prescott (HP) (1997) filter is a popular method to estimate the trend component of univariate time series. It is described as a penalized least squares problem and a special case of the Whittaker–Henderson (WH) method of graduation:[1]

$$
\begin{aligned}
\widehat{\boldsymbol{x}} &= \underset{x_1,\dots,x_T \in \mathbb{R}}{\arg\min} \left[\sum_{t=1}^{T} (y_t - x_t)^2 + \lambda \sum_{t=3}^{T} (\Delta^2 x_t)^2 \right] \\
&= \underset{\boldsymbol{x} \in \mathbb{R}^T}{\arg\min} \left(\|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \|\boldsymbol{D}\boldsymbol{x}\|^2 \right) \\
&= (\boldsymbol{I}_T + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1} \boldsymbol{y},
\end{aligned} \tag{3.1}
$$

where $\boldsymbol{y} = [y_1,\dots,y_T]'$ denotes univariate time series T observations, $\boldsymbol{x} = [x_1,\dots,x_T]'$, $\lambda > 0$ is a smoothing parameter, $\boldsymbol{I}_T$ is the $T \times T$ identity matrix, and $\boldsymbol{D}$ denotes the $(T-2) \times T$ second-order difference matrix such that $\boldsymbol{D}\boldsymbol{x} = [\Delta^2 x_3,\dots,\Delta^2 x_T]'$ with $\Delta^2 x_t = \Delta x_t - \Delta x_{t-1} =$

[1] In the actuarial sciences, the WH method of graduation has been used for mortality table construction. For historical remarks on the filter, see Weinert (2007).

$x_t - 2x_{t-1} + x_{t-2}$ for $t = 3, \ldots, T$. Explicitly, $\boldsymbol{D}$ is the $(T-2) \times T$ Toeplitz matrix of which the first and last rows are $[1, -2, 1, 0, \ldots, 0]$ and $[0, \ldots, 0, 1, -2, 1]$, respectively. The $T \times T$ matrix $(\boldsymbol{I}_T + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (3.1) is referred to as the smoother matrix of the HP filter.

De Jong and Sakarya (2016, Theorem 1) provided an explicit formula for the smoother weights of the HP filter, following which, Cornea-Madeira (2017, Theorem 1) provided a simpler explicit formula. Both of these works applied the Sherman–Morrison–Woodbury (SMW) formula to the following form of matrix:

$$(\boldsymbol{\Omega} + \lambda \boldsymbol{\zeta}_1 \boldsymbol{\zeta}_1' + \lambda \boldsymbol{\zeta}_2 \boldsymbol{\zeta}_2')^{-1}, \tag{3.2}$$

where $\boldsymbol{\Omega}$ is a nonsingular matrix whose inverse is easily obtainable, and both $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ are column vectors. In this paper, we provide a simpler alternative formula for the smoother weights of the HP filter. The reason such a simpler formula is obtainable is that in our approach both $\boldsymbol{\zeta}_1$ and $\boldsymbol{\zeta}_2$ in (3.2) are *unit vectors*.

In addition to the above papers, we list two other papers related to this paper. The first one is Wang et al. (2015), which developed a method for deriving the explicit inverse of a pentadiagonal (five-diagonal) Toeplitz matrix. Our approach may be regarded as an application of Wang et al. (2015). The second one is Yamada and Jahra (2018), which derived explicit formulas for the smoother weights of the exponential smoothing filter (King and Rebelo, 1993), which is also a special case of the WH method of graduation.

The chapter is organized as follows. In Section 3.2, we provide a literature review. In Section 3.3, we show another explicit formula for the smoother weights of the HP filter. Section 3.4 concludes.

3.2 A literature review

In this section, we briefly review two closely related papers: De Jong and Sakarya (2016) and Cornea-Madeira (2017).

3.2.1 De Jong and Sakarya (2016)

Let $\boldsymbol{x} = \boldsymbol{\Gamma\theta}$, where $\boldsymbol{\Gamma}$ is a $T \times T$ nonsingular matrix and $\boldsymbol{\theta}$ is a T-dimensional column vector. Then, the HP filter defined by (3.1) may be represented as $\widehat{\boldsymbol{x}} = \boldsymbol{\Gamma}\widehat{\boldsymbol{\theta}}$, where

$$\begin{aligned}\widehat{\boldsymbol{\theta}} &= \underset{\boldsymbol{\theta}\in\mathbb{R}^T}{\arg\min}\left(\|\boldsymbol{y}-\boldsymbol{\Gamma\theta}\|^2+\lambda\|\boldsymbol{D\Gamma\theta}\|^2\right)\\ &= (\boldsymbol{\Gamma}'\boldsymbol{\Gamma}+\lambda\boldsymbol{\Gamma}'\boldsymbol{D}'\boldsymbol{D\Gamma})^{-1}\boldsymbol{\Gamma}'\boldsymbol{y}. \end{aligned} \tag{3.3}$$

This representation was used in, e.g., Paige and Trindade (2010), which derived a ridge regression (Hoerl and Kennard, 1970) representation of the HP filter.[2] De Jong and Sakarya (2016) considered the case where

$$\boldsymbol{\Gamma} = \begin{bmatrix} \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(1-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(1-0.5)\pi}{T}\right) \\ \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(2-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(2-0.5)\pi}{T}\right) \\ \vdots & \vdots & & \vdots \\ \sqrt{\frac{1}{T}} & \sqrt{\frac{2}{T}}\cos\left(\frac{(2-1)(T-0.5)\pi}{T}\right) & \cdots & \sqrt{\frac{2}{T}}\cos\left(\frac{(T-1)(T-0.5)\pi}{T}\right) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \\ \vdots \\ \boldsymbol{\gamma}_T \end{bmatrix} \tag{3.4}$$

and showed that $\boldsymbol{\Gamma}'\boldsymbol{\Gamma} = \boldsymbol{I}_T$ and $\boldsymbol{\Gamma}'\boldsymbol{D}'\boldsymbol{D\Gamma} = \boldsymbol{\Sigma} - \boldsymbol{p}_1\boldsymbol{p}_1' - \boldsymbol{p}_T\boldsymbol{p}_T'$, where $\boldsymbol{\Sigma} = \mathrm{diag}(0, \sigma_2^2, \ldots, \sigma_T^2)$,

$$\sigma_j = 4\sin^2\left(\frac{(j-1)\pi}{2T}\right), \quad j = 2, \ldots, T,$$

$\boldsymbol{p}_1$ and $\boldsymbol{p}_T$ are T-dimensional column vectors such that $\boldsymbol{p}_1 = (\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)' = [0, p_{1,2}, \ldots, p_{1,T}]'$ and $\boldsymbol{p}_T = (\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1})' = [0, p_{T,2}, \ldots, p_{T,T}]'$, where

$$p_{1,j} = \sqrt{\frac{32}{T}}\sin^2\left(\frac{(j-1)\pi}{2T}\right)\cos\left(\frac{(j-1)\pi}{2T}\right), \quad j = 2, \ldots, T, \tag{3.5}$$

$$p_{T,j} = \sqrt{\frac{32}{T}}\sin^2\left(\frac{(j-1)\pi}{2T}\right)\cos\left(\frac{(j-1)(T-0.5)\pi}{T}\right), \quad j = 2, \ldots, T. \tag{3.6}$$

[2] See also Kim et al. (2009) and Yamada (2015), the former of which gave a lasso (least absolute shrinkage and selection operator) regression (Tibshirani, 1996) representation of the ℓ_1 trend filter and the latter of which provided ridge regression representations of the WH method of graduation.

For the proofs of (3.5) and (3.6), see the Appendix. It is noteworthy that $\boldsymbol{\Gamma}$ is an orthogonal matrix that represents a discrete cosine transformation (DCT-II) (Ahmed et al., 1974).[3] Accordingly, it follows that

$$\widehat{\boldsymbol{\theta}} = (\boldsymbol{A} - \lambda \boldsymbol{p}_1 \boldsymbol{p}_1' - \lambda \boldsymbol{p}_T \boldsymbol{p}_T')^{-1} \boldsymbol{\Gamma}' \boldsymbol{y}, \tag{3.7}$$

where $\boldsymbol{A} = \boldsymbol{I}_T + \lambda \boldsymbol{\Sigma}$. Since $\boldsymbol{A}$ is a diagonal matrix, $\boldsymbol{A}^{-1}$ is easily obtainable. By applying the SMW formula to $(\boldsymbol{A} - \lambda \boldsymbol{p}_1 \boldsymbol{p}_1' - \lambda \boldsymbol{p}_T \boldsymbol{p}_T')^{-1}$ in (3.7), De Jong and Sakarya (2016) derived an explicit formula for the smoother weights of the HP filter.

3.2.2 Cornea-Madeira (2017)

Let $\boldsymbol{Q}_T$ denote the $T \times T$ symmetric tridiagonal Toeplitz matrix where the first row is $[2, -1, 0, \ldots, 0]$, which is a well-known matrix (Strang and MacNamara, 2014), and $\boldsymbol{Q}_m = \boldsymbol{G}_T \boldsymbol{\Lambda}_T \boldsymbol{G}_T'$ denotes its spectral decomposition.[4] Letting $\boldsymbol{q}_1 = [-2, 1, 0, \ldots, 0]'$ be a T-dimensional column vector and $\boldsymbol{q}_T = \boldsymbol{J}_T \boldsymbol{q}_1$, where $\boldsymbol{J}_T$ is the $T \times T$ exchange matrix, it follows that

$$\boldsymbol{D}'\boldsymbol{D} = \boldsymbol{Q}_T^2 - \boldsymbol{q}_1 \boldsymbol{q}_1' - \boldsymbol{q}_T \boldsymbol{q}_T', \tag{3.8}$$

which indicates

$$\widehat{\boldsymbol{x}} = (\boldsymbol{B} - \lambda \boldsymbol{q}_1 \boldsymbol{q}_1' - \lambda \boldsymbol{q}_T \boldsymbol{q}_T')^{-1} \boldsymbol{y}, \tag{3.9}$$

where $\boldsymbol{B} = (\boldsymbol{I}_T + \lambda \boldsymbol{Q}_T^2) = \boldsymbol{G}_T (\boldsymbol{I}_T + \lambda \boldsymbol{\Lambda}_T^2) \boldsymbol{G}_T'$. Since $\boldsymbol{I}_T + \lambda \boldsymbol{\Lambda}_T^2$ is a diagonal matrix and $\boldsymbol{G}_T$ is an orthogonal matrix, $\boldsymbol{B}^{-1} = \boldsymbol{G}_T (\boldsymbol{I}_T + \lambda \boldsymbol{\Lambda}_T^2)^{-1} \boldsymbol{G}_T'$, which is easy to calculate. By applying the SMW formula to $(\boldsymbol{B} - \lambda \boldsymbol{q}_1 \boldsymbol{q}_1' - \lambda \boldsymbol{q}_T \boldsymbol{q}_T')^{-1}$ in (3.9), Cornea-Madeira (2017) derived an explicit formula.

[3] See also Hamming (1973), Bierens (1997), and Strang (1999).

[4] For the explicit forms of $\boldsymbol{\Lambda}_T$ and $\boldsymbol{G}_T$, see (3.16) and (3.17).

3.3 Another explicit formula for the smoother weights of the HP filter

The product of any two tridiagonal Toeplitz matrices is not a pentadiagonal Toeplitz matrix because the first and the last entries in the principal diagonal are different to the other ones (Marr and Vineyard, 1988; Montaner and Alfaro, 1995; Diele and Lopez, 1998; Wang et al., 2015). Accordingly, $\boldsymbol{Q}_{T-2}^2$ is not a pentadiagonal Toeplitz matrix. Explicitly, it is

$$\boldsymbol{Q}_{T-2}^2 = \begin{bmatrix} 5 & -4 & 1 & 0 & \cdots & 0 \\ -4 & 6 & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & 6 & -4 \\ 0 & \cdots & 0 & 1 & -4 & 5 \end{bmatrix}. \tag{3.10}$$

Interestingly, the corresponding pentadiagonal Toeplitz matrix to $\boldsymbol{Q}_{T-2}^2$ is $\boldsymbol{D}\boldsymbol{D}'$ and their relationship may be expressed as

$$\boldsymbol{D}\boldsymbol{D}' = \boldsymbol{Q}_{T-2}^2 + \boldsymbol{U}\boldsymbol{U}', \tag{3.11}$$

where

$$\boldsymbol{U} = [\boldsymbol{e}_1, \boldsymbol{e}_{T-2}]. \tag{3.12}$$

Here, $\boldsymbol{I}_{T-2} = [\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{T-2}]$. Note that (3.11) corresponds to (3.16) of Wang et al. (2015).

By applying the SMW formula to $(\boldsymbol{I}_T + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (3.1), the HP filter may be alternatively expressed as[5]

$$\widehat{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{D}'(\lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{D}\boldsymbol{D}')^{-1}\boldsymbol{D}\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{D}'\boldsymbol{\Psi}^{-1}\boldsymbol{D}\boldsymbol{y}, \tag{3.13}$$

where $\boldsymbol{\Psi} = \lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{D}\boldsymbol{D}'$, which is a pentadiagonal Toeplitz matrix. From (3.11), $\boldsymbol{\Psi}$ may be represented as $\boldsymbol{\Psi} = \boldsymbol{C} + \boldsymbol{U}\boldsymbol{U}'$, where $\boldsymbol{C} = \lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{Q}_{T-2}^2$. As in Wang et al. (2015), applying the SMW formula to $(\boldsymbol{C} + \boldsymbol{U}\boldsymbol{U}')^{-1}$, it follows that

$$\boldsymbol{\Psi}^{-1} = (\boldsymbol{C} + \boldsymbol{U}\boldsymbol{U}')^{-1} = \boldsymbol{C}^{-1} - \boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}. \tag{3.14}$$

It is noteworthy that $\boldsymbol{I}_2 + \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}$ in (3.14) is a 2×2 matrix and hence its inverse is easily obtainable.[6] In addition, for obtaining the explicit formula for $\boldsymbol{C}^{-1}$, it is possible to apply the spectral decomposition of $\boldsymbol{Q}_{T-2}$ as in Cornea-Madeira (2017):

$$\boldsymbol{C}^{-1} = \boldsymbol{G}_{T-2}(\lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}\boldsymbol{G}_{T-2}', \tag{3.15}$$

where $(\lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}$ in $\boldsymbol{C}^{-1} = \boldsymbol{G}_{T-2}(\lambda^{-1}\boldsymbol{I}_{T-2} + \boldsymbol{\Lambda}_{T-2}^2)^{-1}\boldsymbol{G}_{T-2}'$ is a diagonal matrix, where $\boldsymbol{\Lambda}_{T-2} = \mathsf{diag}(\lambda_1, \ldots, \lambda_{T-2})$ is

$$\lambda_j = 4\sin^2\left(\frac{j\pi}{2(T-1)}\right), \quad j = 1, \ldots, T-2, \tag{3.16}$$

and (i, j)-entry of $\boldsymbol{G}_{T-2}$, denoted by $g_{i,j}$, is

$$g_{i,j} = \sqrt{\frac{2}{T-1}}\sin\left(\frac{ij\pi}{T-1}\right), \quad i, j = 1, \ldots, T-2. \tag{3.17}$$

See, e.g., Strang and MacNamara (2014).

We may summarize the above results as follows:

[5]It is of interest that a ridge regression exists in (3.13): $\widehat{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{D}'\widehat{\boldsymbol{\phi}}$, where

$$\widehat{\boldsymbol{\phi}} = \operatorname*{arg\,min}_{\boldsymbol{\phi} \in \mathbb{R}^{(T-2)}} \left(\|\boldsymbol{y} - \boldsymbol{D}'\boldsymbol{\phi}\|^2 + \lambda^{-1}\|\boldsymbol{\phi}\|^2\right) = (\boldsymbol{D}\boldsymbol{D}' + \lambda^{-1}\boldsymbol{I}_{T-2})^{-1}\boldsymbol{D}\boldsymbol{y}.$$

Yamada (2018) listed several penalized/unpenalized least squares problems related to the HP filter.

[6]Likewise, letting $\boldsymbol{V} = [\boldsymbol{q}_1, \boldsymbol{q}_T]$, it follows that $\boldsymbol{V}\boldsymbol{V}' = \boldsymbol{q}_1\boldsymbol{q}_1' + \boldsymbol{q}_T\boldsymbol{q}_T'$, and accordingly, (3.9) becomes $\widehat{\boldsymbol{x}} = (\boldsymbol{B} - \lambda\boldsymbol{V}\boldsymbol{V}')^{-1}\boldsymbol{y}$. The proof of Cornea-Madeira (2017) may become simpler by applying the SMW formula to $(\boldsymbol{B} - \lambda\boldsymbol{V}\boldsymbol{V}')^{-1}$. See the Appendix for details.

Theorem 3.1. $\widehat{\boldsymbol{x}}$ *in* (3.1) *may be expressed as*

$$\begin{aligned}\widehat{\boldsymbol{x}} &= (\boldsymbol{I}_T+\lambda\boldsymbol{D}'\boldsymbol{D})^{-1}\boldsymbol{y}\\ &= \left[\boldsymbol{I}_T-\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}+\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{D}\right]\boldsymbol{y} \qquad (3.18)\\ &= \boldsymbol{y}-\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y}+\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y}, \qquad (3.19)\end{aligned}$$

where $\boldsymbol{U}$ *and* $\boldsymbol{C}^{-1}$ *are defined by* (3.12) *and* (3.15)*, respectively.*

Proof. (3.18) immediately follows from (3.13) and (3.14). □

Remarks. Since $\boldsymbol{U}\boldsymbol{U}'=\boldsymbol{e}_1\boldsymbol{e}_1'+\boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}'$, the trend extracted by the HP filter may be rewritten as $\widehat{\boldsymbol{x}}=\boldsymbol{y}-\boldsymbol{D}'(\boldsymbol{C}+\boldsymbol{e}_1\boldsymbol{e}_1'+\boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}\boldsymbol{D}\boldsymbol{y}$. Then, it is possible to obtain the result in Theorem 3.1 by applying the SMW formula to $(\boldsymbol{C}+\boldsymbol{e}_1\boldsymbol{e}_1'+\boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}$. Nevertheless, in this case, its derivation becomes longer.

Denote (i,j)-entry of $\boldsymbol{C}^{-1}$ in (3.15) by $c_{i,j}$. In addition, let $\boldsymbol{c}_1$ and $\boldsymbol{c}_{T-2}$ denote the first and last column of $\boldsymbol{C}^{-1}$, respectively. Then, since $\boldsymbol{e}_i'\boldsymbol{C}^{-1}\boldsymbol{e}_j=c_{i,j}$ for $i,j=1,T-2$ and $\boldsymbol{C}^{-1}\boldsymbol{U}=[\boldsymbol{C}^{-1}\boldsymbol{e}_1,\boldsymbol{C}^{-1}\boldsymbol{e}_{T-2}]=[\boldsymbol{c}_1,\boldsymbol{c}_{T-2}]$, it follows that

$$(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}=\frac{1}{(1+c_{1,1})(1+c_{T-2,T-2})-c_{1,T-2}c_{T-2,1}}\begin{bmatrix}1+c_{T-2,T-2} & -c_{1,T-2}\\ -c_{T-2,1} & 1+c_{1,1}\end{bmatrix}$$

and we accordingly obtain

$$\begin{aligned}&\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\\ &\quad=\frac{(1+c_{T-2,T-2})\boldsymbol{c}_1\boldsymbol{c}_1'-c_{1,T-2}\boldsymbol{c}_1\boldsymbol{c}_{T-2}'-c_{T-2,1}\boldsymbol{c}_{T-2}\boldsymbol{c}_1'+(1+c_{1,1})\boldsymbol{c}_{T-2}\boldsymbol{c}_{T-2}'}{(1+c_{1,1})(1+c_{T-2,T-2})-c_{1,T-2}c_{T-2,1}}, \qquad (3.20)\end{aligned}$$

where $\boldsymbol{C}^{-1}=[\boldsymbol{c}_1,\dots,\boldsymbol{c}_{T-2}]=[c_{i,j}]_{i,j=1,\dots,T-2}$ is calculated by

$$c_{i,j}=\sum_{k=1}^{T-2}\frac{g_{i,k}g_{j,k}}{\lambda^{-1}+\lambda_k^2},\quad i,j=1,\dots,T-2. \qquad (3.21)$$

From (3.10), $\boldsymbol{Q}_{T-2}^2$ is a centrosymmetric matrix and accordingly $\boldsymbol{C}=\lambda^{-1}\boldsymbol{I}_{T-2}+\boldsymbol{Q}_{T-2}^2$ is also a centrosymmetric matrix. Since the inverse of a nonsingular centrosymmetric matrix is also a centrosymmetric matrix (Graybill, 2001, Theorem 8.15.7), $\boldsymbol{C}^{-1}$ is a centrosymmetric matrix. Then, it follows that $c_{1,1} = c_{T-2,T-2}$, $c_{1,T-2} = c_{T-2,1}$, and $\boldsymbol{c}_{T-2} = \boldsymbol{J}_{T-2}\boldsymbol{c}_1$.

Combining (3.18) and (3.20), it follows that

$$\begin{aligned}\widehat{\boldsymbol{x}} &= \boldsymbol{y} - \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{y} \\ &+ \boldsymbol{D}'\left(\frac{(1+c_{T-2,T-2})\boldsymbol{c}_1\boldsymbol{c}_1' - c_{1,T-2}\boldsymbol{c}_1\boldsymbol{c}_{T-2}' - c_{T-2,1}\boldsymbol{c}_{T-2}\boldsymbol{c}_1' + (1+c_{1,1})\boldsymbol{c}_{T-2}\boldsymbol{c}_{T-2}'}{(1+c_{1,1})(1+c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}}\right)\boldsymbol{D}\boldsymbol{y},\end{aligned}$$

Denote (i,j)-entry of $\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ by $\xi_{i,j}$ for $i,j = 1,\ldots,T$ and i-th entry of $\boldsymbol{D}'\boldsymbol{c}_j$ for $j = 1, T-2$ by $\upsilon_i^{(j)}$ for $i = 1,\ldots,T$. Then, it follows that

$$\begin{aligned}\xi_{i,j} &= \sum_{k=1}^{T-2}\frac{(\Delta^2 g_{i,k})(\Delta^2 g_{j,k})}{\lambda^{-1}+\lambda_k^2}, \quad i,j = 1,\ldots,T, \\ \upsilon_i^{(j)} &= \sum_{k=1}^{T-2}\frac{(\Delta^2 g_{i,k})g_{j,k}}{\lambda^{-1}+\lambda_k^2}, \quad i = 1,\ldots,T, \quad j = 1, T-2,\end{aligned} \tag{3.22}$$

where $g_{-1,j} = g_{0,j} = g_{T-1,j} = g_{T,j} = 0$ for $j = 1,\ldots,T-2$ and these are introduced for notational convenience.

Accordingly, we obtain the following result:

Corollary 3.1. *Let $z_{i,j}$ denote (i,j)-entry of $(\boldsymbol{I}_T + \lambda\boldsymbol{D}'\boldsymbol{D})^{-1}$ in* (3.1). *Then, $z_{i,j}$ is expressed as*

$$z_{i,j} = \delta_{i,j} - \xi_{i,j} + \mu_{i,j}, \quad i,j = 1,\ldots,T, \tag{3.23}$$

where $\delta_{i,j}$ denotes the Kronecker delta, $\xi_{i,j}$ is defined in (3.22)*, and*

$$\mu_{i,j} = \frac{(1+c_{T-2,T-2})\upsilon_i^{(1)}\upsilon_j^{(1)} - c_{1,T-2}\upsilon_i^{(1)}\upsilon_j^{(T-2)} - c_{T-2,1}\upsilon_i^{(T-2)}\upsilon_j^{(1)} + (1+c_{1,1})\upsilon_i^{(T-2)}\upsilon_j^{(T-2)}}{(1+c_{1,1})(1+c_{T-2,T-2}) - c_{1,T-2}c_{T-2,1}}.$$

Remarks. (a) $\sum_{j=1}^{T} z_{i,j} = 1$ for $i = 1,\ldots,T$ because $\boldsymbol{D}\boldsymbol{\iota} = \boldsymbol{0}$. (b) Since $\boldsymbol{D}\boldsymbol{J}_T = \boldsymbol{J}_{T-2}\boldsymbol{D}$ and $\boldsymbol{C}^{-1}$ is a centrosymmetric matrix, it follows that

$$\boldsymbol{J}_T\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{J}_T = \boldsymbol{D}'\boldsymbol{J}_{T-2}\boldsymbol{C}^{-1}\boldsymbol{J}_{T-2}\boldsymbol{D} = \boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}, \tag{3.24}$$

which indicates that $\boldsymbol{D}'\boldsymbol{C}^{-1}\boldsymbol{D}$ is a centrosymmetric matrix. Likewise, since $\boldsymbol{J}_T\boldsymbol{U} = \boldsymbol{U}\boldsymbol{J}_2$, it follows that $\boldsymbol{J}_2\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}\boldsymbol{J}_2 = \boldsymbol{U}'\boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T\boldsymbol{U} = \boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U}$, which indicates $(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}$ is a centrosymmetric matrix. Accordingly, it follows that

$$\begin{aligned}\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1} &= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{J}_T \\ &= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{U}\boldsymbol{J}_2(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{J}_2\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{J}_T \\ &= \boldsymbol{J}_T\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{J}_T,\end{aligned}$$

which indicates that $\boldsymbol{C}^{-1}\boldsymbol{U}(\boldsymbol{I}_2+\boldsymbol{U}'\boldsymbol{C}^{-1}\boldsymbol{U})^{-1}\boldsymbol{U}'\boldsymbol{C}^{-1}$ is a centrosymmetric matrix. From these results, we obtain, e.g., $\xi_{1,1} = \xi_{T,T}$ and $\mu_{1,1} = \mu_{T,T}$ in (3.23). (c) `calc_HP_hat_matrix` in the Appendix is a MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_T + \lambda\boldsymbol{D}'\boldsymbol{D})^{-1}$ in (3.1) based on (3.23).

Finally, we emphasize that our approach leads to a simpler formula because we apply the SMW formula to $(\boldsymbol{C}+\boldsymbol{U}\boldsymbol{U}')^{-1} = (\boldsymbol{C}+\boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_{T-2}\boldsymbol{e}_{T-2}')^{-1}$, where both $\boldsymbol{e}_1$ and $\boldsymbol{e}_{T-2}$ are *unit vectors*. Observe that $\boldsymbol{p}_i$ in (3.7) and $\boldsymbol{q}_i$ in (3.9), where $i = 1, T$, are not unit vectors.

3.4 Concluding remarks

By applying the SMW formula and a discrete cosine transformation matrix, De Jong and Sakarya (2016) derived an explicit formula for the smoother weights of the HP filter. Then, by applying the SMW formula and the spectral decomposition of a symmetric tridiagonal Toeplitz matrix, Cornea-Madeira (2017) provided a simper formula. In this chapter, we provided an alternative simpler formula and explained why our approach leads to a simpler formula. The main result of this chapter is summarized in Theorem 3.1 and Corollary 3.1.

3.5 Numerical example

Consider, we are going to find the inverse of the smoother matrix of the HP filter, $(\boldsymbol{I}_T + \lambda \boldsymbol{D}_2'\boldsymbol{D}_2)^{-1} \in \mathbb{R}^{T\times T}$ where $T = 7$ and $\lambda = 7$. Here, $\boldsymbol{D}_2$ is the second-order difference matrix

$$\boldsymbol{D_2} = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix} \in \mathbb{R}^{5\times 7},$$

and

$$\boldsymbol{D}_2\boldsymbol{D}_2' = \begin{bmatrix} 6 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 6 \end{bmatrix} \in \mathbb{R}^{5\times 5}. \tag{3.25}$$

Suppose $\boldsymbol{Q}_5$ denotes the 5×5 symmetric tridiagonal toeplitz matrix where the first row is $[2, -1, 0, ..., 0]$ and accordingly, $\boldsymbol{Q}_5^2$ is not a pentadiagonal toeplitz matrix because of the first and last elements of the diagonal entries. Explicitly, it is

$$\boldsymbol{Q}_5^2 = \begin{bmatrix} 5 & -4 & 1 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \\ 0 & 1 & -4 & 6 & -4 \\ 0 & 0 & 1 & -4 & 5 \end{bmatrix} \in \mathbb{R}^{5\times 5}. \tag{3.26}$$

Now, from equation (3.25) and (3.26) we get the following expression:

$$\boldsymbol{D}_2\boldsymbol{D}_2' = \boldsymbol{Q}_5^2 + \boldsymbol{U}\boldsymbol{U}', \tag{3.27}$$

where $\boldsymbol{U} = [\boldsymbol{e}_1, \boldsymbol{e}_5]$ and $\boldsymbol{e}_1$, $\boldsymbol{e}_5$ are two unit vectors of the form:

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{e}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Suppose, the spectral decomposition of $\boldsymbol{Q}_5^2$ is $\boldsymbol{G}_5\boldsymbol{\Lambda}_5^2\boldsymbol{G}_5'$. Where, $\boldsymbol{\Lambda}_5^2$ denotes the eigenvector matrix of $\boldsymbol{Q}_5^2$ and its distinct eigenvalues are

$$\lambda_j = 4\sin^2\left(\frac{j\pi}{12}\right), \quad j = 1, ..., 5, \tag{3.28}$$

and the (i, j) th entry of the corresponding eigenvector matrix $\boldsymbol{G}_5$ is

$$g_{i,j} = \left(\frac{1}{3}\right)^{1/2} \sin\left(\frac{\pi i j}{6}\right), \quad i, j = 1, ..., 5. \tag{3.29}$$

From the Woodbury matrix identity and later using equation (3.27), we have

$$\begin{aligned}
(\boldsymbol{I}_7 + 7\boldsymbol{D}_2'\boldsymbol{D}_2)^{-1} &= \boldsymbol{I}_7 - \boldsymbol{D}_2'(\frac{1}{7}\boldsymbol{I}_5 + \boldsymbol{D}_2\boldsymbol{D}_2')^{-1}\boldsymbol{D}_2 \\
&= \boldsymbol{I}_7 - \boldsymbol{D}_2'(\frac{1}{7}\boldsymbol{I}_5 + \boldsymbol{Q}_5^2 + \boldsymbol{U}\boldsymbol{U}')^{-1}\boldsymbol{D}_2 \\
&= \boldsymbol{I}_7 - \boldsymbol{D}_2'(\boldsymbol{C} + \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_5\boldsymbol{e}_5')^{-1}\boldsymbol{D}_2 \\
&= \boldsymbol{I}_7 - \boldsymbol{D}_2'\boldsymbol{\Psi}^{-1}\boldsymbol{D}_2.
\end{aligned} \tag{3.30}$$

Suppose,

$$\boldsymbol{C} = \frac{1}{7}\boldsymbol{I}_5 + \boldsymbol{Q}_5^2, \tag{3.31}$$

$$\boldsymbol{\Psi}^{-1} = (\boldsymbol{C} + \boldsymbol{e}_1\boldsymbol{e}_1' + \boldsymbol{e}_5\boldsymbol{e}_5')^{-1}. \tag{3.32}$$

Using the spectral decomposition of $\boldsymbol{Q}_5^2$ and equations (3.15), (3.16),(3.17), we find

$$\boldsymbol{C}^{-1} = (\frac{1}{7}\boldsymbol{I}_5 + \boldsymbol{Q}_5^2)^{-1} = \begin{bmatrix} 0.72070 & 0.85357 & 0.70783 & 0.47076 & 0.22851 \\ 0.85357 & 1.42853 & 1.32432 & 0.93634 & 0.47076 \\ 0.70783 & 1.32432 & 1.65704 & 1.32432 & 0.70783 \\ 0.47076 & 0.93634 & 1.32432 & 1.42853 & 0.85357 \\ 0.22851 & 0.47076 & 0.70783 & 0.85357 & 0.72070 \end{bmatrix},$$

and let $\boldsymbol{C}^{-1} = [c_{i,j}]_{i,j=1,\ldots,5}$. Applying the SMW formula to equation (3.32) and therefore using equation (3.20) and (3.21) we investigate the value of

$$\boldsymbol{\Psi}^{-1} = \begin{bmatrix} 0.408407 & 0.467981 & 0.363137 & 0.211436 & 0.078564 \\ 0.467981 & 0.929538 & 0.843414 & 0.535559 & 0.211436 \\ 0.363137 & 0.843414 & 1.142960 & 0.843414 & 0.363137 \\ 0.211436 & 0.535559 & 0.843414 & 0.929538 & 0.467981 \\ 0.078564 & 0.211436 & 0.363137 & 0.467981 & 0.408407 \end{bmatrix}. \tag{3.33}$$

Now, from equation (3.30)

$$(\boldsymbol{I}_7 + 7\boldsymbol{D}_2'\boldsymbol{D}_2)^{-1}$$
$$= \begin{bmatrix} 0.5915930 & 0.3488334 & 0.1644177 & 0.0468563 & -0.0188284 & -0.0543080 & -0.0785641 \\ 0.3488334 & 0.3087564 & 0.2188461 & 0.1280182 & 0.0539246 & -0.0040708 & -0.0543080 \\ 0.1644177 & 0.2188461 & 0.2497863 & 0.2024863 & 0.1293674 & 0.0539246 & -0.0188284 \\ 0.0468563 & 0.1280182 & 0.2024863 & 0.2452785 & 0.2024863 & 0.1280182 & 0.0468563 \\ -0.0188284 & 0.0539246 & 0.1293674 & 0.2024863 & 0.2497863 & 0.2188461 & 0.1644177 \\ -0.0543080 & -0.0040708 & 0.0539246 & 0.1280182 & 0.2188461 & 0.3087564 & 0.3488334 \\ -0.0785641 & -0.0543080 & -0.0188284 & 0.0468563 & 0.1644177 & 0.3488334 & 0.5915930 \end{bmatrix}.$$

3.6 Appendix

Proof of (3.5)

From (3.4), $\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2$ is

$$\begin{aligned}\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2 = &\left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(1-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(1-0.5)\pi}{T}\right)\right] \\ &- \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(2-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(2-0.5)\pi}{T}\right)\right].\end{aligned}$$

Let $\beta_j = \pi(j-1)/(2T)$ for $j = 2, \ldots, T$. Then, it follows that

$$\begin{aligned}&\sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(1-0.5)\pi}{T}\right) - \sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(2-0.5)\pi}{T}\right) \\ &= \sqrt{\frac{2}{T}}[\cos(\beta_j) - \cos(3\beta_j)] = \sqrt{\frac{32}{T}}\sin^2(\beta_j)\cos(\beta_j).\end{aligned}$$

The last equality follows from $\cos(\beta_j) - \cos(3\beta_j) = 4\sin^2(\beta_j)\cos(\beta_j)$.

Proof of (3.6)

From (3.4), $\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1}$ is

$$\begin{aligned}\boldsymbol{\gamma}_T - \boldsymbol{\gamma}_{T-1} = &\left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(T-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(T-0.5)\pi}{T}\right)\right] \\ &- \left[\sqrt{\tfrac{1}{T}} \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(2-1)(T-1-0.5)\pi}{T}\right) \quad \cdots \quad \sqrt{\tfrac{2}{T}}\cos\left(\tfrac{(T-1)(T-1-0.5)\pi}{T}\right)\right]\end{aligned}$$

Let $\beta_j = \pi(j-1)/(2T)$ and $\kappa_j = 2T\beta_j = \pi(j-1)$ for $j = 2, \ldots, T$. Then, it follows that

$$\begin{aligned}&\sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(T-0.5)\pi}{T}\right) - \sqrt{\frac{2}{T}}\cos\left(\frac{(j-1)(T-1-0.5)\pi}{T}\right) \\ &= \sqrt{\frac{2}{T}}[\cos(\beta_j(2T-1)) - \cos(\beta_j(2T-3))] = \sqrt{\frac{2}{T}}[\cos(\kappa_j - \beta_j) - \cos(\kappa_j - 3\beta_j)].\end{aligned}$$

Here, since $\sin(\kappa_j) = 0$, it follows that

$$
\begin{aligned}
&\cos(\kappa_j - \beta_j) - \cos(\kappa_j - 3\beta_j) \\
&\quad = \cos(\kappa_j)\cos(\beta_j) + \sin(\kappa_j)\sin(\beta_j) - \cos(\kappa_j)\cos(3\beta_j) - \sin(\kappa_j)\sin(3\beta_j) \\
&\quad = \cos(\kappa_j)[\cos(\beta_j) - \cos(3\beta_j)] + \sin(\kappa_j)[\sin(\beta_j) - \sin(3\beta_j)] \\
&\quad = \cos(\kappa_j)[4\sin^2(\beta_j)\cos(\beta_j)] + \sin(\kappa_j)[4\sin^3(\beta_j) - 2\sin(\beta_j)] \\
&\quad = 4\sin^2(\beta_j)[\cos(\kappa_j)\cos(\beta_j) + \sin(\kappa_j)\sin(\beta_j)] - 2\sin(\kappa_j)\sin(\beta_j) \\
&\quad = 4\sin^2(\beta_j)\cos(\kappa_j - \beta_j),
\end{aligned}
$$

where

$$
\kappa_j - \beta_j = 2T\beta_j - \beta_j = \beta_j(2T-1) = \frac{\pi(j-1)(2T-1)}{2T} = \frac{\pi(j-1)(T-0.5)}{T}.
$$

Application of the SMW formula to $(\boldsymbol{B} - \lambda \boldsymbol{V}\boldsymbol{V}')^{-1}$

As in Cornea-Madeira (2017), by applying the SMW formula to $(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1' - \lambda \boldsymbol{q}_T\boldsymbol{q}_T')^{-1}$, we obtain the following results:

$$
\begin{aligned}
&(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1' - \lambda \boldsymbol{q}_T\boldsymbol{q}_T')^{-1} \\
&\qquad = (\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1')^{-1} + \lambda \frac{(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}\boldsymbol{q}_T\boldsymbol{q}_T'(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}}{1 - \lambda \boldsymbol{q}_T'(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1')^{-1}\boldsymbol{q}_T},
\end{aligned} \tag{3.34}
$$

where

$$
(\boldsymbol{B} - \lambda \boldsymbol{q}_1\boldsymbol{q}_1')^{-1} = \boldsymbol{B}^{-1} + \lambda \frac{\boldsymbol{B}^{-1}\boldsymbol{q}_1\boldsymbol{q}_1'\boldsymbol{B}^{-1}}{1 - \lambda \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_1}. \tag{3.35}
$$

On the other hand, by applying the SMW formula to $(\boldsymbol{B} - \lambda \boldsymbol{V}\boldsymbol{V}')^{-1}$, we obtain

$$
\begin{aligned}
&(\boldsymbol{B} - \lambda \boldsymbol{V}\boldsymbol{V}')^{-1} \\
&\quad = \boldsymbol{B}^{-1} - \boldsymbol{B}^{-1}\boldsymbol{V}(\boldsymbol{V}'\boldsymbol{B}^{-1}\boldsymbol{V} - \lambda^{-1}\boldsymbol{I}_2)^{-1}\boldsymbol{V}'\boldsymbol{B}^{-1}
\end{aligned}
$$

$$= \boldsymbol{B}^{-1} - [\boldsymbol{B}^{-1}\boldsymbol{q}_1, \boldsymbol{B}^{-1}\boldsymbol{q}_T] \begin{bmatrix} \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_1 - \lambda^{-1} & \boldsymbol{q}_1'\boldsymbol{B}^{-1}\boldsymbol{q}_T \\ \boldsymbol{q}_T'\boldsymbol{B}^{-1}\boldsymbol{q}_1 & \boldsymbol{q}_T'\boldsymbol{B}^{-1}\boldsymbol{q}_T - \lambda^{-1} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{q}_1'\boldsymbol{B}^{-1} \\ \boldsymbol{q}_T'\boldsymbol{B}^{-1} \end{bmatrix}. \quad (3.36)$$

By comparing (3.36) with (3.34) and (3.35), it is observable that (3.36) is preferable to (3.34), mainly because (3.36) is symmetric with respect to $\boldsymbol{q}_1$ and $\boldsymbol{q}_T$.

A MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_T + \lambda \boldsymbol{D}'\boldsymbol{D})^{-1}$ in (3.1) based on (3.23)

```
function HP_hat_matrix = calc_HP_hat_matrix(T, lambda)

% T: sample size
% lambda: smoothing parameter

Lam = diag( 4*(sin((1:T-2)*pi/(2*(T-1))).^2) );
G = zeros(T-2,T-2);
for i = 1:T-2
   for j = 1:T-2
      G(i,j) = sqrt(2/(T-1))*sin(i*j*pi/(T-1));
   end
end

invC = zeros(T-2,T-2);
for i = 1:T-2
   for j = 1:T-2
      s = 0;
      for k = 1:T-2
         s = s + G(i,k)*G(j,k)/( (1/lambda)+Lam(k,k)^2 );
      end
      invC(i,j) = s;
   end
end

Xi = zeros(T,T);
```

```
DG = diff([zeros(2,T-2);G;zeros(2,T-2)],2);
for i = 1:T
   for j = 1:T
      s = 0;
      for k = 1:T-2
         s = s+DG(i,k)*DG(j,k)/((1/lambda)+Lam(k,k)^2);
      end
      Xi(i,j) = s;
   end
end

Up1 = zeros(T,1); Up2 = zeros(T,1);
for i=1:T
   s1 = 0; s2 = 0;
   for k = 1:T-2
     s1 = s1+DG(i,k)*G(1,k)/((1/lambda)+Lam(k,k)^2);
     s2 = s2+DG(i,k)*G(end,k)/((1/lambda)+Lam(k,k)^2);
   end
   Up1(i) = s1; Up2(i) = s2;
end

c11 = invC(1,1);
c22 = invC(end,end);
c12 = invC(1,end);
c21 = invC(end,1);
den = (1+c11)*(1+c22)-c12*c21;
Tau = zeros(T,T);
for i=1:T
   for j=1:T
      num = (1+c22)*Up1(i)*Up1(j)-c12*Up1(i)*Up2(j)- ...
              c21*Up2(i)*Up1(j)+(1+c11)*Up2(i)*Up2(j);
      Tau(i,j) = num/den;
   end
end

Z = zeros(T,T);
```

```
I = eye(T);
for i=1:T
   for j=1:T
      Z(i,j) = I(i,j)-Xi(i,j)+Tau(i,j);
   end
end

HP_hat_matrix = Z;

end
```

3.7 References

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Chapter 4

A Discussion on the Whittaker–Henderson Graduation and Bisymmetry of the Related Smoother Matrices

4.1 Introduction

The Whittaker–Henderson (WH) method of graduation is a comprehensive smoothing tool which is widely used in the actuarial literature and in the macroeconomic time series analysis. Although we call WH's graduation method, the method was originally introduced by German scholar George Bohlman in 1899. Whittaker (1923), without knowing about Bohlman's work, published a paper named as *"On a New Mehtod of Graduation"*, where he proposed a method for data smoothing using third order differences $(\Delta^3 x_t = x_t - 3x_{t-1} + 3x_{t-2} - x_{t-3})$. On the other hand, Henderson (1924), published an article about the data smoothing method named as *"A New Mehtod of Graduation"*. According to Joseph (1952), Henderson discovered a factorization formula to calculate the Whittaker's method in a simpler way. Later, the method is known as the Whittaker–Henderson's method of graduation. For the archival assessment of the WH method of graduation, see

Nocon and Scott (2012).[1] Now, the popular Whittaker–Henderson graduation of order p is defined as follows:

$$\begin{aligned}
\widehat{\boldsymbol{x}} &= \underset{x_1,\ldots,x_n \in \mathbb{R}}{\arg\min} \left[\sum_{t=1}^{n} (y_t - x_t)^2 + \lambda \sum_{t=p+1}^{n} (\Delta^p x_t)^2 \right] \\
&= \underset{\boldsymbol{x} \in \mathbb{R}^n}{\arg\min} \left(\|\boldsymbol{y} - \boldsymbol{x}\|^2 + \lambda \|\boldsymbol{D}_p \boldsymbol{x}\|^2 \right) \\
&= (\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} \boldsymbol{y}, \qquad (4.1)
\end{aligned}$$

where, $\boldsymbol{I}_n$ is the identity matrix of size n and $y_1, \ldots, y_n$ denote n observations of a univariate time series and the parameter $\lambda > 0$ is a positive smoothing parameter. Here, $\Delta x_t = (x_t - x_{t-1})$ is called the first order difference and the operator "Δ" represents the backward difference operator. The first term, square of the deviations measures the fidelity to the data and the second term measures the smoothness. The parameter λ is used to control the trade-off between the smoothness of the graduated data and the size of the deviation. The objective function given in (4.1) can also be represented in matrix notation as

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} (\boldsymbol{y} - \boldsymbol{x})'(\boldsymbol{y} - \boldsymbol{x}) + \lambda (\boldsymbol{D}_p \boldsymbol{x})'(\boldsymbol{D}_p \boldsymbol{x}). \qquad (4.2)$$

Here, $\boldsymbol{D}_p \in \mathbb{R}^{(n-p) \times n}$ is the pth order difference matrix such that $\boldsymbol{D}_p \boldsymbol{x} = [\Delta^p x_{p+1}, \ldots, \Delta^p x_n]'$. Explicitly,

$$\boldsymbol{D}_p = \begin{bmatrix} d_0 & \cdots & d_p & 0 & \cdots & 0 \\ 0 & \ddots & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & d_0 & \cdots & d_p \end{bmatrix},$$

where $d_k = (-1)^{p-k} \binom{p}{k}$ for $k = 0, \ldots, p$. In econometrics, when $p = 1$, (4.1) is referred to as the exponential smoothing (ES) filter (King and Rebelo, 1993), when $p = 2$, it is referred to as the Hodrick–Prescott (HP) filter (Hodrick and Prescott, 1997), and when $p = 3$, it is referred to as the HP3rd filter (Reeves et al., 2000). By applying the Sherman–Morrison–

[1] See also Weinert (2007) and Phillips (2010).

Woodbury formula for $(\boldsymbol{I}_n+\lambda\boldsymbol{D}_p'\boldsymbol{D}_p)^{-1}$ in (4.1), it follows that

$$\boldsymbol{y}-\widehat{\boldsymbol{x}}=\boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p}+\boldsymbol{D}_p\boldsymbol{D}_p')^{-1}\boldsymbol{D}_p\boldsymbol{y}. \tag{4.3}$$

In addition, Yamada (2015) showed that the following decomposition holds:[2]

$$\widehat{\boldsymbol{x}}=\boldsymbol{\Pi}_p(\boldsymbol{\Pi}_p'\boldsymbol{\Pi}_p)^{-1}\boldsymbol{\Pi}_p'\boldsymbol{y}+\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'\boldsymbol{y},$$

where $\boldsymbol{\Pi}_p$ is an $n\times p$ matrix of which the t-th row is $[1,t,\dots,t^{p-1}]$ for $t=1,\dots,n$ and $\boldsymbol{F}_p=\boldsymbol{D}_p'(\boldsymbol{D}_p\boldsymbol{D}_p')^{-1}$. Accordingly, it follows that

$$\widehat{\boldsymbol{x}}-\widehat{\boldsymbol{\tau}}=\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'\boldsymbol{y}, \tag{4.4}$$

where $\widehat{\boldsymbol{\tau}}=\boldsymbol{\Pi}_p(\boldsymbol{\Pi}_p'\boldsymbol{\Pi}_p)^{-1}\boldsymbol{\Pi}_p'\boldsymbol{y}$.

Consider a $q\times q$ matrix $\boldsymbol{A}=[a_{i,j}]$.

- $\boldsymbol{A}$ is centrosymmetric if $a_{i,j}=a_{q-i+1,q-j+1}$ for all i,j,
- $\boldsymbol{A}$ is persymmetric if $a_{i,j}=a_{q-j+1,q-i+1}$ for all i,j.

Let $\boldsymbol{T}_q\in\mathbb{R}^{q\times q}$ be the exchange matrix defined as $\boldsymbol{T}_q=[\boldsymbol{e}_q,\dots,\boldsymbol{e}_1]$, where $\boldsymbol{e}_1,\dots,\boldsymbol{e}_q$ are unit vectors such that $\boldsymbol{I}_q=[\boldsymbol{e}_1,\dots,\boldsymbol{e}_q]$. More explicitly, it is

$$\boldsymbol{T}_q=\begin{bmatrix}0&\cdots&0&1\\ \vdots&\iddots&\iddots&0\\ 0&\iddots&\iddots&\vdots\\ 1&0&\cdots&0\end{bmatrix}.$$

Evidently, $\boldsymbol{T}_q$ is a matrix such that $\boldsymbol{T}_q'=\boldsymbol{T}_q$, $\boldsymbol{T}_q^2=\boldsymbol{T}_q'\boldsymbol{T}_q=\boldsymbol{I}_q$, and $\boldsymbol{T}_q^{-1}=\boldsymbol{T}_q$. $\boldsymbol{T}_q$ is a special case of a permutation matrix. Then, $a_{i,j}=a_{q-i+1,q-j+1}$ for all i,j may be repre-

[2]The decomposition is alternatively expressed as $\widehat{\boldsymbol{x}}=\boldsymbol{\Pi}_p(\boldsymbol{\Pi}_p'\boldsymbol{\Pi}_p)^{-1}\boldsymbol{\Pi}_p'\boldsymbol{y}+(\boldsymbol{I}_n+\lambda\boldsymbol{D}_p'\boldsymbol{D}_p)^{-1}(\boldsymbol{y}-\widehat{\boldsymbol{\tau}})$ and it may be derived as in Yamada (2018). Moreover, Yamada (2017) lists several penalized/unpenalized least squares problems related to the HP filter.

sented by $\boldsymbol{A} = \boldsymbol{T}_q \boldsymbol{A} \boldsymbol{T}_q$.[3] Likewise, $a_{i,j} = a_{q-j+1,q-i+1}$ for all i, j may be represented by $\boldsymbol{A} = (\boldsymbol{T}_q \boldsymbol{A} \boldsymbol{T}_q)'$. Accordingly, it immediately follows that if $\boldsymbol{A}$ is a symmetric centrosymmetric matrix, it is also a persymmetric matrix. Similarly if $\boldsymbol{A}$ is a symmetric persymmetric matrix, it is also a centrosymmetric matrix. If $\boldsymbol{A}$ is a symmetric centrosymmetric matrix, it is referred to as a bisymmetric matrix. For example,

$$\begin{bmatrix} \mathbf{5} & 1 & 2 \\ \mathbf{1} & \mathbf{-3} & 1 \\ \mathbf{2} & 1 & 5 \end{bmatrix}$$

is a bisymmetric matrix. As shown above, since it is a bisymmetric matrix, it may be made from the bold-faced 4 entries even though there are 9 entries in it.

Cornea-Madeira (2017) noted that the smoother matrix of the HP filter, $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_2' \boldsymbol{D}_2)^{-1}$, is a bisymmetric matrix and Yamada (2019) for generalizing this proved that this is true for any $p \in \mathbb{N}$ such that $(n-p) > 0$.

The chapter is organized as follows. In Section 4.2, a literature review is discussed and in Section 4.3 using the result of Yamada (2019) we provide formulas for calculating $\widehat{\boldsymbol{x}}$ in (4.1). In Section 4.4, we show that the smoother matrices in (4.3) and in (4.4) are also bisymmetric matrices. Section 4.5 concludes.

[3] Dagum and Luati (2004) referred to the transformation from $\boldsymbol{B}$ to $\boldsymbol{T}_r \boldsymbol{B} \boldsymbol{T}_s$: $t(\boldsymbol{B}) = \boldsymbol{T}_r \boldsymbol{B} \boldsymbol{T}_s$ as t-transformation, where $\boldsymbol{B}$ is a $r \times s$ matrix.

4.2 A literature review

In this section, we briefly review two closely related papers: Yamada (2019) and El-Mikkawy and Atlan (2013).

4.2.1 Yamada (2019)

For proving that the smooother matrix $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$ of WH graduation is a bisymmetric matrix Yamada (2019) first provided the following lemma:

Lemma 4.1. $\boldsymbol{T}_{n-p}\boldsymbol{D}_p = (-1)^p \boldsymbol{D}_p \boldsymbol{T}_n$.

Proof. For $j = 1, \ldots, p$, let

$$\boldsymbol{D}_{(j)} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix} \in \mathbb{R}^{(n-j)\times(n-j+1)},$$

which is a first order difference matrix. Then, since premultiplication (postmultiplication) by an exchange matrix exchanges rows (columns) in reverse order, it follows that

$$\boldsymbol{T}_{n-j}\boldsymbol{D}_{(j)} = \begin{bmatrix} 0 & \cdots & 0 & -1 & 1 \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & -1 & 1 & \iddots & \vdots \\ -1 & 1 & 0 & \cdots & 0 \end{bmatrix} = -\begin{bmatrix} 0 & \cdots & 0 & 1 & -1 \\ \vdots & \iddots & \iddots & \iddots & 0 \\ 0 & 1 & -1 & \iddots & \vdots \\ 1 & -1 & 0 & \cdots & 0 \end{bmatrix} = -\boldsymbol{D}_{(j)}\boldsymbol{T}_{n-j+1}.$$

In addition, by definition of $\boldsymbol{D}_p$, $\boldsymbol{D}_p$ may be represented as follows:

$$\boldsymbol{D}_p = \boldsymbol{D}_{(p)}\boldsymbol{D}_{(p-1)} \times \cdots \times \boldsymbol{D}_{(1)}.$$

Combining these equations yields

$$\boldsymbol{T}_{n-p}\boldsymbol{D}_p = \boldsymbol{T}_{n-p}\boldsymbol{D}_{(p)}\boldsymbol{D}_{(p-1)} \times \cdots \times \boldsymbol{D}_{(1)} = (-1)\boldsymbol{D}_{(p)}\boldsymbol{T}_{n-p+1}\boldsymbol{D}_{(p-1)} \times \cdots \times \boldsymbol{D}_{(1)}$$
$$= (-1)^p\boldsymbol{D}_{(p)}\boldsymbol{D}_{(p-1)} \times \cdots \times \boldsymbol{D}_{(1)}\boldsymbol{T}_n = (-1)^p\boldsymbol{D}_p\boldsymbol{T}_n.$$

□

Now using the above result Yamada provided that, $(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)^{-1}$ in (4.1) is a bisymmetric matrix. Since it is evident that $(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)^{-1}$ is a symmetric matrix, Yamada showed that it is a centrosymmetric matrix. Using $\boldsymbol{T}_{n-p}\boldsymbol{D}_p = (-1)^p\boldsymbol{D}_p\boldsymbol{T}_n$, it follows that

$$\boldsymbol{T}_n\boldsymbol{D}'_p\boldsymbol{D}_p\boldsymbol{T}_n = \boldsymbol{T}_n\boldsymbol{D}'_p(-1)^p(-1)^p\boldsymbol{D}_p\boldsymbol{T}_n = \boldsymbol{D}'_p\boldsymbol{T}_{n-p}\boldsymbol{T}_{n-p}\boldsymbol{D}_p = \boldsymbol{D}'_p\boldsymbol{D}_p,$$

which indicates that $\boldsymbol{D}'_p\boldsymbol{D}_p$ is a centrosymmetric matrix and consequently $(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)$ is also a centrosymmetric matrix. Accordingly, it follows that

$$\boldsymbol{T}_n(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)^{-1}\boldsymbol{T}_n = [\boldsymbol{T}_n(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)\boldsymbol{T}_n]^{-1} = (\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)^{-1}.$$

4.2.2 El-Mikkawy and Atlan (2013)

El-Mikkawy (2013), in their paper, constructed two algorithm for solving centrosymmetric linear system of even and odd order. These two algorithms are described briefly in here.

4.2.2.1 An algorithm for solving centrosymmetric linear system of even order:

Let, an even, n=2m order centrosymmetric partitioned matrix form be as follows:

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{TBT} \\ \hline \boldsymbol{B} & \boldsymbol{TAT} \end{array}\right]\left[\begin{array}{c} \boldsymbol{x}_1 \\ \hline \boldsymbol{x}_2 \end{array}\right] = \left[\begin{array}{c} \boldsymbol{b}_1 \\ \hline \boldsymbol{b}_2 \end{array}\right] \tag{4.5}$$

Here, $\boldsymbol{A}$ is an $m \times m$ matrix of the form $\boldsymbol{A} = [a_{ij}]$ where $i, j = 1, 2, ..., m$ and another matrix $\boldsymbol{B} = [a_{i,j}]$ where $i = m, m-1, ..., 1$ and $j = 2m, 2m-1, ..., m+1$.
$\boldsymbol{x}_1 = [x_1, x_2, ..., x_m]'$, $\boldsymbol{x}_2 = [x_{m+1}, x_{m+2}, ..., x_{2m}]'$

$\boldsymbol{b}_1 = [b_1, b_2, ..., b_m]'$, $\boldsymbol{b}_2 = [b_{m+1}, b_{m+2}, ..., b_{2m}]'$
The system in (4.5) can be written in matrix form as follows:

$$\boldsymbol{Rx} = \boldsymbol{b}, \tag{4.6}$$

where $\boldsymbol{R} = [a_{ij}]_{i,j=1,...,2m}$ is the coefficient matrix of the system (4.5), $\boldsymbol{x} = [x_1, x_2, ..., x_{2m}]'$ and $\boldsymbol{b} = [b_1, b_2, ..., b_{2m}]'$ is the constant vector. Now, let, $\boldsymbol{Q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{I}_m \\ \boldsymbol{T}_m & -\boldsymbol{T}_m \end{bmatrix}$ be an orthogonal matrix such that

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{TBT} \\ \boldsymbol{B} & \boldsymbol{TAT} \end{bmatrix} = \boldsymbol{Q}_1 \begin{bmatrix} \boldsymbol{A}+\boldsymbol{TB} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{TB} \end{bmatrix} \boldsymbol{Q}_1'.$$

Now,

$$\begin{aligned} |\, \boldsymbol{R} \,| &= |\, \boldsymbol{Q}_1 \,| \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}-\boldsymbol{TB} \end{vmatrix} |\, \boldsymbol{Q}_1' \,| \\ &= |\, \boldsymbol{Q}_1 \,||\, \boldsymbol{A}+\boldsymbol{TB} \,||\, \boldsymbol{A}-\boldsymbol{TB} \,||\, \boldsymbol{Q}_1' \,| \\ &= |\, \boldsymbol{A}+\boldsymbol{TB} \,||\, \boldsymbol{A}-\boldsymbol{TB} \,||\, \boldsymbol{Q}_1'\boldsymbol{Q}_1 \,| \\ &= |\, \boldsymbol{A}+\boldsymbol{TB} \,||\, \boldsymbol{A}-\boldsymbol{TB} \,|\,. \end{aligned}$$

Step 1: Construc the $m \times m$ matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ and the m-vectors $\hat{\boldsymbol{b}}$ and $\tilde{\boldsymbol{b}}$ as follows:

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{A}+\boldsymbol{TB} = [a_{ij} + a_{i,2m+1-j}]_{i,j=1,...,m}, \\ \boldsymbol{Q} &= \boldsymbol{A}-\boldsymbol{TB} = [a_{ij} - a_{i,2m+1-j}]_{i,j=1,...,m}, \\ \hat{\boldsymbol{b}} &= [b_1 + b_{2m}, b_2 + b_{2m-1}, ..., b_m + b_{m+1}]', \\ \tilde{\boldsymbol{b}} &= [b_1 - b_{2m}, b_2 - b_{2m-1}, ..., b_m - b_{m+1}]'. \end{aligned}$$

Step 2: Compute $|\boldsymbol{R}|=|\boldsymbol{P}||\boldsymbol{Q}|$. If $|\boldsymbol{R}|=0$ then "No solutions" end if.
Step 3: Solve the two linear systems: $\boldsymbol{Py}=\hat{\boldsymbol{b}}$, and $\boldsymbol{Qz}=\tilde{\boldsymbol{b}}$, for $\boldsymbol{y}=[y_1,y_2,...,y_m]'$ and $\boldsymbol{z}=[z_1,z_2,...,z_m]'$ respectively.
Step 4: The solution vector $\boldsymbol{x}=[x_1,x_2,...,x_{2m}]'$ is given by

$$x_i=\begin{cases}\frac{1}{2}(y_i+z_i) & if\ i=1,2,...,m,\\ \\ \frac{1}{2}(y_{2m+1-i}-z_{2m+1-i}) & if\ i=m+1,m+2,...,2m.\end{cases}$$

4.2.2.2 An algorithm for solving centrosymmetric linear system of odd order:

Let, an odd, n=2m+1 order centrosymmetric partitioned matrix form be as follows:

$$\left[\begin{array}{c|c|c}\boldsymbol{A} & \boldsymbol{v} & \boldsymbol{TBT}\\ \hline \boldsymbol{u}' & q & \boldsymbol{u}'\boldsymbol{J}\\ \hline \boldsymbol{B} & \boldsymbol{Tv} & \boldsymbol{TAT}\end{array}\right]\left[\begin{array}{c}\boldsymbol{x}_1\\ \hline x_{m+1}\\ \hline \boldsymbol{x}_2\end{array}\right]=\left[\begin{array}{c}\boldsymbol{b}_1\\ \hline b_{m+1}\\ \hline \boldsymbol{b}_2\end{array}\right] \tag{4.7}$$

Here, $\boldsymbol{A}$ is an $m\times m$ matrix of the form $\boldsymbol{A}=[a_{ij}]$ where $i,j=1,2,...,m$ and another matrix $\boldsymbol{B}=[a_{i,j}]$ where $i=m,m-1,...,1$ and $j=2m+1,2m,...,m+2$.
$\boldsymbol{v}=[a_{1,m+1},a_{2,m+1},...,a_{m,m+1}]'$, $\boldsymbol{u}=[a_{m+1,1},a_{m+1,2},...,a_{m+1,m}]'$, $q=[a_{m+1,m+1}]$
$\boldsymbol{x}_1=[x_1,x_2,...,x_m]'$, $\boldsymbol{x}_2=[x_{m+2},x_{m+3},...,x_{2m+1}]'$
$\boldsymbol{b}_1=[\boldsymbol{b}_1+\boldsymbol{T}\boldsymbol{b}_2]'=[b_1,b_2,...,b_m]'$, $\boldsymbol{b}_2=[\boldsymbol{b}_1-\boldsymbol{T}\boldsymbol{b}_2]'=[b_{m+2},b_{m+3},...,b_{2m+1}]'$
The system in (4.7) can be written in matrix form as follows:

$$\boldsymbol{Rx}=\boldsymbol{b}, \tag{4.8}$$

where, $\boldsymbol{R}=[a_{ij}]_{i,j=1,...,2m+1}$ is the coefficient matrix of the system (4.7), $\boldsymbol{x}=[x_1,x_2,...,x_{2m+1}]'$ and $\boldsymbol{b}=[b_1,b_2,...,b_{2m+1}]'$ is the constant vector.

Now, let, $\boldsymbol{Q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} \boldsymbol{I}_m & 0 & \boldsymbol{I}_m \\ 0 & \sqrt{2} & 0 \\ \boldsymbol{T}_m & 0 & -\boldsymbol{T}_m \end{bmatrix}$ be an orthogonal matrix such that

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{v} & \boldsymbol{TBT} \\ \boldsymbol{u}' & \boldsymbol{q} & \boldsymbol{u}'\boldsymbol{T} \\ \boldsymbol{B} & \boldsymbol{T}\boldsymbol{v} & \boldsymbol{TAT} \end{bmatrix} = \boldsymbol{Q}_2 \begin{bmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} & 0 \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} & 0 \\ 0 & 0 & \boldsymbol{A}-\boldsymbol{TB} \end{bmatrix} \boldsymbol{Q}_2'.$$

Now,

$$\begin{aligned}
|\,\boldsymbol{R}\,| &= |\,\boldsymbol{Q}_2\,| \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} & 0 \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} & 0 \\ 0 & 0 & \boldsymbol{A}-\boldsymbol{TB} \end{vmatrix} |\,\boldsymbol{Q}_2'\,| \\
&= |\,\boldsymbol{Q}_2\,| \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} & 0 \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} & 0 \\ 0 & 0 & 1 \end{vmatrix} |\,\boldsymbol{A}-\boldsymbol{TB}\,||\,\boldsymbol{Q}_2'\,| \\
&= |\,\boldsymbol{Q}_2\,| \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} \end{vmatrix} |\,\boldsymbol{A}-\boldsymbol{TB}\,||\,\boldsymbol{Q}_2'\,| \\
&= \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} \end{vmatrix} |\,\boldsymbol{A}-\boldsymbol{TB}\,||\,\boldsymbol{Q}_2'\boldsymbol{Q}_2\,| \\
&= \begin{vmatrix} \boldsymbol{A}+\boldsymbol{TB} & \sqrt{2}\boldsymbol{v} \\ \sqrt{2}\boldsymbol{u}' & \boldsymbol{q} \end{vmatrix} |\,\boldsymbol{A}-\boldsymbol{TB}\,|\,.
\end{aligned}$$

Step 1: Construc the matrices $\boldsymbol{P}$, $\boldsymbol{Q}$ of orders $m+1$ and m respectively and the vectors $\hat{\boldsymbol{b}}$ and $\tilde{\boldsymbol{b}}$ of dimensions $m+1$ and m respectively as follows:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}+\boldsymbol{TB} & 2\boldsymbol{v} \\ \boldsymbol{u}' & q \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}+a_{1,2m+1} & a_{1,2}+a_{1,2m} & \cdots & a_{1,m}+a_{1,m+2} & 2a_{1,m+1} \\ a_{2,1}+a_{2,2m+1} & a_{2,2}+a_{2,2m} & \cdots & a_{2,m}+a_{2,m+2} & 2a_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1}+a_{m,2m+1} & a_{m,2}+a_{m,2m} & \cdots & a_{m,m}+a_{m,m+2} & 2a_{m,m+1} \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,m} & a_{m+1,m+1} \end{bmatrix}$$

$$\boldsymbol{Q} = \boldsymbol{A}-\boldsymbol{TB} = [a_{ij}-a_{i,2m+2-j}]_{i,j=1,\ldots,m},$$

$$\hat{\boldsymbol{b}} = [\boldsymbol{b}_1+\boldsymbol{T}\boldsymbol{b}_2, b_{m+1}]' = [b_1+b_{2m+1}, b_2+b_{2m}, \ldots, b_m+b_{m+2}, b_{m+1}]',$$

$$\tilde{\boldsymbol{b}} = [\boldsymbol{b}_1-\boldsymbol{T}\boldsymbol{b}_2]' = [b_1-b_{2m+1}, b_2-b_{2m}, \ldots, b_m-b_{m+2}]'.$$

Step 2: Compute $|\,\boldsymbol{R}\,| = |\,\boldsymbol{P}\,||\,\boldsymbol{Q}\,|$. If $|\,\boldsymbol{R}\,| = 0$ then "No solutions" end if.
Step 3: Solve the two linear systems: $P\boldsymbol{y} = \hat{\boldsymbol{b}}$, and $Q\boldsymbol{z} = \tilde{\boldsymbol{b}}$, for $\boldsymbol{y} = [y_1, y_2, \ldots, y_m, y_{m+1}]'$ and $\boldsymbol{z} = [z_1, z_2, \ldots, z_m]'$ respectively.
Step 4: The solution vector of equation (4.8) is $\boldsymbol{x} = [x_1, x_2, \ldots, x_{2m+1}]'$,given by

$$x_i = \begin{cases} \frac{1}{2}(y_i+z_i) & if\ i = 1,2,\ldots,m, \\ y_{m+1} & if\ i = m+1, \\ \frac{1}{2}(y_{2m+2-i}-z_{2m+2-i}) & if\ i = m+2, m+3, \ldots, 2m+1. \end{cases}$$

4.3 Formulas for calculating $\widehat{\boldsymbol{x}}$ in (4.1)

Let $m = \lfloor n/2 \rfloor$, both $\boldsymbol{H}_{11}$ and $\boldsymbol{H}_{21}$ be $m \times m$ matrices, both $\boldsymbol{w}$ and $\boldsymbol{v}$ be m-dimensional column vectors, and q be a scalar. Then, since $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$ is a centrosymmetric matrix, if n is even,

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{T}_m \boldsymbol{H}_{21} \boldsymbol{T}_m \\ \boldsymbol{H}_{21} & \boldsymbol{T}_m \boldsymbol{H}_{11} \boldsymbol{T}_m \end{bmatrix},$$

and if n is odd,

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{w} & \boldsymbol{T}_m \boldsymbol{H}_{21} \boldsymbol{T}_m \\ \boldsymbol{v}' & q & \boldsymbol{v}' \boldsymbol{T}_m \\ \boldsymbol{H}_{21} & \boldsymbol{T}_m \boldsymbol{w} & \boldsymbol{T}_m \boldsymbol{H}_{11} \boldsymbol{T}_m \end{bmatrix}.$$

For example, see Abu-Jeib (2002, Lemma 2.3). Moreover, $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$ is a symmetric persymmetric matrix, $\boldsymbol{H}_{11} = \boldsymbol{H}_{11}'$, $\boldsymbol{T}_m \boldsymbol{H}_{21} \boldsymbol{T}_m = \boldsymbol{H}_{21}'$, and $\boldsymbol{v} = \boldsymbol{w}$. Accordingly, if n is even,

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{21}' \\ \boldsymbol{H}_{21} & \boldsymbol{T}_m \boldsymbol{H}_{11} \boldsymbol{T}_m \end{bmatrix}, \tag{4.9}$$

and if n is odd,

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{w} & \boldsymbol{H}_{21}' \\ \boldsymbol{w}' & q & \boldsymbol{w}' \boldsymbol{T}_m \\ \boldsymbol{H}_{21} & \boldsymbol{T}_m \boldsymbol{w} & \boldsymbol{T}_m \boldsymbol{H}_{11} \boldsymbol{T}_m \end{bmatrix}. \tag{4.10}$$

For example, when $n = 7$, $p = 3$, and $\lambda = 1$, the smoother matrix is

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$$

$$
= \left[\begin{array}{c:c:c} \boldsymbol{H}_{11} & \boldsymbol{w} & \boldsymbol{H}'_{21} \\ \hdashline \boldsymbol{w}' & v & \boldsymbol{w}'\boldsymbol{T}_m \\ \hdashline \boldsymbol{H}_{21} & \boldsymbol{T}_m\boldsymbol{w} & \boldsymbol{T}_m\boldsymbol{H}_{11}\boldsymbol{T}_m \end{array}\right]
$$

$$
= \left[\begin{array}{ccc:c:ccc} \mathbf{0.8718} & 0.2393 & -0.0342 & -0.0769 & -0.0342 & 0.0085 & 0.0256 \\ \mathbf{0.2393} & \mathbf{0.4302} & 0.3048 & 0.1026 & -0.0285 & -0.0570 & 0.0085 \\ -\mathbf{0.0342} & \mathbf{0.3048} & \mathbf{0.4217} & 0.2821 & 0.0883 & -0.0285 & -0.0342 \\ \hdashline -\mathbf{0.0769} & \mathbf{0.1026} & \mathbf{0.2821} & \mathbf{0.3846} & 0.2821 & 0.1026 & -0.0769 \\ \hdashline -\mathbf{0.0342} & -\mathbf{0.0285} & \mathbf{0.0883} & 0.2821 & 0.4217 & 0.3048 & -0.0342 \\ \mathbf{0.0085} & -\mathbf{0.0570} & -0.0285 & 0.1026 & 0.3048 & 0.4302 & 0.2393 \\ \mathbf{0.0256} & 0.0085 & -0.0342 & -0.0769 & -0.0342 & 0.2393 & 0.8718 \end{array}\right] . \tag{4.11}
$$

Since $(\boldsymbol{I}_n + \lambda \boldsymbol{D}'_p\boldsymbol{D}_p)$ is bisymmetric, when n is even, it may be expressed as

$$
(\boldsymbol{I}_n + \lambda \boldsymbol{D}'_p\boldsymbol{D}_p) = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}'_{21} \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{G}_{11}\boldsymbol{T}_m \end{bmatrix}, \tag{4.12}
$$

where $\boldsymbol{G}_{11}$ is an $m \times m$ matrix. Likewise, when n is odd, it is

$$
(\boldsymbol{I}_n + \lambda \boldsymbol{D}'_p\boldsymbol{D}_p) = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{G}'_{21} \\ \boldsymbol{\alpha}' & r & \boldsymbol{\alpha}'\boldsymbol{T}_m \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{11}\boldsymbol{T}_m \end{bmatrix}, \tag{4.13}
$$

where $\boldsymbol{G}_{11}$ is an $m \times m$ matrix.

Proposition 4.1. *(i) $\boldsymbol{H}_{11}$ and $\boldsymbol{H}_{21}$ in* (4.9) *may be expressed with $\boldsymbol{G}_{11}$ and $\boldsymbol{G}_{21}$ in* (4.12) *as*

$$\boldsymbol{H}_{11} = (\boldsymbol{G}_{11} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21})^{-1}, \tag{4.14}$$

$$\boldsymbol{H}_{21} = -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21}\boldsymbol{H}_{11}. \tag{4.15}$$

(ii) $\boldsymbol{H}_{11}$, $\boldsymbol{H}_{21}$, $\boldsymbol{w}$, and q in (4.10) *may be expressed with $\boldsymbol{G}_{11}$, $\boldsymbol{G}_{21}$, $\boldsymbol{\alpha}$, and r in* (4.13) *as*

$$\boldsymbol{H}_{11} = (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{b}'/c)^{-1}, \tag{4.16}$$

$$\boldsymbol{w} = -\boldsymbol{H}_{11}\boldsymbol{b}/c, \tag{4.17}$$

$$q = c^{-1} + c^{-2}\boldsymbol{b}'\boldsymbol{H}_{11}\boldsymbol{b}, \tag{4.18}$$

$$\boldsymbol{H}_{21} = -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m(\boldsymbol{G}_{21}\boldsymbol{H}_{11} + \boldsymbol{T}_m\boldsymbol{\alpha}\boldsymbol{w}'), \tag{4.19}$$

where

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}' & c \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{\alpha} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{\alpha} \\ \boldsymbol{\alpha}' - \boldsymbol{\alpha}'\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & r - \boldsymbol{\alpha}'\boldsymbol{G}_{11}^{-1}\boldsymbol{\alpha} \end{bmatrix}.$$

Proof. (ii):

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{w} \\ \boldsymbol{w}' & q \end{bmatrix} &= \left(\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' & r \end{bmatrix} - \begin{bmatrix} \boldsymbol{G}'_{21} \\ \boldsymbol{\alpha}'\boldsymbol{T}_m \end{bmatrix} \boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m \begin{bmatrix} \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{\alpha} \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} \boldsymbol{G}_{11} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{\alpha} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{T}_m\boldsymbol{\alpha} \\ \boldsymbol{\alpha}' - \boldsymbol{\alpha}'\boldsymbol{T}_m\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & r - \boldsymbol{\alpha}'\boldsymbol{T}_m\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{T}_m\boldsymbol{\alpha} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \boldsymbol{G}_{11} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{\alpha} - \boldsymbol{G}'_{21}\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{\alpha} \\ \boldsymbol{\alpha}' - \boldsymbol{\alpha}'\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m\boldsymbol{G}_{21} & r - \boldsymbol{\alpha}'\boldsymbol{G}_{11}^{-1}\boldsymbol{\alpha} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{b}' & c \end{bmatrix}^{-1}
\end{aligned}$$

Then, it follows that

$$\boldsymbol{H}_{11} = (\boldsymbol{A} - \boldsymbol{b}\boldsymbol{b}'/c)^{-1}, \quad \boldsymbol{w} = -\boldsymbol{H}_{11}\boldsymbol{b}/c, \quad q = c^{-1} + c^{-1}\boldsymbol{b}'\boldsymbol{H}_{11}\boldsymbol{b}c^{-1}.$$

In addition,

$$\begin{aligned}
\begin{bmatrix} \boldsymbol{H}_{21} & \boldsymbol{T}_m\boldsymbol{w} \end{bmatrix} &= -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m \begin{bmatrix} \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{\alpha} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{w} \\ \boldsymbol{w}' & q \end{bmatrix} \\
&= -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m \begin{bmatrix} \boldsymbol{G}_{21}\boldsymbol{H}_{11} + \boldsymbol{T}_m\boldsymbol{\alpha}\boldsymbol{w}' & \boldsymbol{G}_{21}\boldsymbol{w} + \boldsymbol{T}_m\boldsymbol{\alpha}q \end{bmatrix} \\
&= \begin{bmatrix} -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m(\boldsymbol{G}_{21}\boldsymbol{H}_{11} + \boldsymbol{T}_m\boldsymbol{\alpha}\boldsymbol{w}') & -\boldsymbol{T}_m\boldsymbol{G}_{11}^{-1}\boldsymbol{T}_m(\boldsymbol{G}_{21}\boldsymbol{w} + \boldsymbol{T}_m\boldsymbol{\alpha}q) \end{bmatrix}.
\end{aligned}$$

□

Remarks. (a) A MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_n + \lambda\boldsymbol{D}_p'\boldsymbol{D}_p)^{-1}$ based on (4.14)–(4.19) is provided in the Appendix. It is noteworthy here that (i) even though $(\boldsymbol{I}_n + \lambda\boldsymbol{D}_p'\boldsymbol{D}_p)$ is a $n \times n$ matrix, its inverse is obtainable by inverting $m \times m$ matrices and (ii) since $(\boldsymbol{I}_n + \lambda\boldsymbol{D}_p'\boldsymbol{D}_p)$ is a $(2p+1)$-diagonal matrix, it follows that $\boldsymbol{G}_{21}$ is generally a sparse matrix but $\boldsymbol{G}_{21} \neq \boldsymbol{0}$. For example, when $n = 7$, $p = 3$, and $\lambda = 1$,

$$(\boldsymbol{I}_n + \lambda\boldsymbol{D}_p'\boldsymbol{D}_p) = \left[\begin{array}{c:c:c} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{G}_{21}' \\ \hdashline \boldsymbol{\alpha}' & r & \boldsymbol{\alpha}'\boldsymbol{T}_m \\ \hdashline \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{\alpha} & \boldsymbol{G}_{22} \end{array}\right] = \left[\begin{array}{ccc:c:ccc} 2 & -3 & 3 & -1 & 0 & 0 & 0 \\ -3 & 11 & -12 & 6 & -1 & 0 & 0 \\ 3 & -12 & 20 & -15 & 6 & -1 & 0 \\ \hdashline -1 & 6 & -15 & 21 & -15 & 6 & -1 \\ \hdashline 0 & -1 & 6 & -15 & 20 & -12 & 3 \\ 0 & 0 & -1 & 6 & -12 & 11 & -3 \\ 0 & 0 & 0 & -1 & 3 & -3 & 2 \end{array}\right].$$

Corollary 4.1. *(i) When n is even, letting* $\mathbf{y} = [\mathbf{y}_1', \mathbf{y}_2']'$, *where* $\mathbf{y}_1$ *is a m-dimensional column* *vector, it follows that*

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{11}\boldsymbol{y}_1 + \boldsymbol{H}'_{21}\boldsymbol{y}_2 \\ \boldsymbol{H}_{21}\boldsymbol{y}_1 + \boldsymbol{T}_m\boldsymbol{H}_{11}\boldsymbol{T}_m\boldsymbol{y}_2 \end{bmatrix}, \tag{4.20}$$

where $\boldsymbol{H}_{11}$ *and* $\boldsymbol{H}_{21}$ *are defined in* (4.14) *and* (4.15)*, respectively. (ii) When n is odd, letting* $\boldsymbol{y} = [\boldsymbol{y}'_1, y_{m+1}, \boldsymbol{y}'_2]'$*, where* $\boldsymbol{y}_1$ *is a m-dimensional column vector and* y_{m+1} *is a scalar, it follows that*

$$\widehat{\boldsymbol{x}} = \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \widehat{x}_{m+1} \\ \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{11}\boldsymbol{y}_1 + \boldsymbol{w}y_{m+1} + \boldsymbol{H}'_{21}\boldsymbol{y}_2 \\ \boldsymbol{w}'\boldsymbol{y}_1 + qy_{m+1} + \boldsymbol{w}'\boldsymbol{T}_m\boldsymbol{y}_2 \\ \boldsymbol{H}_{21}\boldsymbol{y}_1 + \boldsymbol{T}_m\boldsymbol{w}y_{m+1} + \boldsymbol{T}_m\boldsymbol{H}_{11}\boldsymbol{T}_m\boldsymbol{y}_2 \end{bmatrix}, \tag{4.21}$$

where $\boldsymbol{H}_{11}$, $\boldsymbol{H}_{21}$, $\boldsymbol{w}$, *and q are defined in* (4.16), (4.17), (4.18), *and* (4.19)*, respectively.*

Remarks. From the centrosymmetry of $(\boldsymbol{I}_n + \lambda\boldsymbol{D}'_p\boldsymbol{D}_p)$, we may obtain an alternative inversion formula to (4.20) by applying El-Mikkawy and Atlan's (2013) CENTROSYMM-I algorithm:

$$\widehat{\boldsymbol{x}}_1 = \frac{1}{2}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2), \tag{4.22}$$

$$\widehat{\boldsymbol{x}}_2 = \frac{1}{2}\boldsymbol{T}_m(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2), \tag{4.23}$$

where $\boldsymbol{\xi}_1 = (\boldsymbol{G}_{11} + \boldsymbol{T}_m\boldsymbol{G}_{21})^{-1}(\boldsymbol{y}_1 + \boldsymbol{T}_m\boldsymbol{y}_2)$ and $\boldsymbol{\xi}_2 = (\boldsymbol{G}_{11} - \boldsymbol{T}_m\boldsymbol{G}_{21})^{-1}(\boldsymbol{y}_1 - \boldsymbol{T}_m\boldsymbol{y}_2)$. It is notable that similarly to (4.20), $\widehat{\boldsymbol{x}}_i$ for $i = 1, 2$ are obtainable by inverting not $n \times n$ matrices but $m \times m$ matrices. Likewise, by applying El-Mikkawy and Atlan's (2013) CENTROSYMM-II algorithm, we may obtain an alternative inversion formula to (4.21) as follows:

$$\widehat{\boldsymbol{x}}_1 = \frac{1}{2}(\boldsymbol{\zeta}_1 + \boldsymbol{\zeta}_2), \tag{4.24}$$

$$\widehat{x}_{m+1} = \zeta_{m+1}, \tag{4.25}$$

$$\widehat{\boldsymbol{x}}_2 = \frac{1}{2}\boldsymbol{T}_m(\boldsymbol{\zeta}_1 - \boldsymbol{\zeta}_2), \tag{4.26}$$

where

$$\begin{bmatrix} \boldsymbol{\zeta}_1 \\ \zeta_{m+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21} & 2\boldsymbol{\alpha} \\ \boldsymbol{\beta}' & r \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{y}_1 + \boldsymbol{T}_m \boldsymbol{y}_2 \\ y_{m+1} \end{bmatrix}$$

and $\boldsymbol{\zeta}_2 = (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}(\boldsymbol{y}_1 - \boldsymbol{T}_m \boldsymbol{y}_2)$. Here, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and r are defined as follows.

$$(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p) = \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{T}_m \boldsymbol{G}_{21} \boldsymbol{T}_m \\ \boldsymbol{\beta}' & r & \boldsymbol{\beta}' \boldsymbol{T}_m \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m \boldsymbol{\alpha} & \boldsymbol{T}_m \boldsymbol{G}_{11} \boldsymbol{T}_m \end{bmatrix},$$

The proofs of (4.22)–(4.23) and (4.24)–(4.26) are provided in the Appendix.

4.4 Bisymmetry of the smoother matrices in (4.3) and (4.4)

We show that similar properties hold for $\boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p} + \boldsymbol{D}_p \boldsymbol{D}_p')^{-1}\boldsymbol{D}_p$ in (4.3) and $\boldsymbol{F}_p(\boldsymbol{F}_p' \boldsymbol{F}_p + \lambda \boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'$ in (4.4).

Corollary 4.2. *$\boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p} + \boldsymbol{D}_p \boldsymbol{D}_p')^{-1}\boldsymbol{D}_p$ in (4.3) is a bisymmetric matrix.*

Proof. From Yamada (2019) we know that,

$$\boldsymbol{T}_n(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}\boldsymbol{T}_n = [\boldsymbol{T}_n(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)\boldsymbol{T}_n]^{-1} = (\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}.$$

It follows that

$$\begin{aligned} \boldsymbol{T}_p \boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p} + \boldsymbol{D}_p \boldsymbol{D}_p')^{-1}\boldsymbol{D}_p \boldsymbol{T}_p &= \boldsymbol{I}_n - \boldsymbol{T}_p(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}\boldsymbol{T}_p \\ &= \boldsymbol{I}_n - (\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1} \\ &= \boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p} + \boldsymbol{D}_p \boldsymbol{D}_p')^{-1}\boldsymbol{D}_p, \end{aligned}$$

which indicates that $\boldsymbol{D}_p'(\lambda^{-1}\boldsymbol{I}_{n-p}+\boldsymbol{D}_p\boldsymbol{D}_p')^{-1}\boldsymbol{D}_p$ is a centrosymmetric matrix. In addition, it is also a symmetric matrix. Thus, it is a bisymmetric matrix. $\square$

Yamada (2019) proved that, $\boldsymbol{T}_{n-p}\boldsymbol{D}_p=(-1)^p\boldsymbol{D}_p\boldsymbol{T}_n$. Similarly, we obtain the following result:

Lemma 4.2. $\boldsymbol{T}_n\boldsymbol{F}_p=(-1)^p\boldsymbol{F}_p\boldsymbol{T}_{n-p}$.

Proof. From Yamada (2019) we know that, $\boldsymbol{T}_{n-p}\boldsymbol{D}_p=(-1)^p\boldsymbol{D}_p\boldsymbol{T}_n$, it follows that

$$\begin{aligned}\boldsymbol{T}_n\boldsymbol{F}_p=\boldsymbol{T}_n\boldsymbol{D}_p'(\boldsymbol{D}_p\boldsymbol{D}_p')^{-1}&=((-1)^p(-1)^p\boldsymbol{D}_p\boldsymbol{T}_n)'\boldsymbol{T}_{n-p}(\boldsymbol{T}_{n-p}\boldsymbol{D}_p\boldsymbol{D}_p'\boldsymbol{T}_{n-p})^{-1}\boldsymbol{T}_{n-p}\\&=(-1)^p(\boldsymbol{T}_{n-p}\boldsymbol{D}_p)'\boldsymbol{T}_{n-p}(\boldsymbol{T}_{n-p}\boldsymbol{D}_p\boldsymbol{D}_p'\boldsymbol{T}_{n-p})^{-1}\boldsymbol{T}_{n-p}\\&=(-1)^p\boldsymbol{D}_p'(\boldsymbol{D}_p\boldsymbol{D}_p')^{-1}\boldsymbol{T}_{n-p}=(-1)^p\boldsymbol{F}_p\boldsymbol{T}_{n-p}.\end{aligned}$$

$\square$

Proposition 4.2. $\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'$ *in* (4.4) *is a bisymmetric matrix.*

Proof. From Lemma 4.2, it follows that

$$\begin{aligned}\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'&=\boldsymbol{F}_p\boldsymbol{T}_{n-p}\boldsymbol{T}_{n-p}(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{T}_{n-p}\boldsymbol{T}_{n-p}\boldsymbol{F}_p'\\&=\boldsymbol{F}_p\boldsymbol{T}_{n-p}(\boldsymbol{T}_{n-p}\boldsymbol{F}_p'\boldsymbol{F}_p\boldsymbol{T}_{n-p}+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{T}_{n-p}\boldsymbol{F}_p'\\&=\boldsymbol{T}_n\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{T}_n\boldsymbol{T}_n\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'\boldsymbol{T}_n\\&=\boldsymbol{T}_n\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'\boldsymbol{T}_n,\end{aligned}$$

which indicates that $\boldsymbol{F}_p(\boldsymbol{F}_p'\boldsymbol{F}_p+\lambda\boldsymbol{I}_{n-p})^{-1}\boldsymbol{F}_p'$ is a centrosymmetric matrix. In addition, it is also a symmetric matrix. Thus, it is a bisymmetric matrix. $\square$

4.5 Concluding remarks

In this chapter, based on the result of Yamada (2019), we presented simple formulas for calculating the smoother matrix of the WH graduation. In addition, we showed some results, which include that two other smoother matrices related with the WH graduation are also bisymmetric. The results obtained in the paper are summarized in Propositions 4.1, and 4.2 and in Corollaries 4.1 and 4.2.

4.6 Appendix

4.6.1 A MATLAB/GNU Octave function to calculate $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)^{-1}$ based on (4.14)–(4.19)

```
function invP = Calc_Hat_WK_Graduation(n, p, lambda)

D = diff(eye(n), p);
P = eye(n)+lambda*D'*D;
m = floor(n/2);
I = eye(m);
T = I(:, m:-1:1);

if mod(n, 2) == 0  % even
   G11 = P(1:m, 1:m);
   G21 = P(m+1:n, 1:m);
   invG11 = inv(G11);
   H11 = inv(G11-G21'*T*invG11*T*G21);
   H21 = -T*invG11*T*G21*H11;
   invP = [H11, H21'; H21, T*H11*T];
else  % odd
   G11 = P(1:m, 1:m);
   alpha = P(1:m, m+1);
   r = P(m+1, m+1);
   G21 = P(m+2:n, 1:m);
   invG11 = inv(G11);
   A = G11-G21'*T*invG11*T*G21;
   b = alpha-G21'*T*invG11*alpha;
   c = r-alpha'*invG11*alpha;
   H11 = inv(A-b*b'/c);
   w = -H11*b/c;
   q = 1/c+(1/c^2)*(b'*H11*b);
   H21 = -T*invG11*T*(G21*H11+T*alpha*w');
   invP = [H11, w, H21'; w', q, w'*T; H21, T*w, T*H11*T];
end
```

```
end
```

4.6.2 Proof of (4.22) and (4.23)

When n is even, $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p' \boldsymbol{D}_p)\widehat{\boldsymbol{x}} = \boldsymbol{y}$ may be expressed as

$$\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{T}_m \boldsymbol{G}_{21} \boldsymbol{T}_m \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m \boldsymbol{G}_{11} \boldsymbol{T}_m \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}. \tag{4.27}$$

Premultiplying (4.27) by $\mathsf{diag}(\boldsymbol{I}_m, \boldsymbol{T}_m)$, we obtain

$$\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{T}_m \boldsymbol{G}_{21} \\ \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{G}_{11} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \boldsymbol{T}_m \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{T}_m \boldsymbol{y}_2 \end{bmatrix}, \tag{4.28}$$

Since

$$\begin{aligned} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{I}_m \\ \boldsymbol{I}_m & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{T}_m \boldsymbol{G}_{21} \\ \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{G}_{11} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{T}_m \boldsymbol{G}_{21} + \boldsymbol{G}_{11} \\ \boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{T}_m \boldsymbol{G}_{21} - \boldsymbol{G}_{11} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{I}_m \\ \boldsymbol{I}_m & -\boldsymbol{I}_m \end{bmatrix}, \end{aligned}$$

premultiplying (4.28) by

$$\begin{bmatrix} \boldsymbol{I}_m & \boldsymbol{I}_m \\ \boldsymbol{I}_m & -\boldsymbol{I}_m \end{bmatrix},$$

it follows that

$$\begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1 + \boldsymbol{T}_m \widehat{\boldsymbol{x}}_2 \\ \widehat{\boldsymbol{x}}_1 - \boldsymbol{T}_m \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 + \boldsymbol{T}_m \boldsymbol{y}_2 \\ \boldsymbol{y}_1 - \boldsymbol{T}_m \boldsymbol{y}_2 \end{bmatrix},$$

which leads to

$$\begin{bmatrix} \widehat{\boldsymbol{x}}_1 + \boldsymbol{T}_m \widehat{\boldsymbol{x}}_2 \\ \widehat{\boldsymbol{x}}_1 - \boldsymbol{T}_m \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} (\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} (\boldsymbol{y}_1 + \boldsymbol{T}_m \boldsymbol{y}_2) \\ (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} (\boldsymbol{y}_1 - \boldsymbol{T}_m \boldsymbol{y}_2) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}.$$

By solving the above simultaneous equations, we obtain (4.22) and (4.23).

Here, we remark that we may apply the inversion formula of centrosymmetric matrix given in Good (1970):

$$\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{T}_m \boldsymbol{G}_{21} \boldsymbol{T}_m \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m \boldsymbol{G}_{11} \boldsymbol{T}_m \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{21} \boldsymbol{T}_m \\ \boldsymbol{T}_m \boldsymbol{K}_{21} & \boldsymbol{T}_m \boldsymbol{K}_{11} \boldsymbol{T}_m \end{bmatrix},$$

where

$$\begin{aligned} \boldsymbol{K}_{11} &= \frac{1}{2}[(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} + (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}], \\ \boldsymbol{K}_{21} &= \frac{1}{2}[(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} - (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}], \end{aligned}$$

Accordingly, it follows that

$$\begin{aligned} \widehat{\boldsymbol{x}}_1 &= \boldsymbol{K}_{11} \boldsymbol{y}_1 + \boldsymbol{K}_{21} \boldsymbol{T}_m \boldsymbol{y}_2 \\ &= \frac{1}{2}[(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} + (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}] \boldsymbol{y}_1 \\ &\qquad + \frac{1}{2}[(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} - (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}] \boldsymbol{T}_m \boldsymbol{y}_2 \\ &= \frac{1}{2}(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2). \end{aligned}$$

Likewise, we may obtain

$$\begin{aligned} \widehat{\boldsymbol{x}}_2 &= \boldsymbol{T}_m \boldsymbol{K}_{21} \boldsymbol{y}_1 + \boldsymbol{T}_m \boldsymbol{K}_{11} \boldsymbol{T}_m \boldsymbol{y}_2 \\ &= \frac{1}{2} \boldsymbol{T}_m [(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} - (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}] \boldsymbol{y}_1 \\ &\qquad + \frac{1}{2} \boldsymbol{T}_m [(\boldsymbol{G}_{11} + \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1} + (\boldsymbol{G}_{11} - \boldsymbol{T}_m \boldsymbol{G}_{21})^{-1}] \boldsymbol{T}_m \boldsymbol{y}_2 \end{aligned}$$

$$= \frac{1}{2}\boldsymbol{T}_m(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2).$$

4.6.3 Proof of (4.24)–(4.26)

When n is odd, $(\boldsymbol{I}_n + \lambda \boldsymbol{D}_p'\boldsymbol{D}_p)\widehat{\boldsymbol{x}} = \boldsymbol{y}$ may be expressed as

$$\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{21}\boldsymbol{T}_m \\ \boldsymbol{\beta}' & r & \boldsymbol{\beta}'\boldsymbol{T}_m \\ \boldsymbol{G}_{21} & \boldsymbol{T}_m\boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{11}\boldsymbol{T}_m \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \widehat{x}_{m+1} \\ \widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ y_{m+1} \\ \boldsymbol{y}_2 \end{bmatrix} \tag{4.29}$$

Premultiplying (4.29) by $\mathsf{diag}(\boldsymbol{I}_m, 1, \boldsymbol{T}_m)$, we obtain

$$\begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{21} \\ \boldsymbol{\beta}' & r & \boldsymbol{\beta}' \\ \boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{\alpha} & \boldsymbol{G}_{11} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1 \\ \widehat{x}_{m+1} \\ \boldsymbol{T}_m\widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1 \\ y_{m+1} \\ \boldsymbol{T}_m\boldsymbol{y}_2 \end{bmatrix} \tag{4.30}$$

Since

$$\begin{aligned} \begin{bmatrix} \boldsymbol{I}_m & 0 & \boldsymbol{I}_m \\ \boldsymbol{0} & 1 & \boldsymbol{0} \\ \boldsymbol{I}_m & 0 & -\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{G}_{11} & \boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{21} \\ \boldsymbol{\beta}' & r & \boldsymbol{\beta}' \\ \boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{\alpha} & \boldsymbol{G}_{11} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m\boldsymbol{G}_{21} & 2\boldsymbol{\alpha} & \boldsymbol{T}_m\boldsymbol{G}_{21} + \boldsymbol{G}_{11} \\ \boldsymbol{\beta}' & r & \boldsymbol{\beta}' \\ \boldsymbol{G}_{11} - \boldsymbol{T}_m\boldsymbol{G}_{21} & \boldsymbol{0} & \boldsymbol{T}_m\boldsymbol{G}_{21} - \boldsymbol{G}_{11} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{G}_{11} + \boldsymbol{T}_m\boldsymbol{G}_{21} & 2\boldsymbol{\alpha} & \boldsymbol{0} \\ \boldsymbol{\beta}' & r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{G}_{11} - \boldsymbol{T}_m\boldsymbol{G}_{21} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_m & 0 & \boldsymbol{I}_m \\ \boldsymbol{0} & 1 & \boldsymbol{0} \\ \boldsymbol{I}_m & 0 & -\boldsymbol{I}_m \end{bmatrix}, \end{aligned}$$

premultiplying (4.30) by

$$\begin{bmatrix} \boldsymbol{I}_m & 0 & \boldsymbol{I}_m \\ \mathbf{0} & 1 & \mathbf{0} \\ \boldsymbol{I}_m & 0 & -\boldsymbol{I}_m \end{bmatrix},$$

it follows that

$$\begin{bmatrix} \boldsymbol{G}_{11}+\boldsymbol{T}_m\boldsymbol{G}_{21} & 2\boldsymbol{\alpha} & \mathbf{0} \\ \boldsymbol{\beta}' & r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{G}_{11}-\boldsymbol{T}_m\boldsymbol{G}_{21} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_1+\boldsymbol{T}_m\widehat{\boldsymbol{x}}_2 \\ \widehat{x}_{m+1} \\ \widehat{\boldsymbol{x}}_1-\boldsymbol{T}_m\widehat{\boldsymbol{x}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{y}_1+\boldsymbol{T}_m\boldsymbol{y}_2 \\ y_{m+1} \\ \boldsymbol{y}_1-\boldsymbol{T}_m\boldsymbol{y}_2 \end{bmatrix},$$

which leads to

$$\begin{bmatrix} \widehat{\boldsymbol{x}}_1+\boldsymbol{T}_m\widehat{\boldsymbol{x}}_2 \\ \widehat{x}_{m+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}_{11}+\boldsymbol{T}_m\boldsymbol{G}_{21} & 2\boldsymbol{\alpha} \\ \boldsymbol{\beta}' & r \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{y}_1+\boldsymbol{T}_m\boldsymbol{y}_2 \\ y_{m+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\zeta}_1 \\ \zeta_{m+1} \end{bmatrix},$$
$$\widehat{\boldsymbol{x}}_1-\boldsymbol{T}_m\widehat{\boldsymbol{x}}_2 = (\boldsymbol{G}_{11}-\boldsymbol{T}_m\boldsymbol{G}_{21})^{-1}(\boldsymbol{y}_1-\boldsymbol{T}_m\boldsymbol{y}_2) = \boldsymbol{\zeta}_2.$$

By solving the above simultaneous equations, we obtain (4.24)–(4.26).

4.7 References

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