HIGHER NASH BLOWUPS OF THE A_3-SINGULARITY

RIN TOH-YAMA

Abstract. We show that the n-th Nash blowup of the toric surface singularity of type A_3 is singular for any $n > 0$. It has been known that the normalization of the n-th Nash blowup of an affine normal toric variety is the toric variety associated to the Gröbner fan of a certain ideal. In our case, we prove that the Gröbner fan contains a certain non-regular cone for any $n > 0$. Thus we conclude that the normalizations are singular, and so are the Nash blowups.

Introduction

Let X be an equidimensional quasi-projective variety over $\mathbb{C}$. The classical Nash blowup of X was defined in [1], and was recently generalized in [2] and [3] independently.

The classical Nash blowup is defined as follows.

Definition 0.1 ([1])**.** Let X be a subvariety of $\mathbb{A}^m$ of dimension r, and $X_{\mathrm{sm}} := X \setminus \mathrm{Sing}\,(X)$. Let $G\,(m,r)$ be the Grassmanian of r-dimensional subspaces of an m-dimensional vector space over $\mathbb{C}$. Then we have a natural morphism

$$\phi : \; X_{\mathrm{sm}} \hookrightarrow X \times G\,(m,r)\,; \; P \mapsto (P, T_P X)$$

where $T_P X$ is the tangent space of X at P. We define $\mathrm{Nash}(X)$ to be the closure of $\phi\,(X_{\mathrm{sm}})$ in $X \times G\,(m,r)$. Moreover we obtain a morphism $\pi : \; \mathrm{Nash}\,(X) \to X$ by restriction of the first projection $X \times G\,(m,r) \to X$ to $\mathrm{Nash}(X)$. The pair $(\mathrm{Nash}(X), \pi)$ is called the Nash blowup of X. For an arbitrary variety, its Nash blowup is defined by gluing the Nash blowups of its affine patches.

The classical Nash blowup is generalized to the n-th Nash blowup for any $n > 0$. Let $\mathcal{M}_{X,P} \subset \mathcal{O}_X$ be the ideal sheaf of a closed point P. In the above definition, we looked at the tangent space at a smooth point P, which can be identified with the dual space of $\mathcal{M}_{X,P}/\mathcal{M}_{X,P}^2$. In other words, we looked at the first infinitesimal neighborhood at P, i.e. the closed subscheme of X associated to the ideal sheaf $\mathcal{M}_{X,P}^2$. The n-th Nash blowup is defined by considering the n-th infinitesimal neighborhoods:

Definition 0.2 ([3])**.** Let $n > 0$ be an integer.

(1) For any closed point $P \in X$, the n-th fat point $P^{(n)}$ is defined to be the closed subscheme of X associated to the ideal sheaf $\mathcal{M}_{X,P}^{n+1}$.

(2) For $N := \binom{\dim X+n}{\dim X}$, let $\mathrm{Hilb}_N(X)$ be the Hilbert scheme of 0-dimensional closed subschemes of length N. We have a natural morphism

$$\phi_n : X_{\mathrm{sm}} \hookrightarrow X \times \mathrm{Hilb}_N(X)\,;\; P \mapsto \left(P, \left[P^{(n)}\right]\right).$$

Then we define $\mathrm{Nash}_n(X)$ to be the closure of $\phi_n(X_{\mathrm{sm}})$ in $X \times \mathrm{Hilb}_N(X)$, and $\pi_n : \mathrm{Nash}_n(X) \to X$ to be the restriction of the first projection $X \times \mathrm{Hilb}_N(X) \to X$ to $\mathrm{Nash}_n(X)$. The pair $(\mathrm{Nash}_n(X), \pi_n)$ is called the n-th Nash blowup of X.

$\mathrm{Nash}_1(X)$ was shown to be isomorphic to the classical $\mathrm{Nash}(X)$ ([3], Proposition 1.8). Moreover, let p be the second projection $X \times \mathrm{Hilb}_N(X) \to \mathrm{Hilb}_N(X)$, and $\mathrm{Nash}'_n(X)$ the closure of

$$p \circ \phi_n(X_{\mathrm{sm}}) = \left\{\left[P^{(n)}\right] \in \mathrm{Hilb}_N(X) \mid P \in X_{\mathrm{sm}}\right\}$$

in $\mathrm{Hilb}_N(X)$. Then $p : \mathrm{Nash}_n(X) \to \mathrm{Nash}'_n(X)$ is an isomorphism ([3], Proposition 1.3), so we identify them.

Now the following questions were raised:

Questions. (1) ([4], Remark 1.5) Is $\mathrm{Nash}_n(X)$ smooth for $n \gg 0$?

(2) ([3], Conjecture 0.2) Let $J^{(\dim X-1)}$ be the $(\dim X-1)$-th neighborhood of the Jacobian subscheme $J \subset X$, i.e. the closed subscheme associated to

$\mathrm{j}_X^{\dim X}$ where j_X is the Jacobian ideal sheaf of X. Let $[Z] \in \mathrm{Nash}_n(X)$ with $Z \nsubseteq J^{(\dim X - 1)}$. Then, is $\mathrm{Nash}_n(X)$ smooth at $[Z]$?

Let X has only finitely many singular points, i.e. J be 0-dimensional. Under this assumption, if the answer to Question (2) is positive, then so is the answer to Question (1): Indeed, any $[Z] \in \mathrm{Nash}_n(X)$ satisfies $Z \nsubseteq J$ for $n \gg 0$ whenever $\mathrm{length}\,(Z) = \binom{\dim X + n}{\dim X} > \mathrm{length}\,(J)$. When X is a curve, it was proved in [3] that the answer to Question (2) is positive, thus so is the answer to Question (1).

If the answer to Question (1) were positive for an arbitrary X, then a resolution of singularities of X could be obtained as $\mathrm{Nash}_n(X)$ for $n \gg 0$. Hence we could resolve singularities without iterations of operations, as with Hironaka's resolution ([5]).

Our main result shows that the answers are negative in general:

Main Theorem. Let $X := \left(z^4 - xy = 0\right) \subset \mathbb{A}^3$ be the toric surface singularity of type A_3. Then $\mathrm{Nash}_n(X)$ is singular for any $n > 0$.

Therefore the A_3-singularity is a counterexample to the above questions, since X has only finitely many singular points. It has been suggested by T. Yasuda ([4], Remark 1.5) that the A_3-singularity might be a counterexample. Moreover, extensive calculations supporting the suggestion were given by D. Duarte ([6], Section 3.5). The current work was motivated by them.

We prove our main theorem in the following way: Duarte's theorem (Theorem 1.2) shows that the normalization $\overline{\mathrm{Nash}_n(X)}$ of $\mathrm{Nash}_n(X)$ is the toric variety associated to the Gröbner fan $\mathrm{GF}\,(J_n)$ of a certain ideal J_n. Thus it is sufficient to show that $\mathrm{GF}\,(J_n)$ contains a non-regular cone. On the other hand, we see that the maximal cones of $\mathrm{GF}\,(J_n)$ are obtained from reduced Gröbner bases of J_n in a certain way. Hence we first give the reduced Gröbner basis of J_n with respect to a certain ordering, and explicitly describe the cone associated to the base. Then the cone is non-regular. Therefore $\overline{\mathrm{Nash}_n(X)}$ is singular, and so is $\mathrm{Nash}_n(X)$.

This paper is organized as follows.

In section 1, we recall the general theory of Gröbner fans of ideals in monomial subalgebras. For any ideal I in a monomial subalgebra, we give a description of the maximal cones of $\mathrm{GF}(I)$ in terms of reduced Gröbner bases of I, as it is more convenient in our exposition.

In section 2, we give a proof of our main theorem. First we give a certain monomial ordering $\preceq$, and then determine the minimal generators of $\mathrm{in}_{\preceq}(J_n)$ for any $n > 0$ where J_n is the ideal introduced in Duarte's theorem (Theorem 1.2). This is the hardest part of the proof, and needs somewhat technical arguments on certain semigroups in $(\mathbb{Z}_{\geq 0})^2$. The minimal generators of $\mathrm{in}_{\preceq}(J_n)$ are exactly leading monomials of elements of the reduced Gröbner basis of J_n. Moreover, for one element of the reduced Gröbner basis, we show that it in addition contains a certain (explicitly given) monomial. These results on elements of the reduced Gröbner basis allow us to explicitly describe the cone in $\mathrm{GF}(J_n)$ associated to the basis. Then our main result is proved.

1. Gröbner fans of ideals in monomial subalgebras

In this section, we recall the theory of Gröbner fans of ideals in monomial subalgebras. Gröbner fans is defined and studied for ideals in polynomial rings, but a very analogous theory can be developed for ideals in monomial subalgebras ([7][8][9][6]).

First of all, let us explain how we are going to use the Gröbner fans. Our setting in this section is as follows.

Notation 1.1. Let $\sigma \subset \mathbb{R}^d$ be a strongly convex full-dimensional rational polyhedral cone, and X the affine toric variety associated to σ.

(1) Let $S := \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^d]$ where $\sigma^\vee$ is the dual cone of σ. Thus $X = \operatorname{Spec} S$.
(2) Let $a_1, \ldots, a_s$ generate $\sigma^\vee \cap \mathbb{Z}^d$, i.e. $\sigma^\vee \cap \mathbb{Z}^d = \mathbb{Z}_{\geq 0} a_1 + \cdots + \mathbb{Z}_{\geq 0} a_s$.
(3) By a coordinate transformation, we can assume that $\sigma^\vee \subset (\mathbb{R}_{\geq 0})^d$. Then S becomes a monomial subalgebra of $\mathbb{C}[x_1, \ldots, x_d]$ in the following way: For

each $a_i = (a_{i1}, \dots, a_{id})$, take the monomial

$$x^{a_i} := x_1^{a_{i1}} \cdots x_d^{a_{id}} \in \mathbb{C}[x_1, \dots, x_d].$$

Then $S = \mathbb{C}[x^{a_1}, \dots, x^{a_s}] \subset \mathbb{C}[x_1, \dots, x_d]$.

With the above notations, we state Duarte's theorem as follows:

Theorem 1.2 ([9], Theorem 2.10)**.** Let $J_n := \langle x^{a_1} - 1, \dots, x^{a_s} - 1 \rangle^{n+1} \subset S$. Then $\overline{\mathrm{Nash}_n(X)}$ is the toric variety associated to the Gröbner fan $\mathrm{GF}\,(J_n)$ of J_n.

By the theorem, we can conclude that $\overline{\mathrm{Nash}_n(X)}$ is singular if $\mathrm{GF}\,(J_n)$ contains a non-regular cone.

Now let us go back to our explanation of the theory of Gröbner fans of ideals in S. Let I be an arbitrary nonzero ideal in S till the end of this section.

Definition 1.3 ([9], Proposition 1.5)**.** Let $w \in \sigma$.

(1) For a nonzero element $f = \sum_{\beta \in \mathbb{N}^d} c_\beta x^\beta$ of S, let $m := \max \left\{ w \cdot \beta \mid x^\beta \in \mathrm{supp}\,(f) \right\}$ where the dot product $\cdot$ denotes the standard inner product on $\mathbb{R}^d$. Then we define the initial form of f with respect to w as

$$\mathrm{in}_w\,(f) := \sum_{w \cdot \beta = m} c_\beta x^\beta.$$

We define $\mathrm{in}_w\,(0)$ to be 0.

(2) $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \mid f \in I \rangle$ is called the initial ideal of I with respect to w.

(3) Let $C[w] := \{ w' \in \sigma \mid \mathrm{in}_{w'}\,(I) = \mathrm{in}_w\,(I) \}$.

Definition and Proposition 1.4 ([9], Proposition 1.6)**.** Let $\overline{C[w]}$ be the closure of $C[w]$ in $\mathbb{R}^d$. Then

$$\mathrm{GF}\,(I) := \left\{ \overline{C[w]} \mid w \in \sigma \right\}$$

forms a polyhedral fan with $|\mathrm{GF}\,(I)\,| = \sigma$. This is called the Gröbner fan of I.

Below we will give an alternative description of the maximal cones of $\mathrm{GF}\,(I)$ to be more suitable for our purpose.

Definition 1.5 ([6], Appendix A, Definition A.1.1)**.** Let $\preceq$ be a total ordering on monomials in S. Then $\preceq$ is called a monomial ordering if it satisfies the following conditions:

(1) Let $x^\alpha, x^\beta \in S$. If x^β divides x^α in S, then $x^\beta \preceq x^\alpha$.
(2) For any $x^\gamma \in S$, $x^\beta \preceq x^\alpha$ implies $x^{\beta+\gamma} \preceq x^{\alpha+\gamma}$.

Remark 1.6. In this paper, divisibility between monomials in S will always mean divisibility in S, not in $\mathbb{C}[x_1, \ldots, x_d]$. For example, in $S = \mathbb{C}\left[u, u^3v^4, uv\right]$, u does not divide u^3v^4.

Definition 1.7 ([9], Definition 1.2, 1.3)**.** Let $\preceq$ be a monomial ordering on S.

(1) A set $\{g_1, \ldots, g_t\}$ of nonzero polynomials in I is called a Gröbner basis of I with respect to $\preceq$ if $\mathrm{lm}_\preceq(g_1), \ldots, \mathrm{lm}_\preceq(g_t)$ generate the ideal

$$\mathrm{in}_\preceq(I) := \langle \mathrm{lm}_\preceq(f) \mid f \in I \rangle$$

i.e. for any $f \in I \setminus \{0\}$ there exists g_i such that $\mathrm{lm}_\preceq(g_i)$ divides $\mathrm{lm}_\preceq(f)$.
(2) We say that a Gröbner basis $\{g_1, \ldots, g_t\}$ is reduced if $\mathrm{lc}_\preceq(g_i) = 1$ for any i and no monomial of g_i is divisible by $\mathrm{lm}_\preceq(g_j)$ for any $i \neq j$.

Theorem 1.8 ([9], Theorem 1.4)**.** Let $\preceq$ be a monomial ordering on S. Then I has a unique reduced Gröbner basis with respect to $\preceq$.

Definition 1.9 (c.f. [10], Chapter 8)**.** (1) Let $\{g_1, \ldots, g_t\}$ be the reduced Gröbner basis of I with respect to $\preceq$. Then

$$\mathbb{G} := \{(g_1, \mathrm{lm}_\preceq(g_1)), \ldots, (g_t, \mathrm{lm}_\preceq(g_t))\}$$

is called the marked Gröbner basis of I with respect to $\preceq$. Note that two distinct monomial orderings may define the same marked Gröbner basis of I. When we do not care about orderings, $\mathbb{G}$ is simply referred to as "a marked Gröbner basis of I".

(2) Let $\mathbb{G} = \{(g_1, x^{\alpha_1}), \ldots, (g_t, x^{\alpha_t})\}$ be a marked Gröbner basis of I. Then we define the cone $C_{\mathbb{G}} \subset \sigma$ by

$$C_{\mathbb{G}} := \left\{ w \in \sigma \mid (\alpha_i - \beta) \cdot w \geq 0 \text{ for any } i \text{ and } x^{\beta} \in \operatorname{supp}(g_i) \right\}.$$

We will see in Theorem 1.17 that the maximal cones of $\mathrm{GF}(I)$ are exactly the cones given as $C_{\mathbb{G}}$.

Lemma 1.10 ([11], Lemma 1, Theorem 4)**.** Let $\preceq$ be a monomial ordering on S, which we regard as a total ordering on $\sigma^{\vee} \cap \mathbb{Z}^d$. Then $\preceq$ extends to a total ordering on $\mathbb{Q}^d$ in a natural way, and there exist row vectors $w_1, \ldots, w_r \in \mathbb{R}^d$ satisfying the following: For any $\alpha, \beta \in \mathbb{Q}^d$, $\beta \preceq \alpha$ if and only if there exists $r_0 \leq r$ such that

$$\forall i < r_0,\ (\alpha - \beta) \cdot w_i = 0 \text{ and } (\alpha - \beta) \cdot w_{r_0} > 0.$$

In this case, we say that $\preceq$ is the monomial ordering associated to the $r \times d$ matrix

$$\begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}.$$

Proof. Let $H \subset \mathbb{Z}^d$ be the abelian subgroup generated by $\sigma^{\vee} \cap \mathbb{Z}^d$. Then $\mathbb{Q} \otimes_{\mathbb{Z}} H = \mathbb{Q}^d$ since $\sigma^{\vee} \subset \mathbb{R}^d$ is full-dimensional.

One can easily check that $\preceq$ extends to a total ordering on H as follows: For any $p, p' \in H$, take expressions $p = p_+ - p_-$ and $p' = p'_+ - p'_-$ by some $p_+, p_-, p'_+, p'_- \in \sigma^{\vee} \cap \mathbb{Z}^d$. Then $p \preceq p'$ if and only if $p_+ + p'_- \preceq p'_+ + p_-$.

Moreover $\preceq$ extends to a total ordering on $\mathbb{Q}^d = \mathbb{Q} \otimes_{\mathbb{Z}} H$ as follows: For any $q, q' \in \mathbb{Q} \otimes_{\mathbb{Z}} H$, there exists $r \in \mathbb{Z}_{>0}$ such that $rq, rq' \in H$. Then $q \preceq q'$ if and only if $rq \preceq rq'$.

Now Robbiano's theorem ([11], Theorem 4) shows that there exists $r > 0$ and a real $r \times d$ matrix M such that the ordering $\preceq$ on $\mathbb{Q}^d$ is associated to M. Then the vectors $w_i :=$ (i-th row of M) satisfy the condition in the assertion. $\square$

Lemma 1.11 ([6], Appendix A, Proposition A.2.2)**.** Let $\preceq, \preceq'$ be monomial orderings on S. Then $\mathrm{in}_{\preceq'}(I) \subset \mathrm{in}_{\preceq}(I)$ implies $\mathrm{in}_{\preceq'}(I) = \mathrm{in}_{\preceq}(I)$.

Lemma 1.12 (c.f. [10], Chapter 8, Theorem 4.7)**.** Let $\mathbb{G} = \{(g_1, x^{\alpha_1}), \ldots, (g_t, x^{\alpha_t})\}$ be a marked Gröbner basis of I. Then we have the following:

(1) $C_{\mathbb{G}}$ is a strongly convex full-dimensional rational polyhedral cone.
(2) For any $w \in \mathrm{int}(C_{\mathbb{G}})$, we have

$$(\alpha_i - \beta) \cdot w > 0 \text{ for any } i \text{ and } x^{\beta} \in \mathrm{supp}(g_i) \setminus \{x^{\alpha_i}\}.$$

Proof. (1) $C_{\mathbb{G}}$ is a rational polyhedral cone because the entries of $\alpha_i - \beta$ are rational. Moreover $C_{\mathbb{G}}$ is strongly convex since $C_{\mathbb{G}} \subset \sigma$ and σ is strongly convex.

Let us show the full-dimensionality of $C_{\mathbb{G}}$. By the definition of $C_{\mathbb{G}}$, it is clear that $C_{\mathbb{G}}$ contains the open subset U of $\mathbb{R}^d$ defined by

$$U := \mathrm{int}(\sigma) \cap \bigcap_{1 \leq i \leq t} \left\{ w \in \mathbb{R}^d \mid (\alpha_i - \beta) \cdot w > 0 \text{ for all } x^{\beta} \in \mathrm{supp}(g_i) \setminus \{x^{\alpha_i}\} \right\}.$$

Therefore it is sufficient to show that $U \neq \emptyset$.

Let $\preceq$ be a monomial ordering on S with respect to which $\mathbb{G}$ is the marked Gröbner basis. By Lemma 1.10, $\preceq$ extends to a total ordering on $\mathbb{Q}^d$ associated to some $r \times d$ matrix M. Let w_i be the i-th row of M, and put $w(\epsilon) := w_1 + \epsilon w_2 + \cdots + \epsilon^{r-1} w_r \in \mathbb{R}^d$ for any $\epsilon \in \mathbb{R}$.

We will see that $w(\epsilon) \in U$ for sufficiently small $\epsilon > 0$.

First, let us remark the following fact: For any $\gamma_1, \gamma_2 \in \sigma^{\vee} \cap \mathbb{Z}^d$ with $x^{\gamma_1} \prec x^{\gamma_2}$, there exists $r_0 \leq r$ such that $(\gamma_2 - \gamma_1) \cdot w_i = 0$ for all $i < r_0$ and $(\gamma_2 - \gamma_1) \cdot w_{r_0} > 0$. Thus, for sufficiently small $\epsilon > 0$, we have $(\gamma_2 - \gamma_1) \cdot (w_{r_0} + \epsilon w_{r_0+1} + \cdots + \epsilon^{r-r_0} w_r) > 0$. This implies that

$$(\gamma_2 - \gamma_1) \cdot w(\epsilon) = (\gamma_2 - \gamma_1) \cdot \left(w_1 + \epsilon w_2 + \cdots + \epsilon^{r-1} w_r \right) > 0.$$

Let L be any ray of $\sigma^{\vee}$ and μ_L its ray generator. Then $1 \prec x^{\mu_L}$. Thus, as we have remarked above, $\mu_L \cdot w(\epsilon) > 0$ for sufficiently small $\epsilon > 0$. By restricting ϵ for all L, we have $w(\epsilon) \in \operatorname{int}(\sigma)$. Moreover, for any $x^\beta \in \operatorname{supp}(g_i) \setminus \{x^{\alpha_i}\}$, we have $x^\beta \prec x^{\alpha_i}$. Thus $(\alpha_i - \beta) \cdot w(\epsilon) > 0$ for sufficiently small $\epsilon > 0$.

Therefore $w(\epsilon) \in U \neq \emptyset$ for sufficiently small $\epsilon > 0$, so (1) holds.

(2) The vectors $\alpha_i - \beta$ in the assertion are contained in $(C_{\mathbb{G}})^\vee$ by Definition 1.9 (2). Thus $F := \{w \in C_{\mathbb{G}} \mid (\alpha_i - \beta) \cdot w = 0\}$ is a face of $C_{\mathbb{G}}$, which is proper since $C_{\mathbb{G}}$ is full-dimensional and $\alpha_i - \beta \neq 0$. Therefore any $w \in \operatorname{int}(C_{\mathbb{G}})$ is not contained in F. □

Definition 1.13 ([9], proof of Proposition 1.5)**.** Let w be an element of σ, and $\preceq$ a monomial ordering on S. Then the w-weighted ordering associated to $\preceq$, denoted by $\preceq_w$, is defined as follows:

$$x^\beta \preceq_w x^\alpha \Leftrightarrow ((\alpha - \beta) \cdot w > 0) \text{ or } \left((\alpha - \beta) \cdot w = 0 \text{ and } x^\beta \preceq x^\alpha\right).$$

One can easily check that $\preceq_w$ is also a monomial ordering on S.

Lemma 1.14 ([6], Appendix A, proof of Proposition A.3.1)**.** Let w be an element of σ, $\preceq$ any monomial ordering on S, and G the reduced Gröbner basis of I with respect to $\preceq_w$. Then

$$C[w] = \{w' \in \sigma \mid \operatorname{in}_{w'}(g) = \operatorname{in}_w(g) \text{ for all } g \in G\}.$$

Corollary 1.15. Let $\mathbb{G} = \{(g_1, x^{\alpha_1}), \ldots, (g_t, x^{\alpha_t})\}$ be a marked Gröbner basis of I, and w an element of $\operatorname{int}(C_{\mathbb{G}})$.

(1) Let $\preceq$ be any monomial ordering on S. Then $\mathbb{G}$ is the marked Gröbner basis of I with respect to $\preceq_w$.
(2) $\overline{C[w]} = C_{\mathbb{G}}$.

Proof. (1) It is sufficient to show that

$$\operatorname{lm}_{\preceq_w}(g_i) = x^{\alpha_i} \text{ for } 1 \leq i \leq t \text{ , and } \operatorname{in}_{\preceq_w}(I) = \langle x^{\alpha_1}, \ldots, x^{\alpha_t}\rangle.$$

By Lemma 1.12 (2), we have $\mathrm{in}_w\left(g_i\right) = x^{\alpha_i}$. Thus, by the definition of $\preceq_w$, one can easily check that $\mathrm{lm}_{\preceq_w}\left(g_i\right) = \mathrm{lm}_{\preceq}\left(\mathrm{in}_w\left(g_i\right)\right) = x^{\alpha_i}$. Hence

$$\langle x^{\alpha_1}, \ldots, x^{\alpha_t} \rangle \subset \mathrm{in}_{\preceq_w}(I).$$

On the other hand, for a monomial ordering $\preceq'$ with respect to which $\mathbb{G}$ is the marked Gröbner basis of I, we have $\mathrm{in}_{\preceq'}(I) = \langle x^{\alpha_1}, \ldots, x^{\alpha_t} \rangle$. Thus $\langle x^{\alpha_1}, \ldots, x^{\alpha_t} \rangle = \mathrm{in}_{\preceq_w}(I)$ by Lemma 1.11. Therefore (1) holds.

(2) By (1), $\mathbb{G}$ is the marked Gröbner basis of I with respect to $\preceq_w$, and Lemma 1.12 (2) shows that $\mathrm{in}_w\left(g_i\right) = x^{\alpha_i}$ for any $1 \leq i \leq t$. Thus, by Lemma 1.14,

$$C[w] = \left\{ w' \in \sigma \mid \mathrm{in}_{w'}(g_i) = x^{\alpha_i} \text{ for all } 1 \leq i \leq t \right\}.$$

This shows that $\mathrm{int}\left(C_{\mathbb{G}}\right) \subset C[w]$ by Lemma 1.12 (2), so $C_{\mathbb{G}} \subset \overline{C[w]}$. On the other hand, this also shows that $C[w] \subset C_{\mathbb{G}}$ by Definition 1.9 (2). Therefore $\overline{C[w]} = C_{\mathbb{G}}$. $\square$

Lemma 1.16. Let $\preceq$ be the monomial ordering associated to a matrix M, and $\mathbb{G}$ the marked Gröbner basis of I with respect to $\preceq$. Then the first row w_1 of M is contained in $C_{\mathbb{G}}$.

Proof. We have $w_1 \in \sigma$ since $1 \preceq x^{\alpha}$ for any x^{α} in S. Thus the assertion follows from the definitions of $C_{\mathbb{G}}$. $\square$

Theorem 1.17. There exists a one-to-one correspondence

$$\{\text{marked Gröbner bases of } I\} \xrightarrow{\cong} \{\text{maximal cones of } \mathrm{GF}(I)\}\,;\ \mathbb{G} \mapsto C_{\mathbb{G}}.$$

Proof. The set $\mathrm{int}\left(C_{\mathbb{G}}\right)$ is nonempty since $C_{\mathbb{G}}$ is full-dimensional by Lemma 1.12 (1). Therefore $C_{\mathbb{G}}$ is a member of $\mathrm{GF}(I)$ by Corollary 1.15 (2), which is maximal because of the full-dimensionality. Thus the correspondence is well-defined.

Let $\preceq$ be any monomial ordering on S (e.g. the lexicographic ordering). One can easily check that the correspondence is injective by Lemma 1.15 (1).

Now fix any maximal cone $C \in \mathrm{GF}(I)$ and take $w \in \mathrm{int}(C)$. Let $\mathbb{G}$ be the marked Gröbner basis of I with respect to $\preceq_w$. We have $w \in C_{\mathbb{G}}$ by Definition 1.13, Lemma 1.10 and Lemma 1.16. Then $C = C_{\mathbb{G}}$: Indeed $\mathrm{GF}(I)$ is a fan and hence $C \cap C_{\mathbb{G}}$ is a face of both C and $C_{\mathbb{G}}$. However $w \in \mathrm{int}(C)$ is contained in $C \cap C_{\mathbb{G}}$, thus $C = C \cap C_{\mathbb{G}} \subset C_{\mathbb{G}}$. Since C is maximal in $\mathrm{GF}(I)$, we have $C = C_{\mathbb{G}}$.

Therefore the correspondence is also surjective, so the assertion holds. □

2. Higher Nash blowups of the A_3-singularity

We give a proof of our main theorem in this section.

Notation 2.1. In this section, let $X := \left(z^4 - xy = 0\right) \subset \mathbb{A}^3$.

(1) Let $\sigma \subset \mathbb{R}^2$ be the cone generated by lattice points $(0,1), (4,-3)$. Then the dual cone $\sigma^\vee \subset \mathbb{R}^2$ is generated by $(1,0), (3,4)$. These cones are strongly convex and full-dimensional.
(2) The monoid $\sigma_{\mathbb{Z}} := \sigma^\vee \cap \mathbb{Z}^2$ is generated by $(1,0), (3,4), (1,1)$.
(3) $S := \mathbb{C}[\sigma_{\mathbb{Z}}] = \mathbb{C}[u, u^3v^4, uv] \subset \mathbb{C}[u,v]$. There is a surjective homomorphism

$$F:\ \mathbb{C}[x,y,z] \twoheadrightarrow S;\ x \mapsto u,\ y \mapsto u^3v^4,\ z \mapsto uv$$

with $\ker F = \langle z^4 - xy \rangle$. Hence X is isomorphic to $\operatorname{Spec} S$, the affine toric variety associated to σ.
(4) For any integer $n > 0$, we define

$$J_n := \langle u-1, u^3v^4 - 1, uv - 1 \rangle^{n+1} \subset S.$$

Then the normalization $\overline{\mathrm{Nash}_n(X)}$ of $\mathrm{Nash}_n(X)$ is the toric variety associated to $\mathrm{GF}(J_n)$ (Theorem 1.2).

Remark 2.2. We will identify elements of $\sigma_{\mathbb{Z}}$ with monomials of $S = \mathbb{C}[u, u^3v^4, uv]$: For example, we identify $(3,4) \in \sigma_{\mathbb{Z}}$ with $u^3v^4 \in S$, and $(1,0) + (1,1)$ with $u \cdot uv$.

Our aim is to find a non-regular cone in $\mathrm{GF}(J_n)$ to prove that $\overline{\mathrm{Nash}_n(X)}$ is singular. As we explained in the previous section, it is sufficient to find a marked Gröbner basis $\mathbb{G}_n$ of J_n such that $C_{\mathbb{G}_n}$ is non-regular.

Definition 2.3. (1) Let $\preceq$ be the monomial ordering on S associated to

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

(see Lemma 1.10 for details).

(2) Let $\mathbb{G}_n$ be the marked Gröbner basis of J_n with respect to $\preceq$.

(3) Let $\mathbb{M}_n := \{\alpha \mid (g,\alpha) \in \mathbb{G}_n\}$. This is the minimal monomial generating set of $\mathrm{in}_{\preceq}(J_n)$.

Below we will describe $C_{\mathbb{G}_n}$ explicitly, and show its non-regularity.

2.1. **Candidate for $\mathbb{M}_n$.** We first study $\mathbb{M}_n$.

Definition 2.4. For each integer $n > 0$, let $\mathcal{P}_n$ be the set consisting of following elements of $\sigma_{\mathbb{Z}}$ (see Figure 1, Figure 2):

For odd n,

$$\begin{aligned}
&p_n := \left(\tfrac{n+3}{2}, 0\right), \\
&q_n^0 := \left(\tfrac{n+3}{2}, 1\right) + \tfrac{n-1}{2}(1,2), \qquad q_n^i := q_n^0 - i(1,2) \ \left(0 \le i \le \tfrac{n-1}{2}\right), \\
&r_n^0 := q_n^0 + (0,1), \qquad\qquad\qquad\quad r_n^j := r_n^0 + j(1,2) \ \left(0 \le j \le \tfrac{n-1}{2}\right), \\
&s_n := \tfrac{n+1}{2}(3,4).
\end{aligned}$$

For even n,

$$\begin{aligned}
&p_n := \left(\tfrac{n+2}{2}, 0\right), \\
&q_n^0 := \left(\tfrac{n+2}{2}, 0\right) + \tfrac{n}{2}(1,2), \qquad q_n^i := q_n^0 - i(1,2) \ \left(0 \le i \le \tfrac{n-2}{2}\right), \\
&r_n^0 := q_n^0 + (0,1), \qquad\qquad\qquad r_n^j := r_n^0 + j(1,2) \ \left(0 \le j \le \tfrac{n}{2}\right), \\
&s_n := \left(\tfrac{n+2}{2}\right)(3,4).
\end{aligned}$$

In the next subsection, we will show $\mathbb{M}_n = \mathcal{P}_n$. Here we prove some properties of $\mathcal{P}_n$ that will be needed later.

The next lemma gives direct description of elements of $\mathcal{P}_n$:

Lemma 2.5. Let $n > 0$ be an integer. If n is odd, then

$$\begin{aligned} p_n &= \left(\tfrac{n+3}{2}, 0\right) = q_n^{\frac{n-1}{2}} - (0,1), & & \\ q_n^0 &= (n+1, n)\,, & q_n^{\frac{n-1}{2}} &= \left(\tfrac{n+3}{2}, 1\right), \\ r_n^0 &= (n+1, n+1)\,, & r_n^{\frac{n-1}{2}} &= \left(\tfrac{3n+1}{2}, 2n\right), \\ s_n &= \tfrac{n+1}{2}(3,4) = r_n^{\frac{n-1}{2}} + (1,2). & & \end{aligned}$$

If n is even, then

$$\begin{aligned} p_n &= \left(\tfrac{n+2}{2}, 0\right) = q_n^{\frac{n-2}{2}} - (1,2), & & \\ q_n^0 &= (n+1, n)\,, & q_n^{\frac{n-2}{2}} &= \left(\tfrac{n+4}{2}, 2\right), \\ r_n^0 &= (n+1, n+1)\,, & r_n^{\frac{n}{2}} &= \left(\tfrac{3n+2}{2}, 2n+1\right), \\ s_n &= \left(\tfrac{n+2}{2}\right)(3,4) = r_n^{\frac{n}{2}} + (2,3). & & \end{aligned}$$

Proof. The assertions follow from direct calculations. □

Now let us explain what $\mathcal{P}_n$ looks like. By the definition of $\mathcal{P}_n$ and Lemma 2.5, we obtain Figure 1 and Figure 2: All q_n^i and r_n^j are lying on the thick-line segments, and conversely all lattice points on the segments are members of $\mathcal{P}_n$. The broken-line segments have lattice points only at the edges. If n is odd, then the slopes of segments $q_n^{\frac{n-1}{2}} q_n^0$ and $r_n^0 s_n$ are both 2. If n is even, then the slopes of $p_n q_n^0$ and $r_n^0 r_n^{\frac{n}{2}}$ are both 2, and the slope of $r_n^{\frac{n}{2}} s_n$ is $\frac{3}{2}$.

Next let us look at how $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ are related.

Lemma 2.6. Let $n > 0$ be an integer.

(1) $\#\mathcal{P}_n = n+3$.

(2) For any distinct members a, b of $\mathcal{P}_n$, we have $b \notin a + \sigma_{\mathbb{Z}}$.

(3) If n is odd, then

$$p_{n-1} + (1,0) = p_n = p_{n+1},\ s_n \neq s_{n+1}.$$

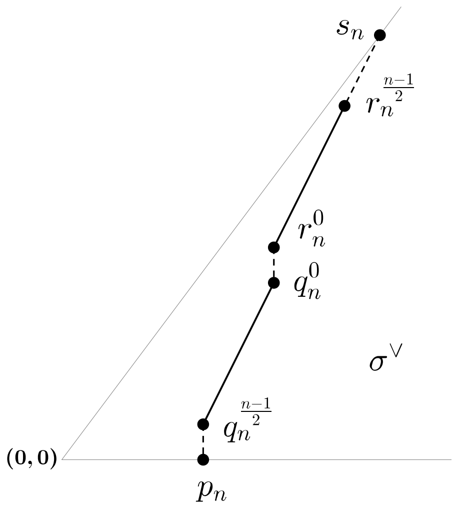

FIGURE 1. $\mathcal{P}_n$ for odd n

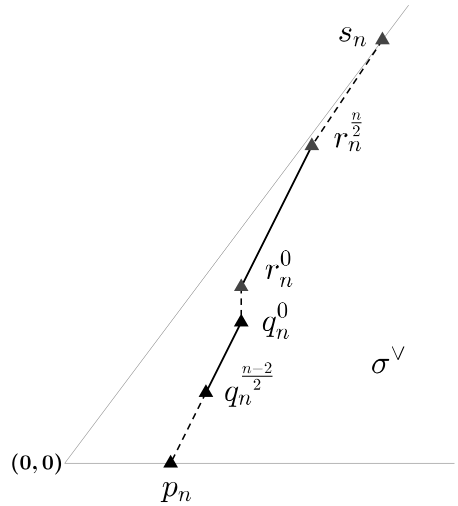

FIGURE 2. $\mathcal{P}_n$ for even n

If n is even, then

$$p_n \neq p_{n+1},\ s_{n-1} + (3,4) = s_n = s_{n+1}.$$

(See Figure 3, Figure 4.)

(4) For the map $\theta : \sigma^\vee \to \sigma^\vee$ defined by $a \mapsto (1,1) + a$, we have

$$\theta\left(q_n^i\right) = q_{n+1}^i,\ \theta\left(r_n^j\right) = r_{n+1}^j$$

and

$$\begin{aligned} \theta(p_n) &= p_{n+1} + (1,1) \text{ and } \theta(s_n) = r_{n+1}^{\frac{n+1}{2}} \quad \text{if } n \text{ is odd,} \\ \theta(p_n) &= q_{n+1}^{\frac{n}{2}} \text{ and } \theta(s_n) = s_{n+1} + (1,1) \quad \text{if } n \text{ is even.} \end{aligned}$$

Thus $\theta(\mathcal{P}_n) \subset \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$.

(5) We have

$$\mathcal{P}_n \cap \mathcal{P}_{n+1} = \begin{cases} \{p_n\} & \text{if } n \text{ is odd,} \\ \{s_n\} & \text{if } n \text{ is even,} \end{cases}$$

and

$$\mathcal{P}_{n+1} = \theta\left(\mathcal{P}_n \setminus \mathcal{P}_{n+1}\right) \sqcup \{p_{n+1}, s_{n+1}\}.$$

(6) We have

$$\mathcal{P}_n + \sigma_{\mathbb{Z}} = (\mathcal{P}_n \setminus \mathcal{P}_{n+1}) \sqcup (\mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}) .$$

Proof. (1) follows from direct counting.

(2) Let $l_1 := (4, -3)$ and $l_2 := (0, 1)$ be the ray generators of σ. Then, for any $a, b \in \mathcal{P}_n$, $b \in a + \sigma_{\mathbb{Z}}$ if and only if $l_1 \cdot (b - a) \geq 0$ and $l_2 \cdot (b - a) \geq 0$. However the following inequalities hold: If n is odd, then

$$\begin{aligned} l_1 \cdot p_n > l_1 \cdot q_n^{\frac{n-1}{2}} > \cdots > l_1 \cdot q_n^0 > l_1 \cdot r_n^0 > \cdots > l_1 \cdot r_n^{\frac{n-1}{2}} > l_1 \cdot s_n, \\ l_2 \cdot p_n < l_2 \cdot q_n^{\frac{n-1}{2}} < \cdots < l_2 \cdot q_n^0 < l_2 \cdot r_n^0 < \cdots < l_2 \cdot r_n^{\frac{n-1}{2}} < l_2 \cdot s_n. \end{aligned}$$

If n is even, then

$$\begin{aligned} l_1 \cdot p_n > l_1 \cdot q_n^{\frac{n-2}{2}} > \cdots > l_1 \cdot q_n^0 > l_1 \cdot r_n^0 > \cdots > l_1 \cdot r_n^{\frac{n}{2}} > l_1 \cdot s_n, \\ l_2 \cdot p_n < l_2 \cdot q_n^{\frac{n-2}{2}} < \cdots < l_2 \cdot q_n^0 < l_2 \cdot r_n^0 < \cdots < l_2 \cdot r_n^{\frac{n}{2}} < l_2 \cdot s_n. \end{aligned}$$

Indeed, we have $l_1 \cdot (1, 2) = -2 < 0$ and $l_2 \cdot (1, 2) = 2 > 0$, thus

$$\begin{aligned} l_1 \cdot q_n^{i+1} = l_1 \cdot \left(q_n^i - (1, 2)\right) = l_1 \cdot q_n^i - l_1 \cdot (1, 2) > l_1 \cdot q_n^i, \\ l_2 \cdot q_n^{i+1} = l_2 \cdot \left(q_n^i - (1, 2)\right) = l_2 \cdot q_n^i - l_2 \cdot (1, 2) < l_2 \cdot q_n^i. \end{aligned}$$

One can easily check the other inequalities by similar calculations.

Then, for any $a \neq b \in \mathcal{P}_n$, we have either $l_1 \cdot (b - a) < 0$ or $l_2 \cdot (b - a) < 0$. Therefore (2) holds.

(3) follows from direct calculations.

(4) One can easily check that $\theta\left(q_n^0\right) = q_{n+1}^0$ for any n by Lemma 2.5. Therefore

$$\theta\left(q_n^i\right) = \theta\left(q_n^0 - i\,(1, 2)\right) = \theta\left(q_n^0\right) - i\,(1, 2) = q_{n+1}^0 - i\,(1, 2) = q_{n+1}^i.$$

The other assertions also follow from similar direct calculations and (3).

(5) In Figure 3 and Figure 4, (4) implies that θ shifts segments as follows:

$$\begin{aligned} \theta\left(q_n^{\frac{n-1}{2}} q_n^0\right) = q_{n+1}^{\frac{n-1}{2}} q_{n+1}^0 \text{ and } \theta\left(r_n^0 s_n\right) = r_{n+1}^0 r_{n+1}^{\frac{n+1}{2}} \text{ if } n \text{ is odd}, \\ \theta\left(p_n q_n^0\right) = q_{n+1}^{\frac{n}{2}} q_{n+1}^0 \text{ and } \theta\left(r_n^0 r_n^{\frac{n}{2}}\right) = r_{n+1}^0 r_{n+1}^{\frac{n}{2}} \text{ if } n \text{ is even}. \end{aligned}$$

This shows that

$$\mathcal{P}_{n+1} \setminus \{p_{n+1}, s_{n+1}\} = \begin{cases} \theta\left(\mathcal{P}_n \setminus \{p_n\}\right) & \text{if } n \text{ is odd,} \\ \theta\left(\mathcal{P}_n \setminus \{s_n\}\right) & \text{if } n \text{ is even.} \end{cases}$$

On the other hand, we have

$$\mathcal{P}_n \cap \mathcal{P}_{n+1} = \begin{cases} \{p_n\} & \text{if } n \text{ is odd,} \\ \{s_n\} & \text{if } n \text{ is even.} \end{cases}$$

Indeed, let n be odd (resp. even). By (3), we have $p_n = p_{n+1} \in \mathcal{P}_n \cap \mathcal{P}_{n+1}$ (resp. $s_n = s_{n+1} \in \mathcal{P}_n \cap \mathcal{P}_{n+1}$). If $\mathcal{P}_n \cap \mathcal{P}_{n+1}$ contains some $a \neq p_{n+1}$ (resp. $a \neq s_{n+1}$), then $a \in \theta\left(\mathcal{P}_n \setminus \{p_n\}\right)$ or $a = s_{n+1} = s_n + (3,4)$ (resp. $a \in \theta\left(\mathcal{P}_n \setminus \{s_n\}\right)$ or $a = p_{n+1} = p_n + (1,0)$). However each of these possibilities contradicts to (2). Therefore (5) holds.

(6) By (3) and (5), it is clear that $\mathcal{P}_{n+1} \subset \mathcal{P}_n + \sigma_{\mathbb{Z}}$. Hence $\mathcal{P}_{n+1} + \sigma_{\mathbb{Z}} \subset \mathcal{P}_n + \sigma_{\mathbb{Z}}$ and it is sufficient to show that

$$\left(\mathcal{P}_n + \sigma_{\mathbb{Z}}\right) \setminus \left(\mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}\right) = \mathcal{P}_n \setminus \mathcal{P}_{n+1}.$$

Fix any $a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1}$.

We have $a \notin \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$: Otherwise $\theta(a) \in \mathcal{P}_{n+1} + (1,1) + \sigma_{\mathbb{Z}}$ but $\theta(a) \in \mathcal{P}_{n+1}$ by (5). This contradicts to (2). Therefore $a \in \left(\mathcal{P}_n + \sigma_{\mathbb{Z}}\right) \setminus \left(\mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}\right)$ so $\left(\mathcal{P}_n + \sigma_{\mathbb{Z}}\right) \setminus \left(\mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}\right) \supset \mathcal{P}_n \setminus \mathcal{P}_{n+1}$.

To see the other inclusion, we will show the following by induction on n:

$$a + (1,0), a + (3,4) \in \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}} \text{ where } a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1}.$$

The case $n = 1$ is easily checked by Figure 5. Let $n > 1$.

Let d be $(1,0)$ or $(3,4)$.

By (5), we have

$$a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1} = \left(\theta\left(\mathcal{P}_{n-1} \setminus \mathcal{P}_n\right) \setminus \mathcal{P}_{n+1}\right) \sqcup \left(\{p_n, s_n\} \setminus \mathcal{P}_{n+1}\right).$$

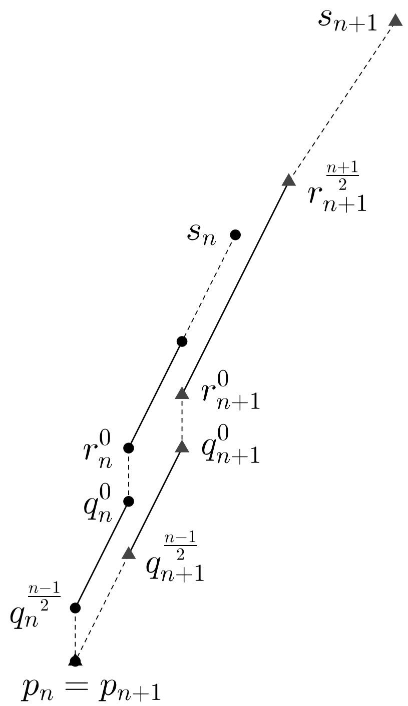

FIGURE 3. $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ for odd n

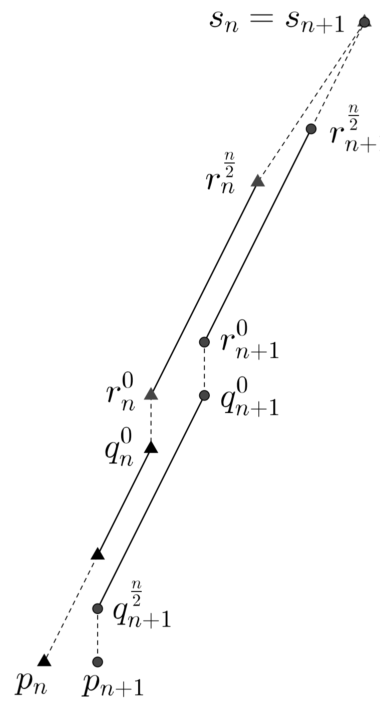

FIGURE 4. $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ for even n

Let a is contained in $\theta(\mathcal{P}_{n-1} \setminus \mathcal{P}_n) \setminus \mathcal{P}_{n+1}$. Then $a = \theta(a')$ for some $a' \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n$, so $a + d = a' + d + (1,1)$. By the induction hypothesis, we have $a' + d \in \mathcal{P}_n + \sigma_{\mathbb{Z}}$ thus $a + d \in \mathcal{P}_n + \sigma_{\mathbb{Z}} + (1,1)$. Since $\mathcal{P}_n + \sigma_{\mathbb{Z}} + (1,1) = \theta(\mathcal{P}_n) + \sigma_{\mathbb{Z}} \subset \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$ by (4), we have $a + d \in \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$.

Let a is contained in $\{p_n, s_n\} \setminus \mathcal{P}_{n+1}$. If n is odd, then $a = s_n$ by (5) and

$$s_n + (1,0) = r_n^{\frac{n-1}{2}} + (1,2) + (1,0) = \theta\left(r_n^{\frac{n-1}{2}}\right) + (1,1) = r_{n+1}^{\frac{n-1}{2}} + (1,1),$$
$$s_n + (3,4) = s_{n+1}.$$

Hence $a + d \in \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$. If n is even, then $a = p_n$ and we have

$$p_n + (1,0) = p_{n+1},$$
$$p_n + (3,4) = q_n^{\frac{n-2}{2}} - (1,2) + (3,4) = \theta\left(q_n^{\frac{n-2}{2}}\right) + (1,1) = q_{n+1}^{\frac{n-2}{2}} + (1,1)$$

thus $a + d \in \mathcal{P}_{n+1} + \sigma_{\mathbb{Z}}$.

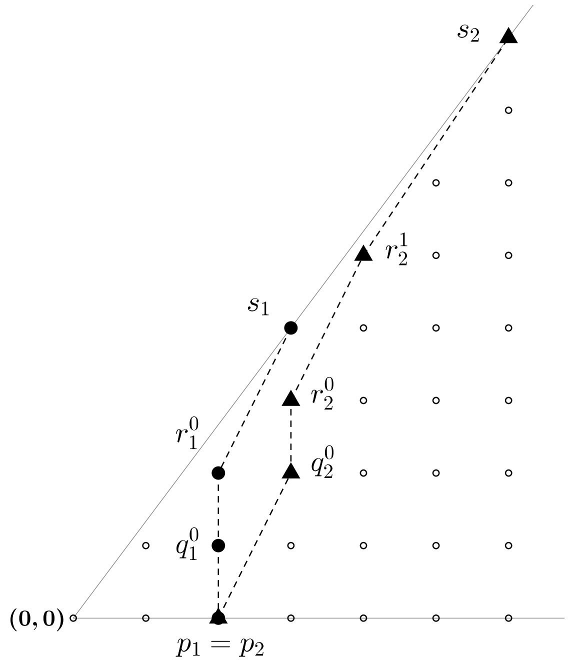

FIGURE 5. $\mathcal{P}_1$ and $\mathcal{P}_2$

Therefore the induction is complete and we have $a+(1,0), a+(3,4) \in \mathcal{P}_{n+1}+\sigma_{\mathbb{Z}}$ for any $a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1}$. Note that $a+(1,1) = \theta\,(a)$ is also contained in $\mathcal{P}_{n+1}+\sigma_{\mathbb{Z}}$ by (4).

Then the other inclusion $(\mathcal{P}_n+\sigma_{\mathbb{Z}}) \setminus (\mathcal{P}_{n+1}+\sigma_{\mathbb{Z}}) \subset \mathcal{P}_n \setminus \mathcal{P}_{n+1}$ holds: For any $b \in (\mathcal{P}_n+\sigma_{\mathbb{Z}}) \setminus (\mathcal{P}_{n+1}+\sigma_{\mathbb{Z}})$, we have $b = a+d'$ for some $a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1}$ and $d' \in \sigma_{\mathbb{Z}}$. Then $d' = 0$: Otherwise, since $\sigma_{\mathbb{Z}}$ is generated by $(1,1), (1,0)$ and $(3,4)$, $b = a+d' \in \mathcal{P}_{n+1}+\sigma_{\mathbb{Z}}$ as we have seen above. Therefore $b = a \in \mathcal{P}_n \setminus \mathcal{P}_{n+1}$. $\square$

Definition 2.7. Let $\mathcal{D}_n := \sigma_{\mathbb{Z}} \setminus (\mathcal{P}_n+\sigma_{\mathbb{Z}})$. In the monomial algebra S, $\mathcal{D}_n$ is the set of monomials which are not divisible by any monomial in $\mathcal{P}_n$. In other words, $\mathcal{D}_n$ consists of all monomials in S not contained in the ideal $\langle\mathcal{P}_n\rangle$ generated by the monomials in $\mathcal{P}_n$.

Definition 2.8. Let $O := (0,0) \in \sigma_{\mathbb{Z}}$. We define the following sets:

$$\begin{aligned}
\mathcal{V}_1 &:= \{O, (1,1), (1,0)\}, \\
\mathcal{V}_2 &:= \{O, s_1, (1,0), q_1^0\}, \\
\mathcal{V}_n &:= \begin{cases} \left\{O, p_{n-1}, q_{n-1}^0, r_{n-1}^{\frac{n-1}{2}}, s_{n-2}\right\} & \text{if } n \geq 3 \text{ is odd,} \\ \left\{O, s_{n-1}, q_{n-1}^0, q_{n-1}^{\frac{n-2}{2}}, p_{n-2}\right\} & \text{if } n \geq 4 \text{ is even.} \end{cases}
\end{aligned}$$

Let $\overline{\mathcal{D}_n}$ be the convex set $\operatorname{Conv}(\mathcal{V}_n)$. We will later show that $\overline{\mathcal{D}_n}$ is the convex hull of $\mathcal{D}_n$ (Lemma 2.10 (5)). The vertices of $\overline{\mathcal{D}_n}$ are exactly elements of $\mathcal{V}_n$, but we do not need this fact. However it is clear that the vertices of $\overline{\mathcal{D}_n}$ are contained in $\mathcal{V}_n$.

Lemma 2.9. Let $n \geq 3$. Then $\overline{\mathcal{D}_n}$ contains the lattice points r_{n-1}^0, q_{n-2}^0 and

$$\begin{cases} q_{n-2}^{\frac{n-3}{2}} & \text{if } n \text{ is odd,} \\ r_{n-2}^{\frac{n-2}{2}} & \text{if } n \text{ is even.} \end{cases}$$

Proof. For each lattice point α in the assertion, we will give a representation of α as follows: $\alpha = c_1\gamma_1 + c_2\gamma_2$ where $\gamma_i \in \mathcal{V}_n$ and $c_i \geq 0$ with $c_1 + c_2 \leq 1$. Then we can conclude that $\alpha \in \overline{\mathcal{D}_n}$.

If n is odd, then

$$\begin{aligned}
r_{n-1}^0 &= \tfrac{n^2-n}{n^2+2n-1} q_{n-1}^0 + \tfrac{2n}{n^2+2n-1} r_{n-1}^{\frac{n-1}{2}}, \\
q_{n-2}^0 &= \tfrac{2}{n^2-1} p_{n-1} + \tfrac{n-2}{n-1} q_{n-1}^0, \\
q_{n-2}^{\frac{n-3}{2}} &= \tfrac{n^2-2n-1}{n^2-1} p_{n-1} + \tfrac{1}{n-1} q_{n-1}^0.
\end{aligned}$$

If n is even, then

$$\begin{aligned}
r_{n-1}^0 &= \tfrac{n}{n+3} q_{n-1}^0 + \tfrac{2}{n+3} s_{n-1}, \\
q_{n-2}^0 &= \tfrac{2}{n^2-n} p_{n-2} + \tfrac{n-2}{n-1} q_{n-1}^0, \\
r_{n-2}^{\frac{n-2}{2}} &= \tfrac{1}{2n} p_{n-2} + \tfrac{2n-3}{2n} s_{n-1}.
\end{aligned}$$

Therefore the assertion holds. $\square$

Lemma 2.10. Let $n > 0$ be an integer.

(1) $\mathcal{D}_1 = \{(0,0), (1,0), (1,1)\}$ and $\mathcal{D}_n = \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n)$ for $n \geq 2$.

(2) $\#\mathcal{D}_n = \frac{1}{2}(n+1)(n+2)$.

(3) $(1,1) + \mathcal{D}_n \subset \mathcal{D}_{n+1}$.

(4) Let $\Phi : \ \sigma_{\mathbb{Z}} \to \mathbb{Z}$ be the map defined by $a \mapsto (-1,1) \cdot a$.

If n is odd, then

$$\Phi\left(\mathcal{D}_n\right) = \left\{-\frac{n+1}{2}, -\left(\frac{n+1}{2} - 1\right), \ldots, -1, 0, 1, \ldots, \frac{n+1}{2} - 1\right\}.$$

If n is even, then

$$\Phi\left(\mathcal{D}_n\right) = \left\{-\frac{n}{2}, -\left(\frac{n}{2} - 1\right), \ldots, -1, 0, 1, \ldots, \frac{n}{2} - 1, \frac{n}{2}\right\}$$

and s_{n-1} is the only member of $\mathcal{D}_n$ which is mapped to $\frac{n}{2}$ by Φ.

(5) $\mathcal{V}_n \subset \mathcal{D}_n$ and $\overline{\mathcal{D}_n}$ is the convex hull of $\mathcal{D}_n$.

(6) p_n is strictly bigger than any element of $\mathcal{D}_n$ with respect to $\preceq$.

(7) Let $n \geq 2$. We define $\Psi_n : \ \sigma_{\mathbb{Z}} \to \mathbb{Z}$ by $a \mapsto l_n \cdot a$ where

$$l_n := \begin{cases} (2n-2, -n+2) & \text{if } n \text{ is odd,} \\ (2n, -n+1) & \text{if } n \text{ is even.} \end{cases}$$

Then, if n is odd, we have

$$\max \Psi_n\left(\mathcal{D}_n\right) = \Psi_n\left(r_{n-1}^{\frac{n-1}{2}}\right)$$

and this is equal to

$$\min \Psi_n\left(\mathcal{P}_n\right) = \Psi_n\left(q_n^{\frac{n-1}{2}}\right).$$

If n is even, we have

$$\max \Psi_n\left(\mathcal{D}_n\right) = \Psi_n\left(s_{n-1}\right)$$

and this is equal to

$$\min \Psi_n\left(\mathcal{P}_n\right) = \Psi_n\left(p_n\right).$$

Proof. (1) The case $n=1$ follows from Figure 5. Let $n \geq 2$. Then Lemma 2.6 (6) implies $\mathcal{P}_n + \sigma_{\mathbb{Z}} \subset \mathcal{P}_{n-1} + \sigma_{\mathbb{Z}}$, thus the following equalities hold:

$$\begin{aligned} \mathcal{D}_n &= \sigma_{\mathbb{Z}} \setminus (\mathcal{P}_n + \sigma_{\mathbb{Z}}) \\ &= (\sigma_{\mathbb{Z}} \setminus (\mathcal{P}_{n-1} + \sigma_{\mathbb{Z}})) \sqcup ((\mathcal{P}_{n-1} + \sigma_{\mathbb{Z}}) \setminus (\mathcal{P}_n + \sigma_{\mathbb{Z}})) \\ &= \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n). \end{aligned}$$

Hence (1) holds.

We prove (2)-(5) by induction on n.

(2) The case $n=1$ follows from (1). Let $n \geq 2$. We have $\#(\mathcal{P}_{n-1} \setminus \mathcal{P}_n) = n+1$ by Lemma 2.6 (1) and Lemma 2.6 (5). Hence (1) implies that

$$\#\mathcal{D}_n = \#\mathcal{D}_{n-1} + \#(\mathcal{P}_{n-1} \setminus \mathcal{P}_n) = \frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2).$$

(3) Let $n=1$. In Figure 6, it is clear that $\mathcal{D}_1$ consists of lattice points of $\overline{\mathcal{D}_1}$. Moreover $\mathcal{D}_2 = \mathcal{D}_1 \sqcup (\mathcal{P}_1 \setminus \{p_1\})$ by (1) and Lemma 2.6 (5). Thus one can see by Figure 6 that $\mathcal{D}_2$ also consists of lattice points of $\overline{\mathcal{D}_2}$. Then it is easily checked that $(1,1) + \mathcal{D}_1 \subset \mathcal{D}_2$.

Let $n \geq 2$. By (1) we have

$$(1,1) + \mathcal{D}_n = ((1,1) + \mathcal{D}_{n-1}) \cup ((1,1) + (\mathcal{P}_{n-1} \setminus \mathcal{P}_n)).$$

Now (1) also shows that $\mathcal{D}_n \subset \mathcal{D}_{n+1}$. Thus, by the induction hypothesis, we have

$$(1,1) + \mathcal{D}_{n-1} \subset \mathcal{D}_n \subset \mathcal{D}_{n+1}.$$

Moreover Lemma 2.6 (5) shows that

$$(1,1) + (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) = \theta(\mathcal{P}_{n-1} \setminus \mathcal{P}_n) \subset \mathcal{P}_n \setminus \mathcal{P}_{n+1}$$

and $\mathcal{P}_n \setminus \mathcal{P}_{n+1} \subset \mathcal{D}_{n+1}$ by (1). Therefore $(1,1) + \mathcal{D}_n \subset \mathcal{D}_{n+1}$.

(4) The case $n=1$ follows from direct calculations.

Let $n \geq 2$. By (1) we have $\Phi(\mathcal{D}_n) = \Phi(\mathcal{D}_{n-1}) \cup \Phi(\mathcal{P}_{n-1} \setminus \mathcal{P}_n)$.

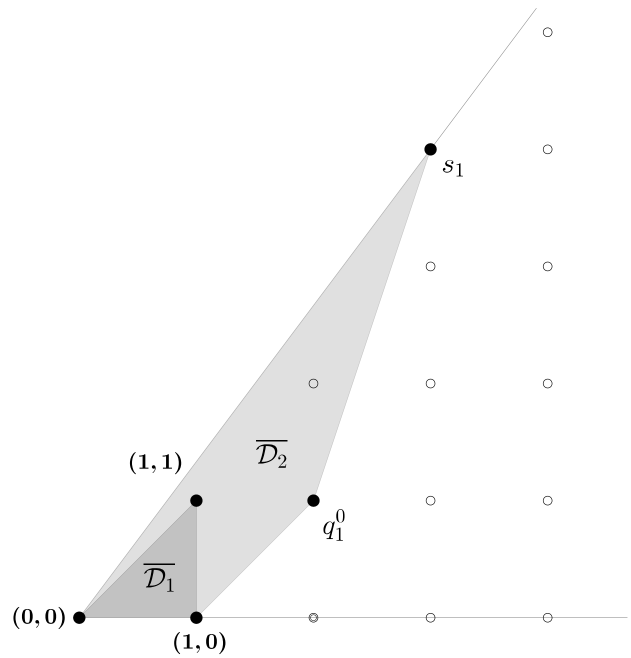

FIGURE 6. $\overline{\mathcal{D}_1}$ and $\overline{\mathcal{D}_2}$

Let n be even. By the induction hypothesis, $\Phi\left(\mathcal{D}_{n-1}\right)$ consists of

$$-\frac{n}{2}, -\left(\frac{n}{2}-1\right), \ldots, -1, 0, 1, \ldots, \frac{n}{2}-1.$$

Moreover $\Phi\left(\mathcal{P}_{n-1} \setminus \mathcal{P}_n\right)$ consists of

$$\Phi\left(q_{n-1}^i\right) = -1-i,\ \Phi\left(r_{n-1}^j\right) = j \text{ and } \Phi\left(s_{n-1}\right) = \frac{n}{2}$$

where $0 \leq i, j \leq \frac{n-2}{2}$. Hence the assertion holds for even n.

Let n be odd. By the induction hypothesis, $\Phi\left(\mathcal{D}_{n-1}\right)$ consists of

$$-\left(\frac{n+1}{2}-1\right), -\left(\frac{n+1}{2}-2\right), \ldots, -1, 0, 1, \ldots, \frac{n+1}{2}-2, \frac{n+1}{2}-1.$$

Moreover $\Phi\left(\mathcal{P}_{n-1} \setminus \mathcal{P}_n\right)$ consists of

$$\Phi\left(p_{n-1}\right) = -\frac{n+1}{2},\ \Phi\left(q_{n-1}^i\right) = -1-i,\ \Phi\left(r_{n-1}^j\right) = j$$

where $0 \leq i \leq \frac{n-3}{2}$ and $0 \leq j \leq \frac{n-1}{2}$. Hence (4) holds.

(5) In the proof of (3), we have already shown the cases $n = 1, 2$: In Figure 6, $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) consists of lattice points of $\overline{\mathcal{D}_1}$ (resp. $\overline{\mathcal{D}_2}$).

Let $n \geq 3$. It is sufficient to show that $\mathcal{V}_n \subset \mathcal{D}_n \subset \overline{\mathcal{D}_n}$.

Let n be odd. First let us show $\mathcal{D}_n \subset \overline{\mathcal{D}_n}$. By (1), we have $\mathcal{D}_n = \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n)$. One can check $\mathcal{D}_{n-1} \subset \overline{\mathcal{D}_n}$ as follows. If $n = 3$, we have $\overline{\mathcal{D}_2} \subset \overline{\mathcal{D}_3}$: Indeed

$$\mathcal{V}_2 = \left\{O, s_1, (1,0), q_1^0\right\}$$

and $O, s_1 \in \mathcal{V}_3 \subset \overline{\mathcal{D}_3}$. Since $p_2 = (2,0) \in \mathcal{V}_3$, we have $(1,0) \in Op_2 \subset \overline{\mathcal{D}_3}$. Furthermore $q_1^0 \in \overline{\mathcal{D}_3}$ by Lemma 2.9. Thus $\mathcal{V}_2 \subset \overline{\mathcal{D}_3}$, so $\overline{\mathcal{D}_2} \subset \overline{\mathcal{D}_3}$. If $n > 3$, we also have $\overline{\mathcal{D}_{n-1}} \subset \overline{\mathcal{D}_n}$: Indeed

$$\mathcal{V}_{n-1} = \left\{O, s_{n-2}, q_{n-2}^0, q_{n-2}^{\frac{n-3}{2}}, p_{n-3}\right\}$$

and $O, s_{n-2} \in \mathcal{V}_n \subset \overline{\mathcal{D}_n}$. Lemma 2.9 shows that $q_{n-2}^0, q_{n-2}^{\frac{n-3}{2}} \in \overline{\mathcal{D}_n}$. Moreover $p_{n-3} \in Op_{n-1} \subset \overline{\mathcal{D}_n}$ since $p_{n-1} \in \mathcal{V}_n$. Thus $\mathcal{V}_{n-1} \subset \overline{\mathcal{D}_n}$, so $\overline{\mathcal{D}_{n-1}} \subset \overline{\mathcal{D}_n}$.

Therefore $\mathcal{D}_{n-1} \subset \overline{\mathcal{D}_{n-1}} \subset \overline{\mathcal{D}_n}$ for odd $n \geqq 3$ by the induction hypothesis.

To see $\mathcal{P}_{n-1} \setminus \mathcal{P}_n \subset \overline{\mathcal{D}_n}$ for odd $n \geqq 3$, note that the following equalities hold:

$$\mathcal{P}_{n-1} \setminus \mathcal{P}_n = \mathcal{P}_{n-1} \setminus \{s_{n-1}\} = \left\{\text{lattice points on } p_{n-1}q_{n-1}^0 \text{ and } r_{n-1}^0 r_{n-1}^{\frac{n-1}{2}}\right\}$$

(see Figure 4). Now $p_{n-1}, q_{n-1}^0, r_{n-1}^{\frac{n-1}{2}} \in \mathcal{V}_n$, and $r_{n-1}^0 \in \overline{\mathcal{D}_n}$ by Lemma 2.9. Thus $\mathcal{P}_{n-1} \setminus \mathcal{P}_n \subset \overline{\mathcal{D}_n}$.

Therefore $\mathcal{D}_n = \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) \subset \overline{\mathcal{D}_n}$ for odd $n \geqq 3$.

Next let us show that $\mathcal{V}_n = \left\{O, p_{n-1}, q_{n-1}^0, r_{n-1}^{\frac{n-1}{2}}, s_{n-2}\right\} \subset \mathcal{D}_n$ for odd $n \geqq 3$. By the induction hypothesis, $O, s_{n-2} \in \mathcal{V}_{n-1} \subset \mathcal{D}_{n-1}$. Moreover $p_{n-1}, q_{n-1}^0, r_{n-1}^{\frac{n-1}{2}} \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n$ by Lemma 2.6 (5). Therefore $\mathcal{V}_n \subset \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) = \mathcal{D}_n$ by (1).

Hence (5) holds for odd n.

Similar arguments prove the cases for even $n \geqq 4$: To show $\mathcal{D}_n = \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) \subset \overline{\mathcal{D}_n}$, we will check that $\mathcal{D}_{n-1} \subset \overline{\mathcal{D}_n}$ and $\mathcal{P}_{n-1} \setminus \mathcal{P}_n \subset \overline{\mathcal{D}_n}$ respectively:

For the set

$$\mathcal{V}_{n-1} = \left\{O, p_{n-2}, q^0_{n-2}, r^{\frac{n-2}{2}}_{n-2}, s_{n-3}\right\},$$

$O, p_{n-2} \in \mathcal{V}_n \subset \overline{\mathcal{D}_n}$. Moreover $q^0_{n-2}, r^{\frac{n-2}{2}}_{n-2} \in \overline{\mathcal{D}_n}$ by Lemma 2.9. We also have $s_{n-3} \in Os_{n-1} \subset \overline{\mathcal{D}_n}$ since $s_{n-1} \in \mathcal{V}_n$. Thus $\mathcal{V}_{n-1} \subset \overline{\mathcal{D}_n}$, so $\overline{\mathcal{D}_{n-1}} \subset \overline{\mathcal{D}_n}$. Then, by the induction hypothesis, $\mathcal{D}_{n-1} \subset \overline{\mathcal{D}_{n-1}} \subset \overline{\mathcal{D}_n}$. On the other hand

$$\mathcal{P}_{n-1} \setminus \mathcal{P}_n = \mathcal{P}_{n-1} \setminus \{p_{n-1}\} = \left\{\text{lattice points on } q^{\frac{n-2}{2}}_{n-1} q^0_{n-1} \text{ and } r^0_{n-1} s_{n-1}\right\}$$

(see Figure 3). Then $q^{\frac{n-2}{2}}_{n-1}, q^0_{n-1}, s_{n-1} \in \mathcal{V}_n$ and $r^0_{n-1} \in \overline{\mathcal{D}_n}$ by Lemma 2.9, so $\mathcal{P}_{n-1} \setminus \mathcal{P}_n \subset \overline{\mathcal{D}_n}$.

Therefore $\mathcal{D}_n = \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) \subset \overline{\mathcal{D}_n}$.

Let us check that $\mathcal{V}_n = \left\{O, s_{n-1}, q^0_{n-1}, q^{\frac{n-2}{2}}_{n-1}, p_{n-2}\right\} \subset \mathcal{D}_n$. By the induction hypothesis, $O, p_{n-2} \in \mathcal{V}_{n-1} \subset \mathcal{D}_{n-1}$. Moreover $s_{n-1}, q^0_{n-1}, q^{\frac{n-2}{2}}_{n-1} \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n$ by Lemma 2.6 (5). Therefore $\mathcal{V}_n \subset \mathcal{D}_{n-1} \sqcup (\mathcal{P}_{n-1} \setminus \mathcal{P}_n) = \mathcal{D}_n$, so (5) holds.

To prove (6) and (7), let us remark that, for any $l \in \mathbb{R}^2$, the function $f : \overline{\mathcal{D}_n} \ni a \mapsto l \cdot a \in \mathbb{R}$ attains the maximum value at the vertices of $\overline{\mathcal{D}_n}$. Therefore, by (5), we have $\max f(\mathcal{D}_n) = \max f(\mathcal{V}_n)$.

(6) By the definition of $\preceq$ (Definition 2.3), we only have to show that

$$(2, -1) \cdot p_n > (2, -1) \cdot a \text{ for all } a \in \mathcal{D}_n.$$

Let n be odd. The case $n = 1$ is easily checked. Let $n \geq 3$. To determine $\max\{(2, -1) \cdot a \mid a \in \mathcal{D}_n\}$, we give the following calculations for elements of $\mathcal{V}_n$:

$$\begin{aligned}
(2, -1) \cdot p_{n-1} &= (2, -1) \cdot \left(\tfrac{n+1}{2}, 0\right) = n + 1, \\
(2, -1) \cdot q^0_{n-1} &= (2, -1) \cdot (n, n-1) = n + 1, \\
(2, -1) \cdot r^{\frac{n-1}{2}}_{n-1} &= (2, -1) \cdot \left(\tfrac{3n-1}{2}, 2n - 1\right) = n, \\
(2, -1) \cdot s_{n-2} &= (2, -1) \cdot \tfrac{n-1}{2}(3, 4) = n - 1.
\end{aligned}$$

Hence $\max\{(2, -1) \cdot a \mid a \in \mathcal{D}_n\} = n + 1 < (2, -1) \cdot p_n = n + 3$.

Let n be even. The case $n=2$ is easily checked. Let $n \geq 4$.

$$\begin{aligned}
(2,-1)\cdot s_{n-1} &= (2,-1)\cdot \tfrac{n}{2}(3,4) = n,\\
(2,-1)\cdot q_{n-1}^{0} &= (2,-1)\cdot (n,n-1) = n+1,\\
(2,-1)\cdot q_{n-1}^{\frac{n-2}{2}} &= (2,-1)\cdot \left(\tfrac{n+2}{2},1\right) = n+1,\\
(2,-1)\cdot p_{n-2} &= (2,-1)\cdot \left(\tfrac{n}{2},0\right) = n.
\end{aligned}$$

Hence $\max\{(2,-1)\cdot a \mid a \in \mathcal{D}_n\} = n+1 < (2,-1)\cdot p_n = n+2$. Therefore (6) holds.

(7) The case $n=2$ is easily checked. Let $n \geq 3$ be odd. To determine $\max \Psi_n(\mathcal{D}_n)$, we give the following calculations for elements of $\mathcal{V}_n$:

$$\begin{aligned}
\Psi_n(p_{n-1}) &= (2n-2,-n+2)\cdot\left(\tfrac{n+1}{2},0\right) = n^2-1,\\
\Psi_n\left(q_{n-1}^{0}\right) &= (2n-2,-n+2)\cdot(n,n-1) = n^2+n-2,\\
\Psi_n\left(r_{n-1}^{\frac{n-1}{2}}\right) &= (2n-2,-n+2)\cdot\left(\tfrac{3n-1}{2},2n-1\right) = n^2+n-1,\\
\Psi_n(s_{n-2}) &= (2n-2,-n+2)\cdot\tfrac{n-1}{2}(3,4) = n^2-1.
\end{aligned}$$

Therefore $\max \Psi_n(\mathcal{D}_n) = \Psi_n\left(r_{n-1}^{\frac{n-1}{2}}\right) = n^2+n-1$.

To determine $\min \Psi_n(\mathcal{P}_n)$, note that $\Psi_n(1,2) = 2 > 0$. Then

$$\Psi_n\left(q_n^{i}\right) = \Psi_n\left(q_n^{i+1}+(1,2)\right) = \Psi_n\left(q_n^{i+1}\right) + \Psi_n(1,2) > \Psi_n\left(q_n^{i+1}\right).$$

Hence $\min \Psi_n\left(q_n^{\frac{n-1}{2}} q_n^{0} \cap \mathcal{P}_n\right) = \Psi_n\left(q_n^{\frac{n-1}{2}}\right)$. Similar calculations show that $\min \Psi_n\left(r_n^{0} s_n \cap \mathcal{P}_n\right) = \Psi_n\left(r_n^{0}\right)$ (see Figure 1). Therefore

$$\min \Psi_n(\mathcal{P}_n) = \min\left\{\Psi_n(p_n), \Psi_n\left(q_n^{\frac{n-1}{2}}\right), \Psi_n\left(r_n^{0}\right)\right\}.$$

Thus

$$\min \Psi_n(\mathcal{P}_n) = \Psi_n\left(q_n^{\frac{n-1}{2}}\right) = n^2+n-1$$

by direct calculations, and (7) holds for odd n.

Let $n \geq 4$ be even. Then

$$\begin{aligned}
\Psi_n\left(s_{n-1}\right) &= (2n, -n+1) \cdot \tfrac{n}{2}(3,4) = n^2 + 2n,\\
\Psi_n\left(q_{n-1}^{0}\right) &= (2n, -n+1) \cdot (n, n-1) = n^2 + 2n - 1,\\
\Psi_n\left(q_{n-1}^{\frac{n-2}{2}}\right) &= (2n, -n+1) \cdot \left(\tfrac{n+2}{2}, 1\right) = n^2 + n + 1,\\
\Psi_n\left(p_{n-2}\right) &= (2n, -n+1) \cdot \left(\tfrac{n}{2}, 0\right) = n^2.
\end{aligned}$$

and therefore $\max \Psi_n\left(\mathcal{D}_n\right) = \Psi_n\left(s_{n-1}\right) = n^2 + 2n$.

On the other hand, $\Psi_n(1,2) = 2 > 0$, thus we have

$$\min \Psi_n\left(\mathcal{P}_n\right) = \min\left\{\Psi_n\left(p_n\right), \Psi_n\left(r_n^{0}\right), \Psi_n\left(s_n\right)\right\}$$

(see Figure 2). By direct calculations, we have

$$\min \Psi_n\left(\mathcal{P}_n\right) = \Psi_n\left(p_n\right) = n^2 + 2n.$$

Hence (7) holds. This completes the proof. □

2.2. **Proof of $\mathbb{M}_n = \mathcal{P}_n$.** As we remarked in Definition 2.3 (3), $\mathbb{M}_n$ generates $\mathrm{in}_{\preceq}\left(J_n\right)$. The key to the proof of the equality $\mathbb{M}_n = \mathcal{P}_n$ is to show that $\mathcal{P}_n$ also generates $\mathrm{in}_{\preceq}\left(J_n\right)$. We prepare some lemmas.

Lemma 2.11 ([6], Appendix A, Proposition A.2.1)**.** For any ideal I of S, the monomials of S not contained in $\mathrm{in}_{\preceq}(I)$ form a $\mathbb{C}$-basis of S/I. Therefore we have

(1) $\dim_{\mathbb{C}} S/J_n = \dim_{\mathbb{C}} S/\mathrm{in}_{\preceq}\left(J_n\right)$,

(2) $\dim_{\mathbb{C}} S/\langle\mathcal{P}_n\rangle = \#\mathcal{D}_n$.

Lemma 2.12. (1) $\dim_{\mathbb{C}} S/\mathrm{in}_{\preceq}\left(J_n\right) = \frac{1}{2}(n+1)(n+2) = \dim_{\mathbb{C}} S/\langle\mathcal{P}_n\rangle$.

(2) $\left(J_n : uv - 1\right)_S = J_{n-1}$.

(3) $\dim_{\mathbb{C}} \mathrm{in}_{\preceq}\left(J_{n-1}\right)/\mathrm{in}_{\preceq}\left(J_n\right) = n+1$. Furthermore if a set of monomials generates $\mathrm{in}_{\preceq}\left(J_{n-1}\right)$ as an ideal, then the set also generates $\mathrm{in}_{\preceq}\left(J_{n-1}\right)/\mathrm{in}_{\preceq}(J_n)$ as a vector space over $\mathbb{C}$.

Proof. (1) By Lemma 2.11 (1), we can consider $\dim_{\mathbb{C}} S/J_n$ instead of $\dim_{\mathbb{C}} S/\mathrm{in}_{\preceq}\left(J_n\right)$.

Let $J_0 := \langle u-1, u^3v^4-1, uv-1\rangle$. Then $J_0 = \langle u-1, uv-1\rangle$ since the following equality holds:

$$u^3v^4 - 1 = \left(u^3v^3 + u^2v^2 + uv + 1\right)(uv-1) - u^3v^4(u-1).$$

Moreover S_{J_0} is a regular local ring of dimension two because J_0 is the maximal ideal in S corresponding to the smooth point $(1,1,1)$ of $X = (z^4 - xy = 0)$.

Now consider $\mathrm{gr}_{J_0}(S) = \bigoplus_{\nu=0}^{\infty} J_0^{\nu}/J_0^{\nu+1}$. Then we obtain an isomorphism of graded $\mathbb{C}$-algebras

$$\mathbb{C}[x_1,x_2] \xrightarrow{\cong} \mathrm{gr}_{J_0 S_{J_0}}(S_{J_0}) \cong \mathrm{gr}_{J_0}(S);$$
$$x_1 \mapsto \left[u-1 \bmod J_0^2\right],\ x_2 \mapsto \left[uv-1 \bmod J_0^2\right].$$

Hence

$$\dim_{\mathbb{C}} S/J_n = \dim_{\mathbb{C}} \mathbb{C}[x_1,x_2]/\langle x_1,x_2\rangle^{n+1} = \frac{1}{2}(n+1)(n+2).$$

Lemma 2.10 (2) and Lemma 2.11 (2) show the last equality in the assertion.

(2) $(J_n : uv-1)_S \supset J_{n-1}$ is obvious for any $n > 0$ by the definition of J_n. We will show the other inclusion by induction on n.

Fix any $f \in (J_n : uv-1)_S$. Then $f \in \left(J_0^{n+1} : uv-1\right)_S$ since $J_n = J_0^{n+1}$.

Assume that $n = 1$. Let us consider $\mathrm{gr}_{J_0}(S)$ again. From $f \in \left(J_0^2 : uv-1\right)_S$, it follows that

$$\left[uv-1 \bmod J_0^2\right] \cdot [f \bmod J_0] = \left[(uv-1)f \bmod J_0^2\right] = 0.$$

However $\mathrm{gr}_{J_0}(S)$ is an integral domain and $\left[uv-1 \bmod J_0^2\right]$ is a nonzero elements as we have seen above, so $[f \bmod J_0] = 0$, i.e. $f \in J_0$.

Assume that $n > 1$. $J_n \subset J_{n-1}$ and hence $(J_n : uv-1)_S \subset (J_{n-1} : uv-1)_S$. Therefore, by the induction hypothesis, we have $f \in J_{n-2} = J_0^{n-1}$. Then

$$\left[uv-1 \bmod J_0^2\right] \cdot [f \bmod J_0^n] = \left[(uv-1)f \bmod J_0^{n+1}\right] = 0.$$

Thus $[f \bmod J_0^n] = 0$, i.e. $f \in J_0^n = J_{n-1}$. Hence (2) holds.

(3) By (1), we have

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{in}_{\preceq}(J_{n-1})/\mathrm{in}_{\preceq}(J_n) &= \dim_{\mathbb{C}} S/\mathrm{in}_{\preceq}(J_n) - \dim_{\mathbb{C}} S/\mathrm{in}_{\preceq}(J_{n-1}) \\ &= \frac{1}{2}(n+1)(n+2) - \frac{1}{2}n(n+1) \\ &= n+1.\end{aligned}$$

For the last assertion, let $\{m_1, \ldots, m_r\}$ be any set of monomials which generates $\mathrm{in}_{\preceq}(J_{n-1})$ as an ideal. It is obvious that the vector space $\mathrm{in}_{\preceq}(J_{n-1})/\mathrm{in}_{\preceq}(J_n)$ is generated by monomials in $\mathrm{in}_{\preceq}(J_{n-1})$. Let m be any monomial in $\mathrm{in}_{\preceq}(J_{n-1})$. Then m is divisible by some m_i. If $m \neq m_i$, then there exists $u^a v^b \in \left\{u, u^3v^4, uv\right\}$ such that m is divisible by $m_i(u^a v^b)$. However one can find $f \in J_{n-1}$ with $\mathrm{lm}_{\preceq}(f) = m_i$ and obtain $g := (u^a v^b - 1)f \in J_n$. Then $m_i(u^a v^b) = \mathrm{lm}_{\preceq}(g) \in \mathrm{in}_{\preceq}(J_n)$ and hence $m \equiv 0 \bmod \mathrm{in}_{\preceq}(J_n)$. Therefore $\mathrm{in}_{\preceq}(J_{n-1})/\mathrm{in}_{\preceq}(J_n)$ is generated by $\{m_1, \ldots, m_r\}$ as a vector space. $\square$

The following proposition determines the marked Gröbner basis $\mathbb{G}_1$ of J_1 with respect to $\preceq$:

Proposition 2.13. The reduced Gröbner basis of J_1 with respect to $\preceq$ consists of the following polynomials:

$$\begin{aligned}&\underline{u^3v^4} + u - 4uv + 2, \\ &\underline{u^2v^2} - 2uv + 1, \\ &\underline{u^2v} - u - uv + 1, \\ &\underline{u^2} - 2u + 1,\end{aligned}$$

where the underlined monomials are the leading terms with respect to $\preceq$. Therefore $\mathbb{M}_1$ coincides with $\mathcal{P}_1 = \{(3,4), (2,2), (2,1), (2,0)\}$.

Proof. First we will show that the polynomials are contained in J_1.

Let $g_1 := u^3v^4 + u - 4uv + 2$. Then

$$g_1 = \left((uv)^2 + 2uv + 3\right)(uv-1)^2 - (u-1)\left(u^3v^4 - 1\right) \in J_1.$$

Moreover

$$(2,-1)\cdot(3,4) = 2 \geq (2,-1)\cdot(1,0) = 2 > (2,-1)\cdot(1,1) = 1,$$
$$(1,1)\cdot(3,4) = 7 > (1,1)\cdot(1,0) = 1,$$

and hence $\mathrm{lm}_{\preceq}(g_1) = u^3v^4$. Furthermore

$$g_2 := u^2v^2 - 2uv + 1 = (uv-1)^2 \in J_1,$$
$$g_3 := u^2v - u - uv + 1 = (u-1)(uv-1) \in J_1,$$
$$g_4 := u^2 - 2u + 1 = (u-1)^2 \in J_1.$$

Therefore the polynomials are contained in J_1 and their leading terms are the ones in the assertion.

Next we will show that

$$\mathrm{in}_{\preceq}(J_1) = \langle u^3v^4, u^2v^2, u^2v, u^2\rangle.$$

The right hand side is obviously contained in the left hand side, and coincides with $\langle \mathcal{P}_1\rangle$. Then $\mathrm{in}_{\preceq}(J_1) = \langle \mathcal{P}_1\rangle$ since $\dim_{\mathbb{C}} \mathrm{in}_{\preceq}(J_1)/\langle \mathcal{P}_1\rangle = 0$ by Lemma 2.12 (1). Therefore $\{g_1, g_2, g_3, g_4\}$ is a Gröbner basis of J_1 with respect to $\preceq$. It is easy to check that no monomial of $\mathrm{supp}(g_i)$ is divisible by $\mathrm{lm}_{\preceq}(g_j)$ for $j \neq i$. Hence the basis is reduced. $\square$

We need the following lemma for the cases $n > 0$.

Lemma 2.14. Consider the homomorphism $\mathbb{C}[u,v] \to \mathbb{C}[\lambda, \lambda^{-1}]$ of $\mathbb{C}$-algebras given by $u \mapsto \lambda^{-1}, v \mapsto \lambda$. By restricting to $S \subset \mathbb{C}[u,v]$, we obtain

$$\phi : S \to \mathbb{C}[\lambda^{\pm}];\ u \mapsto \lambda^{-1},\ u^3v^4 \mapsto \lambda,\ uv \mapsto 1.$$

Then we have the following:

(1) ϕ is surjective and $\ker(\phi) = \langle uv - 1\rangle$.

(2) Let $n > 0$ be even. Then

$$B := \left\{\lambda^{-\frac{n}{2}}, \lambda^{-\left(\frac{n}{2}-1\right)}, \ldots, \lambda^{-1}, 1, \lambda, \ldots, \lambda^{\frac{n}{2}-1}, \lambda^{\frac{n}{2}}\right\} \subset \mathbb{C}[\lambda^{\pm}]$$

is a $\mathbb{C}$-basis of $\mathbb{C}[\lambda^{\pm}]/\phi(J_n)$.

(3) Let $n > 0$ be even, and let f be an element of S satisfying that $\mathrm{lm}_{\preceq}(f) \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n$. Then

$$\mathrm{supp}\,(\phi(f)) \subset B.$$

Moreover, if $f \in J_n$, then $\phi\,(f) = 0$.

Proof. (1) ϕ is obviously surjective. Let $F : \mathbb{C}[x, y, z] \twoheadrightarrow S$ be the homomorphism introduced in Notation 2.1 (3). Then one can easily see that $\ker(\phi \circ F) = \langle xy - 1, z - 1\rangle$. Therefore

$$\ker\phi = F\left(\langle xy - 1, z - 1\rangle\right) = \langle u^4v^4 - 1, uv - 1\rangle = \langle uv - 1\rangle$$

because $u^4v^4 - 1 = \left(u^3v^3 + u^2v^2 + uv + 1\right)(uv - 1)$.

(2) One can easily check that

$$\phi\,(u - 1) = -\lambda^{-1}\phi\left(u^3v^4 - 1\right) \text{ and } \phi\,(uv - 1) = 0.$$

Thus $\phi(J_n) = \langle\phi\left(u^3v^4 - 1\right)\rangle^{n+1} = \langle\lambda - 1\rangle^{n+1}$.

It is clear that $1, \lambda, \ldots, \lambda^n$ form a $\mathbb{C}$-basis of $\mathbb{C}[\lambda]/\langle\lambda - 1\rangle^{n+1} = \mathbb{C}[\lambda^{\pm}]/\phi(J_n)$. Now λ is a unit element of this ring. Hence, by multiplying $\lambda^{-\frac{n}{2}}$, we obtain a $\mathbb{C}$-basis $B = \left\{\lambda^{-\frac{n}{2}}, \lambda^{-\left(\frac{n}{2}-1\right)}, \ldots, \lambda^{-1}, 1, \lambda, \ldots, \lambda^{\frac{n}{2}-1}, \lambda^{\frac{n}{2}}\right\}$ of $\mathbb{C}[\lambda^{\pm}]/\phi(J_n)$.

(3) Fix any $m \in \mathrm{supp}\,(f)$ and let us show $\phi\,(m) \in B$.

Suppose that $\phi(m) \notin B$, i.e. $\phi\,(m) = \lambda^d$ or $\phi\,(m) = \lambda^{-d}$ for some $d \geq \frac{n}{2} + 1$.

If $\phi(m) = \lambda^d$ for $d \geq \frac{n}{2} + 1$, then m is divisible by $\left(u^3v^4\right)^d$: Indeed m can be written as $m = u^a\left(u^3v^4\right)^b(uv)^c$ for some $a, b, c \geq 0$, and $\phi\,(m) = \lambda^{-a}\lambda^b1^c$. Therefore $b \geq d$.

Now $\mathrm{lm}_{\preceq}(f) \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n = \mathcal{P}_{n-1} \setminus \{p_{n-1}\}$ by Lemma 2.6 (5), hence direct calculations show that

$$\mathrm{lm}_{\preceq}(f) \cdot (2,-1) = \begin{cases} n+1 & \text{if } \mathrm{lm}_{\preceq}(f) = q^i_{n-1}, \\ n & \text{if } \mathrm{lm}_{\preceq}(f) = r^i_{n-1} \text{ or } s_{n-1}. \end{cases}$$

On the other hand, $m \cdot (2,-1) = 2a + 2b + c \geq 2b \geq 2d \geq n+2$. These calculations show that $\mathrm{lm}_{\preceq}(f) \prec m$. This is a contradiction.

If $\phi(m) = \lambda^{-d}$ for $d \geq \frac{n}{2} + 1$, then m is divisible by u^d and hence $m \cdot (2,-1) \geq 2a \geq 2d \geq n+2$. Thus we also have $\mathrm{lm}_{\preceq}(f) \prec m$, a contradiction.

Therefore $\phi(m) \in B$, so $\mathrm{supp}(\phi(f)) \subset B$.

Assume $f \in J_n$. If $\phi(f) \neq 0$, then $\mathrm{supp}(\phi(f)) \neq \emptyset$. However $\phi(f) \equiv 0 \bmod \phi(J_n)$, so there exists a non-trivial relation between elements of $\mathrm{supp}(\phi(f))$ in $\mathbb{C}[\lambda^{\pm}]/\phi(J_n)$. This contradicts to (2). $\square$

The following proposition is the first consequence of the above lemmas.

Proposition 2.15. Let $n > 0$ be an integer. Then $\mathbb{M}_n$ coincides with $\mathcal{P}_n$.

Proof. We prove the assertion by induction on n. The case $n = 1$ has already been done (Proposition 2.13).

Let $n \geq 2$. The arguments will go as follows: We will show $\mathcal{P}_n \subset \mathrm{in}_{\preceq}(J_n)$. Then we can conclude that $\langle \mathbb{M}_n \rangle = \mathrm{in}_{\preceq}(J_n)$ is also equal to $\langle \mathcal{P}_n \rangle$ by Lemma 2.12 (1). Therefore $\mathcal{P}_n \subset \mathbb{M}_n$: Indeed any monomial $x^\alpha \in \mathcal{P}_n$ is divisible by some $x^\beta \in \mathbb{M}_n$, and this x^β is also divisible by some $x^{\alpha'} \in \mathcal{P}_n$. Then Lemma 2.6 (2) implies that $x^\alpha = x^{\alpha'} = x^\beta$. On the other hand, since $\mathbb{G}_n$ is reduced, similar arguments for any $x^\alpha \in \mathbb{M}_n$ show the other inclusion. Therefore $\mathcal{P}_n = \mathbb{M}_n$.

Let us show $\mathcal{P}_n \subset \mathrm{in}_{\preceq}(J_n)$. By Lemma 2.6 (5), we have

$$\mathcal{P}_n = \begin{cases} \theta(\mathcal{P}_{n-1} \setminus \{s_{n-1}\}) \sqcup \{p_n, s_n\} & \text{if } n \text{ is odd}, \\ \theta(\mathcal{P}_{n-1} \setminus \{p_{n-1}\}) \sqcup \{p_n, s_n\} & \text{if } n \text{ is even}. \end{cases}$$

Let n be odd. To see $\theta(\mathcal{P}_{n-1} \setminus \{s_{n-1}\}) \subset \mathrm{in}_{\preceq}(J_n)$, fix any $\alpha \in \mathcal{P}_{n-1}$ and let us show that $\theta(\alpha) \in \mathrm{in}_{\preceq}(J_n)$. By the induction hypothesis, we have $\alpha \in \mathbb{M}_{n-1}$. Thus there exists $(f, \alpha) \in \mathbb{G}_{n-1}$. Now $f \in J_{n-1}$ and hence $(uv-1)f \in J_n$. Therefore $\theta(\alpha) = \mathrm{lm}_{\preceq}((uv-1)f) \in \mathrm{in}_{\preceq}(J_n)$.

To see $s_n \in \mathrm{in}_{\preceq}(J_n)$, let us remark that J_1 has $g := u^3v^4 + u - 4uv + 2$ with $\mathrm{lm}_{\preceq}(g) = (3,4)$ (Proposition 2.13). Then $g^{\frac{n+1}{2}} \in (J_1)^{\frac{n+1}{2}} = J_n$, so $s_n = \frac{n+1}{2}(3,4) = \mathrm{lm}_{\preceq}\left(g^{\frac{n+1}{2}}\right) \in \mathrm{in}_{\preceq}(J_n)$.

Furthermore $p_n \in \mathrm{in}_{\preceq}(J_n)$: Indeed, by the induction hypothesis, $p_{n-1} \in \mathbb{M}_{n-1}$. Hence there exists $(h, p_{n-1}) \in \mathbb{G}_{n-1}$. Then $(u-1)h \in J_n$ and hence $p_n = p_{n-1} + (1,0) = \mathrm{lm}_{\preceq}((u-1)h) \in \mathrm{in}_{\preceq}(J_n)$ by Lemma 2.6 (3).

Therefore $\mathcal{P}_n \subset \mathrm{in}_{\preceq}(J_n)$ for odd $n > 0$.

Let n be even. One can see that $\theta(\mathcal{P}_{n-1} \setminus \{p_{n-1}\}) \subset \mathrm{in}_{\preceq}(J_n)$ by arguments similar to the above. Moreover $s_n \in \mathrm{in}_{\preceq}(J_n)$: Indeed, by the induction hypothesis, $s_{n-1} \in \mathbb{M}_{n-1}$. Hence one can find $(g, s_{n-1}) \in \mathbb{G}_{n-1}$. Then $(u^3v^4 - 1)g \in J_n$ and $s_n = s_{n-1} + (3,4) = \mathrm{lm}_{\preceq}\left((u^3v^4-1)g\right) \in \mathrm{in}_{\preceq}(J_n)$ by Lemma 2.6 (3).

Now we will show $p_n \in \mathrm{in}_{\preceq}(J_n)$. This is somewhat harder.

Lemma 2.12 (3) shows that $\mathrm{in}_{\preceq}(J_{n-1}) / \mathrm{in}_{\preceq}(J_n)$ is generated by $\mathbb{M}_{n-1} = \mathcal{P}_{n-1}$ as a vector space, and $\dim_{\mathbb{C}} \mathrm{in}_{\preceq}(J_{n-1}) / \mathrm{in}_{\preceq}(J_n) = n+1$. However $\#\mathcal{P}_{n-1} = n+2$ by Lemma 2.6 (1). Therefore there exists a non-trivial relation between monomials in $\mathcal{P}_{n-1}$ in the vector space $\mathrm{in}_{\preceq}(J_{n-1}) / \mathrm{in}_{\preceq}(J_n)$. Hence precisely one element $\alpha \in \mathcal{P}_{n-1}$ is contained in $\mathrm{in}_{\preceq}(J_n)$: Otherwise the existence of such relation contradicts to Lemma 2.11.

Let us show that this α is p_{n-1}. Assume the contrary, $\alpha \in \mathcal{P}_{n-1} \setminus \{p_{n-1}\}$. Since $\alpha \in \mathrm{in}_{\preceq}(J_n)$, there exists $f \in J_n$ such that $\mathrm{lm}_{\preceq}(f) = \alpha$. Then $\mathrm{lm}_{\preceq}(f) = \alpha \in \mathcal{P}_{n-1} \setminus \{p_{n-1}\} = \mathcal{P}_{n-1} \setminus \mathcal{P}_n$ and hence $f \in \ker \phi$ by Lemma 2.14 (3). Thus $f = (uv-1)h \in J_n$ for some $h \in S$ by Lemma 2.14 (1). Now $h \in J_{n-1}$ by Lemma 2.12 (2), so $\mathrm{lm}_{\preceq}(h) = \alpha - (1,1) \in \mathrm{in}_{\preceq}(J_{n-1})$. However this leads to a contradiction: By the induction hypothesis, there exists $\alpha' \in \mathcal{P}_{n-1}$ such that $x^{\alpha'}$ divides $x^{\alpha-(1,1)}$, i.e. $\alpha \in \alpha' + (1,1) + \sigma_{\mathbb{Z}}$. However $\alpha, \alpha' \in \mathcal{P}_{n-1}$ and this contradicts to Lemma 2.6 (2).

Thus $\alpha = p_{n-1}$ is contained in $\mathrm{in}_{\preceq}(J_n)$, i.e. $p_n = p_{n-1} \in \mathrm{in}_{\preceq}(J_n)$ by Lemma 2.6 (3). Therefore $\mathcal{P}_n \subset \mathrm{in}_{\preceq}(J_n)$. This completes the proof. □

2.3. **Non-regularity of $C_{\mathbb{G}_n}$.** Our next task is to show the non-regularity of $C_{\mathbb{G}_n}$. We have already seen that $C_{\mathbb{G}_n}$ is a 2-dimensional strongly convex cone (Lemma 1.12 (1)), so $C_{\mathbb{G}_n}$ has two rays.

Next lemma explains our strategy for determining the rays:

Lemma 2.16. Let $w \neq (0,0)$ be a lattice point of $C_{\mathbb{G}_n}$. If there exists $(g, \alpha) \in \mathbb{G}_n$ and $\beta \in \mathrm{supp}\,(g) \setminus \{\alpha\}$ satisfying that $(\alpha - \beta) \cdot w = 0$, then $\mathbb{R}_{\geq 0} w$ is a ray of $C_{\mathbb{G}_n}$.

Proof. By Definition 1.9 (2), $\gamma := \alpha - \beta$ is contained in $(C_{\mathbb{G}_n})^{\vee}$. Thus γ defines a face $\tau := \{a \in C_{\mathbb{G}_n} \mid \gamma \cdot a = 0\}$ of $C_{\mathbb{G}_n}$. Since $C_{\mathbb{G}_n} \subset \mathbb{R}^2$ is a strongly convex cone of dimension 2 and $\gamma \neq 0$, τ must be a proper face, i.e. $\tau = \{(0,0)\}$ or τ is a ray. We have $(0,0) \neq w \in \tau$ by the hypothesis, so τ is the ray $\mathbb{R}_{\geq 0} w$. □

Therefore we only have to find appropriate $w \in C_{\mathbb{G}_n}$ and $(g, \alpha) \in \mathbb{G}_n$.

Lemma 2.17. Let $f \in J_n$ satisfy $\mathrm{lc}_{\preceq}(f) = 1$. Then $(f, \mathrm{lm}_{\preceq}(f)) \in \mathbb{G}_n$ if and only if $\mathrm{lm}_{\preceq}(f) \in \mathcal{P}_n$ and $\mathrm{supp}\,(f) \setminus \{\mathrm{lm}_{\preceq}(f)\} \subset \mathcal{D}_n$.

Proof. Let $\alpha := \mathrm{lm}_{\preceq}(f)$. If $(f, \alpha) \in \mathbb{G}_n$, then $\alpha \in \mathcal{P}_n$ by Proposition 2.15. Moreover, any monomial m in $\mathrm{supp}\,(f) \setminus \{\alpha\}$ is not divided by $\mathcal{P}_n$, i.e. $m \in \mathcal{D}_n$.

Conversely, by Proposition 2.15, $\alpha \in \mathcal{P}_n$ implies that there exists $g \in J_n$ such that $(g, \alpha) \in \mathbb{G}_n$. Since $\mathrm{supp}\,(f) \setminus \{\alpha\} \subset \mathcal{D}_n$, no monomial in $\mathrm{supp}\,(f) \setminus \{\alpha\}$ is divisible by any monomial in $\mathcal{P}_n$. This implies that $\{(f, \alpha)\} \cup (\mathbb{G}_n \setminus \{(g, \alpha)\})$ is also the marked Gröbner basis of J_n with respect to $\preceq$. By the uniqueness of the reduced Gröbner basis (Theorem 1.8), we have $f = g$. □

Proposition 2.18. $L_1 := \mathbb{R}_{\geq 0}(2, -1)$ is a ray of $C_{\mathbb{G}_n}$.

Proof. Let $w := (2, -1)$. Then $C_{\mathbb{G}_n}$ contains w by Lemma 1.16.

As we have seen in Proposition 2.13, the reduced Gröbner basis of J_1 contains

$$g_1 := u^3v^4 + u - 4uv + 2$$

where $\mathrm{lm}_{\preceq}(g_1) = (3,4) = s_1$. Now let $g_n := (uv-1)^{n-1} g_1$.

For any $f, g \in S$, one can easily see that $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\,\mathrm{in}_w(g)$. Thus $\mathrm{in}_w(g_n) = (uv)^{n-1}(u^3v^4+u)$ since $\mathrm{in}_w\left((uv-1)^{n-1}\right) = (uv)^{n-1}$ and $\mathrm{in}_w(g_1) = u^3v^4+u$. Therefore we have

$$\alpha_n := (uv)^{n-1} u^3v^4,\ \beta_n := (uv)^{n-1} u \in \mathrm{supp}(g_n).$$

It is clear that $\mathrm{lm}_{\preceq}(g_n) = \mathrm{lm}_{\preceq}\left((uv-1)^{n-1}\right)\mathrm{lm}_{\preceq}(g_1) = \alpha_n = (n-1)(1,1)+s_1$. Then, by Lemma 2.6 (4), we have

$$\begin{aligned}
&\alpha_1 = s_1 \in \mathcal{P}_1,\\
&\alpha_2 = (1,1) + s_1 = r_2^1 \in \mathcal{P}_2,\\
&\alpha_n = (1,1) + \alpha_{n-1} = r_n^1 \in \mathcal{P}_n \text{ for } n \geqq 3.
\end{aligned}$$

Thus α_n is a member of $\mathcal{P}_n$.

Moreover one can show $\mathrm{supp}(g_n)\setminus\{\alpha_n\} \subset \mathcal{D}_n$ by induction on n: The case $n=1$ is easily checked. Let $n>1$. Then any $m \in \mathrm{supp}(g_n)$ can be written as $m = (uv)^d m'$ for some $m' \in \mathrm{supp}(g_{n-1})$ and $d \in \{0,1\}$. If $m' \neq \alpha_{n-1}$, then $m' \in \mathcal{D}_{n-1}$ by the induction hypothesis. Since $\mathcal{D}_{n-1} \subset \mathcal{D}_n$ by Lemma 2.10 (1) and $(1,1)+\mathcal{D}_{n-1} \subset \mathcal{D}_n$ by Lemma 2.10 (3), we have $m \in \mathcal{D}_n$. If $m' = \alpha_{n-1}$ and $d=0$, then $m = \alpha_{n-1}$. In this case, by the above equations, we have $m = s_1 \in \mathcal{D}_2$ when $n=2$ by Lemma 2.10 (5), and $m = r_{n-1}^1 \in \mathcal{P}_{n-1}\setminus\mathcal{P}_n \subset \mathcal{D}_n$ when $n \geq 3$ by Lemma 2.6 (5) and Lemma 2.10 (1).

Therefore $(g_n, \alpha_n) \in \mathbb{G}_n$ by Lemma 2.17.

Now the vector

$$\alpha_n - \beta_n = (3,4) - (1,0) = (2,4)$$

satisfies $(2,4)\cdot w = 0$. Hence L_1 is a ray of $C_{\mathbb{G}_n}$ by Lemma 2.16. $\square$

Lemma 2.19. Let $n \geq 2$ be an integer. Then $\mathbb{G}_n$ contains (g, α) with the following property: If n is odd,

$$\alpha = q_n^{\frac{n-1}{2}} \text{ and } r_{n-1}^{\frac{n-1}{2}} \in \operatorname{supp}(g).$$

If n is even,

$$\alpha = p_n \text{ and } s_{n-1} \in \operatorname{supp}(g).$$

Proof. Let n be even. Let $\phi: S \to \mathbb{C}[\lambda^{\pm}]$ be the homomorphism in Lemma 2.14, and

$$\overline{f} := \lambda^{-\left(\frac{n}{2}+1\right)} (1-\lambda)^{n+1}.$$

Then $\overline{f}$ is the image of $(-1)^{\frac{n}{2}} (u-1)^{\frac{n}{2}+1} \left(u^3v^4-1\right)^{\frac{n}{2}} \in J_n$ by ϕ, so $\overline{f} \in \phi(J_n)$.

It is clear that $\lambda^{-\left(\frac{n}{2}+1\right)}$, $\lambda^{\frac{n}{2}} \in \operatorname{supp}\left(\overline{f}\right)$ and

$$\operatorname{supp}\left(\overline{f}\right) \subset C := \left\{\lambda^{-(\frac{n}{2}+1)}, \lambda^{-\frac{n}{2}}, \ldots, \lambda^{-1}, 1, \lambda, \ldots, \lambda^{\frac{n}{2}}\right\}.$$

Now note that the monoid homomorphism Φ in Lemma 2.10 (4) can be identified with the restriction of ϕ on the monomials of S and $\mathbb{C}[\lambda^{\pm}]$. Hence, by Lemma 2.10 (4), one can see that any monomial in $C \setminus \left\{\lambda^{-(\frac{n}{2}+1)}\right\}$ has a preimage by ϕ in $\mathcal{D}_n$, and in particular, $\lambda^{\frac{n}{2}}$ has a unique preimage $s_{n-1} \in \mathcal{D}_n$. In addition, $\lambda^{-(\frac{n}{2}+1)}$ has a preimage p_n. Hence one can obtain a preimage f of $\overline{f}$ such that

$$p_n, s_{n-1} \in \operatorname{supp}(f) \text{ and } \operatorname{supp}(f) \setminus \{p_n\} \subset \mathcal{D}_n.$$

The coefficient of p_n in f is one since so is the coefficient of $\lambda^{-\left(\frac{n}{2}+1\right)}$ in $\overline{f}$.

Since $\phi(f) = \overline{f} \in \phi(J_n)$, there exists $\Delta \in \ker(\phi)$ such that $f + \Delta \in J_n$. Let $\{g_1, \ldots, g_t\}$ be the reduced Gröbner basis of J_n with respect to $\preceq$. Then, by the division algorithm ([6], Appendix A, Theorem A.1.4), Δ has the following representation:

$$\Delta = \sum_{i=1}^{t} q_i g_i + r \text{ where } \operatorname{supp}(r) \subset \mathcal{D}_n.$$

Now r also satisfies $g := f + r \in J_n$ because $g = (f + \Delta) - \sum q_i g_i$ is a difference of elements of J_n. Let us show that this g satisfies the condition in the assertion.

It is clear that $\operatorname{supp}(g) \setminus \{p_n\} \subset \mathcal{D}_n$ because $\operatorname{supp}(f) \setminus \{p_n\}$ and $\operatorname{supp}(r)$ are contained in $\mathcal{D}_n$. Moreover $p_n \in \operatorname{supp}(g)$: Indeed, $p_n \in \operatorname{supp}(f)$ and $p_n \notin \operatorname{supp}(r)$ since $\operatorname{supp}(r) \subset \mathcal{D}_n$ and $p_n \notin \mathcal{D}_n$. Thus $g = f + r$ contains p_n.

Then $\operatorname{lm}_{\preceq}(g) = p_n$ by Lemma 2.10 (6). Now $\operatorname{lc}_{\preceq}(g) = 1$ since the coefficient of p_n in f is one, and therefore we have $(g, p_n) \in \mathbb{G}_n$ by Lemma 2.17.

To see $s_{n-1} \in \operatorname{supp}(g)$, it is sufficient to check that $s_{n-1} \notin \operatorname{supp}(r)$ since $s_{n-1} \in \operatorname{supp}(f)$. Furthermore the coefficient of s_{n-1} in r coincides with the coefficient of $\phi(s_{n-1}) = \lambda^{\frac{n}{2}}$ in $\phi(r)$ because s_{n-1} is a unique element of $\operatorname{supp}(r) \subset \mathcal{D}_n$ which is sent by ϕ to $\lambda^{\frac{n}{2}}$ (Lemma 2.10 (4)). However one can see $\phi(r) = 0$ as follows: $\phi(\mathcal{D}_n)$ can be identified with $\Phi(\mathcal{D}_n)$ in Lemma 2.10 (4), and then one can see that $\Phi(\mathcal{D}_n)$ is the set B in Lemma 2.14 (2). Now $\operatorname{supp}(r) \subset \mathcal{D}_n$, so $\phi(r)$ is a linear combination of monomials in B. On the other hand, since $\Delta \in \ker \phi$ and $g_i \in J_n$, we have $\phi(r) = \phi(\Delta - \sum q_i g_i) \equiv 0 \bmod \phi(J_n)$. If $\phi(r) \neq 0$, then $\phi(r)$ gives a non-trivial linear relation between monomials in B in $\mathbb{C}[\lambda^{\pm}]/\phi(J_n)$. This contradicts to Lemma 2.14 (2), so $\phi(r) = 0$. Therefore the coefficient of s_{n-1} in r must be zero, i.e. $s_{n-1} \notin \operatorname{supp}(r)$. Thus $s_{n-1} \in \operatorname{supp}(g)$ and the assertion holds for even n.

Let $n \geq 3$ be odd. By the case of even n, there exists $(h, p_{n-1}) \in \mathbb{G}_{n-1}$ such that $s_{n-2} \in \operatorname{supp}(h)$. Let us show that $g := (uv - 1)h \in J_n$ satisfies the condition in the assertion.

It is clear that $\operatorname{lm}_{\preceq}(g) = p_{n-1} + (1,1) = q_n^{\frac{n-1}{2}} \in \mathcal{P}_n$ by Lemma 2.6 (4). Moreover $\operatorname{supp}(g) \setminus \{\operatorname{lm}_{\preceq}(g)\} \subset \mathcal{D}_n$: Indeed, $\operatorname{supp}(h) \setminus \{p_{n-1}\} \subset \mathcal{D}_{n-1} \subset \mathcal{D}_n$ by Lemma 2.10 (1), and $p_{n-1} \in \mathcal{P}_{n-1} \setminus \mathcal{P}_n \subset \mathcal{D}_n$ by Lemma 2.6 (5) and Lemma 2.10 (1). Thus $\operatorname{supp}(h) \subset \mathcal{D}_n$. Furthermore $(1,1) + (\operatorname{supp}(h) \setminus \{p_{n-1}\}) \subset (1,1) + \mathcal{D}_{n-1} \subset \mathcal{D}_n$ by Lemma 2.10 (3). Hence $\operatorname{supp}(g) \setminus \{\operatorname{lm}_{\preceq}(g)\} \subset \mathcal{D}_n$.

Therefore $\left(g, q_n^{\frac{n-1}{2}}\right) \in \mathbb{G}_n$ by Lemma 2.17.

Now $s_{n-2} \in \operatorname{supp}(h)$ and $r_{n-1}^{\frac{n-1}{2}} = s_{n-2} + (1,1)$ by Lemma 2.6 (4). Thus $r_{n-1}^{\frac{n-1}{2}} \in \operatorname{supp}(uvh)$. On the other hand, $\operatorname{supp}(h) \setminus \{p_{n-1}\} \subset \mathcal{D}_{n-1}$ by Lemma 2.17. Since $r_{n-1}^{\frac{n-1}{2}} \notin \mathcal{D}_{n-1}$, we have $r_{n-1}^{\frac{n-1}{2}} \notin \operatorname{supp}(h)$. Therefore $g = (uv-1)h$ contains $r_{n-1}^{\frac{n-1}{2}}$. $\square$

Proposition 2.20. Let $L_2 := \begin{cases} \mathbb{R}_{\geq 0}(2n-2, -n+2) & \text{if } n \text{ is odd,} \\ \mathbb{R}_{\geq 0}(2n, -n+1) & \text{if } n \text{ is even.} \end{cases}$

Then L_2 is a ray of $C_{\mathbb{G}_n}$.

Proof. Let l_n be the one in Lemma 2.10 (7):

$$l_n := \begin{cases} (2n-2, -n+2) & \text{if } n \text{ is odd,} \\ (2n, -n+1) & \text{if } n \text{ is even.} \end{cases}$$

Then $L_2 = \mathbb{R}_{\geq 0} l_n$.

First, let us show $l_n \in C_{\mathbb{G}_n}$. It is easy to check $l_n \in \sigma$, so it is sufficient to check that $l_n \cdot (\alpha - \beta) \geq 0$ for any $(g, \alpha) \in \mathbb{G}_n$ and $\beta \in \operatorname{supp}(g) \setminus \{\alpha\}$ by the definition of $C_{\mathbb{G}_n}$. In Lemma 2.10 (7), we have already seen that $\Psi_n : \sigma_{\mathbb{Z}} \ni a \mapsto l_n \cdot a \in \mathbb{R}$ satisfies

$$\max \Psi_n(\mathcal{D}_n) = \min \Psi_n(\mathcal{P}_n).$$

Therefore $l_n \cdot (\alpha - \beta) \geq 0$ since $\alpha \in \mathcal{P}_n$ and $\beta \in \mathcal{D}_n$ by Proposition 2.15. Thus $l_n \in C_{\mathbb{G}_n}$.

By Lemma 2.19, $\mathbb{G}_n$ contains (g, α) such that

$$\alpha = q_n^{\frac{n-1}{2}} \text{ and } r_{n-1}^{\frac{n-1}{2}} \in \operatorname{supp}(g) \text{ if } n \text{ is odd,}$$

$$\alpha = p_n \text{ and } s_{n-1} \in \operatorname{supp}(g) \text{ if } n \text{ is even.}$$

Now let

$$v_n := \begin{cases} q_n^{\frac{n-1}{2}} - r_{n-1}^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ p_n - s_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Then $l_n \cdot v_n = 0$ by Lemma 2.10 (7), so L_2 is a ray of $C_{\mathbb{G}_n}$ by Lemma 2.16. □

As a consequence of the above arguments, we have a complete description of $C_{\mathbb{G}_n}$: $C_{\mathbb{G}_n}$ is the 2-dimensional cone whose rays are $L_1 = \mathbb{R}_{\geq 0}(2, -1)$ and $L_2 = \mathbb{R}_{\geq 0} l_n$

where

$$l_n := \begin{cases} (2n-2, -n+2) & \text{if } n \text{ is odd,} \\ (2n, -n+1) & \text{if } n \text{ is even.} \end{cases}$$

One can easily check that l_n is the primitive ray generator of L_2. It was suggested in [6] that $\mathrm{GF}\,(J_n)$ might contain this cone.

Now we are ready to prove our main result.

Theorem 2.21. For any $n > 0$, $\overline{\mathrm{Nash}_n(X)}$ has a singular point of type A_1, and $\mathrm{Nash}_n(X)$ is also singular.

Proof. To see the non-regularity of $C_{\mathbb{G}_n}$, let N be the sublattice of $\mathbb{Z}^2$ generated by $w := (2, -1)$ and l_n. Then $N \neq \mathbb{Z}^2$ since

$$\det \begin{pmatrix} w \\ l_n \end{pmatrix} = 2.$$

Hence $C_{\mathbb{G}_n}$ is non-regular. Moreover this calculation shows that the affine toric variety associated to $C_{\mathbb{G}_n}$ is the A_1-singularity $\left(z^2 - xy = 0\right) \subset \mathbb{A}^3$. Thus $\overline{\mathrm{Nash}_n(X)}$ has a singular point of type A_1 by Theorem 1.2, so $\mathrm{Nash}_n(X)$ is also singular: Otherwise $\overline{\mathrm{Nash}_n(X)} = \mathrm{Nash}_n(X)$ and one has a contradiction. □

Acknowledgments. I would like to thank Nobuyoshi Takahashi for many valuable advice. Some computations (by Macaulay2 [12]) suggested by him gave me fundamental ideas for the proof.

References

[1] Nobile, A. (1975). Some properties of the Nash blowing-up. *Pacific J. of Math.* 60(1): 297-305.

[2] Oneto, A., Zatini, E. (1991). Remarks on Nash blowing-up. *Rend. Sem. Mat. Univ. Torino* 49(1): 71-82.

[3] Yasuda, T. (2007). Higher Nash blowups. *Compositio Math.* 143(6): 1493-1510.

[4] Yasuda, T. (2012). Universal flattening of Frobenius. *American J. of Math.* 143(2): 349-378.

[5] Hironaka, H. (1964). Resolution of singularities of an algebraic variety over a field of charactaristic zero II. *Ann. of Math.* 79(2): 205-326.

[6] Duarte, D. (2014). Nash modification on toric surfaces and higher Nash blowup on normal toric varieties, Ph.D. dissertation. Univ. of Toulouse, Toulouse, France.

[7] Ollivier, F. (1991). Canonical bases: relations with standard bases, finiteness conditions, and applications to tame automorphisms, in: Mora, T., Traverso, C. eds. *Effective Methods in Algebraic Geometry*: Birkhäuser, pp. 379-400.

[8] Sturmfels, B. (1995). *Gröbner Bases and Convex Polytopes*, University Lecture Series, Vol. 8: American Math. Soc.

[9] Duarte, D. (2014). Higher Nash blowup on normal toric varieties. *J. of Algebra* 418: 110-128.

[10] Cox, D. A., Little, J., O'Shea, D. (2005). *Using Algebraic Geometry*, Graduate Text in Math., Vol. 185, 2nd ed: Springer.

[11] Robbiano, L. (1985). Term orderings on the polynomial ring. In: Caviness B.F. eds. *EURO-CAL '85*, Lecture Notes in Computer Science, Vol. 204: Springer, pp. 513-517.

[12] Eisenbud, D., Grayson, D., Stillman, M. (2016). *Macaulay2*, Version 1.9.2: A computer algebra system. Available at http://www2.macaulay2.com/Macaulay2/.