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A barrier against indirect proof:

Focusing on proof of the converse of the inscribed angle theorem by conversion

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1. Introduction

There is a traditional field of mathematics education research on indirect proof. Freudenthal (1973), a leading mathematician in mathematics education research, addressed this topic in the early stage of this research field. He characterized indirect proof as having the role of a heuristic device and argued that students must experience it as a useful pattern of thinking before encountering it as a form of writing. Since then, it has been commonly recognized that students find it difficult to learn indirect proof. There is a longstanding debate regarding the major factors influencing this difficulty: the special structure of argumentation of indirect proof (Antonini & Mariotti, 2008; Leron, 1985) and its mathematical contents (Barnard & Tall, 1997; Dawkins & Karunakaran, 2016). Researchers continue to explore how students can overcome the difficulty of indirect proof with the support of their teachers (cf. Hayata, Hakamata, & Uegatani, 2018). However, most current studies in this field are biased toward particular types of indirect proof, namely proof by contradiction and by contraposition. Although all Japanese mathematics textbooks for eighth grade students prove the converse of the Pythagorean theorem by coincidence and the converse of the inscribed angle theorem by conversion, there are only a few studies on these types of indirect proof (e.g., Byham, 1969; Uegatani, Hayata, & Hakamata, 2019). There is thus a need for more studies on these two specific topics.

This paper aims to report how students engage in a mathematical activity, especially to construct proof by conversion in a mathematical lesson, and to hypothesize a potential barrier against indirect proof. The structure of this paper is as follows. In Section 2, we will review the existing literature. In Section 3, we will propose two hypotheses and develop our research questions based on the results of the literature review. Further, in Section 4, we will describe the design of an experimental lesson and its expected results from our hypotheses. Section 5 will explain our method, and Section 6 will report its actual results. Subsequently, in Section 7, we will discuss and hypothesize a potential barrier against indirect proof by conversion in particular and indirect proof in general. Finally, in the last section, we will provide a conclusion, stating that there is an interwoven relationship between insufficient recognition of the goal of proving and ineptitude of using local mathematical techniques for proving.

2. Literature review

2.1. What is proof by conversion?

According to a famous Japanese glossary of terms in mathematics education, proof by conversion is defined as follows:

> Suppose that there is a cluster of theorems. Suppose also that the assumptions of these theorems cover all possible cases of a topic and that any two conclusions of the theorems do not hold at the same time (i.e., they are mutually exclusive). Then, a method of proving that all the converses of these theorems hold is called proving by conversion.
>
> (Hasegawa, 2000, p. 296; originally written in Japanese and translated into English by us for this paper)

Let us present this definition more formally. Suppose that there is a cluster of theorems: $P_1 \to Q_1, P_2 \to Q_2, \cdots, P_n \to Q_n$. Suppose also that $P_1, P_2, \cdots, P_n$ are collectively exhaustive and that $Q_1, Q_2, \cdots, Q_n$ are mutually exclusive. Then, we can say that all the converses of these theorems, that is, $Q_1 \to P_1, Q_2 \to P_2, \cdots, Q_n \to P_n$, hold.

The validity of proof by conversion can be proved by contradiction as follows.

Assumptions:

$$
\begin{aligned}
&P_1 \to Q_1 && \cdots\ [1]\\
&P_2 \to Q_2 && \cdots\ [2]\\
&\ \vdots\\
&P_n \to Q_n && \cdots\ [n]\\
&P_1 \lor P_2 \lor \cdots \lor P_n = \mathrm{T} && \cdots\ [*]\\
&Q_i \land Q_j = \mathrm{F}\ (\forall i, j: i \neq j) && \cdots\ [**]
\end{aligned}
$$

(T means true; F means false)

Conclusions:

$$
\begin{aligned}
&Q_1 \to P_1\\
&Q_2 \to P_2\\
&\ \vdots\\
&Q_n \to P_n
\end{aligned}
$$

Proof: Suppose, for the sake of contradiction, that there exists an integer $k\ (1 \leq k \leq n)$ such that $\neg(Q_k \to P_k)$ holds. Then, $\neg(Q_k \to P_k) \Leftrightarrow \neg(\neg Q_k \lor P_k) \Leftrightarrow Q_k \land \neg P_k\ \cdots$ [***].
By the assumption [*], at least one of $P_1, P_2, \cdots,$ or P_n holds.
Let i be an integer such that P_i holds.
If $i = k$, P_i contradicts the supposition [***].
Hence, we have $i \neq k$, and Q_i holds by the assumption $[i]$.
Since $Q_i \land Q_k$ does not hold by the assumption [**], Q_k does not hold.
This contradicts the supposition [***].
Therefore, for any integer $k\ (1 \leq k \leq n)$, $Q_k \to P_k$.

2.2. The inscribed angle theorem, its converse, and their proofs in Japanese textbooks

Generally, in Japanese ninth grade mathematics textbooks, the inscribed angle theorem is described as follows: 1) an inscribed angle that subtends a circular arc is half of the central angle that subtends the same circular arc; 2) inscribed angles that subtend the same circular arc are equal. On the other hand, the expressions of the converse of the inscribed angle theorem are complicated: suppose that two points C and P are on the same side of a straight line AB; if $\angle APB = \angle ACB$, then the four points A, B, C, and P lie on a common circle. From a mathematical point of view, there are two problems about how to deal with these two theorems in Japanese junior high school mathematics textbooks.

First, the converse of the inscribed angle theorem is not really a converse in the strict sense. Japanese junior high school students learn the converse of the proposition "if P, then Q" as the proposition "if Q, then P." The abovementioned converse of the inscribed angle theorem does not fit into this definition of a converse. We can, for example, better describe the inscribed angle theorem and its converse as follows. The inscribed angle theorem is that, suppose that two points C and P are on the same side of a straight line AB; the four points A, B, C, and P lie on a common circle, then $\angle APB = \angle ACB$. Its converse is that, suppose that two points C and P are on the same side of a straight line AB; if $\angle APB = \angle ACB$, then the four points A, B, C, and P lie on a common circle. However, since these new descriptions do not mention the relationship between the inscribed angles and the central angle, they are still problematic.

Second, by the definition of proof by conversion presented in the previous section, it is wrong in the strict sense to say that we prove the converse of the inscribed angle theorem by conversion. Proof by conversion can be applied to a cluster of theorems. Hence, for example, it can be applied to the following cluster of theorems: suppose that two points C and P are on the same side of a straight line AB; 1) if $\angle APB < \angle ACB$, then P lies outside of the circumscribed circle of the triangle ABC; 2) if $\angle APB = \angle ACB$, then P lies on the circumscribed circle of the triangle ABC; 3) if $\angle APB > \angle ACB$, then P lies inside the circumscribed circle of the triangle ABC. In Japanese junior high school textbooks, the cluster of these three theorems are virtually proved by conversion, and only the second theorem is displayed as the converse of the inscribed angle theorem.

2.3. Cognitive unity of a theorem

An influential study on indirect proof by Antonini and Mariotti (2008) argued about the potential of the idea of the *cognitive unity of a theorem* (Garuti, Boero, & Lemut, 1998). It is based on the continuity between conjecturing and proving. Garuti, Boero, and Lemut (1998) proposed the following tentative hypothesis:

> *the greater is the gap between the exploration needed to appropriate the statement and the proving process, the greater is the difficulty of the proving process*
>
> (p. 347, italics in the original)

A practical implication for teaching proving in general can be drawn from this hypothesis. A mathematics teacher should provide her students with opportunities to explore a target mathematical topic, to produce a conjecture, and to prove it in sequence in a lesson or in some lessons. A traditional style of proving tasks "prove that…" is less informative for students in terms of what they should do. Students obtain not only a conviction of what statement is true but also a clue for constructing its proof, through a sequence of exploring and conjecturing activities.

Recent development of research on cognitive unity has focused on rationality from Habermas' perspective. Boero (2017) analyzed students' think-aloud solving processes and identified a need of cognitive unity on epistemic, teleological, and communicative rationality between conjecturing and proving. Furthermore, it is worth mentioning the theoretical construct of *meta-cognitive unity* in indirect proof proposed by Arzarello and Sabena (2011). They argued that the difficulty of proving in an indirect way lies in the rupture between a teleological aspect at the metalevel and an epistemic one at the ground-level. Indirect argumentations often inform students why a statement holds and why its negation does not hold. Thus, after they make indirect argumentations, they lose their motivation toward the next proving process.

Over twenty years after Garuti, Boero, and Lemut's (1998) proposal, some readers of the current paper, who acknowledge mathematics learning through activities, may think that a didactic approach like the cognitive unity approach is trivial. It is certainly easier for students to construct a proof if the two activities of conjecturing and proving are unified in a sense. However, Antonini and Mariotti's (2008) implication goes beyond this triviality. It is especially with regard to proving by contradiction that they argue the efficiency of the cognitive unity approach. Given a task of proving a statement P, students can never know where a contradiction comes from in advance. For this reason, if they want to prove the given statement P, they must temporarily shift from the current proving stage to the exploring and conjecturing stages before successfully proceeding to proof construction. For example, in the case of proof by contradiction, students should conjecture what a statement R is such that both R and $\neg R$ simultaneously hold. Since there is a large gap between an explicit requirement to construct a proof of the given statement P and an implicit requirement to conjecture the statement R different from P, the construction of an indirect proof (not an indirect argument) seems to be unnatural for students and not spontaneous.

3. Hypotheses and research questions: Teacher's role and a shift of attention to a complementary set

Based on the result of the literature review presented in the previous section, we believe that junior high school students seldom spontaneously prove the converse of the inscribed angle theorem by conversion. On the other hand, we also believe that direct teaching of proving by conversion is less fruitful. If students cannot understand the motivation toward proving by conversion, they cannot acknowledge this method in general. At most, they understand that the method is only for the converse of the inscribed angle theorem. They are unlikely to become willing to use this method.

To avoid the problem that students cannot acknowledge the method of proving by conversion, it seems important to create the missing cognitive unity in proof by conversion, which is a role that mathematics teachers should fulfill. If students solely engage in mathematical activities for exploring a given topic, making a conjecture, and proving it, then indirect proof construction can never occur spontaneously because there would be cognitive ruptures. Hence, our first hypothesis is that *if the teacher bridges such ruptures suitably and at an appropriate time, then students can engage in such mathematical activities with cognitive unity*.

Moreover, we also propose a potential difficulty in constructing indirect proof: *a shift of attention to a complementary set*. This idea can be compared with the idea of *a shift of attention from the particular to the general* proposed by Mason (1989). For students to construct direct proof through the cognitive unity approach, they require a shift of attention from the particular to the general. For example, suppose that students explore a property of the sums of two odd numbers. Then, they may interpret Fig 1(A) as Fig 1(B) through activities. This change of interpretation is characterized as a shift of attention from the particular to the general. This type of shift of attention can occur even if they consider only pairs of odd numbers. There is no need to consider pairs of odd and even numbers nor pairs of even numbers, that is, elements in the complement of the set of all the pairs of odd numbers. On the other hand, the construction of indirect proof requires a different shift of attention. When we want to prove the converse of the inscribed angle theorem by conversion, we must consider not only the case of $\angle APB = \angle ACB$ but also the cases of $\angle APB < \angle ACB$ and $\angle APB > \angle ACB$. This extension of the scope of consideration can be described as a shift of attention to the complement of the initial scope. When we want to prove a statement by contradiction or by contraposition, we must also explore under the condition that the negation of the initial conclusion holds. This extension can also be described as the same type of shift of attention. We thus characterize such a flexible way of exploration as a shift of attention to a complementary set.

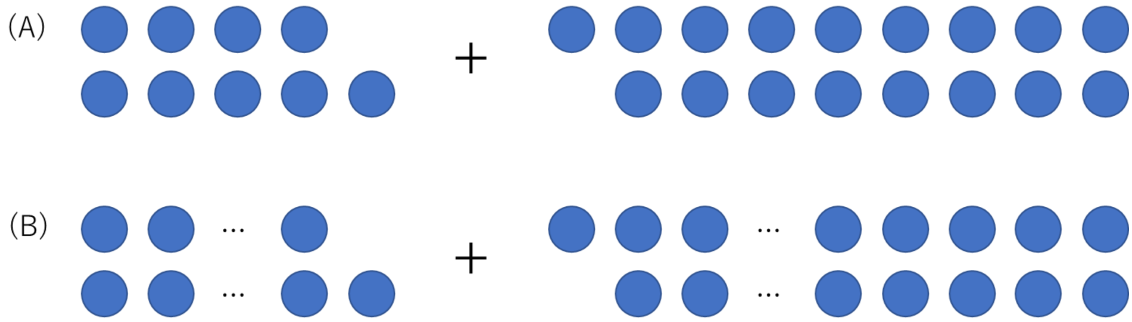

Fig 1: An example of shifts of attention from the particular (A) to the general (B)

As described by Mason (1989), a shift of attention from the particular to the general is a "delicate" shift. The structural similarity between *before* and *after* holds. However, this similarity implies the existence of cognitive unity between the particular and the general statements. On the other hand, a shift of attention to a complementary set is a kind of turning back. It means that we expand our scope, discarding the narrower scope. Choosing this strategy is choosing a roundabout route. Therefore, an indirect way of proving is somehow artificial and is unlikely to be spontaneous. Following the abovementioned first hypothesis, we therefore propose *the second hypothesis* that *this cognitive rupture between going in a direct way and in a roundabout way should be bridged by the teacher.*

Now, we present our research questions, which are as follows: *What didactic supports by the teacher establish her students' cognitive unity between direct and indirect exploration, especially when students are expected to prove by conversion? How do students engage with such supports?*

4. Lesson design

To explore the two abovementioned research questions, we designed an experimental mathematics lesson for ninth grade students and observed the implemented lesson. The lesson consists of four stages: exploring, conjecturing, proving, and summarizing.

First, students engage in a mathematical activity of exploring the position of a point P. We interpret the converse of the inscribed angle theorem as a property of the single point P, rather than as a property of the four points A, B, C, and P. At this stage of the lesson, the teacher poses the following task: suppose that there is a triangle ABC such as that shown in Fig 2; now, we want to get a point P, where $\angle APB = \angle ACB$; then, where is P? Since students know the inscribed angle theorem, they conjecture a set of all possible points such as those shown in Fig 3 or Fig 4. In both figures, the bold lines are potential candidates for the location of P. Because we aim to emphasize "the same side of the line AB," it is not assumed that the two points C and P are on the same side of the line AB.

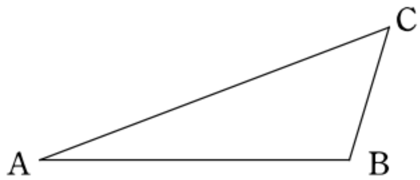

Fig 2: A triangle ABC

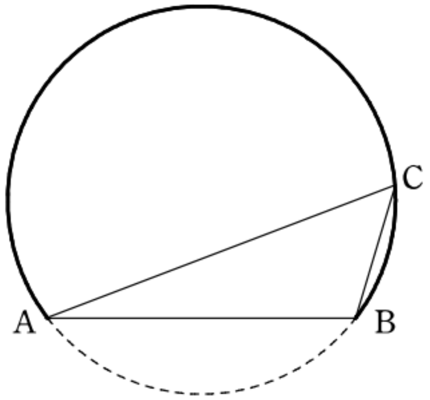

Fig 3: The first conjecture of students' responses

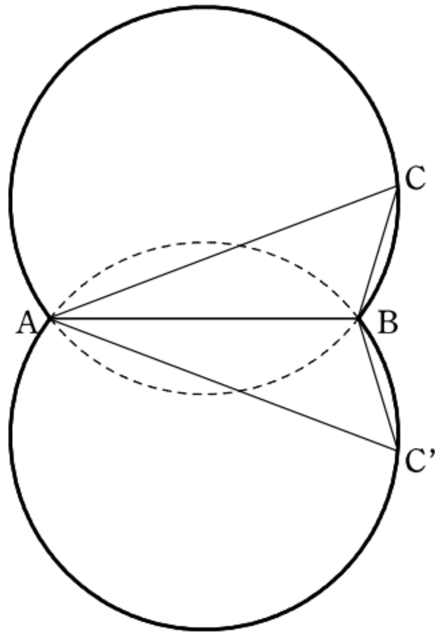

Fig 4: The second conjecture of students' responses (triangles ABC and ABC' are congruent)

Second, students engage in a classroom discussion to formulate a conjecture. After sharing students' answers to the task in the first stage, the teacher asks them if the point P cannot be on the circumscribed circles of the triangle ABC or ABC'. This question is expected to play the role of *shifting students' attention to the complement of the initial scope*. Since they have already finished the initial task, their answers to this question are likely to be yes. However, we conjecture that students cannot provide mathematically rigorous reasons for their affirmative answer. They are likely to recognize that $\angle APB > \angle ACB$ if P is inside the circumscribed circle of the triangle ABC, and that $\angle APB < \angle ACB$ if P is outside the circumscribed circle of the triangle ABC, but their understanding will probably remain intuitive. They are unlikely to immediately construct the proof. Thus, this stage is a conjecturing stage. The teacher summarizes students' conjecture and asks the following key question: *How different are $\angle APB$ and $\angle ACB$ if P is inside or outside the circumscribed circle of the triangle ABC?* This question is needed, based on the second hypothesis presented in the previous section. A shift of students' attention to the complement of the initial scope is insufficient to motivate them to prove in a mathematically rigorous manner, because they may obtain satisfaction from their intuitive understanding. We conjecture that this question plays a role of orienting them toward the construction of proof, that is, it succeeds in bridging the cognitive rupture.

Third, students try to conjecture their answers to the abovementioned key question and to prove them. They are expected to smoothly construct the proof because of the established cognitive unity.

Fourth, the teacher shares students' way of proving and summarizes their proved conclusion as the converse of the inscribed angle theorem. This is the goal of this experimental lesson.

5. Method

The experimental lesson was conducted in 2019 in a ninth-grade class at a Japanese junior high school attached to a national university. It was a fifty-minute-long lesson. Twenty female students and twenty-one male students participated in the lesson. The teacher of this lesson, who regularly teaches mathematics in this school, is the first author of this paper. We video-recorded the lesson and obtained a copy of students' worksheets after the lesson. Two video cameras, one at the front and the other at the rear, captured the classroom discussion.

6. Result of the experimental lesson

The lesson proceeded based on the abovementioned design. At the exploring stage, referring to Fig 2, students explored the position of a point P. About three minutes after the start of exploration, because several students could not find a possible location of the point P, the teacher suggested a possible location on the blackboard (Fig 5). About two more minutes later, the teacher advised all students to plot some potential candidates for grasping a full picture. The teacher drew the circumscribed circle of the triangle ABC on the blackboard about three minutes before the conjecturing stage, so that students could refer to it at that stage. The exploring stage lasted for about eleven minutes.

The conjecturing stage started when the teacher asked students to stop exploring. Students were asked about the possible position of P. One student answered as follows:

S1: We can put (point P) everywhere on the circle (drawn on the blackboard).
T: Everywhere? (Pointing to a point on the minor arc) Can we put it here?
S1: Ah, on the circle, except on the shorter arc AB.

The teacher asked a different student who thought like S1 to explain why.

S2: Using what was learned in the last class, that inscribed angles that subtend the same arc are equal, the angles (∠APB) are the same wherever (the point P is).

The teacher then asked if there were students who chose different locations.

S3: The arc AB …. Under the arc AB. The arc AB is on the upper side.
T: Do you mean that because the arc AB is on the upper side, the circle is on the lower side (of the arc AB)? What circle did you draw?
S3: Passing through A and B….
T: What circle? We can draw various sizes of circles passing through the two points A and B.
S3: A circle, which is the same size as the circle now on the blackboard.

After this interaction, the teacher confirmed that the inscribed angle theorem is also applicable to this case.

The teacher then asked all students if that was all and encouraged them to discuss with each other. About one minute later, the teacher indicated a point and asked if we could put point P at that point (Fig 6). This question seemed to play the role of shifting students' attention to the complement of the initial scope. Two students, S4 and S5, said no, but they could not provide sufficient reasons. They only argued that it was because the point was not on

Fig 5: The teacher's suggestion

Fig 6: The teacher's question: Can we put the point P here?

the circle. The teacher thus additionally asked how large ∠APB is if ∠APB ≠ ∠ACB. A student answered as follows:

S6: It becomes smaller.

T: Oh, OK? When we put point P outside the circle, is ∠APB smaller?

The teacher continued to ask what would happen when we put point P inside the circle. A different student answered the following:

S7: It becomes larger.

After this confirmation, the teacher summarized the discussion and formulated a fact and two conjectures. The fact was as follows: by the inscribed angle theorem, when we put point P on the circle, ∠APB = ∠ACB. The conjectures are as follows: 1) when we put point P inside the circle, ∠APB > ∠ACB; and 2) when we put point P outside the circle, ∠APB < ∠ACB.

The teacher moved toward the next proving stage, giving the following instruction: "If we can show how different ∠APB and ∠ACB are, then we can find that these are not equal. […] Divide them into inside and outside cases and then consider each case. Please write down your explanation." This instruction corresponded to the designed key question. Students subsequently started to prove. About five minutes later, the teacher again asked all students how different ∠APB and ∠ACB are, and advised them, "it is fine if we can show that there is a difference between this and this; it is fine if we can show that the difference is the same size as that in the figure." After that, the teacher encouraged students to discuss with each other. About three minutes later, the teacher added another advice: "if it is difficult for you to directly compare [the angle P] with the angle C, then construct an angle somewhere the same size as the angle C; it is fine if we can show that the angle P is evidently larger or smaller than the angle." After this final advice, the teacher found that at least two students constructed the expected proof.

About twelve minutes after the start of the proving stage, the teacher decided to share the expected proof with all students, starting with the case that point P was inside the circle. The teacher confirmed that if we could show that ∠APB = ∠ACB + "a positive number," then it would be evident how large ∠APB is in comparison with ∠ACB. Showing Fig 7 to students, the teacher asked a student which angle was the same size as the difference between

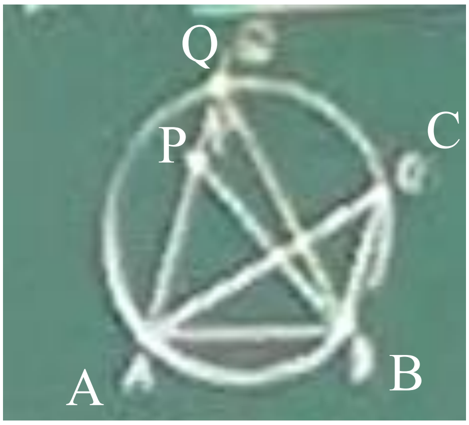

Fig 7: The shared figure for proving

(the label is added by us to enhance clarity of the image)

∠APB and ∠ACB. The student seemed confused for a moment but soon responded as follows:

S8: Which angle? … Ah, (∠APB is equal to) ∠PQB + ∠QBP.

Based on this response, the idea of applying the exterior angle theorem was shared in the classroom. Additionally, the teacher and students constructed a similar proof for the case that point P was outside the circle. Reflecting on these two proofs, the teacher formulated the converse of the inscribed angle theorem as an implication.

Finally, after completing the lesson, the teacher summarized the contents of the lesson.

7. Discussion

In this section, we reflect on students' actual responses in the experimental lesson from the perspective of cognitive unity. We do not intend to discuss how deeply each student understood the method of proving by conversion. Rather, we intend to analyze the classroom situation and, based on that, to derive some implications. We discuss two implications here.

First, students cannot explicitly shift their attention to the points inside or outside the circle by themselves. Two students, S4 and S5, could not provide a sufficient reason when the teacher asked if we could put point P inside the circle. They could not even focus on the difference between ∠APB and ∠ACB. We can interpret their quandary as follows: although they implicitly consider the complement of the initial scope, they do not feel the need to explicitly consider the difference between ∠APB and ∠ACB. As we hypothesized, there is a cognitive rupture between focusing on the initial scope and its complement. Mathematics teachers should bridge this rupture at least when they first encounter a new type of indirect proof.

Second, it is noteworthy that students did not seem to express reluctance to understand the logic of proof by conversion; rather, they expressed their confusion because they could not grasp what they should show. Even if they could retrospectively understand that the application of the exterior angle theorem is useful for showing the difference between the two angles, only a few students were able to prospectively apply the theorem without the teacher's advice. This suggests that there is a content-specific barrier against indirect proof: an interwoven relationship between insufficient recognition of the goal of proving and ineptitude of using local mathematical techniques for proving. As Arzarello and Sabena (2011) argued, we can also argue that this is a kind of meta-cognitive rupture. However, our claim is slightly different from theirs. We argue that this is an issue of mathematical mindset rather than that of a meta-cognitive rupture. The two elements of the barrier we proposed are interwoven. On the one hand, if students were skilled with the application of the exterior angle theorem to proving the difference between two angles, then they could easily understand what they should show. On the other hand, if they understood that the goal of proving was the graphical illustration of the difference between the two angles, then they could come up with the application of the exterior angle theorem in that context. It is suggested that proof by conversion and the sub-proofs themselves are not difficult for students, but that their ineptitude of using local mathematical techniques for proving prevents them from naturally completing proof construction. Because Japanese mathematics curriculum does not strongly intend to cultivate students' skills related to local mathematical techniques, this is an environmental barrier against indirect proof. This is also a new barrier since it is different from the difficulty in understanding the structure of indirect proof as well as in understanding the contents of the proof.

8. Conclusion

This paper reported how students engage in an experimental lesson and hypothesized a potential barrier against indirect proof. Through the lesson, on the one hand, we realized a didactic support for shifting students' attention to the complement of the initial scope; one the other hand, as a potential environmental barrier against indirect proof, we indicated an interwoven relationship between insufficient recognition of the goal of proving and ineptitude of using local mathematical techniques for proving. Based on this observation, we found potential answers to our two research questions. First, a didactic support for shifting students' attention to a complementary set can bridge the cognitive rupture between direct and indirect exploration. Second, if they are skilled with using local mathematical techniques, students can smoothly complete the construction of proof by conversion. Our findings therefore suggest that, to develop a better lesson design, it can be useful to identify local mathematical techniques necessary for the construction of indirect proof and to analyze how students learn those techniques. Such an analysis can help teachers conjecture when and why students cannot smoothly continue proof construction. However, our claims are only based on a single classroom observation, and the generalizability of our findings are thus limited. Further empirical research on this topic is therefore necessary.

Acknowledgements

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Summary in Japanese

間接証明に対する障壁

転換法による円周角の定理の逆の証明に焦点を当てて

1. 序論

数学教育研究においては，古くから間接証明の研究がなされている．何が主要な間接証明学習の困難性であるかについては，見解が 2 つに分かれており，間接証明の構造に由来するという主張 (Antonini & Mariotti, 2008; Leron, 1985) と，間接証明で扱われる数学的内容にあるという主張 (Barnard & Tall, 1997; Dawkins & Karunakaran, 2016) がある．しかしながら，間接証明研究は，背理法や対偶証明法という特定のタイプの間接証明に偏っている．日本の数学の教科書では，転換法と同一法という 2 つのタイプの間接証明法も扱われているが，これらの研究は限られている (数少ない例としては，例えば，Byham, 1969; Uegatani, Hayata, & Hakamata, 2019). そこで本稿は，特に転換法に焦点を当て，数学の授業で転換法を構成するための数学的活動において，生徒達がどのように振る舞うかを報告し，間接証明に対する障壁について仮説を立てることを目的とする．結論として本稿では，証明する目標の不十分な認識と証明のための局所的な数学的技術の未熟の間の，相互依存的関係の存在を，間接証明に対する障壁として提起する．

2. 文献レビュー

2.1. 転換法とは何か？

一群の定理 $P_1 \to Q_1, P_2 \to Q_2, \cdots, P_n \to Q_n$ があり，$P_1, P_2, \cdots, P_n$ はすべての場合を尽くしており，かつ，$Q_1, Q_2, \cdots, Q_n$ は相互排他的であるとする．このとき，一群の定理すべての逆定理 $Q_1 \to P_1, Q_2 \to P_2, \cdots, Q_n \to P_n$ が成り立つ．この事実を利用した証明法を転換法という．

2.2. 円周角の定理，その逆，日本におけるそれらの証明

日本における円周角の定理とその逆の記述については，微妙な問題をはらんでいる．それは，円周角の定理の逆が，中学 2 年生で習う「逆」の意味では，逆になっていない点にある．$P \to Q$ に対して，正確に $Q \to P$ の形になっていないのである．

また，一般に，転換法が用いられているとされているが，これも正確には転換法になっていない．転換法は，一群の定理に対して使用される証明法なのであって，円周角の定理単体に適用できる証明法ではないからである．

2.3. 定理の認知的統一性

影響力の高い間接証明研究の 1 つ，Antonini and Mariotti (2008) は，「定理の認知的統一性」(Garuti, Boero, & Lemut, 1998) というアイディアが，実践的に示唆に富むという．定理の認知的統一性とは，その言明を利用するために必要な探究と，その証明の過程にある溝が大きければ大きいほど，証明の過程における困難性も大きくなるという仮説を基礎に据え，探究過程と証明過程は認知的に統一的な過程であることが望ましいとする理論である．近年では，Boero (2017) がハーバーマスの合理性の観点から，認知的統一性が合理性において保たれている必要性を明らかにしたり，Arzarello and Sabena (2011) が，間接証明に固有の認知的統一性として，メタ認知的統一性を指摘したりしている．

数学的活動の重要性が叫ばれる昨今において，認知的統一性のアイディアは比較的自明に見えるようにもなってきた．しかしながら，間接証明においては，本来証明すべき事柄と実際に証明すべき事柄がずれているため，その実際に証明すべき事柄が，証明活動に入る前から明確であるとは限らない (例えば，背理法では，矛盾をどこから導出すべきかが明確ではない)．そのため，間接証明においては，この認知的統一性がより一層重要になると考えられる．

3. 仮説とリサーチ・クェッション：教師の役割と補集合への注意の移行

先行研究から示唆される仮説は，次の 2 つである．第一に，教師が探究過程と証明過程の認知的断絶を適切な方法・適切なタイミングで架橋するなら，生徒達は認知的統一性を保った数学的活動に取り組むことができる．第二に，間接証明は回り道的な探究なので，真っ直ぐに探究せずに回り道して解決するという着想を得るには，教師による橋渡しが必要である．特に第二の仮説は，直接的探究における Mason (1989) の「特殊から一般への注意の移行」論と対比すれば，間接的探究は「初期の探究範囲からその補集合への注意の移行」として特徴付けられる．これらの仮説を踏まえ，我々は次のリサーチ・クェッションを立てる．とりわけ転換法による証明が期待される場合に，教師のどんな手立てが直接的探究と間接的探究の認知的統一性を確立するのか？ また，そうした手立てを伴い，生徒達はどのように探究に取り組むのか？

4. 授業デザイン

リサーチ・クェッションの検証のため，実験的な授業を設計した．授業は，探究，予想，証明，まとめの 4 つの段階からなる．探究の段階では，Fig 2 の三角形 ABC において，∠APB ＝∠ACB となる点 P の位置を探究する．円周角の定理を知っている生徒達なら，Fig 3 や Fig 4 を予想すると目される．予想の段階では，教室全体で，探求の段階で得られた予想について検討する．教師は，補集合への注意の移行を促すために，点 P の位置が，三角形 ABC の外接円の内側や外側ではあり得ないのかを問う．その場合，生徒達は，∠APB が ∠ACB よりも大きくなったり小さくなったりしてしまうということを答えると考えられるので，そのタイミングで次を問う．点 P の位置が，三角形 ABC の外接円の内側や外側であった場合，∠APB と ∠ACB は，どれくらい違うのか？ この問いが，転換法による証明活動の契機となると考える．円周角の定理より，外接円上の場合は証明済みなので，証明の段階では，点 P が外接円の内部にある場合と外部にある場合に ∠APB と ∠ACB がどれくらい違うのかを証明する．

5. 方法

実験授業は，2019 年に，日本のある国立大学附属中学校の 3 年生に対して実施された (授業時間 50 分，女子 20 名，男子 21 名)．この授業の教師は，このクラスで普段から数学を教えており，本稿の第一著者である．授業はビデオカメラ 2 台で教室の前方と後方から撮影され，生徒達のワークシートのコピーが授業後に収集された．

6. 実験授業の結果

実験授業は，概ねデザイン通りに進行した．探究の段階では，事前の予想通り，Fig 3 や Fig 4 の解答が現れた．続く予想の段階で，教室での議論を通じて，円周角の定理より，条件を満たすのは，Fig 3 の範囲だけでなく，Fig 4 の範囲が妥当であるであろうことが確認された．その上で，教師は，生徒達全員にこれですべてかどうかを問い，互いに議論することを奨励した．1 分間議論させた後，教師は三角形 ABC のの外接円の内部の一点を指差し，点 P はそこに取り得るかを問うた (Fig 6)．この問いは，補集合への注意の移行を促したように思われる．しかし，指名された生徒 2 名は，円上にないから条件を満たせない旨を主張するのみであった．そこで教師は，追加して，∠APB ≠∠ACB なら，∠APB はどれくらい大きいのかを問うた．指名された生徒は，「小さくなる」と答えた．円の外側についても，同様に確認された．この確認の後，次のように教師から予想が示された．1) 点 P が円の内部にあるとき，∠APB ＞∠ACB である．2) 点 P が円の外部にあるとき，∠APB ＜∠ACB である．

この予想の形成を経て，証明の段階に移行した．教師からは，円の内部と外部に分けて，∠APB と ∠ACB がどれくらい異なるのかを明らかにし，証明するように指示が与えられた．そして，生徒達は各々証明を始めた．途

中，証明活動開始から約 5 分後，教師は図中の何かと同じ大きさだけ違うことが示せたらよいことを教室全体に助言し，周囲と議論するように促した．さらに約 3 分後，教師はさらに，角 P と角 C が直接比較しにくいときは，どこかに角 C と同じ大きさの角を構成し，その角と角 P を比較して大小を述べればよいことを全体に助言した．最後の助言の後，教師は，少なくとも 2 人の生徒が，期待された証明を構成しているのを見出した．

証明の段階に移行してから約 12 分後，教師は期待された証明を生徒全員と証明するよう試みた．Fig 7 を示しながら，教師が，∠APB と ∠ACB との違いは，どの角と同じ大きさかを問うたところ，指名された生徒は，一瞬戸惑いを見せたものの，教師の意図を理解し，∠APB が ∠PQB ＋∠QBP と等しいことを述べた．

この応答に基づき．外角定理を応用するアイディアが共有された．加えて，教師と生徒は．円の外部に点 P が位置する場合についても同様の証明を構成した．構成した 2 つの証明を振り返りながら，教師は，その含意として，円周角の定理の逆を定式化した．最後に，教師はその授業の内容を要約して授業を終えた．

7. 考察

実験授業を振り返ることで得られる示唆が 2 つある．第一に，生徒達は自ら円の内部や外部の点へと注意を明示的に移行することはできないようであった．彼らは，∠APB と ∠ACB の大きさの違いを，明示的に議論する必要性を感じないようであった．仮説として提起していたように，最初の考察範囲への焦点化と，その補集合への焦点化との間の認知的断絶があるようである．数学の教師は，少なくとも，生徒達が新しいタイプの間接証明に出会うときは，その断絶を橋渡しすべきである．

第二に，生徒達は，間接証明法の論理の理解には抵抗を示さなかったように思われる．むしろ，何を示すべきかがつかめなかったという点で，混乱を示した．外角定理の適用が有用であることが，遡及的に理解できたとしても，教師の助言に先立って外角定理を適用できた生徒はわずかであった．このことは，間接証明に対する内容固有の障壁の存在を示唆する．すなわち，証明する目標の不十分な認識と証明のための局所的な数学的技術の未熟の間の，相互依存的関係が，障壁となっている．これは，Arzarello and Sabena (2011) が述べたメタ認知的断絶の一種としても捉えられるが，我々の主張は，彼らとは若干異なる．我々は，メタ認知的断絶というよりも，数学的な傾向性の問題であると考える．現在の日本の中学校の数学のカリキュラムにおいては，角の大きさの違いを証明することに外角定理を活用するというテクニックの指導が強力に意図されているわけではない．そのため，生徒達にとって間接証明が困難であるというよりは，生徒達が間接証明を円滑に構成できない環境的障壁が存在すると考えるほうが適切であるように思われる．これは，間接証明の困難性としてこれまで指摘されてきた，間接証明の構造の理解とも，間接証明の内容の理解とも異なる，第三の障壁である．

8. 結論

本稿では，補集合への注意の移行を促す教授学的手立てを実現した一方で，間接証明に対する潜在的な環境的障壁として，証明する目標の不十分な認識と証明のための局所的な数学的技術の未熟の間の，相互依存的関係を示した．実験授業の観察結果に基づくと，本稿のリサーチ・クェッションへの解答は，次のように述べることができよう．第一に，補集合への注意の移行を促す手立ては，直接的探究と間接的探究の間の認知的断絶を架橋することができる．第二に，生徒達が局所的な数学的テクニックに習熟しているなら，彼らは円滑に転換法の証明を構成することができる．我々の成果は，数学の教師にとって，期待される証明の構成に必要な局所的な数学的テクニックを分析しておくことが有用であるということを示唆する．しかしながら，我々の主張は，単一の授業観察にのみ基づいている．成果の一般化のために，一層の経験的研究が必要である．