Doctoral Dissertation

Mongolian Secondary School Teachers' Mathematical Knowledge for Teaching with a Reference to Geometry

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

MKT. Mathematical Knowledge for Teaching

SKT. Statistical Knowledge for Teaching

SMK. Subject Matter Knowledge

PCK. Pedagogical Content Knowledge

CCK. Common Content Knowledge

SCK. Specialized Content Knowledge

KCS. Knowledge of Content and Students

KCT. Knowledge of Content and Teaching

KCC. Knowledge of Content and Curriculum

CICD. Concept Image and Concept Definition

MKTCI. Mathematical Knowledge for Teaching Concept Image

MKTCD. Mathematical Knowledge for Teaching Concept Definition

CCA. Canonical Correlation Analysis

MSUE. Mongolian State University of Education

NUM. National University of Mongolia

KHU. Khovd University

MECS. Ministry of Education, Culture and Science

MPIA. Mongolian Professional Inspection Authority

EEC. Education Evaluation Center

IRD. "Ireedui" secondary school

MonGeni. "Mongene" secondary schools

ABSTRACT

This research aims to identify Mongolian secondary school teachers' mathematical knowledge for teaching with a particular reference to geometry, and to reveal how teacher beliefs and context where teachers are trained and work is likely to situate and contribute to teachers’ MKT. To achieve the aims, the following questions are set:

1. What is Mongolian secondary school teachers’ MKT geometry?
2. How do these teachers believe about the nature of geometry and its teaching and learning?
3. How is teacher beliefs associated with their MKT geometry?
4. How does school context situate secondary school teacher MKT geometry?
5. How is pre-service teacher education context in Mongolia likely to contribute to teachers’ MKT geometry?

In order to answer the questions, an extensive literature including Mongolian secondary education geometry curriculum and mathematics teacher review was carried targeting teachers’ MKT, belief and context. Review of the secondary school geometry curriculum identified the focus of content of geometry; such as the plane shape and symmetry concepts. Based on the literature review and findings from the curriculum, a conceptual framework of the research is developed consisting of several components. First of all, the framework is built on Ball et al's (2008**)** MKT model, nevertheless, it does not include HCK because KCC deals with teachers’ knowing of how a particular mathematics curriculum content are taught preceding years and will be taught later years. By Hill, Ball and Shilling (2008), the framework focuses on the plane shape and symmetry concepts applying CICD theory (Tall & Vinner, 1986). A rationale why applying this theory is that there is a tension between the concept image and concept definition in the shape; and this is a cognitive theory that can be applied in learning concept image and definition. Teachers’ belief is not distinguishable from teacher MKT as it affects teacher choices of instructional tasks, representations, approaches

in geometry teaching. For teacher beliefs in this research, a particular attention is to teachers' belief about nature of school geometry. The belief in this research adapts Beswick (2011) conceptualization that teachers' beliefs about the nature of school mathematics may be separated from beliefs about the nature of geometry as a science discipline. It is important to know what kind of mathematical knowledge teachers hold and why that knowledge is important in the particular countries in which they teach (Stylianides and Delaney, 2011). They noted that teachers' MKT is shaped in in the context of teacher education program. Thus, this research considers context in teacher education comprises from pre-service teacher education as it establishes fundamental knowledge for teacher MKT, and secondary school context which is represented by situated aspect of teacher MKT. At the *school context*, an aspect of situated-ness is investigated through teachers' individual and collaborative reflections.

Based on the conceptual framework, instruments are developed for teachers. The questionnaire for teachers' MKT has 2 category of items such as MKTCI and MKTCD referencing the plane shapes. Teacher belief questionnaire consists of 2 parts such as teachers' belief about the nature of school and belief about discipline geometry (Beswick, 2011). Teachers' belief about the learning geometry reflects characteristics of three different views of belief about the learning geometry – content focused with performance, content-focused with understanding, and learner-focused. As for school context, teacher individual and collaborative reflections based on Turner (2008) are investigated focusing on certain aspects of geometry teaching and learning.

Using the instruments, data was collected during December 2014 when geometry is widely taught to secondary school students. The instruments are administered to 57 secondary school mathematics teachers who work in Ulaanbaatar, a capital city, and Khovd aimag, a rural area, of Mongolia.

Data was analyzed using a combination of quantitative and qualitative methods. The analysis revealed several results for teachers' MKTCI and MKTCD, beliefs, and context. In Mongolia, teachers' MKT can be characterized by limited KCT and KCC, and some inconsistencies between the concept image and formal definitions for the shapes in terms of CCK, SCK and KCS. These teachers hold Platonist view of belief about the nature of geometry, and learner-focused view of belief about the geometry learning. A certain views of teachers' belief are negatively associated with SCK and KCS sub-domains of teachers' MKT. Teachers who hold stronger Problem solving and Platonist view of belief possess less SCK. Teachers' MKT is likely to be positively situated in school context through teachers' individual and collaborative reflections, however, how and what to reflect is a matter. Pre-service teacher training is likely to contribute school teachers' KCTCD developing students with better CCKCD and enabling them to develop KCSCD, KCTCD and KCCCD.

Based on the results and findings, recommendations for the improvement of teachers MKT geometry emphasizing the concept image of the plane shapes, and pre-service teachers training and schools context can be provided; as a result, secondary school students' achievement in geometry can be potentially increased.

CHAPTER ONE. BACKGROUND AND PROBLEM STATEMENT

1.1 Research Background

Fundamental to the achievement of EFA, is the concept of quality. Even if all children get into school by 2015 what is really important in terms of long term poverty reduction and the enhancement of the quality of their lives is that: (a) they manage to stay in school and complete the education cycle and (b) that they receive a quality education experience which is sufficient to enable them to become independent lifelong learners as a result of having been in school. One of the overarching goals for education strategy for after 2015 is set as teachers in 6th goal (UNESCO, 2014). By this goal, "*all governments ensure that all learners are taught by qualified, professionally trained, motivated and well-supported teachers as of 2030*" (UNESCO, 2014). In the implementation of this strategic goal, focus is on empowering teacher professional development and teaching quality (UNESCO, 2014). Even though there are some of the 'tools for conviviality', teachers remain central to the achievement of a quality education (Chris, 2007, p.2). For example, an analysis of TIMSS 2011 results for grade 4 from 45 countries found that, across the countries, the better the teacher quality, the less the incidence of low achievement (UNESCO, 2014). Thus, the achievement of quality education for all calls for more and better trained teachers as subject knowledge and pedagogical processes lie at the heart of quality education (UNESCO, 2012). For instance, the quality of school mathematics teaching depends critically on the subject-related knowledge that teachers are able to bring to bear on their work of teaching (Rowland & Ruthwen, 2011). A certain lack of teachers' knowledge of mathematics is associated with less successful teaching and lower student attainment in the subject (Askew, 2008, p.18). Policymakers can change textbooks, they can change tests, and they can recommend classroom activities or approaches, but changing mathematics teaching must involve teachers in more fundamental ways (Tutak, 2009). Therefore, improving student outcomes is also about improving the quality of the teaching workforce. Teacher quality is an important factor in determining gains in student achievement, even after accounting for prior student learning and family background characteristics (Darling-Hammond, 2000; Hanushek, Kain and Rivkin, 1998; Muñoz, Prather and Stronge, 2011; Wright, Horn and Sanders, 1997).

In the context of Mongolia, along with the goal, the government policy for general school education during 2014-2024 emphasizes that "teacher professional competence that is defined as subject and pedagogical knowledge for teaching, are crucial for quality education; and it must be strengthened". Nevertheless, as it is reported by World Bank (2014), several key challenges in education of Mongolia are identified; and one of them is the quality of teaching in secondary schools of the country. In particular, secondary school mathematics in Mongolia has been facing a problem; and its cause is critically judged by lack of teachers' mathematics content and teaching knowledge.

Results of international and national evaluations support the above conclusion by providing evidences on students' low achievement. Trends in International Mathematics and Science Study (TIMSS, 2007) reported results of Mongolian grade 4 and 8 students' achievement in mathematics subject (Table 1).

Table 1. Mathematics Achievement TIMSS 2007

Grade	Mean	Content Domain				Int'l average
		Number	Algebra	Geometry	Data chance	
Grade 4	436	463	NA	390	424	500
Grade 8	432	447	435	413	418	500

Note. Adapted from TIMSS Report, 2007

By Table 1, Mongolian students' performance in mathematics is lower than international average, specifically; their performance in geometry is the lowest. By TIMSS (2007, p.29), geometry items for grade 8 deals with the properties and characteristics of a variety of plane shapes and transformations as well as measurement, location and movement of the shapes. It requires students to recall and use attributes of the shapes such as triangles, quadrilaterals, and other common polygons and geometrical figures, including lengths of sides and sizes of angles, and to provide explanations based on geometric relationships.

It is wise to consider national assessment to verify the result of the TIMSS. In April 2013, Mongolian Professional Inspection Authority (MPIA) evaluated school students' achievement in several subjects; and lowest performance (in average, 47% performance) subject is estimated as geometry again. In the assessment, the plane shape-problems are widely used. After the evaluation of MPIA, Education Evaluation Center (EEC) of Mongolia (2014) investigated the problem taking into consideration of possible factors those affect student' low achievement in geometry. This investigation concludes that teacher quality is the most influential factor. In the investigation, teacher quality is operationally defined as attributes related to geometry content and pedagogical knowledge, and teaching.

The results of TIMSS, MPIA and EEC investigation raised a debate on the quality of school mathematics education among politicians, policy makers, educators, teachers and parents; and one of the causes for the low achievement is judged as teachers' poor knowledge of subjects they teach (Javzmaa, 2009). In Mongolia, it is a well known fact that school teachers greatly rely on textbooks, in particular to geometry, they struggles to teach this subject because they face difficulties to understand the subject matters of geometry, solve common geometry problems, to represent essential ideas in geometry content, and to promote students' understanding; and as a consequence, it affects the quality of teaching and learning geometry.

Indeed, geometry is the foundation of mathematics as we know it today; it was developed to explain phenomena and solve problems that bore directly on daily life (NCTM, 2006). It acts as a bridge between events in daily life and mathematical concepts, geometry has a crucial importance for mathematics learning. It is used for solving problems associated with other branches of mathematics besides its usage for solving problems about daily life and utilization in other disciplines such as art for different purposes (Biber et al, 2013). In particular, geometry develops students' spatial thinking that is strongly related students' well performance in mathematics and other subjects (NCTM, 2006). Spatial thinking plays an important role in learning of arithmetic, word problems, measurement, algebra and calculus (NCTM, 2006).

Although the above problem and situation are faced to secondary mathematics in Mongolia; and benefit of having teachers with quality professional knowledge is highly appreciated by scholars, educators and practicioners, and teaching-related initiatives to reform teacher education in Mongolia is prioritized; there is a limited number of the quality, substantive studies on teachers that have been conducted in Mongolia (Gita, 2012, p.3). There are few studies in this field, yet, mainly did not apply scientifically research methodologies and latest perspectives and theories in education research. In order to reveal evidence on how much content related knowledge Mongolian secondary school teachers have acquired, as well as to understand how this knowledge is accumulated through teacher development panel, to identify fundamental issues for the teacher related reform, and to recommend potential treatments to improve the quality of teaching, a study on teacher subject specific knowledge is urgently needed for further improvement of the quality mathematics education of Mongolia.

1.2 Problem statement

The substantial efforts to trace the effects of teacher knowledge on student learning, and the problem of what constitutes important knowledge for teaching (Ball, 2002) leads to discuss about what it means to know content for teaching. A significant contribution to the discussion on mathematics content knowledge that is needed for teaching is made by Ball and her colleagues (2008) conception of the mathematical knowledge for teaching (MKT). MKT is defined as the particular form of mathematical knowledge which is useful for and usable in the work that teachers do as they teach mathematics to their students (Ball & Bass, 2000). It has 2 main components – subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK consists of 3 sub-domains: *common content knowledge, specialized content knowledge, and horizon content knowledge*. PCK has 3 sub-domains namely *knowledge of content.*

This model is recognized as the strongest conceptualization of mathematical knowledge for teaching that is defined as the particular form of mathematical knowledge which is useful for and usable in, the work that teachers do as they teach mathematics to their students (Ball & Bass, 2000; Adler & Davis, 2006).

Although it is influential, it misses some of the important aspects that are likely to affect teachers' knowledge for teaching. By Goulding et al (2002) teachers' *beliefs* about the nature of mathematics may be tied up with subject matter knowledge in the way in which teachers approach mathematical situations. If teachers believe that mathematics is principally a subject of rules and routines which have to be remembered, then their own approach to unfamiliar problems will be constrained, and this may impact on their teaching (Petrou & Goulding, 2011). Barkatas and Malone (2005) indicated that mathematics teachers' belief about mathematics could not be separated from their belief about teaching and learning mathematics.

Moreover, teacher knowledge is shaped by the *context* of national education systems, particular types of school and teacher education institutions (Ruthwen & Rowland, 2011). Putnam and Borko (2000, p.3) argue that this professional knowledge is stored together with charactersitic features of classrooms, and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations. Williams (2011) noted that teacher knowledge is distributed; rather than being held "in the head" of any individual teacher, such knowledge is held by teachers collectively – in a school or in the profession (Stylianides & Delaney, 2011, p.186). It is evidenced in Cobb, Yackle and Wood (1991) work that sees knowledge learning is happening not in isolated classrooms but within a teacher professional community within a particular school and teacher training institution.

1.3 Research aims and questions

Based on the above discussion, aims of this research are (1) to identify Mongolian secondary school teachers' mathematical knowledge for teaching with a particular reference to geometry, (2) how teachers' beliefs and context, where teachers are trained and work, are likely to shape teachers' MKT, and (3) to provide recommendations to the teachers to improve their mathematical knowledge for teaching geometry.

The aim of the research leads to investigate and answer to the following questions:

1. What is Mongolian secondary school teachers' MKT geometry?
2. How do these teachers believe about the nature of geometry and its teaching and learning?
3. How are teacher beliefs associated with their MKT geometry?
4. How does school context situate secondary school teacher MKT geometry?
5. How is pre-service teacher education context in Mongolia likely to contribute to teachers' MKT geometry?

Answers to these questions will provide some recommendations for improving the quality of secondary mathematics teacher' MKT, furthermore, secondary school geometry teaching in Mongolia.

1.4 Research Significance

At *educational practice* level, first of all, it will help to understand current status of Mongolian prospective and practicing teachers' knowledge that is needed for effective teaching. More general, the research will provide some significant implications to improve practicing and prospective teachers' MKT, specifically, in geometry. The implications will make a contribution to reform pre-service education curricula, especially professional didactics courses, teaching practices, as well as curricula for in-service teacher training. It is reasonable to use the implications to set out criteria for teaching certification exams and practicing teacher evaluation. Moreover, the implications can provide important supports to professional developments of individual teachers; improve the quality of teachers, ultimately, quality of teaching and learning of mathematics in secondary schools.

As for *scholar level*, it will make contribution to conceptualization of teacher knowledge, particularly, MKT by taking into consideration of how the MKT can be "situated in a particular context" (Pepin, 2011) which is different from where it is developed. In addition, it contributes to an effort to understand distinctive belief about the nature of schools and discipline geometry. Moreover, it will help to understand and extend how a particular context in which teaching and learning take place (Petrou & Goulding, 2011), teacher beliefs and mathematics curriculum (Andrews, 2011) play role in teacher subject specific knowledge for teaching. It will also make a contribution to understand how professional community within a school plays role in the teacher mathematical knowledge for teaching.

1.5 Limitations of the research

The research limits the teacher training context as pre-service teacher training, and does not consider the in-service teacher training in Mongolia. Rationale of this limitation can be interpreted as that during the beginning of 1990s to 2013, there was not nation-wide in-service training system for school teachers. During this period, all short-term in-service trainings were delivered by projects and programmes funded by donor organizations in education sector. Therefore, content of these trainings are very varied; and impossible to collect all the training curricula and investigate how these training influenced on school teachers' knowledge for teaching.

Moreover, another limitation of the research is related to the sample teachers those mainly selected from secondary schools in Ulaanbaatar, a capital city of Mongolia. Few percentage of teachers are selected from Khovd aimag, western province of Mongolia. It indicates that the obtained result and conclusions may not represent a picture of the whole population of secondary school mathematics teachers of Mongolia.

CHAPTER TWO. SCHOOL MATHEMATICS EDUCATION IN MONGOLIA

2.1 Secondary School Mathematics Teachers

At present, there are two generations of mathematics teachers in Mongolian secondary schools – trained before and after 1990. In 1990, Mongolian education system is fundamentally shifted from socialism to democratoc ideology; thus, teachers have different beliefs and practice.

Before 1990s, for pedagogy, teacher pre-service curriculum content heavily depended upon education theories and perspectives developed by Soviet scholars. These references promoted various ideas of geometry teaching and learning such as ways of representations of the subject matters, and why a particular representation works in the situation (Markushevich, 1982), exercise sequence (Gusev et al., 1979), difficult problems whose solution calls for non-standard approaches (Gusev et al., 1979). However, the most influential theory in education was the theory of Vygotsky. It is seen from the literatures, more instrumentalist view about mathematics and about teaching and learning were observed for teachers in that time.

After 1990, more student-centered teaching is promoted; and one of the fundamental changes in mathematics teaching is recognized as constructivist perspective. Secondary school pre-service training curriculum introduced ideas of more child centered, participatory approach, diversity of students' needs; particularly, curriculum development has changed from a theoretical toward a pragmatic approach (Johnny, 2011). Teachers are expected to shift their belief into more pragmatic view, and apply more child-centered, constructive teaching.

In Mongolia, schools have traditionally customed professional activities; and one of the most common activity is a methodology unit assembly. All mathematics teachers at a secondary school belong to subject-specific methodology units; and unit assembly is regulated in the school policy. Teachers are obliged to attend in the unit assembly. Main functions of the unit are to develop mathematics syllabi (mainly outline of mathematics topics that should be annually taught to all grades), discuss about mathematics teaching and learning, prepare for professional (teaching and subject Olympiads) competitions and develop teachers' knowledge and skills for teaching mathematics. However, depending upon the school context, the unit activities could be varied. Recently, the Japanese lesson plan has taken place in some urban schools.

Teacher pre-service training institutions

There are 3 leading institutions for pre-service teacher training for secondary school mathematics teachers in Mongolia.

Mongolia State University of Education (MSUE) is the largest teacher pre-service training institution specialized in purely education. MSUE is the only institution that trains teachers for all levels of education: pre-school, primary, secondary schools and higher education.

Second largest and oldest is National University of Mongolia (NUM) which is traditionally more academic-oriented institution. It trains not only secondary school teachers but also scientists, engineers, economist, etc. Third is Khovd University (KHU), one of the higher education institutions in rural area, located in western part of Mongolia. Main function of this institution is to train secondary school teachers. More than 90% of teachers in Mongolian secondary schools are trained in these 3 institutions.

By the Education Law (2016), all primary and secondary schools have to employ teachers with professional background and pedagogical training. Secondary school teachers should hold a bachelor‘s degree from the teacher education institutions and earn the teaching certificate from Institute for Teacher Professional Development (ITPD). Teaching job is accounted as a profession in the country.

As for entry requirement for teacher training institution, the most important requirement is a good result of a mathematics test in the higher education entrance examination. It is a high-stake examination for the school graduates. There are several options for the mathematics test; however, those who consider to pursue teacher education take a core mathematics test. Although the requirement emphasizes high score in the core test, most high achievers in this test pursue non-teaching professions in higher education. Teacher profession has been second or third choice of the graduates for last 3 decades. Fortunately, since 2014, thanks to the government scholarship system for Teacher University, many top-achieving students in mathematics have been attracted to study in teacher training institutions.

In-service training for secondary mathematics teachers

ITPD is a responsible institution to train in-service mathematics teachers, and established in 2012. The institute delivers 2 types of in-service trainings – mandatory for every 5 years and optional for every year. The mandatory in-service training has different purposes focusing on target teachers. For example, as for 2016, in general, the compulsory in-service training for mathematics teachers with 5-year teaching experience aimed to develop teachers' methodology and skills on professional theories and supporting children learning, to apply information technology, and to acquire ability to collaboratively learn at workplace within the scope of the national curriculum implementation.

10-year teachers' training in 2016 aimed to enable teachers to share their experiences of teaching, to learn latest trends in professional theories and teaching methodology, independently develop themselves using information technology and to improve their research and consulting skills. In particular, as for sharing the experience and collaborative learning, Japanese lesson study has been taking place in some schools.

2.2 School mathematics curriculum: Geometry focus

Schooling in Mongolia consists of 12(5+4+3) - year system. Primary grades are 1 to 5, secondary 6 to 9, and high school is 10 to 12. When talk about secondary school, it is about grades from 6 to 9. As common sense, primary, secondary and high school have respective national curricula of mathematics. School mathematics curriculum is revised during 2012-2014; and being implemented since 2013.

In national mathematics curriculum, geometry is one of the content domains. The below, each level of school geometry is described in content, as well as some implications for geometry teaching and learning.

Primary school geometry

National curriculum for primary mathematics aims to enable students to acquire basic skills in mathematics, develop their language and thinking and to be motivated in mathematics learning. This curriculum content is divided into 2 sections based on students' ability, learning skill, and their physical and cognitive development. First section is content for grades 1 - 3, second is for grades 4 - 5.

Table 2. Primary school geometry content

Grades 1 -3	Grade 4-5
I. Understanding of measurement	
Length, weight, volume, time units, operation on these units, calendar, thermometer, conversion, area of square and rectangle by counting unit squares	Length, weight, time, angle, temperature units, area and volume of the shapes, time zone, thermometer, angle measurement, perimeter and area of the shapes, volume and surface area of cube, regular parallelepiped
II. Investigation of shapes	
Object concepts: Rectangle, square, triangle, line, angles, cube, regular parallelepiped Relation concepts: Perpendicular, parallel-ness, basic symmetry through folding the shapes Activity: Observing, measuring, folding, composing, simple construction	Object concept: angle, regular and non-regular polygon, circle, round shapes, sphere, prism, cylinder, pyramid, conus Relation concept: Perpendicularity, parallel-ness, basic symmetry through folding the shapes, Activity: Observing, measuring, folding, composing, simple construction

By Table 2, at grades 1-3, students acquire the understanding of the basic shapes through observing, measuring, folding shapes of objects. Some basic ideas of symmetry are introduced when students fold papers to create symmetrical shapes; and it focuses on developing students' intuitive images of properties of the shapes using the surrounding objects. By grades 1-3 textbooks, most geometrical shape related problems are mainly about measuring, drawing shapes of the objects, calculating their perimeter, area and surface area of the shapes. Few simple construction problems on perpendicular and parallel lines are included.

By Table 2, at grades 4-5, students deepen their understanding about the shapes and their attributes using positional relationships through symmetrical, parallel and perpendicular properties of the objects. Symmetry is not formally defined here. This is learnt through observing, measuring, calculating, composing and constructing basic shapes; and it enables students to understand essential attributes and components of the shapes. However, they do not learn relationships between the attributes of shapes. At these grades, students learn some basic ideas about the classifying the shapes as regular and irregular polygons, congruence shapes using their attributes. In grades 4-5 textbooks, problems of the shapes are mainly about this calculation of perimeters, areas and inner angles of the shapes.

Secondary school geometry

Aim of the secondary mathematics is to develop students' skills to work with numbers, shapes, measurement, and data, apply these skills to solve problems faced in their life and to be motivated in learning mathematics and developing creative thinking.

In the curriculum, the following implications are recommended to teachers for geometry teaching and learning in secondary schools:

- Encourage students' activities such as drawing, measuring, identifying, constructing, recognizing, exploring, reflecting, reasoning, representing,
- Emphasize students' active participation and activities,
- Consider students' prior knowledge, abilities, and identify their difficulties and common misconceptions occurred during the learning,
- Focus more on developing students' understanding of essential properties of concepts at grades 6 to 7 through simple geometrical constructions, and gradually, formal definition of the concepts at grades 8 to 9,
- Selecting sound representations, tasks, and activities in the teaching,

- Develop students' team work and independent learning skills,

Geometry content for secondary school is summarized in the following Table 3.

Table 3. Secondary school geometry content

Grade 6	Grade 7	Grade 8	Grade 9
I. Object concept			
Angles, triangle, quadrilateral, polygon, cube, prism	Angle, quadrilaterals, triangles, circle, cube, prism, parallelepiped, trapezoid, parallelogram	Angles, regular polygon, triangle, circle, prism, cylinder, pyramid, conuses, parallelepiped,	Triangles, regular polygons, circle, prism, cylinder, pyramid, conuses, parallelepiped, vector,
	Informal definitions for the concepts	Formal definitions, its structure, application of the definitions, Pythagorean and other theorems, proof	
II. Relation concept			
Parallel, perpendicular symmetry, rotation, translation, geometrical construction Classification of the shapes,	Parallel, perpendicularity, symmetry, sequential transformation in translation, rotation, and symmetry, coordinate system, geometrical construction Classifications of triangles,	Congruence, symmetry, rotation, translation and homothetic, invariant properties, coordinate system, geometrical construction	Similarity, congruence symmetry, intersections of figures, motions, transformations, coordinate system, Positional relationship between circles, and circle, trigonometry relationship
III. Measurement			
Length, mass, time, measurement tools, units, conversion of units, area, surface area, volume, relationship between one and two dimensional units of measurement	Measurement units in length, mass, area and volume, traditional measurement units in Mongolia, area, perimeter of triangle, trapezoid and parallelogram**,** length of a circle, area of round shape, relationship between one, two and three dimensional shape measurement units	Speed and complex unit in daily life, conversion of speed units in relation to physics, area of ***simple plane shape,*** length of a circle, edge length of prism and height, surface area and volume of cylinder	Time zones, area and perimeter of triangle, parallelogram, trapezoid, regular polygon, and compounded ***plane shapes,*** circle length, area of round shape, sector area, length of arc, volume of prism, cylinder, pyramid, conuses, area of intersections in figures
IV. Activity			
Drawing, observing, measuring, constructing, recognizing, exploring, reflecting, reasoning, representing			

From Table 3, the plane shape is the main concept in secondary geometry. Relation concepts such as symmetry, motions, congruence and similarity are used to deepen their understanding about the shapes, identify the properties and verify the properties of the shapes through observing, drawing, measuring, constructing, recognizing, exploring, reflecting and reasoning the concepts. Coordinate system is also introduced to develop students understanding about the positional relationships between the shapes. The same as primary grades, measurement is also to be learned.

At grades 6 and 7, the triangle is a main object concept in geometry. Students are expected to perceive properties of the shapes and identify interrelationships among the properties using the symmetry and motions in geometry. Students learn this by constructing and exploring relationships among the shapes. This learning enables students to develop evoked images of the shapes and their properties; and to classify the shapes based on the properties they have. In addition, one of the significant features of geometry at these grades is informal definitions for the concepts. At these grades, students are not specifically introduced the necessary and sufficient conditions of the definitions, but, it includes a list of attributes or properties of the defined concepts. For example, definition (grade 7 textbook, p. 134) for the symmetry is as follow:

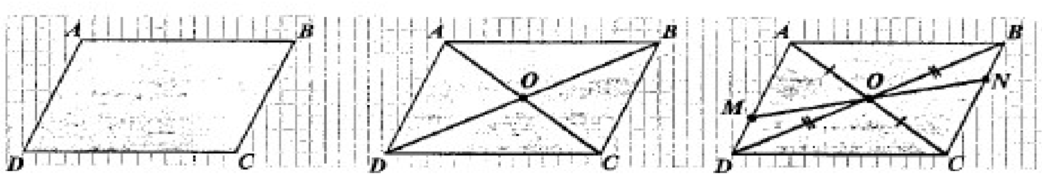
> *If line "a" crosses through the midpoint of AB segment, and this line is perpendicular with the segment, points A and B are symmetrical along the line "a". Line "a" is a mirror line of the symmetry.*

Furthermore, one of the essential points at this grade is that a way of developing geometry concepts through geometrical constructions is an idea of practical exercises. Practical exercises enables students to understand properties of the shapes; then, gradually, to develop property-based informal definitions_for the shapes. It encourages defining processes. Displaying informal definitions in the textbook may presents that discussion on informal definitions posed by students is expected to take place at a classroom, and compare it with the textbook definitions. Moreover, ordinary tasks play a significant role for formation of the plane shape understanding at grade 7. These tasks usually focus on constructing the shapes, generating hypothesis based on the explored properties, verifying the hypothesis and making conclusions. At this level, analogy and inductive reasoning is more applied in geometry. However, the curriculum does not clearly represents why geometrical construction, symmetry and motions as well as parallel and perpendicular are intended in the curriculum along with the shape concept.

Geometry content in textbook

Grade 7 textbook content shows that there are four categories of assignments, namely, practical works, examples and ordinary tasks. Practical works are usually at the beginning of the concept; it lists a set of activities like placing points, drawing and measuring shapes that leads students to explore properties of the concept or generate hypothesis and draw conclusions. It sometimes lists steps of activities for geometrical constructions. This is especially significant to understand the properties of the shapes for defining the concepts. For example, grade 7, under the point symmetry, the following practical work is placed; and it is followed by the definition of the point symmetry:

Figure 1. Practical work: Constructing the point symmetrical shapes



1. Draw a parallelogram and label as A, B, C, and D letters.

2. Link A to C, and B to D as line segments and place O letter on the intersected point.

3. Construct symmetrical points of the parallelogram points along O point.

As for this practical work, last three questions reflect curriculum objective that is "to represent the properties in mathematical words", so, this work is followed by the definition for the point symmetry. It encourages students to understand properties of the point symmetry; and it is assumed that based on these properties, they are expected to figure out the definitions for the point symmetry. In order word, practical work guides students to defining the concepts and understanding the properties; and it may be utilize an inductive method.

Examples show students how to solve a particular problem. For example, in order to guide students how to calculate length of a line segment, an example problem and how to solve the problem are described. It also represents how to understand problem settings in geometry.

The most common type of task in the textbook is ordinary geometry problems with assignment to construct the shapes, verify the properties, calculate the area, length or volumes, find out the symmetry point or mirror line; and a few number practical problems like to estimate how many meter of cotton is needed to cover a rectangle-shaped box, etc.

Table 4. Numbers of practical works and tasks in grade 7 textbook

	Basic concepts	Plane shapes		Coordinate system	Solid shapes	Total
		Triangle	Quadrilaterals			
1. Practical work	[3]+{3}	[3]	[1]	1+[1]	(1)	13 (1); [8]; {3}
2. Example			{1}+[15]	6	3	25 [15]; {1}
3. Ordinary task	[17]+{10}	2+[6]+{1}	[5]+{1}+1	14	16	73 [28]; {12}
Total	33 [20];{13}	12 [9]; {1}	24 [21];{2}	22 [1]	20 (1)	111 (1); [51]; {16}

Note: "**[]**" indicates problems in symmetry concept; "**{}**" shows problems in parallel and perpendicular line; "**()**" represents problems in geometrical constructions;

In Table 4, the "example" represents kinds of introductions and explanations of how to solve geometrical problems. "Ordinary tasks" represents ordinary geometrical problems to be solved referencing the examples. For example, calculate a sum of inner angles of polygon, add up the given angles, etc. Table 4 presents that most of tasks belongs to "others" which includes problems in basic concepts like points, lines, segments, angles, and perpendicular and parallel lines. However, the number figure indicates most problems in this category relate to the symmetry. Moreover, it is seen that there are more ordinary tasks on solid figures, however, if we see the nature of these tasks, most of them are about volume and surface areas of the figures; and all the calculation relies on areas of quadrilaterals. Thus, it is reasonable to deduce that the shape concept is more emphasized in grade 7 geometry. The symmetry seems the most used geometrical approach to explore relationships between two shapes while working on ordinary tasks.

At grades 8 and 9, the shapes, in particular, triangle is a main concept; and understanding of this concept and its properties through symmetry, motions, geometrical constructions, congruence is an essential subject of geometry. As for properties of the concept practical works are reduced, instead, formal definitions take more places. Understanding of properties is developed through practical work at grade 7 is now shifted to the understanding through meaning of formal definitions. That is why there are more definitions in the textbook. Geometrical thinking at this grade is grounded at more deductive reasoning. Abstract definitions for the concept can be comprehended. The proofs are given on the definitions using the properties of relation concepts. By Dalaijamts et al (2010), developers of grade 8 mathematics curriculum, "features of geometry content are considered as students will encounter the fundamental definitions, axioms and theorems (first time), a concept of the triangle, its similarity and congruence that is foundation to solve geometry problems, proof in plane geometrical shapes' properties, and initial understanding of proof by contradiction. Grade 9 geometry content focuses on the understanding of quadrilaterals, their properties and partition classifications of the shapes through formal definitions. As more theorems including Pythagorean Theorem are emphasized; verification of the properties and proofs take place in the learning. At grade 7, symmetry is introduced through the construction, however, at this grade, congruence, symmetry, similarity, and motions are defined formally and mathematically. The necessary and sufficient conditions are presented in the definitions. Grade 8 textbook presents the emphasis of the curriculum clearly.

Table 5. Numbers of practical works and tasks in grade 8 textbook

	Others*	Triangle	Circle &:		Total
			line	triangle	
1. Practical work	2+{1}	4+<8>	1	1	13 <4>; {1}

2. Definitions	16	10	2	2	30
3. Example	6+{1}	12+<6>			25 <6>; {1}
4. Theorem/proof	4+{3}	8+<6>			21 <6>; {3}
5. Ordinary task	66+(14)	68+<49>+(15)	(1)		216 <49>; (30)
Total	113 {5}; (14)	186 <69>; (15)	4 (1)	3	309 <69>; {5}; (30)

Note: "[]" indicates problems in symmetry concept; "{}" shows problems in parallel and perpendicular; "()" represents problems in geometrical constructions; "<>" shows congruence and similarity; "*" cell includes problems in lines, segments, and angle;

Table 5 shows that most problems are ordinary tasks; and they are related to triangle concept; and within these problems, most of them are about the congruence. Thus, it is reasonable to think out that the essential concept as this grade geometry is the triangle and congruence and other motions to understand the relationships between 2 different shapes. At grade 8, the definitions for triangle concept is formally introduced, so that, understanding the meaning of the definition is promoted through practical works. It may imply that inductive method is highly emphasized for defining the concepts. Practical works enable to explore properties of the triangle; and this is an essential for developing conceptual image of the concept. In other word, at this grade, students shift from the perceptual image to conceptual image of a triangle, furthermore arrives at formal definitions. In addition, grade 8 students encounter many theorems and its proofs, while most theorems are related to congruence of triangles. Formal proofs of the theorems apply the definitions.

The ordinary tasks reinforce formation of the concept; yet, tasks focus more on application in abstract geometry. One of the emphases for these tasks is that problem setting is differentiated as "*givens*", "*conditions for...*" and "*what should be calculated or found*". It may relate to van Hiele level 4.

Results of the curriculum analysis

Geometry is one of the domains in school mathematics curriculum (SSMC, 2009) from primary through grade 12 in Mongolia. At primary level (grades 1-5), by the curriculum, a main concept of geometry is intended to be the basic shapes developing students' mental images of the shapes using the surrounding objects. By the geometry curriculum, students at lower primary grades recognize shapes according to how they look like using visual prototypes in surrounding environment such as doors and desks. Their thinking of the shapes is characterized at visual level via standard examples. At upper primary grades, it is expected that the level of thinking involves aspects of the descriptive level, so, students deal with some properties of the shapes by observing, measuring and drawing. However, by the curriculum and textbooks, it can be seen that students still work with the prototype, standard orientation examples, but, no opportunity to deal with different shapes, for example in non-standard orientations. At the end of the primary school, students' all mental images of attributes of the shapes are characterized by these prototype examples.

By the curriculum analysis, teachers are expected to know about the above interpretation. They need to understand why certain concepts and topics are embedded in the curriculum, how the relation concept such as symmetry is included along with the shapes, what properties can be extracted from the shapes if the symmetry is utilized, why properties of the shapes need to be exposed to students and how it supports students' mental images of the shapes. In particular, teachers need to understand how to represent the shape concepts to develop students' mental images of the shapes.

At secondary level (grades 6-9), an essential concepts are the shapes and along with symmetry that helps students to classify objects and shapes according to the properties; and explore more properties and relationships among the shapes. As it is cited in Melih (2014), the concept of symmetry plays a key role in the comprehension of reflection, rotation, translation, and reflective translation, which are subjects that are included in the transformation geometry. Without proper understanding about the symmetry, reflection, rotation, translation as well as congruence and similarity are likely to be challenge for students. In addition, two of the significant feature of this geometry is the classification of the shapes and proofs that verifies the properties of the shapes. Verification of the properties demands deductive reasoning. By Isoda (2010), as for the deductive reasoning, foundation is to understand the meaning of definition. Both classification and proofs apply the definitions, thus, unless students properly understand the concept definition, it is likely to be difficult to do proofs in geometry. Formal definitions for the shapes are expected to be developed based on students' informal definitions built on prior grades, so, understanding of, for instance, equivalent definitions are crucial in the instructions.

By the curriculum analysis, secondary mathematics teachers are required to know:

- How students' mental images affect students' understanding of the formal definitions,
- How they evoked during the learning,
- What the structure of formal definitions is,
- How the shapes are classified by the definitions,
- How 2 types of classifications are different,
- How the representation for the formal concept definition of the shapes is different from a representation to develop students' mental images of the shapes,
- Why the knowing properties of the shapes are important.

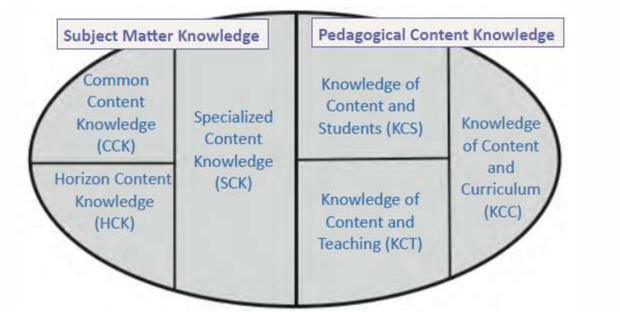
At the end, it can be summarized that current practice of geometry teaching and learning indicates that there is a mismatch between concept image and concept definitions for the shapes including symmetry as it is a supporting concept for formation of the shape concept. By the curriculum and textbook, Mongolian primary and lower secondary school students (grades 1-7) are not given richer experiences of images of the shapes properly thus, students' mental images of the shapes are limited and problematic at these grades. One of the significant evidences is that geometry content of grades 1-7 textbooks always represent the most prototype examples of the shapes (regular, convex, and gravity-based), yet, miss critical non-examples of the shapes. Findings from many research studies conclude that students limit concepts to studied examplars and consider inessential but common features as essential to the concept (Burger & Shaughnessy, 1986; Fisher, 1978; Ward, 2004). Later at grades 8-10, Mongolian students face difficulties to learn the formal concept definition of the shapes. By van Hiele (1981), if the foundation of the concept image is problematic, it is likely to be challenging to learn formal concept definitions of the shapes later on. In the research literature, there is a "tension" which exists between the concept image and concept definition in various mathematical context (Bingolbali & Monaghan, 2008; Even & Tirosh, 1995; Tall & Vinner, 1981; Tirosh et al., 2011). This situation demands Mongolian secondary mathematics teachers to have special knowledge for tackling the tension, developing students' image of the shapes properly, bridging their image with formal definitions of the shape effectively, and forming definitions of the shape with understanding. Without this knowledge, teachers are likely to fail establishing appropriate geometry understanding in students along the van Hiele levels through grades.

CHAPTER THREE. LITERATURE REVIEW

3.1 Litearture Review on Mathematical Knowledge for Teaching

Teacher mathematical knowledge for teaching is highly valued in school mathematics teaching and learning. Initial development of this perspective is introduced by Shulman in his notion of PCK; and it is widely recognized as knowledge which has received the most attention in the mathematics education research and professional development literature in the last years (Chick et al, 2006; Hill, Rowan, & Ball, 2005). The strongest conceptualization is recognized as Ball and her colleagues' (2008) model of mathematical knowledge for teaching that is defined as the particular form of mathematical knowledge which is useful for and usable in, the work that teachers do as they teach mathematics to their students (Ball & Bass, 2000; Adler & Davis, 2006). They have studied teacher subject matter knowledge but also special forms of knowledge that are particular to the profession of teaching (Ball, Hill, & Bass, 2005). They take care of theoretical and empirical justification to delineate its boundaries and relationship to other constructs (Hill, Ball, & Schilling, 2008). The study aimed to understand and measure mathematical knowledge for teaching – the mathematical knowledge that teachers use in classrooms to produce instruction and student growth (Hill, Ball, & Schilling, 2008). When develop the framework, Ball et al. (2004) referred to a specific mathematics topic: they begin their paper with the analysis of the specific mathematical topic of multiplication of decimals. They then refer to teacher's mathematical knowledge of the content (Huillet, 2009). A model of mathematical knowledge for teaching (MKT) is developed by practice-based investigation; and it can be used to demonstrate both

Figure 2. Domain map for MKT



subject matter knowledge and PCK (Hill, Ball, & Schilling, 2008, p.377). The left side of the oval, labeled "subject matter knowledge," contains three strands that lie outside Shulman's popular conceptualization of SMK; *common content knowledge* (CCK), roughly described as knowledge that is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics. For example, the correct drawing of a rectangle is a matter of CCK (Ball, 2014). A word "common" does not mean that everyone has this knowledge rather, this is knowledge of a kind used in a wide variety of settings, not unique to teaching. S*pecialized content knowledge* (SCK) is the mathematical knowledge unique to teaching; and it is mathematical knowledge not typically needed for purposes other than teaching. For example, an accurate choice of the shapes to present for assessing students' understanding of what a rectangle is, and why is a choice, whether an unusual method proposed by a student would work in general or in particular geometrical problems serve for SCK (Ball, 2014). This work involves an uncanny kind of unpacking of mathematics that is not needed in setting other than teaching. *Horizon content knowledge* (HCK) is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum, and a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires. HCK can be considered if a teacher is aware of, for instance, if it is okay to shade the given shape and what issues are involved with the fact that children learn about rectangles before polygons (Ball, 2014). *Knowledge of content and students* (KCS) is knowledge that combines knowing about students and knowing about mathematics. Central to this knowledge is about common student conceptions and misconceptions about particular mathematical content. For example, what students likely to know about what is a rectangle, what are common difficulties with and misconceptions in learning about rectangle, and why it is difficult are aspects of KCS (Ball, 2014). *Knowledge of content and teaching* (KCT) combines knowing about teaching and knowing about mathematics. In other words, KCT is an amalgam involving a particular mathematical idea or procedure and familiarity with pedagogical principles for teaching that particular to content to be taught. For example, KCT deals with how would a teacher sequences the given shapes to discuss the concept of a rectangle, how to represent a rectangle using various examples and non-examples of the shape, instructional advantages and disadvantages of representations used to teach a rectangle, and which one would be good to discuss essential attributes of a rectangle (Ball, 2014). *Knowledge of content and curriculum* (KCC) is the curricular knowledge of mathematics content needed for teaching. It includes knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, and determining learning goals about a particular topic for a particular activity. Since the development of the MKT, many numbers of scholars and researchers have investigated teacher subject knowledge using the framework, and interpreted the sub-domains of the MKT in same as well different ways. However, the most common features of the interpretations stand on CCK, SCK, KCT and KCS, yet, some controversies in HCK.

Depaepe et al. (2013) cite three clear merits of MKT: that it was borne out of empirical research on the knowledge teachers require to teach mathematics; that MKT took Shulman's (1986) heuristic and turned it into a valid measure of teachers' mathematical knowledge for teaching; and lastly, that it provides empirical evidence of a positive relationship between student learning and teachers' PCK.

Research literature in Mongolian secondary school teachers' MKT is almost none. In 2018, initial effort to investigate teachers' MKT was done by Itgel (2018) who was a director of National Institute for Educational Research. His study aimed to investigate the most reliable and fit competence model for secondary school mathematics teachers in Mongolia using a confirmatory factor analysis. In the study, he defined the professional competence as consisting of beliefs (nature of mathematics, mathematics learning and teaching students as learners), knowledge (SMK, PCK, Curriculum Knowledge and Knowledge of Student Learning), practice (learning environment, planning for learning, teaching in action and student assessment) and attitude toward mathematics and its teaching and learning. The study concluded that in Mongolian context, the most reliable model for secondary school mathematics teachers' professional competence had 3 components excluding attitudes. This study has few interesting findings. Firstly, by his study, Mongolian secondary school mathematics teachers' beliefs is more likely to be characterized by belief about mathematics teaching, meantime, their knowledge is more loaded by CCK, KCT and KCC. Second, the study had results that present mild relationship between teachers' knowledge and belief as well as moderate relationship between teachers' knowledge and practice, in particular teaching in action.

Gaps in MKT model

Although work of Ball and her colleagues has been particularly influential in the study of the mathematics for teaching (Askew, 2008, p.18), it misses some important aspects that are likely to affect teachers' knowledge for teaching. Thus, inclusion of these aspects in research can be complementary parts of the framework.

By Helen (2008), looking at the mathematics that teachers draw on in mathematics lessons could be as much as measure of their **beliefs** about the role of mathematics in the curriculum as about their knowledge of mathematics per se. By Goulding et al (2002) teachers' beliefs about the nature of mathematics may be tied up with subject matter knowledge in the way in which teachers approach mathematical situations. If teachers believe that mathematics is principally a subject of rules and routines which have to be remembered, then their own approach to unfamiliar problems will be constrained, and this may impact on their teaching (Petrou & Goulding, 2011). Barkatas and Malone (2005) indicated that mathematics teachers' belief about mathematics could not be separated from their belief about teaching and learning mathematics. For example, researchers have identified teachers with good understanding of mathematics but who still adopted "transmission" style teaching approaches rather than work on crafting student explanation (Ball, 1991).

At another hand, knowledge is social and contextualised rather than individual and general, whilst knowledge about mathematics teaching is less about general principles and more about intertwined collections of more specific patterns that hold across variety situations (Lave & Wenger, 1998; Putnam & Borko, 2000). Teacher knowledge is shaped by the context and cultures not just of national education systems but of particular types of school and teacher education institutions (Ruthwen & Rowland, 2011). While the personal knowledge of individual teachers remains a central focus of work in this field, a broadening of perspective has recognized significant ways in which mathematical knowledge is situated within teaching and distributed across pedagogical resources and professional communties (Ruthwen & Rowland, 2011). Viewing knowledge as situated, social and distributed places greater emphasis on the communities in which mathematics teachers are engaged rather than on individual knowledge. Building on Spillane's (1999) work, Millet, Brown and Askew (2004) highlight the importance of the professional community of teachers in a school.

3.2 Literature review on teacher beliefs

By Ernest (1989), for study of teachers' knowledge, it is necessary to consider beliefs to account for the differences between mathematics teachers. It is possible for two teachers to have very similar knowledge, but for one to teach mathematics with a problem solving orientation, whilst the other has a more didactic approach.

Ernest (1989) described three categories of teacher beliefs about the nature of mathematics that have been widely adopted and used (e.g Beswick, 2005, 2009).The *first* is the instrumentalist view that sees mathematics as, "an accumulation of facts, skills, and rules to be used in the pursuance of some external end." (Ernest, 19879, p.250 cited by Beswick, 2011,p.129). According to this view of mathematics the various topics that comprise the discipline are unrelated (Beswick, 2011, p,129). The *second* category is the Platonist view in which mathematics is seen as a static body of unified, pre-existing knowledge awaiting discovery. In this view the structure of mathematical knowledge and the interconnections between various topics are of fundamental importance (Beswick, 2011, p,129). Ernest's (1989) *third* category is the problem solving view in which mathematics is regarded as a dynamic and creative human invention; a process, rather than a product (Ernest, 1989), and the view that best reflects relatively recent changes in the way that mathematicians view their discipline (Cooney & Shealy, 1997 cited by Beswick, 2011, p.129).

Recent studies (Beswick, 2011) suggest that teacher beliefs about nature of mathematics as the discipline may be separate from beliefs about the nature of mathematics as the school subject. Reason of this is that "school mathematics is different from mathematical activity by mathematicians". Beliefs related to specific aspects of the particular context in which a teacher is working, for example about specific students' interests and abilities, can also influence which of their other beliefs are most influential in terms of shaping their practice in that context (e.g. Beswick, 2004). There is some (limited) empirical evidence that teachers can indeed construct separate belief clusters about school mathematics and the discipline in order to teach the subject in a manner consistent with their beliefs about mathematics teaching (Beswick, 2011, p.132). There is a recognition that "at least some teachers have different beliefs about the nature of school and mathematicians' mathematics may go some way to explaining apparent inconsistencies among teachers' beliefs about mathematics and its teaching and learning and the apparent instability of beginning teachers' commitments to contemporary mathematics teaching (e.g. Ball, 1990 cited by Beswick, 2011, p.146). Thus, by Beswick (2011), this is theoretically possible for teachers to hold differing beliefs about the nature of mathematics depending whether they are considering it as a discipline or as a school subject (p.132).

Based on this perspective, Beswick (2011) proposed a framework combining beliefs about the nature of mathematics as the discipline and the school subject.

Table 6. Combinations of beliefs about school and discipline mathematics

		Beliefs about nature of discipline mathematics		
		Instrumentalist	Platonist	Problem solving
Beliefs about the nature of school mathematics	Instrumentalist	School mathematics is about learning basic skills that students will need in everyday life.	School mathematics is about learning basic skills that will allow understanding of higher level more interesting mathematics later	Mathematics can be creative but you need to have a set of basic skills first. Mathematical creativity is not for school
	Platonist	School mathematics is a body of hierarchical interconnected knowledge that seeds to be learned so that it can be applied to practical situations.	School mathematics is part of a body of hierarchical interconnected knowledge understanding of which forms the basis on which some will learn higher level mathematics	School mathematics is part of a body of hierarchical interconnected knowledge understanding of which will enable the gifted few eventually to be mathematically creative
	Problem solving	Learner/process focus is aimed at motivating students to learn the skills they need in everyday life.	Learner/process focus is aimed at motivating students so that they come to understand more of the body of hierarchical interconnected knowledge that is mathematics.	Learner/process focus is aimed at helping students to appreciate mathematics as a powerful and creative process.

Referenced from (Beswick, 2011)

Table 6 proposes theoretically reasonable implications for teaching of each possible combination of Ernest's (1989) three categories of beliefs about the nature of mathematics in relation to mathematics as a school subject and as a discipline. The descriptions of practice contained in the matrix cells are intended to be consistent with beliefs about mathematics teaching and learning corresponding to the undifferentiated views of the nature of mathematics shown in Table 7. Beswick (2011) suggests that more attention needs to be paid to the beliefs about the nature of mathematics that the teachers have constructed as a result of the cumulative experience of learning mathematics in primary and secondary school, an university, and, for experienced teachers, from years of involvement in the profession (p.145).

Teacher belief about the nature of mathematics cannot be separated by belief about mathematics teaching and learning. Beswick (2005) summarized connections among Ernest's (1998) categories of beliefs about the nature of mathematics, an adaptation of the corresponding categories that he proposed for beliefs about mathematics learning and Van Zoest, Jones and Thornton's (1994) categories relating to mathematics teaching. It is summarized in Table 7, in which beliefs in same row are considered theoretically consistent, and those in the same column have been regarded by some researchers as constituting a continuum (Beswick, 2011, p.129).

Table 7. Connection between teacher belief about mathematics and its teaching and learning

Beliefs about nature of mathematics (Ernest, 1989)	Beliefs about mathematics teaching and learning (Van Zoest et al. 1994)	Beliefs about mathematics learning (Ernest, 1989)
Instrumentalist	Content focussed with an emphasis on performance	Skill mastery, passive reception of knowledge
Platonist	Content focused with an emphasis on understanding	Active construction of understanding
Problem solving	Learner focussed	Autonomous exploration of own interests

From Beswick (2011, p.130) (This is originally imported from Beswick, 2005, p.40)

By Table 7, theoretically, teachers, who hold Instrumentalist belief that mathematics is an accumulation of facts, rules and skills to be used in the pursuance of some external end, tend to teach the content emphasizing students' performance using algorithms, procedures and rules. Therefore, these teachers believe to promote memorizing, focusing correct answers and learning under teacher's direct explanation in students. In other hands, Platonist teachers teach the content focusing on students' understanding of the subject matter; thus, they are likely to emphasize students' active learning and constructing the understanding. Platonist teachers believe that students need to dig into the subject matter by investigating why certain procedures and solutions work in a particular problem, and exploring different ways to solve a particular problem. Moreover, these teachers pay attention on teacher's explanation in students' learning, nevertheless, not as direct as instrumentalist teachers believe. Problem solving teacher are more likely to be constructivist that encourages students to explore knowledge and interest in the subject matter. They ideally emphasize students' ways to solve a problem independently without teacher's direct help.

3.3 Literature review on context related to teacher knowledge

In pursuing the work on PCK, a number of questions, not all related have arisen; one of them is that context plays a signifcant role in teacher PCK. However, there are varieties in context related to teacher knowledge. Some researchers investigated classroom context where actual teaching happens, while others investigate more broad professional community context. However, all they agree that knowledge for teaching is "situated", "shaped by particular contexts" (Putnam & Borko, 2000 cited by Pepin, 2011).

Putnam and Borko (2000, p.3) argue that this professional knowledge is developed in context, stored together with charactersitic features of classrooms, and activities, organized around the tasks that teachers accomplish in classroom settings, and accessed for use in similar situations. Fennema and Franke (1992) were critical to the work of mathematical knowledge for teaching as knowledge needed in teaching is interactive and dynamic in nature. They developed a model of teacher knowledge that centres on teacher knowledge as it occurs in the context of the classroom. The model includes knowledge of the content, knowledge of pedagogy, knowledge of students' cognition and teachers' beliefs. They see that knowledge is developed in a specific context and often develops through interactions with the subject matter and the students in the classroom. They also claim that the key to understanding teacher knowledge, researchers need to carefully take into account the context in which teachers work. Pursuing the work of Fennema and Franke, Cambridge university team led by Rowland (2011) developed a framework for mathematics teacher knowledge needed in teaching; and their model is named as Knowledge Quartet - mathematical knowledge in teaching. A main argument for the development of their model is that classroom teaching is the most valuable context where teacher knowledge is truly exposed and reliable moment that what content knowledge for teaching teachers possesses. They categorized situations from classrooms where mathematical knowledge surfaces in teaching, and proposed that the framework has 4 dimensions, namely, *Foundation* (knowledge, belief, and understanding), *Transformation* (knowledge-in-action as demonstrated both in planning to teach, and in the act of teaching), *Connection* (choices and decisions that made for the more or less discrete parts of mathematical content) and *Contingency* (classroom events impossible to plan) (Rowland et al, 2011). The framework is developed from the investigation of the mathematical content knowledge of pre-service elementary school teachers in England and Wales using video taped lessons taught by the students. Rowland et al (2011) noted that the framework can be used as a tool for classifying ways in which the pre-service teachers' SMK and PCK come into play in the classroom. Advantages of the framework can be accounted as it captures how different components of teachers' knowledge is integrated and come into play in the classroom; and how pre-service teachers develop different forms of PCK depending upon on the knowledge and views they bring into their training (Meredit, 1995).

Another dimension of the context related to teacher knowledge is school context. Williams (2011) noted that teacher knowledge is distributed; rather than being held "in the head" of any individual teacher, such knowledge is held by teachers collectively – in a school or in the profession (Stylianides & Delaney, 2011, p.186). By Hodgen (2011), "the communities in which mathematics teachers are engaged rather than on individual knowledge" can make a major contribution to understand teacher knowledge. It is evidenced in Cobb, Yackle and Wood (1991) work that sees knowledge learning is happening not in isolated classrooms but within a teacher professional community within a school and institution. Teachers' knowledge to be gained is situated in the practice of community and can only be gained through participation in that professional community (Turner, 2009); yet, teachers' reflection plays a significant roles. By Turner (2009), teachers' growth of knowledge for mathematics teaching has been influenced by individual reflection as well as by participation in communities of practice, with the interaction between the two being dependent on individual contexts. He studied 2 mathematics teachers for 5 years to invetsigate how their reflection and participation in professional community within a school is related to the development of their mathematics subject knowledge and pedagogical knowledge. Participant teachers were asked to wrote regular reflections on their mathematics teaching which were taped and sent to the researcher. They were also recommended to interact with other teachers of the school. By the study, it is resulted that mathematics subject and pedagical knowledge of a teacher who regularly reflected and actively participated in the professional community within a school has well developed. Meantime, the knowledge of a teacher, who had a less constructive relationship with collegues and little reflection on the what is discussed in the community, tends to be different from the previous teacher and lags behind.

Jeremy (2011) studied how teacher mathematics teacher knowledge is situated in the context in which a teacher acts and interacts in the community she/he belongs. The study investigated an individual primary teacher, Alexandra, and her knowledge of proportional reasoning. The study found out that Alexandra's knowledge in context appeared stronger than her knowledge out of context. In another word, Alexandra's mathematical knowledge for teaching had developed in large part within the context of teaching, teacher education and curriculum development. Her knowledge for teaching was supported by the social communities and relationship in which how she acts and interacts in that community. Jeremy (2011) claims that the communities in which mathematics teachers are engaged rather than on individual knowledge can make a major contribution to understand teacher knowledge. In addition, teachers teaching within a school setting might be seen to form a community of mathematical teaching practice with its own norms and expectations and ways of being and doing (White et al, 2013, p.404).

Pre-service teacher education context has a crucial role in teacher knowledge for teaching. The models of subject thinking and learning that prospective teachers have developed as students are also well known to constitute an important base for the forms of teacher knowledge and practice that they go on to develop (Ruthven, 2011, p.95). During the teacher training course (at pre-service level) the assumption seems to be that the SMK relevant to the phase that the trainee will be teaching will be unpacked and that PCK will be developed by the training course by elements of both university based and school based work including the use of assignments (Goulding & Petrou, 2011). For example, abstract algebra should contribute to the mathematical knowledge of future teachers by assuring that they understand why number systems and algebraic structures operate as they do (Ticknor, 2012, p.308). The ideas of the above arena with questions about what is taught and learned in abstract algebra; how does what was learned contribute to the overall mathematical knowledge of preservice teachers, and do those students make connections between the mathematics they learn in the university classroom and the mathematics they will teach in secondary school? (Ticknor, 2012). This perspective framed the analysis of the mathematical content knowledge that the preservice teachers encountered in abstract algebra setting. It was expected that the course would address the knowledge of mathematics that would be considered CCK. Yet, we still do not know much about the effects of prospective teacher preparation in content and methods on their classroom instruction (Yeping et al, 2008).

3.4 Literature review on geometry teaching and learning

Geometry is the foundation of mathematics as we know it today; it was developed to explain phenomena and solve problems that bore directly on daily life (NCTM, 2006). It acts as a bridge between events in daily life and mathematical concepts, geometry has a crucial importance for mathematics learning. It is used for solving problems associated with other branches of mathematics besides its usage for solving problems about daily life and utilization in other disciplines such as art for different purposes (Biber et al, 2013). In particular, geometry develops students' spatial thinking that is strongly related students' well performance in mathematics and other subjects (NCTM, 2006). Spatial thinking plays an important role in learning of arithmetic, word problems, measurement, algebra and calculus (NCTM, 2006). Nevertheless, at the school, it is a subject that students are likely to struggle and perform poor.

As it is referenced from NCTM (1999), the content of geometry is appropriate both for the development of lower level of mathematical thinking, (i.e. the recognition of shape), as well as for higher order thinking, (i.e. the discovery of the properties of shapes), the construction of geometrical models and the solution of mathematical problems (as cited in Mattheou, 2009). The representation of geometrical objects and the relationships between geometrical objects and their representations constitute important problems in geometry (as cited Mattheou, 2009).

By Mongolian secondary school curriculum analysis done in Chapter 2, the most significant concept in geometry is the plane shape; and teaching and learning of plane shape concept image; and it can be justified that a main weakness in Mongolian secondary geometry is a mismatch between concept image and concept definition for the shapes. Studies (Bingolbali & Monaghan, 2008; Even & Tirosh, 1995; Gray, Pinto, Pitta & Tall, 1999; Levenson, Tsamir & Tirosh, 2007; Schwarz & Hershkowitz, 1999; Vinner & Tall, 1999; Levenson, Tsamir &

Tirosh, 2007; Schwarz & Hershkowitz, 1999; Vinner & Dreyfus, 1989) have shown that tension exists between students' concept images and concept definitions within various mathematical contexts in primary and secondary education. In order to examine the concept image and definitions for the plane shapes, further, to tackle with the tension, it is important to study theories in geometry learning at first.

There are two influential theories in geometry. By Piaget and Inhelder (1967), children's ideas about shapes do not come from passive looking. Instead, they come as children's bodies, hand, eyes and minds engage in active construction. Children need to explore shapes extensively to fully understand them; merely seeing and naming pictures is insufficient. They have to explore the parts and attributes of shapes (Clements, 2011, p.152).

Another influential theory on children geometrical thinking is the van Hiele's theory. Researchers generally supports that the van Hiele levels are useful in describing students' geometric concept development. By van Hiele's theory:

> *Level 1*: it is the *visual* level, in which students can recognize shapes only as wholes and cannot form mental images of them. Students do not think about the attributes, or properties, of shapes, students at this level included imprecise visual qualities and irrelevant attributes, such as orientation, in describing the shapes while omitting relevant attributes
>
> *Level 2*: this is *descriptive/analytic* level, students recognize and characterize shapes by their properties. Students establish properties experimentally by observing, measuring, drawing and model making. Students at this level do not see relationships between classes of figures.
>
> *Level 3*: this is the *informal deduction/relational* level that students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand and sometimes even provide logical arguments in the geometric domain. They can classify figures hierarchically by ordering their properties and can give informal arguments to justify their classifications. They can discover properties of classes of figures by informal deduction. One property can

signal other properties, so definitions can be seen not merely as descriptions but as ways of logically organizing properties. The students still do not grasp that logical deduction is the method for establishing geometric truths.

Level 4: this is the *formal deduction* level. Students can establish theorems within an axiomatic system. They recognize the difference among undefined terms, definitions, axioms and theorems and are capable of constructing original proofs that is, producing a sequence of statements that logically justifies a conclusion as a consequence of the "givens".

Level 5: this is *rigor mathematical* level. Students are able to transfer understanding and compare different axiomatic systems such as non-Euclidean geometries.

Different levels of geometrical thinking are sequential, invariant, and hierarchical. Each level has its own language and way of thinking; teachers unaware of this hierarchy of language and concepts can easily misinterpret students 'understanding of geometric ideas (Clements, 2011, p.152). A construct that has been applied extensively to geometric thinking and learning is that of the *concept of image*. Students often use concept images rather than definitions of concepts in their reasoning. However, the concept images are adversely affected by inappropriate instruction (Clements, 2011, p.155).

Concept image

Students recognize some components and attributes of shapes and describe shapes in a variety of ways that are not necessarily "mathematical" (Clements, 2011, p.153). Students may have problems due to somehow diluted image of concept, thus "all mental attributes associated with a concept should be included in the concept image (Tall & Viller, 1981)". All mental pictures (pictorial, symbolic and others), all mental attributes (concsious, or unconsious) and associated processes are included in the concept image (Semadeni, 2008, p.4). Attributes may be critical or non-critical. In mathematics, critical attributes (for example, as for polygon, these attributes include (a) closed figure, (b) four sides, (c) four vertices, (d) four angles) stem from the concept definition. Non-critical attributes include the overall size, of the figure (large or small) and orientation (horizontal base) (Tsamir et al., 2008, p.83). Individuals who base their reasoning on critical attributes may at the very least be operating at the second van Hiele level. If the students point out that a figure is a quadrilateral because it has four sides therefore it also has four angles and vertices, then that child may be operating at the third van Hiele level (Tsamir et al., 2008, p.83). All examples of a concept must contain the entire set of critical attributes for that concept (Tsamir et al., 2008, p.83). Burger and Shaughnessy (1986) claimed that an individual's reference to non-critical attributes has an element of visual reasoning. Thus they further claimed that a child using this reasoning may either be at van Hiele level one or two, as he is pointing to a specific attribute, and not judging the figure as whole (Tsamir et al., 2008, p.83). As it cited by Tsamir et al (2008), Herskowitz (1989) claimed that in addition to the necessary and sufficient (critical) attributes that all examples share, prototypical examples of a shape have special (non-critical) attributes "which are dominant and draw our attention". In geometry, a non-example of a concept is an instance which is missing at least one critical attribute of the concept being considered (Tsamir et al., 2008, p.92).

In order to encourage student concept image, they must be given richer experiences so that they are able to form a more coherent concept; it involves a balance between the variety of examples and non-examples necessary to gain coherent images and the complexity which may increase the cognitive demand to unacceptable level (Tall, 1988). Many research studies conclude that students limit concepts to studied examplars and consider inessential but common features as essential to the concept (Burger & Shaughnessy, 1986; Fisher, 1978; Fuys, Geddes, & Tischler, 1988; Zykova, 1969 cited by Ward, 2004, p.42). To broaden the scope of concept images, students should view, for example, polygons that are not always "traditional" in appearance; i.e., regular, convex, and gravity-based. By allowing students to experience polygons that are irregular, asymmetric, convex as well as concave, and those that are not necessarily gravity-based, students will develop more well-rounded concept image of a polygon and, more importantly one that is in harmony with the mathematical definition (Ward, 2004, p.53). Instructors of mathematics content courses should encourage students to verbalize and describe their collection of concept images of polygons and provide their definitions as well, as a means to assess the breadth and depth of their conceptions, to clarify misconceptions, and then assist students in developing a deep, connected understanding of concepts (Ward, 2004, p.53). Activities such as flips, slides, and turns will assist them in creating mental images; that is concepts images of shapes in various orientations (Ward, 2004, p.53). It is often argued that instruction of geometrical concepts should include more than mere exposure to prototypical examples of the concept (Clements et al. 1999; Hershkowitz, 1989 cited by Tsamir et al. 2008, p.93). Tsamir et al. (2008) suggest that geometry instruction include exposure to a variety of nonexamples, and not merely intuitive examples (p.93). In geometry specifically we allow that visual judgement may be a necessary first level, but that analytical judgement based on critical attributes should follow. Intituitive nonexamples of triangle encourage visual reasoning rather than analytical thinking (Tsamir et al. 2008). Differentiating intitutive and non-intituitive non-examples enables teachers to understand how they impact on childrens thinking (Tsamir et al. 2008).

Concept definition and its relation to classifications of the shapes

Having precise definitions for mathematical concepts ensures mathematical coherence and provides the foundation for building mathematical theories (Tirosh et al., 2011, p.233). As it is cited in Habila and Simon (2015) from Marchis (2012), students have misconceptions in geometry because of concept definition; and this concept image may not develop in some students, and in others, it may not be related to the formal definition. As for the classifications, there are two types of definitions for the plane shapes; this is related to the classifications. According to Jones (2000), following De Villiers, classifications can be hierarchical or partitional. Hierarchical classifications use inclusive definitions such as specifying that trapezoid is a quadrilateral with at least one pair of sides parallel-which means that a parallelogram is a special type of trapezoid. Partitional classifications use exclusive definitions such as specifying that a trapezoid is a quadrilateral only one pair of sides parallel-which excludes parallelograms as trapezoids (Battista, 2007, p.865). In general, in mathematics, inclusive definitions (and thus hierarchical classifications) are preferred, although exclusive definitions and partitional classifications are certainly not mathematically incorrect. As it cited by Battista (2007), a number of studies show that many students have great difficulty with the hierarchical classifications of quadrilaterals (De Villiers, 1994; Jones, 2000). Many mathematicians (among them: Khinchin, 1968; Solow, 1984; Vinner, 1991) refer to some logical principles that must be met when defining any mathematical concept. Among others, they point out the following:

1. *Defining is giving a name*. The name of the new concept is presented in the statement used as a definition and appears only once in this statement;
2. For defining the new concept, only *previously defined concepts* may be used;
3. A definition establishes *necessary and sufficient conditions for the concept*;
4. The set of conditions shoud be *minimal*;

In didactically suitable definition of a new concept, only concepts already known to the learner should be included (Landman & Leikin, 2000). The level of development of the learner is defined by his/her current knowledge and by the knowledge which is in their 'zone of proximal development' (ZPD), which determines the dynamics of the development of the learner To enable the learner's intellectual development, the teacher may relate only to concepts that belong to the ZPD of the learner (Vygotsky, 1982 cited by Landman & Leikin, 2000).

In order to form the concept images and definitions, a teacher should start with various examples and non-examples by means of which the concept image will be formed (Vinner, 1991, p.80). A teacher should point at the conflicts between the concept image and the formal definition and deeply discuss the weird examples (Vinner, 1991, p.80). The definition of a concept, once determined in a curriculum, influences the approach to teaching mathematics, the learning sequence, the set of theorems and proofs. Consequently, definitions, and the ways in which they are presented to students, shape the relationship between a concept image and a concept definition, forming an essential part of one's knowledge structure that affects the learner's thinking processes (Tall and Vinner, 1981; Vinner, 1991 cited by Zazkis & Leikin, 2008, p.133). Zazkis and Leikin (2008) consider that definitions of mathematical concepts, the underlying structures of the definitions and the process of defining are fundamental components of teachers of mathematics. Their study claims that teacher's personal knowledge of mathematical definitions affects (a) their curricular decisions regarding the way mathematical concepts are taught, and (b) their pedagogical conception of the ways in which students may or may not learn these concepts (p.133). Didactically suitable definition of a new concept should rely on the concepts known to the learner (Zazkis & Leikin, 2008, p.133).

Examples of each strand are drawn using a topic of *rectangle* (Ball, 2011). While the drawing of a rectangle is matter of CCK, an accurate choice of figures to present for assessing whether students understand what a rectangle is, and why is a choice serves for SCK. Moreover, knowledge on definition of a rectangle and why it is accurate belong to teacher's SCK, so, language of the definition takes a part on SCK of teachers. Identifying a mathematically accurate definition of rectangle that is usable by particular grade students, what they likely to know about a rectangle, as well as common difficulties with and misconceptions about rectangles and why it is difficult are aspects of KCS. KCT deals with sequence of the tasks. For instance, how would a teacher sequences the given figures to discuss the concept of a rectangle, what task would be created using the given figures to set up a productive discussion aimed at developing a definition, and which one would be good to discuss in a whole-class discussion are attributes of the KCT. HCK can be considered if a teachers are aware of, for instance, if it is okay to shade the given figure and know what the issues are involved with the fact that children learn about rectangles before polygons.

Concept image and concept definition theory

In 1981, Tall and Vinner developed theory called as Concept Image and concept definition (CICD) in advanced mathematics setting. By Tall and Vinner (1981, p.152), the term *concept image* is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes"; and the *concept definition* refers to "a form of words used to specify that concept" (Tirosh et al., 2011, p.233). The portion of the concept image which is activated at a particular time is the *evoked concept image*. At different times, seemingly conflicting images may be evoked. There is a part of the concept image or concept definition which may conflict with another part of the concept image or concept definition; it is a *potential conflict factor*. Such factors need never be evoked in circumstances which cause actual cognitive conflicts but if they are so evoked the factors concerned will be then called *cognitive conflict factors*. A formal concept definition is a definition accepted by the mathematical community whereas personal concept definition may be formed by the individual and change with time and circumstance (Tall & Vinner, 1981, pp.151-152).

Because the concept image actually contains a conglomerate of ideas, some of these ideas may coincide with the definition while other may not (Tirosh et al., 2011, p.233). When a problem is posed to an individual, there are several cognitive paths that may be taken, considering both the concept image and concept definition. At times, although the individual may have been presented with the definition, this particular path may be bypassed. According to Vinner (1991), an intuitive response is one where "everyday life thought habits take over and respondent is unaware of the need to consult formal definition" (p.73). It does not always promote the logical and deductive reasoning necessary for developing formal mathematical concepts. By Fischbein (1993), the image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions (Tirosh et al., 2011, p.234). By cited by Bingolbali & Monagham (2008, p. 30), Vinner (1992, p.20) anticipated our work:

> *The concept image... is shaped by common experience, typical examples, class prototypes, etc. with a given textbook and a given teaching, one can predict the outcoming concept images and can predict also the results of cognitive tasks posed to the students.*

Within the set of examples, a prototypical example is intuitively accepted as representative of the concept (Tsamir et al., 2008, p.82). Although Tall and Vinner (1991) introduced their theory within the context of advanced mathematical thinking, the interplay between concept definition and concept image is part of the process of concept formation at any age and any mathematical context (Tirosh et al., 2011, p.234). In the context of teacher knowledge, knowing aspects of CICD theory may enlighten teachers to the tension which exist between the concept image and concept definition and inform their teaching in ways that promote children advancement along the van Hiele levels of thinking (Tirosh et al., 2011, p.234).

3.5 Literature review on combination of MKT and CICD in geometry

Tirosh et al. (2011) used CICD theory and MKT framework in the research on building kindergarden-teachers' MKT in geometry. They claim that teachers must be able to explain why a figure is, or is not a triangle. They also need to know effective ways of presenting figures to their students so that they too will be able to differentiate between triangles and non-triangles (Tirosh et al., 2011, p.232). Knowing that the diagonals of a parallelogram are not necessarily perpendicular may be considered knowledge typical of anyone who knows mathematics (CCK). Knowing "how the mathematical meaning of edge is different from the everyday reference to the edge of a table" (p.400) is an example of SCK. Knowing which shapes young students likely to identify as triangles, and that confusion between area and perimeter may lead to erroneous answers, are examples of KCS. Knowing how to sequence the presentation of examples and which examples may deepen students' conceptual knowledge is KCT (Tirosh et al., 2011, p.233). They have proposed the framework with four strands of MKT, namely, CCK, SCK, KCS and KCT.

Table 8. Tirosh et al (2011) framework for MKT in CICD theory

	Domains of teacher knowledge			
Domains of mathematical thinking	CCK	SCK	KCS	KCT
Concept image	Cell 1	Cell 2	Cell 3	Cell 4
Concept definition	Cell 5	Cell 6	Cell 7	Cell 8

Reference: (Tirosh et al., 2011, p.233).

Cell 1: CCK-Image. Here we address the common knowledge of a concept's image. This includes knowing to draw examples and non-examples of triangles.

Cell 2: SCK-Image. Here we address the specialized knowledge of a concept's image necessary for teaching. This includes a rich concept image of triangles which incorporates scalene and obtuse triangles with different orientations and not just equilateral and isosceles triangles. It may also include a broad image of non-examples for triangles beyond circles and squares (Tsamir, Tirosh, & Levenson, 2008).

Cell 3: KCS-Image. Here we address knowledge related to students and concept image. This includes knowing that the equilateral triangle is a prototypical triangle (Herschkowitz, 1990) and that young children may not identify as a triangle a long and narrow triangle such as the scalene triangle, even when admitting that it has three points and lines (Shaugnessy & Burger, 1985). We also include in this cell knowledge of the van Hiele model (e.g., van Hiele & van Hiele, 1958) for students' geometrical thinking and being able to recognize, for example, that a student's concept image at the most basic level takes in the whole shape without considering its components. As such, this cell includes knowing that a rounded 'triangle' is often identified as a triangle (Hasegawa, 1997) because children take in the likeness of the whole shape, ignoring that the shape is missing vertices.

Cell 4: KCT-Image. Here we address knowledge related to teaching and concept images. This includes knowing which examples and non-examples to present a student which will broaden his or her concept image of a triangle to include, for example, triangles with different orientations.

Cell 5: CCK-Definition. Here we address common knowledge related to concept's definition. It includes knowing that a triangle may be defined as a polygon with three straight sides.

Cell 6: SCK-Definition. Here we address the specialized knowledge of a concept's definition. In mathematics, definitions are apt to contain only necessary and sufficient conditions required to identify the concept. Other critical attributes may be reasoned out from the definition. Thus, this cell includes knowing that defining the triangle as a three-sided polygon implies that it must be a closed figure with three vertices. It includes knowing that the triangle may be defined as a three-sided polygon, or a polygon with three angles, or a polygon with three vertices and that all three definitions are equivalent.

Cell 7: KCS-Definition. Here we address knowledge related to students and concept definitions. It includes knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level because they do not necessarily perceive that a polygon with three sides must have three vertices. For example, research has suggested that for young children, the association between a triangle and the attribute of "threeness" may be stronger that the necessity for it to be closed or for its vertices to be pointed (Tsamir et al., 2008).

Cell 8: KCT-Definition. Here we address knowledge related to teaching and concept definition. It includes speaking to children with precise language, calling the vertices of a triangle by their proper name as opposed to referring to them as corners. It also includes knowing which examples and non-examples of a triangle to present to children which may encourage children's use of concept definitions and promote their advancement along the van Hiele levels of geometrical thinking. For example, presenting non-examples of a triangle which are not intuitively recognized as such, may encourage children to refer back to the concept definition when identifying the figure as a non-example of a triangle (Tsamir et al., 2008).

Tirosh et al (2011) interpreted that Cell 1 is teachers' common knowledge of concept image includes the drawing examples and non-examples of triangles. Cell 2 is teachers' specialized content knowledge that includes a rich concept image of triangles incorporating different triangles with different orientations. It also includes a broad image of non-examples of triangles beyond circles and squares. Cell 3 is teachers' knowledge related to students and concept images including a prototypical triangle that young children may not identify as a triangle. Cell 4 is teachers' knowledge related to teaching and concept images that includes knowing which examples and non-examples to present to students which will broaden their concept image of a triangle considering triangles with different orientations. Cell 5 is teachers' common knowledge that a triangle may be defined as a polygon. Cell 6 is teachers' specialized knowledge of a concept definition. It includes knowing three different definitions for the triangle. Cell 7 is teachers' knowledge related to students and concept definitions. It is the knowing that a minimalist definition may not be appropriate for young students at the first or second van Hiele level. Cell 8 is teachers' knowledge related to teaching and concept definitions. It includes knowing which examples and non-examples of a triangle to present to encourage children's use of concept definitions and promote their advancement along the van Hiele levels.

Cell-based above interpretations are mainly focused on examples and non-examples of a triangle, and structure of minimalist definition for the triangle concept. Their study was qualitative and only used teachers' responses of face-to-face interviews. They found out that in order to strengthen teachers' MKT, a tool with combination of MKT and CICD could be effectively used. However, in the combination, the mathematical context used to illustrate of this tool and the grade-level at which teachers taught must be taken into consideration (Tirosh et al, 2011). This study concludes and recommends that "regarding mathematical context, geometry is a natural venue for discussing images and definitions. In addition, the combination of the theories has potential to be used for investigating and promoting teachers' MKT in both elementary and secondary grades, but not high grades.

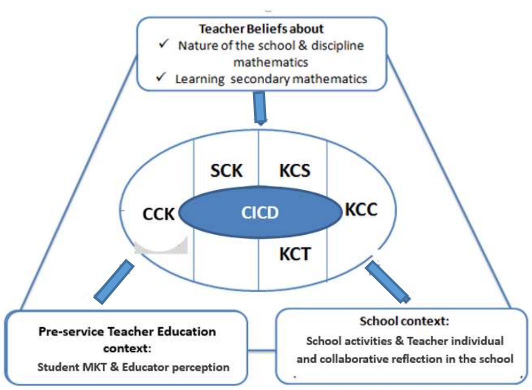
CHAPTER FOUR. RESEARCH DESIGN AND METHODOLOGY

4.1 Research Framework

The framework for the research is illustrated as Figure 3. Based on geometry situation in Mongolian schools and findings from literature review, the following points are considered in development of the research framework:

- The framework is built on Ball et al's (2008) **MKT** model. However, teachers' MKT geometry in this research does not include HCK. Because HCK refers to "teachers' familiarity with the topics and issues during the preceding and later years" (Fernandez & Figueiras, 2014). Indeed, by Hill, Ball and Shilling (2008), KCC is also some extent deals with teachers' knowing of how a particular mathematics content are taught preceding years and will be taught later years. Therefore, referencing the above interpretations, in this research, it is reasoned out overlapping ideas of these two sub-domains can be investigated under one of them, KCC, when teachers are asked to explain why a particular content is appropriate with particular grade students.

Figure 3. Conceptual Framework



The research framework focuses on the **plane shape and symmetry** those are main concepts in primary through secondary geometry, applying **CICD theory**. Symmetry helps students to understand relationships between 2 different shapes; and as it is cited in Melih (2014), the concept of symmetry plays a key role in the comprehension of reflection, rotation, translation, and reflective translation. A rationale why applying this theory is that:

- By Mongolian secondary school mathematics curriculum, there is a tension between the concept image and concept definition in the shape.

- MKT has 2 aspects: mathematics content and teaching that takes into account of learning. As it is a cognitive theory in mathematics, it is sound to apply mathematics theory in learning concept image and definition.

- Teacher **beliefs** about the nature of school mathematics, and teaching and learning the geometry must be considered in the analysis as it affects teacher choices of instructional tasks, representations, approaches in geometry teaching. For teacher beliefs in this research, a particular attention is to teachers' belief about the nature of school geometry. The belief in this research adapts Beswick (2011) idea of that teachers' beliefs about the nature of school mathematics including geometry may be separated from beliefs about the nature of geometry as a science discipline.
- By Stylianides and Delaney (2011) it is important to know what kind of mathematical knowledge teachers hold and why that knowledge is important in the particular countries in which they teach. They noted that teachers' MKT is shaped in the context of teacher education program. Thus, this research considers c**ontext** in teacher education comprises from pre-service teacher education and secondary school context. As for pre-service teacher education context, its students' possession of MKT, beliefs and educators' perception about MKT are subjects of the investigation. This invetsigation enables to understand and analyze how content and method in pre-service teacher education in Mongolia is likely contribute to teachers' MKT in secondary geometry. At the school context, situated aspect is usually determined as teachers' individual and collaborative reflections. Individual reflections will be feature through teacher's own reading of research references, books, observing own classroom teaching, and listening to others, on the other hand, collaborative reflection is identified through teachers' participation in professional community activities such as discussions after open lessons, the lesson

study, interactions in methodology unit teachers. These reflections are investigated in the research in order to how the school context tend to influence on teacher MKT.

Thorough investigation enables to analyze what knowledge and skills high school graduates bring in pre-service teacher education, how subjects in the pre-service teacher education contribute to develop student teacher MKT, how student teacher MKT transits to school context, how teachers' beliefs and MKT is likely to be influenced by school curriculum as well as situated aspect in a school context, and how Mongolian teachers' beliefs about the nature of (the school) geometry and teaching and learning is related will be systematically analyzed, and will provide effective implications to improve secondary teacher quality in Mongolia.

Conceptualization of MKT

This research operationalizes teachers' MKT as it is shown in Table 9. The research framework includes all sub-domains except HCK which refers to "teachers' familiarity with the topics and issues during the preceding and later years (Fernandez & Figueiras, 2014). Indeed, by Hill, Ball and Shilling (2008), KCC is also some extent deals with teachers' knowing of how a particular mathematics content are taught preceding years and will be taught later years. Therefore, it is necessary to include KCC which is not a cell of Tirosh et al (2011) framework in Table 8.

Table 9. Conceptualization of MKT

Sub-domain	Conceptualization	Examples
CCK	Knowledge that is used in wide variety of settings, not unique to teaching - common with how it is used in many other professions or occupations that also use mathematics	What is a triangle, and correct drawing of a given triangle
SCK	Knowledge that is unique to teaching and allows teachers to engage in particular teaching tasks and knowing if the given statements are mathematically true or not.	If it's mathematically true that If a line of symmetry cuts through a side then it makes a right angle with that side
KCS	A combination of knowledge of students and mathematics content - familiarity with, and anticipation of, students' conception and	What students likely to know about what is a rectangle, what are common difficulties with and

	misconceptions about a particular mathematics content and causes of these misconceptions	misconceptions in learning about rectangle
KCT	Knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations	How to represent a rectangle using various examples and non-examples of the shape, and how would a teacher sequences the given shapes to discuss the concept of a rectangle
KCC	Knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, determining learning objectives for a particular activity	What content is appropriate to grade 7 students, why, prior and after knowledge according to the curriculum

Operationalization of MKTCI and MKTCD

Figure 4. MKT and CICD

The research interprets MKT sub-domains with CICD in the plane shape as it is shown in Table 10. There are 5 sub-domains for MKTCI (CCKCI, SCKCI, KCSCI, KCCCI, KCTCI) and MKTCD (CCKCD, SCKCD, KCSCD, KCCCD, KCTCD) respectively. Interpretations for each sub-domains are described in Table 10.

Sub-domains of MKT applying CICD of the plane shape does not directly adapt Tirosh et al (2011) interpretation, because of facts that their research is contextualized in kindergarten geometry, and single triangle concept is stressed in the content. However, some essential ideas from Tirosh et al (2011) are used in the development of CICD in this research. Important ideas in Tirosh et al interpretation are inclusion of examples and non-examples of the shapes for promoting not only concept image but also concept definitions for the shapes. By Tall (1988), in order to encourage student concept image, they must be given richer experiences so that they are able to form a more coherent concept; it involves a balance between the variety of examples and non-examples necessary to gain coherent images and the complexity which may increase the cognitive demand. Another idea is that what attributes of the shape must be reflected in necessary and sufficient condition of the concept definition. Zazkis and Leikin (2008) consider that definitions of mathematical concepts, the underlying structures of the definitions and the process of defining are fundamental components. In addition, this research focuses on not only the triangle but also other shapes with supporting concepts such as angle, symmetry. It means that levels of teacher' MKT, geometry content and students learning are broader than those in kindergarten.

Therefore, this research focuses on not only quadrilaterals but also symmetry and related concepts. For the concept image, critical attributes, examples and non-examples of the shapes and symmetrical properties of the shapes are emphasized. Meantime, concept definition mainly prefers examples and non-examples that highlight essential attributes of the shapes, necessary and sufficient condition for the definition, and how classification of the quadrilaterals reflects inclusive and exclusive definitions.

Table 10. Operationalization of sub-domains in MKTCI & MKTCD

MKT	MKTCI	MKTCD
CCK	**CCKCI**: Common knowledge of quadrilateral images when symmetry is involved	**CCKCD**: Common knowledge that rectangle is formally defined as a parallelogram.
SCK	**SCKCI**: Specialized content knowledge of images polygons with particular symmetrical properties that is not commonly discussed and knowing if the given statements about the polygon images are mathematically true or not. It also includes knowledge of critical attributes of the polygon.	**SCKCD**: Specialized knowledge of choosing mathematically correct formal definition of the concept of rectangle and recognizing what is involved (excluded or included classifications of shapes) in the various definitions. It also includes knowledge of structure of (necessary and sufficient condition) a formal definition of the shape concept.
KCS	**KCSCI**: Knowledge of students' common misconception related to quadrilateral images. It also includes causes of students' misconception on inner angles of quadrilaterals.	**KCSCD**: Knowledge related to students and concept definition. It includes knowing what is confusing in their ideas related to the formal definition of inscribed angles and students' incomplete interpretation of this definition.
KCT	**KCTCI**: Knowledge related to teaching and triangle concept images that includes knowing advantages and disadvantages of the given representations. The given representations consist of examples and non-examples of a triangle that highlight critical attributes of the shape. It also includes choice of the representations to teach the triangle concept image.	**KCTCD**: Knowledge related to teaching and concept definition for a triangle. It includes selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation. It also includes knowledge of how to use examples and non-examples in the representation to define the triangle concept.

KCC	**KCCCI**: Knowledge of what grade students should be taught the symmetrical property of triangle through geometrical construction and what learning goals can be set for this activity of construction.	**KCCCD**: Knowledge of curriculum and formal concept definition of symmetry. It includes knowing at what grade level students are typically taught the formal definition of symmetry and students' familiarity (previous and after knowledge related to definition) with the definitions.

Conceptualization of teacher's beliefs

The research adapts Beswick (2011) conceptualization of the combinations frame of beliefs about the discipline and school mathematics described in Table 11. The adaptation is done by changing mathemaics into geometry as follow:

Table 11. Teachers' belief about nature of school and discipline geometry

		Beliefs about nature of discipline geometry		
		Instrumentalist	Platonist	Problem solving
Beliefs about the nature of school geometry	Instrumentalist	School geometry is about learning basic skills that students will need in everyday life.	School geometry is about learning basic skills that will allow understanding of higher level more interesting mathematics later.	School geometry can be creative but you need to have a set of basic skills first. Mathematical creativity is not for schools.
	Platonist	School geometry is a body of hierarchical interconnected knowledge that seeds to be learned so that it can be applied to practical situations.	School geometry is a part of a body of hierarchical interconnected knowledge understanding of which forms the basis on which some will learn higher level mathematics.	School geometry is part of a body of hierarchical interconnected knowledge understanding of which will enable the gifted few eventually to be mathematically creative.
	Problem solving	Learner's process focus is aimed at motivating students to learn the skills they need in everyday life.	Learner/process focus is aimed at motivating students so that they come to understand more of the body of hierarchical interconnected knowledgeof geometry.	Learner/process focus is aimed at helping students to appreciate geometry as a powerful and creative process.

Table 11 proposes theoretically reasonable implications for teaching of each possible combination of Ernest's (1989) three categories of beliefs about the nature of mathematics in relation to mathematics as a school subject and as a discipline. The descriptions of practice contained in the matrix cells are intended to be consistent with beliefs about mathematics (geometry) teaching and learning corresponding to the undifferentiated views of the nature of school mathematics (geometry) (Beswick, 2011).

In addition, teacher belief about the learning of geometry in the research adapts a frame developed in TEDS-M (2008), because they are internally consistent with the model developed by Beswick (2011).

4.2 Research Methodology

Research conceptual framework is presented in Figure 1. The research applies quantitative and qualitative methods.

4.2.1 Research locale and sample

The research sampled 57 secondary mathematics teachers of Mongolia. For the sampling, at first, 6 secondary schools in Mongolia are selected in convenient way. Moreover, to select the schools, how to maximize opportunities to uncover variations among the school context, and teachers' beliefs and MKT are emphasized. Schools are selected based on if they have various professional community activities, and if their locations are in Ulaanbaatar, a capital city, and Khovd aimag, a rural area, of Mongolia. It must be noted that there is no dedicated analysis on geographical locations. A reason is that urban schools have teachers have more opportunity to develop their professional knowledge and competence by attending various trainings and accessing information, while, rural schools have teachers whose situation is a quite different; and they do not have the same opportunity as urban teachers have. It is reasoned out that this condition may cause varieties among teachers in terms of MKT and beliefs as well as school context. All secondary mathematics teachers at the selected schools are sampled.

Table 12. Demographic data of the sample teachers

Gender		Teacher age		Teaching experience		Education level	
F (%)	M (%)	Age interval	%	Year interval	%	Degree	%
78.9%	21.1%	(23-28) (29-34) (35-40) (41-46)	22.8% 12.3% 14% 21.1%	(1-5) (6-11) (12-17) (18-23)	21.1% 17.9% 17.9% 8.9%	Diploma Bachelor Master	19.3% 54.4% 26.3%

		(47-52) (53 and above)	24.6% 5.3%	(24-29) (30 and above)	25% 8.9%		

By Table 12, most (78.9%) teachers are female, aged more than 35 years old (65%). They had mathematics including geometry teaching experience in secondary schools more than 11 years (60.7%). Most (73.7%) teachers including those with diploma graduated from teacher pre-service education institutions and earned the degree of bachelor in mathematics teaching. It must be explained that diploma teachers graduated from 5-year pre-service teacher education institutions before 1990. During this time, bachelor and master degrees were not introduced to higher education system of Mongolia, thus, the diploma is considered as an equal degree to bachelor in secondary mathematics teaching.

4.2.2 Research instruments

The research applies a questionnaire to investigate teachers' MKT of CICD of the shapes, beliefs and context. Moreover, interview is as a triangulation tool in that they were described as being to confirm that teachers' responses were consistent with their actual classroom practices and their knowledge and belief.

Table 13. Research instruments

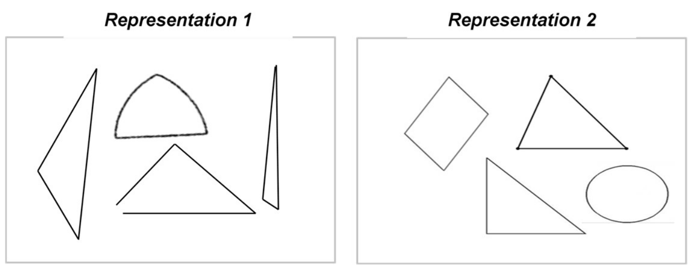
Instrument	Purpose of the instrument
MKT Questionnaire	- To investigate teachers' MKTCI and MKTCD for the shapes and its characteristics; - To investigate prospective teachers' MKTCI and MKTCD for the shapes
Belief Questionnaire	- To identify teachers' belief about nature of school and discipline mathematics, and belief about geometry learnings;
Reflection questionnaire	- To identify teachers' reflections within their schools;
Interview	- To investigate educators' perception of secondary school mathematics teachers' MKT - To confirm the consistency of teachers' responses and dig into teachers' responses;

MKT Questionnaire

Questionnaire items reflect interpretations of MKTCI and MKTCD which are shown in Table 10 in previous section. In the development of items, two aspects are concerned. One is the setting of an item which concerns what context should be given, and how it can be asked. Mongolian secondary mathematics teachers' familiarity with the item setting contributes to the validity of the items. Another is appropriateness with the interpretations given in Figure 5. For example, development of items of Q7 and Q8 (see Figure 5*)* can be interpreted.

Figure 5. Example item of MKT questionnaire

Q7. At professional development workshop, teachers are given assignment to develop representations to teach a triangle to students. They have developed the following two different representations with example and non-example sets on the topic.



Please answer which representation would you choose (Tick as ✓)?

A. Representation 1; B. Representation 2; C. Both; D. I am not sure

Q8. Please write up all advantages and disadvantages for the representation.

These 2 items are for measuring teachers' KCTCI. By Table 10, KCTCI is interpreted as "*knowledge related to teaching and triangle concept images that includes knowing advantages and disadvantages of the given representations. The given representations consist of examples and non-examples of a triangle that highlight critical attributes of the shape. It also includes choice of the representations to teach the triangle concept image*".

By this interpretation, the items need to reflect kinds of representations that are to build up mental images of triangle concept through examples and non-examples of the shape, as well as it must include knowledge of advantages and disadvantages of the representations. The item setting describes teachers' assignment at professional training workshop which is a context in the item. Key words of the items are given as "to choose" and "write up".

Responding to these items requires from teachers the following knowledge, and subject matters in these knowledge are considered as expected responses from teachers:

- knowledge of essential or critical attributes of the triangle for promoting proper image of the shape,
- knowledge of the representation lacks of examples and non-examples for proper image of the triangle,
- Advantages and disadvantage of the representations including consideration of student deeper understanding of the triangle concept image, prototype examples and non-examples of the shape, emphasizing attributes of the triangle,

All items are developed in the similar way as how it is described the above.

Table 14. Relevance between item content and curriculum including originality of items

Item	MKT	Curriculum relevance	Resource
Q1	CCKCI	Quadrilaterals and interrelations among the quadrilaterals	(LMT, 2008)
Q2-Q5	SCKCI	Polygons with symmetrical properties	(LMT, 2008)
Q6	KCSCI	Inner and outer angles of irregular quadrilaterals	(LMT, 2008)
Q7-Q8	KCTCI	Triangles and its attributes	Self-developed referencing from curriculum and textbooks
Q9-Q11	KCCCI	Triangle and symmetry; geometrical construction through practical exercise	Self-developed referencing from curriculum and textbooks
Q12	CCKCD	Rectangle definition and critical attributes for the definition	Self-developed referencing from Silfverberg and Matsuo (2008)
Q13	SCKCD	Quadrilateral classification (hierarchical and partitional) and related definitions	(LMT, 2008)
Q14	KCSCD	Definition of the inscribed angle supported by an idea of triangle	(Somayajulu, 2012)
Q15-Q16	KCTCD	Triangle definition and understanding of structure of this definition (necessary and sufficient condition)	Self-developed referencing from curriculum and textbooks
Q17-Q18	KCCCD	Symmetry definition and features of the definition	Self-developed referencing from curriculum and textbooks

Table 14 presents cohesion between questionnaire items and curriculum. It covers the plane shapes, angle, symmetry and classification of the shapes content of secondary geometry. It however misses the content in the motions and proof. Because it is conceptualized that the motions and proofs are likely to be challenge for students without proper development of the concept image and definition for the shapes and symmetry. This conceptualization also applies to teachers. If teachers have proper knowledge about the concept image and definition of the shapes, symmetry and classification of the shapes; then, they can deal with the motions and proofs. This is a main reason why the research only focuses on the shapes, symmetry, classifications of the shapes and angles.

Table 14 also presents the originality of the items. Some items are adapted from the released items of Learning Mathematics for Teaching (LMT) project conducted by Michigan University researchers in 2008, and some of them are self-developed referencing from the curriculum, textbook and other researchers.

In order to ensure the content validity of the questionnaire, peer-review is conducted. Just after the instrument development, without showing the item assumptions, this questionnaire is given to peer-reviewers to evaluate its match to the relevant sub-domain of MKT. Peer-reviewers evaluated all items and reported which item belongs to which sub-domain. Peer-reviewers are 2 groups of individuals. One of the groups is peer students who have mathematics teaching and research experience in Japan, Cambodia and Rwanda. Another group is Mongolian teachers (2) who have 18 years of mathematics teaching experience to upper and lower secondary schools in Mongolia. This evaluation provided the same result as it is assumed in the research design. It must be remarked that this questionnaire is also given to non-teaching Mongolian individuals, who are not mathematics teachers but have English language background, to check out spelling, wording, and other minor issues. In addition, Mark Scheme will provide more evidences for content validity. All expected responses from the sample teachers are intended in the mark scheme, thus, the fit of teacher actual responses to the mark scheme can also ensure the validity of the instrument.

Belief Questionnaire

Teacher beliefs questionnaire consists of 2 parts. First part is to reveal teachers' belief about the nature of school and discipline geometry which is based on Beswick (2011) conceptualization. Items of this part of questionnaire directly apply Beswick (2011) conceptualization in Table 11. For example, first statement in Table 11 is "*School geometry is about learning basic skills that students will need in everyday life*". To write up first item (B11) of the questionnaire, this statement directly applied.

Table 15. Items for teacher belief about nature of school and discipline geometry

		Beliefs about the nature of the discipline geometry		
Beliefs about the nature of school geometry		Instrumentalist	Platonist	Problem solving
	Instrumentalist	B11	B12	B13
	Platonist	B14	B15	B16
	Problem solving	B17	B18	B19

Note: As for B12 - B (belief), 1(belief about nature of school and discipline geometry), 2(second item)

These items used 6 scale Likert scheme from strongly disagree to strongly agree. In total 9 items are written up.

Second part is about teachers' belief about the learning geometry. In the item development, items should reflect characteristics of three different views of belief about the learning geometry. Characteristics of the views of belief are described in Table 16.

Table 16. Items for teacher belief about learning geometry

Views of belief	Characteristics of belief about the learning	Item
Content-focused on performance	✓ Memorizing formulas ✓ Getting correct answers, and procedures, ✓ Speed in the classrooms ✓ Learning under teacher's direct explanation	B21-B24, B26, B29-B210
Content–focused with understanding	✓ Attending to the teachers' explanations ✓ Understanding why certain procedures and answers are there, ✓ Investigating why certain solution works for a particular problem, ✓ Exploring different ways to solve a particular problems	B25, B27, B211

Learner-focused	✓ Emphasis on students' own ways to solve a problem, ✓ Solving a problem without teacher's direct help,	B28, B212, B213, B214

Note: As for B211 - B (belief), 2(belief about learning of geometry), 11(eleventh item)

Mongolian teachers' familiarity with views reflected in belief items is crucial for obtaining valid results from the instrument. Mongolian teachers had instrumentalist view of belief about nature of discipline mathematics. However, during 1990s, since the introduction of student-centred perspective of teaching and learning as well as constructivism in mathematics education to pre- and in-service curriculum, teachers became familiar with the aspects discussed in the Platonist and problem-solving views of belief.

School context questionnaire

As for school context, teacher individual and collaborative reflections based on Turner (2008) conceptualization are investigated focusing on certain aspects of geometry teaching and learning. Individual reflections are activities of individual readings, observations and listening, while collaborative reflection is group discussions among peer teachers. Each question of the reflection questions will have certain relevance to SMK and PCK in MKT framework. It is shown in Table 10. Item development of school context questionnaire focus on (1) how teachers reflect, and (2) what aspects of geometry teaching and learning they reflect. For individual reflection, teachers reflect through reading, observing own students in classes and listening to what other teachers tell. These three are how teachers reflect. In practice, teachers discuss about many aspects related to teaching and learning geometry. This research pays attention certain aspects of teaching and learning which are likely to influence on sub-domains of teachers' MKT.

Table 17. Items for teachers' individual reflection

Way of individual reflection	Aspects discussed	Items
Reading the references	✓ how to accurately represent a subject matter to students and unusual solution methods in problems related to these subject matters; ✓ how students likely to think about the subject matters;	R11 R12 R13

	✓ what is the most or least effective representations to develop student understanding of the subject matters and why;	
Observing students in their teaching and other teachers' classrooms	✓ how other teachers tackle with student unusual methods and errors at geometry lessons; ✓ how other teachers represent the shapes to students at geometry lessons; ✓ what common errors of my students likely to repeat during my teaching; ✓ how my students imagine and define the shapes during my teaching;	R14 R15 R16 R17
Listening to what other teachers tell	✓ what other teachers discuss about effectiveness of their chosen representations of the subject matters without saying my ideas; ✓ discuss about student common errors, misconceptions related to the subject matter without saying my ideas	R18 R19

Each aspect in Table 17 reflected in each item of the questionnaire. For example, R11 item reflects first aspect in Table 17, and it is developed as:

> "R11: From various references excluding the curriculum and textbooks, I read about how to accurately represent a subject matter to students and unusual solution methods in problems related to these subject matters"

This item specifically highlights how teachers reflect (*I read about…*) and what aspects related to MKT is reflected (*accurately represent a subject matter to students* – related to KCT*; and unusual solution methods in problems related to these subject matters* – related to SCK). Then, in total, 9 items are developed with 4-level Likert scale from never to often. At a school, teachers share a lot among peer teachers. However, items on collaborative reflection questionnaire focus on teachers' shared discussion aspects that are likely to be relevant with MKT. For example, it is anticipated that teachers' shared discussion on what are possible representations of the subject matter of geometry, and which one is the most appropriate and why it is likely to encourage teachers' PCK, in particular, teachers' knowledge of content and teaching. This aspect is reflected in R21 item - R*21: We discuss what possible representations of the subject matters, which is the most appropriate one and why.*

Table 18. Items for teachers' collaborative reflections

	Aspects discussed	Item
Teachers' shared discussion among peer teachers at a school	✓ how to cover intended content in curriculum within teaching hours ✓ what are possible representations of the subject matter, which is the most appropriate and why ✓ what are student common errors and misconceptions related to specific topics of geometry ✓ how to develop alternative learning activities to tackle with student difficulty or misconceptions in geometry lessons ✓ what is the most or less difficult part of specific topic teaching geometry ✓ what is the most essential subject matter in the geometry topic	R21 R22 R23 R24 R25 R26

All aspects in Table 18 put in items the same way as it is described for R21 item. In total, 6 items are developed.

Interview

Interview is used to reveal educators' perception of secondary school mathematics teachers' MKT. It enables to view a general picture of how secondary school mathematics teachers are trained and what is emphasized in the training. To dig into teachers' responses and results the data analysis, focus group interview is taken place. Structure and questions of the interview is developed in accordance to what should be dogged into the results of the quantitative analysis.

4.2.3 Validity of instruments

As for the reliability of the instruments, internal consistency of the questionnaires are estimated by using Cronbach reliability coefficient alpha because it is by far the most commonly employed indicator of the reliability of a instruments in the social sciences (Knapp & Mueller, 2011). In the social and behavorial sciences, reliability coefficients in the .50 to .60 or above range are often considered acceptable with values below these cut-offs being acknowledged as study limitations.

Moreover, one that could threaten the validity was the translation of the items into Mongolian language. Items are translated into Mongolian language, first, by a researcher and second, by non-teacher Mongolian translator. Afterward, wording and spelling are compared to know how different the translations were. Except some technical terms, in overall, they have translated in the same meanings. For example, as for MKT questionnaire, Q14 includes a term “opening”. There were 2 kinds of translation for this term. Both translation was linguistically sound, however, mathematically different meaning. Peer-translator translated the “opening” as a Mongolian term that commonly used in the open set in topology. Yet, the researcher translated it as a term that means “open-wide”. After the discussion with a school teacher, it is decided to use “open-wide”. Another term translated differently was “counterexample”. It was translated by 2 translators as “surug jishee” and “esreg jishee” which could be confusing to school teachers. The researcher consulted with a school teacher; and she recommended to use “esreg jishee-counter examples” because teachers used to this term.

In addition, validity of the all instrument items is estimated using the exploratory factor analysis.

4.3 Data collection

Data collection and administration

Using this instrument, data was collected during December 2014 when geometry is widely taught to secondary students. The instruments are sent to head teachers of the selected schools. Head teachers administered the instruments with assistance of an individual who represents the researcher. Teachers are provided enough time to understand and respond to items of the questionnaires. Before it takes place, brief instruction on how to respond to the questionnaires is given to the sample teachers.

Scales and scoring for quantitative data

Data on teacher background consists of teacher gender, age, teaching experience, graduated pre-service institution, and graduated year. The scale of these items is mainly nominal.

Data on teacher reflection items have four-choice responses; and the scales of these responses are based on ordinal indicators on a four-point scales ranging from *always* to *never*.

Data on teacher belief scales used six-point scales which based on ordinary indicators ranging from *strongly disagree* to *strongly agree*.

For data on teacher MKT, in scoring teachers' responses to MKT related items, the focus is on the correctness of the response and on the thinking process and depth of knowledge demonstrated with regard to that response. Scoring takes into account the variability with which knowledge of mathematics and mathematics teaching [and for the national option general knowledge for teaching] can be expressed. Therefore, aside from those constructed responses where there is one and only one correct score, a number of different responses to an item may receive full credit and still others may receive partial credit. Therefore, *Mark/scoring Scheme* are developed before to administer the questionnaire.

Table 19. Mark/scoring scheme for teachers' responses to MKT items

Items	Expected response	MKT	Marks	Max	Descriptions of the marks
Q1	A or D	CCKCI	If: A or D - 2; A, D, B, C, - 1; E - 0	2	A or D – 2 marks are given for knowing of (1) *properties of symmetry in quadrilaterals* and (2) *all quadrilaterals*; B, C – 1 mark is given for improper knowledge that is knowing one of the above points; E - Lack of both kinds of knowledge.
Q2	T	SCKCI	If: Correct - 1; Incorrect - 0;	1	Correct T & F – 1 mark is given for mathematically true knowledge of properties of symmetries in quadrilaterals; incorrect T & F – no mark for some lack of the above knowledge,
Q3	T			1	
Q4	T			1	
Q5	F			1	
Q15	D	KCTCD	If: D - 4; A - 3; C - 2; B - 1;	4	D – 4 marks are given for proper knowledge of *(1) prototype example and non-examples of triangle, (2) essential and non-essential properties of triangle, (3) representations of the triangle definition (4) teaching concept definition for the triangle and its structure*; A – 3 marks are given for knowledge that misses *(3) representations of the triangle definition* ; C – 2 marks are given for knowledge that lacks *(1) prototype example and non-examples of triangle, (4) teaching concept definition for the triangle and its structure*; B – 1mark is given for

					improper knowledge that lacks (*1) prototype example and non-examples of triangle, (3) representations of the triangle definition (4) teaching concept definition for the triangle and its structure*
Q16	Open		If two points are mentioned - 2; one - 1; no - 0;	2	2 marks are given if the explanation includes *(1) essential properties of triangle for its definition in the representation, and (2) necessary and sufficient condition for defining the triangle concept in the representation,* 1 mark is given if the explanation includes one of the above points;
Q6	C	KCSCI	If: C - 5; A - 4; B - 3; D - 2; E - 1;	5	C – 5 marks are given for knowledge of students' misconception in *(1) regular and irregular quadrilateral, (2) counterexample of quadrilateral, (3) inner and outer angles of regular quadrilateral, (4) inner and outer angles of irregular quadrilateral, (5) angle sum formula of the angles in quadrilateral*; A – 4 marks are given for knowledge of students' misconception that misses *(4) inner and outer angles of irregular quadrilateral*; B – 3 marks are given for knowledge of students' misconception that lacks *(1) regular and irregular quadrilateral* and *(4) inner and outer angles of irregular quadrilateral;* D – 2 marks are given for knowledge of students' misconception that lacks *(1) regular and irregular quadrilateral, (3) inner and outer angles of regular quadrilateral, (4) inner and outer angles of irregular quadrilateral;* and E – 1 mark is given for knowledge that only knows students' misconception on *(1) regular and irregular quadrilateral.*
Q14	Open	KCSCD	If three points are mentioned - 3; two - 2; one - 1; no - 0;	3	3 marks are given for knowledge of students' misconception of (1) *the same chord and the same amount of the circumference, (2) the opening - mathematical term; (3) inscribed angle;* 2 marks are given if the explanation lacks one of the above points; and 1 mark is given if the explanation mentions only 1 point.
Q12	F	CCKCD	If: F - 4; A - 3; E, C, D - 2; B, G - 1;	4	F – 4 marks are given for knowledge of *(1) all properties of the shape, (2) understanding of a structure of the concept definition, (3) essential attributes of the shape for its definition and structure of the concept definition, and (4) definition through 'sibling' concept*; A – 3 marks are given for knowledge that misses (*2) understanding of a structure of the concept definition;* E, C, D – 2 marks are given for knowledge that lacks *(2) understanding of a structure of the concept definition, and (4) definition through 'sibling' concept*; B, G – 1 mark is given for knowledge that lacks *(2) understanding of a structure of the concept definition, (3) essential attributes of the shape for its definition and structure of the concept*

					definition, and (4) definition through 'sibling' concept;
Q17	B	KCCCD	If: B - 2; A - 1; C - 0;	2	B – 2 marks are given for knowledge of *(1) how the concept is formally defined in the curriculum and (2) which grade the given definition is intended*; A – 1 mark is given for knowledge that lacks *(2) which grade the given definition is intended*; C - No knowledge;
Q18	Open		If five points - 5; any four - 4; any three - 3; any two - 2; any one - 1; no - 0;	5	5 marks are given if the explanation includes points of (1) *symmetry concept intention in grade 7 curriculum, (2) content integration in different grade curriculum, (3) specific features of the definitions (grade 8, the definition includes a characteristic of the symmetry line itself), (4) textbook content; (5) student intended proficiencies in the curriculum;* 4 marks are given if the explanation includes 4 out of 5 points; 3 marks are given for 3 out of 5 points; 2 marks are given for 2 out of 5 points; and 1 marks is given for 1 out of 5 points
Q7	A	KCTCI	If: A - 3; C - 2; B - 1; D - 0;	3	A – 3 marks are given for knowledge of *(1) essential properties of the shape for promoting proper image of the shape, (2) examples and non-examples of the shape, (3) selecting appropriate representation taking into account of developing proper image of the triangle shape*; C – 2 marks are given for knowledge that misses *(2) examples and non-examples of the shape; and* B – 1 mark is given for knowledge that lacks *(2) examples and non-examples of the shape, (3) selecting the most appropriate representation taking into account of developing proper image of the triangle shape*
Q8	Open		If four points - 4; any three - 3; any two - 2; any one - 1; no - 0;	4	4 marks are given if the explanation points out *(1) consideration of student deeper understanding of the triangle concept image, (2) emphasis of attributes of the shape, (3) examples including prototype of a triangle & non-examples (intuitive and non-intuitive) of a triangle, (4) student common misconceptions;* 3 marks are given for 3 out of 4 points; 2 marks are given for 2 out of 4 points; and 1 marks is given for 1 out of 4 points
Q9	B	KCCCI	If: A - 1; B - 2; others - 0;	2	B – 2 marks are given for knowledge of *(1) which grade uses the given practical work practical work and (2) a difference between practical works in grades 7 & 8*; A – 1 mark is given for knowledge that lacks *(2) a difference between practical works in grades 7 & 8;*
Q10	Open		If four points - 4; any three - 3; any two - 2; any one - 1; no - 0;	4	4 marks are given if the explanation includes *(1) intention of practical exercise in grade 7 curriculum including textbook; (2) topic (geometrical construction of symmetry) intention in grade 7 curriculum including content integration through grades 6-9; (3) nature of the last question in the practical exercise - congruent*

					triangles; (4) particular proficiencies of grade 7 students; 3 marks are given for 3 out of 4 points; 2 marks are given for 2 out of 4 points; and 1 marks is given for 1 out of 4 points
Q11	Open		If four points - 4; any three - 3; any two - 2; any one - 1; no - 0;	4	4 marks are given if the explanation includes *(1) geometrical constructions of symmetrical triangles using the tools; (2) understanding of the reflection; (3) properties of the congruent triangles; (4) developing informal definitions for the symmetry;* 3 marks are given for 3 out of 4 points; 2 marks are given for 2 out of 4 points; and 1 marks is given for 1 out of 4 points
Q13	A	SCKCD	If: A - 3; D - 2; B, C, E - 1; F - 0;	3	A – 3 marks are given for knowledge of *(1) inclusive and exclusive definitions for the quadrilaterals (2) how the exclusive and inclusive definition is related to the classification of the shapes, (3) mathematically correct definitions*; D – 2 marks are given for knowledge that misses *(1) inclusive and exclusive definitions for the quadrilaterals*; and B, C, E – 1 mark is given for knowledge that lacks *(1) inclusive and exclusive definitions for the quadrilaterals (2) how the exclusive and inclusive definition is related to the classification of the shapes*
Total marks				**51**	

Scoring for qualitative/open data

There are two types of data – quantitative and qualitative. Most items on teacher MKT are multiple choices that represent levels of teacher knowledge. The response choices to these items are scored using the previously developed mark scheme which differentiates the responses as knowledge level. However, there are few open items that teachers are asked to write up the responses in words which mean qualitative data. In order to score the qualitative data on these items, beforehand, the responses to open items are coded using the microanalysis technique adapted from the grounded theory (Strauss & Corbin, 1998); and it is used to translate qualitative data into quantitative as well as provide qualitative interpretation. By Strauss and Corbin (1998), the microanalysis is the detailed line-by-line analysis to generate initial categories with their categories and dimensions and to suggest relationships among categories; a combination of open and axial coding (p.57). Originally, this technique is developed in the context of developing a new theory building in social qualitative research, however, it is reasoned out that especially, ideas of identifying dimensions and establishing categories are effective to deal with qualitative data of the research.

The open coding is an analytic process through which concepts are identified and their properties and dimensions are discovered in data; meantime, the axial coding is a process of relating categories to their subcategories, termed "axial" because coding occurs around the axis of a category, linking categories at the level of properties and dimensions (Strauss & Corbin, 1998). This research applies the open coding to identify the dimension of the data using data to data comparison. After the open coding took place, the axial coding is utilized for establishing categories of the data organizing the dimensions into conceptual clusters based on the mark scheme. The clusters enable to score the data using the mark scheme as well as interpret it in qualitative manner. For example, Q18 is an open response item. One of the teachers responded the "*Definition A*" is the most appropriate for grade 7.

Figure 6. Q18 item and teacher's response

Q18. Why do you think your selected definition is appropriate with grade 7?
Definition A: *If line "a" crosses through the midpoint of AB segment; and this line is perpendicular with the segment, points A and B will be the symmetrical along the line "a". Line "a" is called as a mirror line of the symmetry*.
Teacher's actual response: *Because, I know, this topic is intended in the grade 7 curriculum. In addition, this is a basic definition for the symmetry.*

By Figure 6, teacher's actual response is analyzed line by line, and labeled ideas in the response. This is a simple case to code the response openly. Here, three clear concepts mentioned, thus, categories are created as "*curriculum intention", "level of the concept definition",* and *"definition for the symmetry*". These 3 categories are compared to other responses and analyzed taking into consideration of aspects in research framework. In order to conduct axial coding, 3 categories need to be linked categories at the level of properties. Obviously, *"level of the concept definition",* and *"definition for the symmetry*" categories need to be analyzed to identify how they are linked. In context of secondary school geometry, a level of concept definition is related to a level of content of a concept that reflected in a definition. These 2 categories are emerged under "*a level of the symmetry content given in the definition*". Now, there are 2 clusters as "*grade 7 curriculum intention of the symmetry*" and "*a level of the symmetry content given in the definition*". By using the scoring scheme (Table 19), these two categories are given marks in number values. It is a sample of how quantification process is occurred in qualitative data.

Data analyzing method

Quantitative and qualitative analysis are performed in the data analysis. Qualitative analysis is also done in order to interpret what is beyond results in numbers. However, before to start the analysis, data screening took place focusing on data missing, no-responses and attrition. After the screening is completed, numerical analysis on teacher demography, teacher reflections, MKT and beliefs are done.

Descriptive statistics are estimated in the mean, standard deviation, frequencies, including normality check – Shapiro-Wilk test, the test for univariate normality are sensitive to even slight departures from a normal distribution (Lix & Keselma, 2010), the reliability that presents in Cronbach or Split-halves coefficients. The reliabilities of questionnaires on teacher reflections, beliefs and MKT items are estimated. The reliability estimation is done using Cronbach alpha coefficient. Construct validity of the research instruments is estimated using exploratory factor analysis. Exploratory factor analysis is conducted by discovering common patterns in a set of items or variables. This factor analysis enables to cluster the variables or the questionnaire items based on the shared variances. In other words, it clusters which variables go together. In general, as for the factor analysis, bigger sample size is recommended; however, Guadagnoli and Velicer (1988) proposed that it could be used for a smaller sample size research.

Teacher' MKT is analyzed in simple Excel sheet applying frequencies of teachers' responses. Characteristics of teachers' MKT is squeezed form this frequency table. In order to verify the results of MKT, interrelations among the sub-domains of MKT as well as CI and CD items are analyzed using the canonical correlation analysis (CCA) and respective coefficients. The general goal of CCA is to uncover the relational patterns between two sets of variables. CCA has often conceptualized as a unified approach to various univariate an multivariate parametric (Knapp, 1978; Thompson, 1991 cited by Fan & Konold, 2010) and some non-parametric statistical procedures (Fan, 1996; Knapp, 1978; cited by Fan & Konold, 2010). This analytical approach can be used in a wide range of substantive issues in education (Fan & Konold, 2010). One of the concerns for using CCA is that there should be a reasonable expectation that the two set of variables are substantively related, and that the relation between the two sets of variables is of potential research interest (Fan & Konold, 2010). One of the advantage of using CCA is that it gives more parsimonious understanding about the relation patterns between the sets of variables. Characteristics of teacher MKT will be identified by picking up the most frequency responses to the MKT questionnaire items.

Teacher beliefs, school contexts, and pre-service teacher education context are analysed using PivotTable in Excel, and additional analysis on relathionships are done estimating simple correlations among the variables in Excel.

In order to have a deeper analysis, quantitative analysis is followed by the interview with focus groups; and it aims to support the interpretation of the results of quantitiave analysis by digging into what is beyond the responses and results of data analysis.

CHAPTER FIVE. RESULTS, FINDINGS AND DISCUSSION

Data is analyzed using quantitative and qualitative methods, however, it must be noted that qualitative analysis is done using quantitative analysis. Results of the quantitative analysis are interpreted and deeply discussed using data from the interview with groups of the sample teachers. Data is analyzed in accordance to the research questions. Moreover, data analysis of the respective instruments estimates data distribution tendency as well as the reliability of the instruments through descriptive statistics.

5.1 Teacher MKT geometry

This analysis deals with first research question to reveal Mongolian secondary school teachers' MKT with reference to the plane shape concept. This analysis reveals what MKT the sample teachers have, how they performed on the questions, what responses they presented more or less, what interrelations are within and between MKT subdomains and MKT in concept image and concept definition, what are differences among teachers in terms of their MKT and characteristics of their MKT.

Reliability and validity of MKT questionnaire

At the beginning, overall behavior of the data is described in Table 20 as of means, standard deviation, and normality check using Shapiro-Wilk. The normality check provides opportunity to apply the appropriate statistical methods in data analysis. The reliability of the questionnaire is estimated as Cronbach alpha coefficient.

Table 20. Descriptive statistics and reliability for MKT data

Min	Max	Mean	Standard deviation	Shapiro-Wilk Test			Reliability
				Statistics	df	Sig.	
10	29	21.30 CI(20.10; 22.50)	4.512	.984	57	.281	.656

The descriptive statistics in Table 20 summarizes the data behaviour as follow. It is evidenced by Shapiro-Wilk statistics (.984 at p value of .281) that the test is normally distributed. The reliability of the test is estimated by Cronbach coefficient at .656 which indicates the moderate reliability, however, it can be interpreted by small sample size. In general, for the research context, this level of reliability is acknowledged considering the research limitations.

In order to ensure the validity of the questionnaire, the exploratory factor analysis is conducted in questionnaire data. Based on the theoretical perspective of the research, 5 factors are given in the estimation; and rotated eigenvalues are used in the analysis. Since the purpose of the analysis is to validate the questionnaire, we will see how the items of the questionnaire is clustered based on the rotated factor loadings. It must be noted that by Kline (1994), the signs of the loadings do not affect the interpretation of the magnitude of the factor loading. The factor analysis is done using SPSS software; and it provided the following results (Table 21).

Table 21. SPSS output for Rotated factor matrix after Varimax rotation

Item/variable	Factor				
	1	2	4	5	6
Q1 : CCK/CI				.682	
Q12: CCK/CD				.591	
Q2 : SCK/CI			.579		
Q3 : SCK/CI			.798		
Q4 : SCK/CI			.675		
Q5 : SCK/CI			.611		
Q13: SCK/CD			.598		
Q6 : KCS/CI					.608
Q14: KCS/CD					.509
Q7 : KCT/CI		.612			
Q8 : KCT/CI		.691			
Q15: KCT/CD		.517			
Q16: KCT/CD		.751			
Q9 : KCC/CI	.749				
Q10: KCC/CI	.571				
Q11: KCC/CI	.551				
Q17: KCC/CD	.536				
Q18: KCC/CD	.618				

Extraction method: Principal Axis Factoring; Rotation Method: Varimax with Kaiser Normalization;

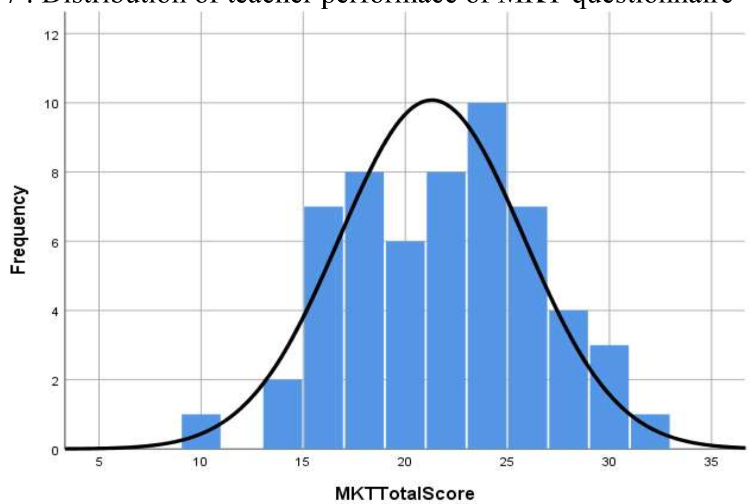
By above Table 21, initially, 6 actors are loaded; however, factor 3 does not show significant loadings. The factor loadings show that 5 factors have at least 2-5 variables or items those loadings are more than .511. This result indicate that items are exactly clustered into 5 factors that can be named as KCC for factor 1, KCT for factor 2, SCK for factor 4, CCK for factor

5 and KCS for factor 6; and all the items within a particular factor share the similar variances. Thus, it is statistically reasonable to say that that the questionnaire items measure what it is intended to measure.

Teacher MKT performance

To see general performance of teachers, total scores on the questionnaire is illustrated in Figure 7.

Figure 7 . Distribution of teacher performace of MKT questionnaire



The highest mark of teachers' performance is expected at 51; yet, the mean indicates 21.5marks. To identify the acceptable marks for teachers' marks, the cut-off point is estimated using Nedelsky's item-based method that borderline teacher responds to a multiple-choice question eliminating the answers that is recognized as wrong and then guessing at random from the remaining answers. By this method, the cut-off point is estimated as 12; teachers who scored above 12 point are considered as acceptable teachers in terms of MKT in CICD. By the data analysis, most (98.2%) teachers passed the cut-off point, and only one teacher was lower than the cut-off point. All teachers except lower point are marked between 14 to 31 points, and no one did approach the highest mark.

Teachers' MKTCI

Teacher responses to MKTCI items are summarized in the following Table 22.

Table 22. Teacher responses to MKTCI items

Question	Expected response	MKT sub-domain	Teachers' responses	Response %
Q1	A or D	CCKCI	A	48.3%
			B	1.7%
			C	8.3%
			D	41.7%
			E	-
Q2	T	SCKCI	T	53.8%
			F	38.5%
			U	7.7%
Q3	T	SCKCI	T	84.8%
			F	11.3%
			U	3.9%
Q4	T	SCKCI	T	94.4%
			F	3.7%
			U	1.9%
Q5	F	SCKCI	T	64.8%
			F	26.0%
			U	9.2%
Q6	C	KCSCI	A	41.1%
			B	12.5%
			C	37.5%
			D	7.1%
			E	1.8%
Q7	A	KCTCI	A	50.8%
			B	27.1%
			C	20.3%
			D	1.8%
Q8	Interpretations to why Representation 1 (A) is chosen	- The shapes that is not related to the triangle may confuse students; and convex quadrilateral is not studied at grade 7, however, it is good to compare the shapes - Obtuse, isosceles, and equilateral triangles are missing - Rectangle and polygon are not included, some kinds of triangles are missing, yet, it is a good example to explain various traingles - I am not sure how to explain circle to students as for this lesson - More shapes must be included like rhombus, square etc		
Q9	B	KCCCI	A	20.7%
			B	60.3%
			C	5.7%
			D	5.7%
			E	3.8%
			F	3.8%
Q10	Interpretation to why the given work is appropriate to Grade 7 student (B) - KCCCI	*This work is appropriate with Grade 7 students, because:* - It is appropriate with age and thinking of Grade 7 students - Content of grades 7 and 8 has this concept - it is related to symmetry; easy to consider the symmetry - It can be used in the symmetry teaching and learning		

		- At middle grades, students study the symmetry first time after the concept of congruent triangles is studied - Indeed, this work is appropriate with grade 6 to 8. The symmetry is taught to grade 6 students in relation to the coordinate system - In accordance to school curriculum - The reflection is a topic for grade 7, to give concept about symmetry - At grade 7, students are first time introduced the symmetry - In order to teach the coordinate planes, this practical exercise can be used - The content of the work is in line with the curriculum
Q11	Interpretation to what learning objectives could be set up the given exercise for students - KCCCI	*When I use this practical exercise for teaching the triangle concept, I would set up the following learning objectives*: - The objective can be set at advanced thinking level, real objects, - To transform as for the line symmetry - To learn how to find symmetrical lines; to construct symmetrical shapes - To place points in the coordinate system - To give understanding about the reflection transformation - To recognize congruent and non-congruent triangles - To find out, draw point coordinates and do practical exercise - To teach mirror lines, reflection lines and line reflection - Using the measurements, to be able to understand that it measures a distance between two points, as well as length is kept by reflection,

From Table 22, by Q1, majority of the teachers (90%) do know that all quadrilaterals whose two diagonals are both lines of symmetry are square and rhombi. It indicates that they have improper CCKCI of properties of symmetry for quadrilaterals. As for SCKCI (Q2-Q4), majority of teachers have mathematically sound knowledge of some symmetrical properties of polygons, nevertheless, they lack of knowledge that rectangle is a quadrilateral with exactly two lines of symmetry (Q5). Responses to KCSCI (Q6) question indicate that 37.5% of teachers did know students' misconceptions related to inner angles of the shape, and the remaining (62.5%) of teachers misinterpreted students' misconception. They lack of knowledge of the most precise appraisal of students' misconception that is about the meaning of inner angles in a case of convex quadrilateral. Because a student argued that *angle A* is about a right angle, *angle C* is only slightly larger than *angle A*. Here, a main cause of the misconception is that a student picks *outer angle C*, but not *inner angle C*. Teachers did not recognize this cause. Responses to KCTCI (Q7) question presents that teachers chose reasonably appropriate representation, nevertheless, by responses to Q8, their interpretation of why they chosen *REPRESENTATION 1* is related to the classification of triangle. None of them considered the significance of examples and non-examples which promote essential attributes of the shapes. It is an indication that they lack of knowledge of identifying significance of examples and non-examples of triangle. Examples and non-examples of triangle are necessary to gain coherent images and prevent potential conflict factors which cause students' further misconception in learning of triangle concept. Results of teachers' KCCCI (Q9) indicate that teachers are good at pointing which content is appropriate with grade 7 student; however, by Q10, they lack of knowledge to interpret why the given exercise or task is appropriate with certain grade and as Q11 responses, what learning objectives could be set.

Teacher MKTCD

Teachers' MKTCD for the plane shape is measured 7 items; and the most common responses of teachers are summarized in the following Table 23:

Table 23. Teacher responses to MKTCD items

Question	Expected response	MKT sub-domain	Teachers' responses	Response %
Q12	F	CCKCD	A	36.5%
			B	-
			C	5.8%
			D	3.8%
			E	3.8%
			F	44.3%
			G	5.8%
Q13	A	SCKCD	A	31.8%
			B	22.7%
			C	-
			D	22.7%
			G	11.4%
			F	11.4%
Q14	Open	KCSCD	*The complicating idea for given statement of a student would be related*: - Inscribed angles cannot take up the same amount circumference - Teachers should not accept this answer. The complicating idea is the term "opening"	

<table>
<tr><td></td><td></td><td></td><td colspan="2">- The complicating idea is "the openings are the same"
- The opening is the complicating idea, however, I am not sure what lines (parallel or intersected) are discussed here
- The complicating idea is "to take up the same amount circumference". It is supposed to be equal length circumference
- What the opening means is not clear, as well, the explanation does not use the definition
- No complicated idea
- Triangle areas do not have to be equal when angles are the same and its inverse sides
- Angle definition must be learnt well
A student has misconception about the angle</td></tr>
<tr><td rowspan="4">Q15</td><td rowspan="4">D</td><td rowspan="4">KCTCD</td><td>A</td><td>11.9%</td></tr>
<tr><td>B</td><td>6.8%</td></tr>
<tr><td>C</td><td>54.2%</td></tr>
<tr><td>D</td><td>27.1%</td></tr>
<tr><td>Q16</td><td>Interpretation to advantages of Representation C (C)</td><td colspan="3">Because REPRESENTATION C enables to:
- To classify and name the triangle by angles
- To discuss about classification of triangles, to give the understanding and reinforce about the structure of triangle, however, equilateral triangle needs to be added
- To give understanding about the triangle shape, classified triangle shapes need to be shown
- To give more understanding about various triangles
- Appropriate with improving the understanding of a triangle. Seems, it is appropriate to the classifying triangles as angles and sides
- Various types of triangles are shown, To create various triangles and determine its properties
- Shapes of triangles (isolesces, right, acute and obtuse) are well seen, however, equilateral triangle must be included
- All types of triangle is included, and it is appropriate to explain that triangles are classified as its edges and angles.
- Angles are varied; orientations are varied; right and isosceles triangles can be seen
- A triangle is drawn when three points that do not lie on the same plane are connected by straight line. Thus, depending on the position of three points that do not lie on the same plane, various triangles can be created. REPRESENTATION C enables to show it to students
- It is effective to explain about angles and sides in more descriptively
- What are common aspects in all the shapes can be discussed
- I consider a reinforcing lesson, so, I would represent various shapes for next lessons
- This representation shows right, acute, obtuse, equilateral, isosceles, and scalene triangles
- It shows almost all types of triangles, so, it is appropriate to inference and define triangle</td></tr>
<tr><td rowspan="3">Q17</td><td rowspan="3">B</td><td rowspan="3">KCCCD</td><td>A</td><td>22.6%</td></tr>
<tr><td>B</td><td>62.3%</td></tr>
<tr><td>C</td><td>9.4%</td></tr>
</table>

			D	-
			E	1.9%
			F	3.8%
Q18	Interpretation to why the given definition can be taught to Grade 7 (B) students	*A reason why it can be used for Grade 7 is that*: - It is intended in the standards, It is intended in this grade content - Appropriate to grade 7 students' age and thinking - The curriculum indicates that this topic for grade 7, as well, this is a basic definition of symmetrical shapes - Appropriate to give understanding related to reflection lines - At grade 7, students start to learn the transformation - Grade 6 and 7, it is good to have understanding when coordinate system is studied - At grade 7, in order to learn the coordinate system, students deal with finding out symmetrical points, and symmetry transformation, this work must be done before the coordinate system - This will be explained in relation to the coordinate system - At grade 7, symmetry is introduced, however, it could be used any grades - During study of the positional relationship between points and lines, understanding of the symmetry is given - Midpoint of the segment; constructing a shape in the coordinate system - Notation of A and B, fundamental understanding is provided at grade 7		

Table 23 shows that teachers have better knowledge of concept definition (CD) of the plane shapes compared to its concept image (CI). As for CCKCD (Q12), only 44.3% of teachers have knowledge of the definition for parallelogram and its structure. The remaining 55.7% of teachers do lack of knowledge that a formal concept definition in geometry establishes necessary and sufficient conditions for the concept, and the set of conditions should be minimal. By the school textbook, the rectangle is defined the parallelogram, mean, this is supposed to be known all the teachers. SCKCD (Q13) question intended to reveal teacher knowledge inclusive and exclusive definitions for the quadrilaterals. Inclusive definition specifies, for example, trapezoid is a quadrilateral with at least one pair of sides parallel which means that a parallelogram is a special type of trapezoid. Exclusive definition specifies, for example, a trapezoid is a quadrilateral only one pair of sides parallel which excludes parallelogram as trapezoids. Only one third of teachers have mathematically correct knowledge of two types of definitions which are related to the classification of quadrilaterals. The remaining teachers lack of SCKCD in inclusive and exclusive definitions for quadrilaterals. KCSCD (Q14) question is related to the definition for inscribed angles and student complicating ideas. Three points could be discussed here:

- The same chord and the same amount of the circumference;
- The opening – mathematical term and inscribed angles.

Most teachers recognized the above points when they figure out students' complicating ideas in the statement. For KCTCD (Q15), 54.2% of teachers responded that to teach the structure of definition for triangle, they would use *REPRESENTATION C* because it presents all types of triangle (Q16). In other words, they emphasize the triangle classification in the representation. Most teachers do not know to consider how the chosen representation illustrates the structure (necessary and sufficient condition) of the definition in the interpretation. Responses to KCCCD (Q17) question illustrate that teachers know the curriculum intention of topics. Open response to Q18 in KCCCD intended to reveal teachers' knowledge of symmetry concept in grade 7 curriculum, content integration in different grade curricula, specific features of the given definition (at grade 8, the definition includes characteristics of the symmetry line itself), and students' competences in the curriculum. Teachers' interpretation was not very precise, and their interpretation is limited by which topic should be taught to which grade students; nevertheless, lack of knowledge to interpret why the given task is appropriate to certain grade students.

In general, based on the results in Table 22 and 23, sampled teachers' MKT geometry can be characterized by CCK, KCT and KCC, however, some inconsistency in SCK and KCS; however, it is likely to depend on if it is about the concept image or concept definition of the shape. For concept image of the shapes, teachers have proper CCK, SCK and reasonable KCT. It may be due to a fact that the curriculum including textbook do not have proper content for developing students' mental image of the shape concept. They lack of KCS of the plane shapes and symmetry.

As for the CICD for the shapes, they lack essential knowledge of teaching the concept image; and it is limited by formal exclusive definition for the concept missing inclusive definition for the shapes.

Based on the results in Tables 22 and 23, the following characteristics can be identified for teachers' MKTCI and MKTCD. Findings for each sub-domain of teachers' MKTCI and MKTCD referencing the plane shape are presented in Table 24.

Table 24. Characteristics of teachers' MKTCI and MKTCD

MKTCI	MKTCD
CCKCI: ***Proper common*** knowledge of quadrilateral images when symmetry is involved	**CCKCD**: ***Limited common knowledge*** of the formal concept definition of the shapes.
SCKCI*: Proper specialized* content knowledge of images polygons with particular symmetrical properties that is not commonly discussed and knowing if the given statements about the polygon images are mathematically true or not	**SCKCD**: ***Lack of specialized knowledge*** of choosing mathematically correct definition of rectangle concept taking into consideration of exclusive and inclusive and exclusive classifications of shapes. Their knowledge is limited by the formal definitions do not pay attention on structure of (necessary and sufficient condition) the concept definition of the shapes
KCSCI: ***Limited knowledge*** about students' common misconception related to quadrilateral images and causes of students' misconception on inner angles of quadrilaterals.	**KCSCD**: ***Proper knowledge*** related to students and concept definition is identified as sound. They know what is confusing in their ideas related to the definition of inscribed angles.
KCTCI: ***Limited knowledge*** about the choice of representation for teaching triangle concept images. To make a choice of the representation, they picked the most appropriate representation, yet, focus of the choice was on the classification of the shapes, therefore, their interpretation of instructional advantages and disadvantages of the chosen representation did not consider examples and non-examples of a triangle that highlight critical attributes of the shape for developing students' images of triangle concept.	**KCTCD**: ***Limited knowledge*** for representing the concept definition for a triangle to students. Their interpretation of the instructional advantages and disadvantages of the chosen representation is deviated due to their emphasis on the triangle classification. The interpretation is supposed to deal with how to use examples and non-examples in the representation to define the triangle concept.
KCCCI: ***Knowledge*** of what grade students should be taught the symmetrical property of triangle through geometrical construction, their prior and after knowledge in the curriculum, yet, ***limited knowledge*** of reflecting the curriculum and what learning goals can be set for this activity of construction.	**KCCCD**: ***Knowledge*** of curriculum content of the concept definition for quadrilateral symmetry, ***limited knowledge*** at what grade level students are typically taught the formal definition of symmetry and students' familiarity (previous and after knowledge related to definition) with the definitions.

5.2 Teacher beliefs

This analysis deals with the sample teachers' beliefs about the nature of school geometry and the learning of geometry which is an issue of the second research question. The analysis digs into what belief about the nature of school geometry the sample teachers have, what are the most and least believed view (Instrumentalist, Platonist, Problem solving), what they believe about the learning of geometry and what are the most or least believed perspective of the geometry learning. It also analyzes what relationships exist between beliefs about the nature of school geometry and the learning of geometry.

Descriptive statistics including reliability

Descriptive statistics provides general trends in the data distribution. Table 25 shows this trend via mean, standard deviation, normality (Shapiro-Wilk test) and reliability (Cronbach alpha) coefficients with related 95% confidence intervals.

Table 25. Descriptive statistics and reliability for belief about the nature of geometry

Mean	Standard deviation	Shapiro-Wilk Test			Reliability
		Statistics	df	Sig.	
89.67 95% CI (86; 92)	11.246	.865	56	.000	.791

The descriptive statistics indicate that the questionnaire's mean is 89.67 with 95% confidence interval (86; 92). Standard deviation is estimated at 11.246. The Cronbach coefficient (.791) presents that it is reasonable to rely on the questionnaire results.

The same as teachers' MKT questionnaire, construct **validity** of belief questionnaire is ensured using exploratory factor analysis. The factor analysis is conducted using SPSS and provided the next results (Table 26) in the same way as MKT questionnaire. Only difference was that this factor analysis did not provide a number of factors, instead, the analysis naturally clustered the factors. There are two different beliefs – belief about nature of school and discipline geometry; and belief about geometry learning. Thus, the factor analysis is separately done.

Table 26. SPSS output for rotated factor matrix on belief about nature of geometry

Item/variable	Factor		
	1	2	3
B11: Instrumentalist & Instrumentalist	.879		
B12: Instrumentalist & Platonist	.858		
B13: Instrumentalist & Problem solving			
B14: Platonist & Instrumentalist			.869

B15: Platonist & Platonist			.745
B16: Platonist & Problem solving			.700
B17: Problem solving & Instrumentalist		.797	
B18: Problem-solving & Platonist		.770	
B19: Problem solving & Problem solving			

Note: Extraction method: Principal Axing Factoring; Rotation method: Varimax with Kaiser Normalization

By Table 26, the items or variables on belief about the nature of school and discipline geometry are loaded on 3 factors. Factor 1 mainly shares the same variance with items or variables more about instrumentalist view of belief about the nature of school geometry; and factor 2 shares with variables about problem solving view of belief about the nature of school geometry. Factor 3 is more likely to include Platonist view of belief. Variable within the factors have the similar loadings. This result indicates that questionnaire items on instrumentalist and Platonist views of belief about the nature of school and discipline geometry may have the similar pattern. Items on problem solving is distinctive pattern from the previous 2 views of belief, however, B11 and B19 items do not share the variance with three factors. Thus, it can be said that it behaves quite differently from other, thus, items B11 and B19 were not included in the data analysis. It can be noted that the questionnaire items, in some extent, except B11 and B19, can provide valid results.

Belief about nature of school and discipline geometry

Teachers' belief about the nature of school geometry is investigated through questions from B11 to B19 in the belief questionnaire. Teachers' responses to the questionnaire items are summarized via descriptive statistics (means and standard deviations) as following Table 27.

Table 27. Means and standard deviations for teachers' belief about the nature of geometry

		Beliefs about nature of discipline geometry					
		Instrumentalist (Ins)		Platonist (Plat)		Problem solving (PSol)	
		Mean	SD	Mean	SD	Mean	SD
Beliefs about the nature of school geometry	Ins	4.45	1.159	4.77	1.009	2.98	1.272
	Plat	4.85	.97	5	.915	4.16	1.424
	PSol	4.09	1.297	4.04	1.22	NA	NA

The results in Table 27 indicate that the sample teachers agree stronger in Platonist& Platonist view of belief about the nature of school and discipline geometry. Platonist& Platonist view presents that school geometry is a part of a body of hierarchical interconnected knowledge of understanding of which forms the basis on which some will learn higher level geometry. It is probably an indication that teachers do not hold distinctive belief about the nature of school geometry and discipline geometry.

In contrary, teachers hold weaker view of Instrumentalist& Problem solving belief which presents that school geometry is just for basic skills, so it is impossible to have creative ideas. By means of the views, these teachers strongly agree on Platonist view of the school geometry; and weaker agree on Problem solving view of school geometry.

Belief about geometry learning

At the beginning, it is necessary to discuss the validity of questionnaire items on belief about geometry learning. The research applied the exploratory factor analysis to validate these items (B21-B214) of the belief questionnaire. In total 14 items are analyzed. The same principle and method are applied to the factor analysis as MKT questionnaire. In brief, items from B21 to B214 are factorized using rotated factor loadings in SPSS software.

Table 28. SPSS output for rotated factor matrix on belief about the geometry learning

Item/variable	Factor			
	1	2	3	4
B21: Content-focused on performance				
B22: Content-focused on performance		.655		
B23: Content-focused on performance	.840			
B24: Content-focused on performance		.710		
B25: Content-focused with understanding		.819		
B26: Content-focused on performance	.815			
B27: Content-focused with understanding				
B28: Learner focused			.807	
B29: Content-focused on performance	.767			
B210: Content-focused on performance				
B211: Content-focused with understanding				.862
B212: Learner focused			.670	
B213: learner focused			.648	
B214: Content-focused with understanding			.759	

Note: Extracted method: principal axing factoring; Rotation method: Varimax with Kaiser Normalization

Table 28 shows that all variables/items except B21, B27 and B210 are clustered into 4 factors consisting of several variables with the shared variances. B21, B27 and B210 behave differently from other variables. By interview with a group of teachers, it was revealed that B210 question is differently understood by teachers.

By the loadings of the variables, as of Factor 1, B23, B26 and B29 share the same variances. These 3 questions indeed deal with the learning that promotes to getting correct answers. Meantime, B22, B24 and B25 share the same variances in Factor 2 loading. These are intended to measure view of belief about the geometry learning that focuses on content with correct procedures and speed. Thus, as it is planned in the methodology, it is reasonable to consider B22, B23, B24, B25, B26 and B29 items under the view of content-focused on performance. It should be noted that B25 is not supposed to be an item for content-focused on performance. It is planned to be an item for content-focused with understanding. Nevertheless, during the interview with a group of teachers, it is understood as about to follow teacher's explanation regarding correct procedures. Factor 3 loading clearly indicate the view of the learning focused on learners; and Factor 4 loading presents the content-focused with understanding view of geometry learning. By this analysis, except B21, B27, and B210, the questionnaire data is able to provide valid results, thus, data analysis is not include these 3 variables.

Table 29. Means and standard deviations for teachers' belief about the geometry learning

View	Aspects of view	Mean	SD
Content-focused on performance	✓ Memorizing formulas ✓ Getting correct answers, and procedures, ✓ Speed in the classrooms ✓ Learning under teacher's direct explanation of procedures	2.84	1.168
Content-focused with understanding	✓ Attending to the teachers' explanations ✓ Understanding why certain procedures and answers are there, ✓ Investigating why certain solution works for a particular problem,	4.25	1.199

Learner-focused	✓ Emphasis on students' own ways to solve a problem, ✓ Exploring different ways to solve a particular problems ✓ Solving a problem without teacher's direct help,	4.4	1.054

By the definition, Platonist teachers tend to hold a belief focusing on subject content with students' understanding. Results of descriptive statistics in Table 29 show a bit different scenario. By Table 29, the teachers have more agreement on learner-focused learning of geometry which enables students to figure out own ways to solve a problem and to explore different ways to solve it without teachers' direct help. Here, most commonly agreed statement is that "it is helpful for pupils to discuss different ways to solve particular problems in geometry" which promotes the understanding of the content. Moreover, teachers commonly believe that they should allow students to figure out their own ways to solve geometrical problems. Interestingly, teachers do not agree that "it does not really matter if you understand a geometrical problem, if you can get the right answer". Moreover, teacher disagree with that "when pupils are working geometry problems, more emphasis should be put on getting correct answer than on the process followed", "non-standard procedures should be discouraged because then can interfere with learning the correct procedure", and "hand-on geometry experiences aren't worth the time and expense". In general, it is reasonable to say that these teachers tend to hold belief about geometry learning focused on learners/students, but not merely content.

Relationship between belief about the nature of school/discipline geometry and its learning

In order to verify the result of the belief analysis, the relations between B15 (Platonist and Platonist view) and variables (B28, B212, B213, B214) are estimated in SPSS using Pearson Correlation with sig. 2-tailed. By the analysis, there were not statistically significant relationships between B15 and B212, B213. It indicates that teachers' Platonist & Platonist view of belief about the nature of school and discipline geometry is not correlated with view of geometry learning that emphasizes:

1. Pupils can figure out a way to solve geometrical problems without a teacher's help
2. Teachers should encourage pupils to find their own solutions to geometrical problems even if they are inefficient

These teachers do not believe that geometry learning happen efficiently with some help of teachers. Table 30 presents the statistically significant relationships.

Table 30. SPSS output for correlations of B15 with B28 & B214

		Belief about geometry learning – Learner focused	
		B28: Teachers should allow pupils to figure out their own ways to solve geometrical problems	B214: It is helpful for pupils to discuss different ways to solve particular problems
B15: Belief about nature of geometry – Platonist & Platonist view	Pearson Correlation Sig. (2-tailed)	.468** .000	.597** .000

By Table 30, there is a moderate relationship (Pearson correlation at .468 with Sig. 000) between Platonist & Platonist view of belief (B15) and learner focused belief about the geometry learning (B28). It means that as for Mongolian secondary school teachers' context, as teachers with Platonist belief about the nature of geometry tend to allow students to figure out their own ways to solve geometrical problems. This is a quite interesting result, because, by definition, Platonist teachers tend to emphasize the content with students' understanding. Moreover, a moderate relation is also estimated between B15 and B214. It indicates that teachers who agree stronger in Platonist view of belief are likely to support students' learning to discuss different ways to solve particular problems in geometry.

5.3 Association of teacher beliefs with MKT

By the research framework, third research question deals with how teachers' belief about the nature of school and discipline geometry is associated with their MKT. In order to seek an answer to this question, relationship between teachers' belief about the nature of geometry and their MKT is identified using Pearson correlation in SPSS. By the estimation, there is not positive relation between the belief and MKT, however, few negative correlations are estimated as it is shown in Table 31.

Table 31. Pearson correlation coefficients in teacher beliefs and MKT

	B12	B13	B14	B15	B16	B17	B18
CCK	.057	-.053	-.078	-.087	.098	-.049	-.053
SCK	-.145	.029	.030	-.138	-.150	-.305*	-.432**
KCS	.133	-.207	.158	.083	-.373*	.201	-.323*
KCT	.002	-.028	.060	.111	-.054	-.143	.218
KCC	.018	.060	-.038	.092	.100	-.022	-.062

Note: * Correlation is significant at the .05 level; ** significant at .01 level

Coefficients in Table 31 indicate that there are mainly statistically not significant and few significant relationships between teachers' belief and MKT. Teachers' KCS and SCK is negatively associated with some views of belief about the nature of school and discipline geometry; and this relationship is statistically significant. By a moderate, negative relationship ($r=-.305$, $p<.05$) between B17 and SCK, teachers' Problem solving view of belief about the nature of discipline geometry and Instrumentalist view about the nature of school geometry (B17) is negatively associated with teachers' SCK. In other words, teachers who strongly agree with this view of belief are likely to possess less SCK; and teachers who disagree with this view tend to have more SCK. By definition, Problem solving and Instrumentalist teachers tend to emphasize students' motivation to learn basic skills. By mild, negative correlations ($r=-.432$, $p<.01$) between B18 and SCK, teachers' Problem solving view of belief about the nature of discipline geometry and Platonist view about the nature of school geometry (B18) is also inversely associated with their SCK, statistically. These teachers focus on students' motivation in the learning of a body of hierarchical interconnected knowledge understanding of which forms the basis on which some will learn higher level geometry content; and they have less SCK that is a unique knowledge for teaching profession. Relationship between KCS and B16 ($r=-.373$, $p<.05$) presents that teachers' Platonist view of belief about the nature of discipline geometry and Problem solving view about the school geometry is negatively associated with their KCS. These teachers believe that school geometry is part of a body of hierarchical interconnected knowledge understanding of which enables the gifted few eventually to be mathematically creative. By correlation coefficient ($r=-.323,\ p<.05$), teachers' Problem solving view of belief about the nature of discipline geometry and Platonist view about the nature of school geometry (B18) is also negatively associated with their KCS. Indeed, these teachers believe that learner focus aims to motivate students so that they come to understand more of the body of hierarchical interconnected knowledge of geometry.

5.4 Situated-ness of teachers' MKT in school context

Situated aspect teachers' MKT in school context is investigated through professional activities within a school, teachers' individual and collaborative reflections. Beforehand, descriptive statistics are estimated to see general tendency and reliability of the data collected on the reflections. The same as previous analysis, the validity of the questionnaire is estimated applying exploratory factor analysis. Then, professional community activities at the schools, teachers' individual and collaborative reflections are analyzed clustering the by the schools. Teachers' MKT is also clustered by the schools.

Descriptive statistics including reliability

As it is discussed in the methodology section, school context is investigated through teacher collaborative and individual reflections. Teacher individual reflection is studied by R11-19 in the second part of the questionnaire. Collaborative reflection is investigated R21-R26 questions in the questionnaire. General charactersitics of the questionnaire for the reflections are illustrated in the descriptive statistics.

Table 32. Descriptive statistics for the reflections

Mean	Standard deviation	Shapiro-Wilk Test			Reliability
		Statistics	df	Sig.	
47.49	6.478	.903	56	.000	.832

Mean and standard deviation of the data are 47.49 and 6.478. Descriptive statistics present that the questionnaire reliability is estimated as Cronbach coefficient at .832, which indicates reliable instrument. Shapiro-wilk test coefficient shows that the data is normally distributed.

Professional community activities at schools

Professional community activities at each school is investigated using PivotTable in Excel. Teachers are asked to choose what extent they are involved in professional community activities in their schools. Techers' responses are measured Likert-scales, and averages of the scales are estimated to create the Pivot Table.

Table 33. Averages of teachers' responses on school level activities

	Unit assembly	Math Olympiads	Discussion with peer teachers	Lesson study	Pilot team meeting	Open lesson
School	Q171	Q172	Q173	Q174	Q175	Q176
School #2	3.75	2.88	2.50	2.75	1.88	2.25
School #20	3.88	3.00	2.88	3.75	3.50	3.00
School #45	3.60	2.60	2.80	2.80	3.20	2.40
School IRD	4.00	3.13	3.50	3.63	3.00	3.50
School Kho	3.90	3.29	2.90	3.19	2.81	3.05
School MoGe	3.80	2.60	2.90	2.30	2.10	2.10
Grand Total	3.85	3.00	2.92	3.08	2.72	2.78

Table 33 shows that mathematics teaching methodology unit assembly is the most common professional activity not only within an individual school but also among the schools. In order to dig into why this is the most common among the schools, head teachers and teachers of the schools are interviewed. It is identified that the unit assembly is regulated by the policy of all schools, thus, it is obliged to all schools to run the unit assembly.

Since the unit assembly is common among the school, in order to specify their features, second common activities are looked at. By averages of the responses in Table 33, the following results are appeared:

Table 34. Second common activities in schools

School	**Common professional collaborative activity among your school teachers is…**
School #2	Mathematics Olympiads among teachers as well as students
School #20	Lesson Study

School #45	Meetings on the piloting of curriculum and textbook
School IRD	Lesson Study
School Kho	Mathematics Olympiads among teachers as well as students
School MoGe	Conversation among peer teachers about geometry instruction

By Table 34, schools #2 and Kho conduct mostly Mathematics Olympiads among teachers as well as students. By interview with teachers from these 2 schools also verify this result; and they mainly focus on training secondary grade students to succeed at the Olympiads. The Olympiads ask students to solve more challenging mathematics problems, thus, reasonably, teachers of the schools are expected to be knowledgeable in challenging math problems. Schools 20 and IRD are likely to deliver Lesson study among teachers.

School #45 runs meetings on the piloting curriculum and textbooks. By interview with teachers, during the meeting, teachers mainly discuss about what challenges they faced to teach the content and how to improve the content of curriculum and textbooks reflecting the challenges.

School MoGe is a type of school where students' talents in art is signified. By interview results, teachers of the school tend to discuss more about how to teach the geometry within the allocated teaching hours. Novice or younger teachers like to discuss about the teaching the most difficult topics of geometry for teaching.

Teacher individual reflections

Firstly, to estimate the validity of individual reflection questionnaire, the factor analysis is also conducted in SPSS.

Table 35. SPSS outputs for rotated factor matrix on individual reflections

Item/variable	Factor		
	1	2	3
R11: Reflection by reading	.842		
R12: Reflection by reading	.865		
R13: Reflection by reading	.801		
R14: Reflection by observing		.733	
R15: Reflection by observing		.839	
R16: Reflection by observing		.674	
R17: Reflection by observing		.626	
R18: Reflection by listening to peers			.822
R19: Reflection by listening to peers			.858

Table 35 precisely presents that variables are loaded by 3 variables based on their shared variances. R11, R12 and R13 are loaded in the same variances, mean, they have the similar behavior. This behavior is regarded as teachers' reflection by reading. In the analysis R14, R15, R16 and R17 share the same variances. R18 and R19 express teachers' reflection by listening to peer teachers' discussion. The above result ensures that the data on teachers' reflection can provide valid results.

Table 36. Averages of teachers' responses on individual reflection

School	Reflection by reading				Reflection by observing			Reflection by listening	
	R11	R12	R13	R14	R15	R16	R17	R18	R19
School #2	2.75	2.75	2.25	2.13	2.38	3.25	3.75	2.75	3.00
School #20	3.38	3.38	3.38	3.13	3.38	4.00	4.00	3.63	3.63
School #45	3.20	3.40	3.40	3.00	3.20	4.00	3.80	3.60	3.40
School IRD	3.25	3.63	3.00	3.00	3.13	3.88	4.00	3.38	3.13
School Kho	3.29	3.29	3.14	2.81	3.19	3.74	3.76	3.48	3.33
School MG	2.90	2.80	2.30	2.70	3.10	3.60	3.20	3.00	3.10
Total	3.15	3.20	2.92	2.78	3.08	3.72	3.73	3.32	3.27

By Table 36, all school teachers reflect through observation. Teachers in schools #2, IRD, Kho tend to individually reflect observing how their students develop the image and definition for the shapes during the teaching in classrooms. Meanwhile, teachers in school #45 and MG individually reflect observing what common errors of their students are likely to repeat during the teaching in classrooms. Teachers in school #20 individually reflect observing students' image and definition development at the same time their common errors during the classroom teaching.

Teacher collaborative reflections

Teacher collaborative reflections are identified through R22 to R26 items of the questionnaire. These questions deal with teachers' collaborative reflections occurred at schools.

Table 37. Means and standard deviations of teachers' collaborative reflections

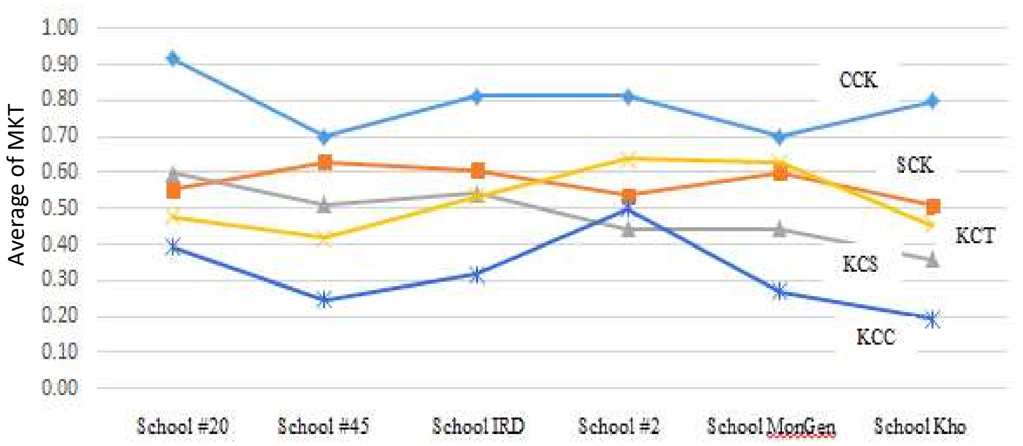
School	Possible representations of the subject matter, which is the most appropriate and why	Student common errors and in specific topics of geometry	Alternative learning activities to tackle with student difficulty or misconceptions in geometry	Most or least difficult part of specific topic teaching geometry	Essential ideas in the geometry topic
	R22	R23	R24	R25	R26
School #2	3.13	3.00	2.88	2.88	2.88
School #20	3.25	3.25	3.13	3.25	3.13
School #45	2.60	2.40	2.60	2.80	3.20
School IRD	3.63	3.13	3.25	3.38	3.63
School Kho	3.14	2.86	2.57	2.86	2.71
School MoGe	3.10	2.70	2.70	2.60	2.90
Grand Total	3.17	2.90	2.80	2.93	2.98

In Table 37, underlined values present the most common professional activities within a school. By Table 37, all schools, except school #45, tend to promote teachers' collaborative reflections on possible representations to teach, the most appropriate way of the representations and why it is appropriate. In addition, teachers of school #20 also collaboratively reflect common errors of students and essential ideas in a certain concepts in geometry. Teachers of school #45 as well as school IRD are likely to reflect the most or least difficult part of specific topic in teaching geometry. In overall, reflection of possible representations to teach and the most appropriate way of the representation is the most common collaborative reflection in the schools.

Teachers' MKT as of the schools

Situated aspect teachers' MKT in school context is investigated through teachers' individual and collaborative reflections. Therefore, how teachers' MKT is varied as schools is a part of the analysis. Teachers' MKT is clustered by the schools using PivotTable in Excel. The pivot tables and graphs are created using the averages of sub-domains of MKT. Here, sub-domains are not differentiated as CI and CD.

Figure 8. Teachers' MKT as of schools



By Figure 8, there are some minor variances in their MKT. For first 3 schools, teachers' CCK, SCK and KCS are better than KCT and KCC. Meantime, as for school #2 and MonGen, teachers are more knowledgeable in CCK, KCT and SCK. School 2MonGeni teachers have limited KCC; and school #2 teachers have limited KCS. As for school Kho, teachers' MKT has the similar pattern with MonGeni.

In general, among the schools, School #20 teachers are more knowledgeable in CCK and KCS, school #45 and IRD teachers in SCK, school #2 and MonGeni in KCT. Kho school teachers have limited MKT.

Table 38. Summary of school context

School	Common professional activity	Teacher individual reflection	Teacher collaborative reflection
School 20	Lesson Study	✓ Observation of how students develop the image and definition for the shapes during the teaching ✓ Observation of what common errors students are likely to repeat during the teaching	✓ Discussion on what are possible representation for geometry content, the most appropriate ones and why ✓ Discussion on what are student common errors and misconceptions related to specific geometry content ✓ Discussion on the most essential ideas in specific geometry content
School #45	Meetings for piloting of curriculum and textbook	✓ Observation of what common errors students are likely to repeat during the teaching	✓ Discussion on most or least difficult part of teaching specific content for teaching
School IRD	Lesson Study	✓ Observation of how students develop the	✓ Discussion on what are possible representation for

		image and definition for the shapes during the teaching	geometry content, the most appropriate ones and why ✓ Discussion on most or least difficult part of teaching specific content for teaching
School #2	Mathematics Olympiads	✓ Observation of how students develop the image and definition for the shapes during the teaching	✓ Discussion on what are possible representation for geometry content, the most appropriate ones and why
School MoGe	Conversation about geometry teaching	✓ Observation of what common errors students are likely to repeat during the teaching	✓ Discussion on what are possible representations for geometry content, the most appropriate ones and why
School Kho	Mathematics Olympiads	✓ Observation of how students develop the image and definition for the shapes during the teaching	✓ Discussion on what are possible representation for geometry content, the most appropriate ones and why

Relationships among the context aspects and MKT

In order to understand how school context situates teachers' MKT, relationships among the most common school-level professional community activities, the reflections and MKT sub-domains need to be analyzed. The relationships among these variables are estimated using simple correlations in Excel.

Table 39. Correlation matrix: MKT, reflections and professional community activities

		CCK	SCK	KCS	KCT	KCC
Professional activity	Mathematics Olympiads	.197	.122	-.234	.087	.017
	Lesson study	.108	.053	.008	.192	.088
	Pilot the curriculum and textbook team meetings	.083	.007	188	*118*	-.314*
Individual reflections	Reflection through reading about the representations and students' thinking				*-.353**	
	Reflection through observation on how my students image and define the shapes during my teaching	.095	.056	.190	.043	.288*
	Reflection through listening to other teachers about effectiveness of representations and students' misconceptions	.186	.066	.075	-.409**	*.171*
Collaborative reflections	Discussion on how to develop alternative learning activities to tackle with student difficulty or misconceptions in geometry	.084	.013	.276*	.046	.038

Note: ** correlation is significant at the 0.01 level, * significant at the 0.05 level

Table 39 presents that there are some weak/mild relationships (*underlined values*) among the variables. Mathematics Olympiads activity at schools weakly ($r=.197$) related to teachers' CCK and ($r=.122$) SCK. Lesson study activity has also weak ($r=.108$) relationship with teachers' CCK and KCT. However, these relationships are not statistically significant. Statistically significant, negative relationship ($r=-.314$, $p<.05$) is estimated between the school activity on the pilot the curriculum and textbook team meeting with teachers' KCC. It implies teachers in the schools, where more this kind of activity is held, are likely to associate with teachers' less KCC.

As for individual reflection, there is a statistically significant relationships between teachers' individual reflections and one of the sub-domains of MKT. Positive, mild relationship ($r=288$, $p<.01$) is calculated between teachers' observation on how students image and define the shapes during the teaching and their KCC. It means that how teachers know about the concept image and definition is associated with how it is reflected in the curriculum. In contrary, teachers' reflection through reading about the representations and students' thinking is negatively related ($r=-.353$, $p<.05$) to their KCT. Statistically significant negative relation ($r=-.409$, $p<.01$) is also calculated between teachers' reflection through listening to other teachers about effectiveness of representations and students' misconceptions and their KCT. This means that the listening to peer-teachers without contributing to the discussion is associate with less KCT of teachers.

Teachers' collaborative reflection is mildly related ($r=.276$, $p<.05$) to teachers' KCS. It indicates that teachers' discussion on how to develop alternative learning activities to tackle with student difficulty or misconceptions in geometry relate to what teachers know in KCS.

5.5 Pre-service teacher training context and MKT

In order to investigate how pre-service teacher education context is likely to contribute teachers' MKT, first of all, teachers' MKT is clustered pre-service teacher education institutions where the sample teachers graduated from.

Teacher pre-service teacher training institutions

MSUE is, as mentioned in Chapter 2, the largest teacher training institution in Mongolia. Most secondary school teachers are graduated from this institution. MSUE is much more oriented to education; thus, their curriculum is more likely to focus on teaching and learning. NUM is the largest and oldest science institution of Mongolia, and it focuses on more academic disciplines. However, secondary teacher training curriculum is a part of the education service of this institution. NUM is the institute where majority of curriculum authors work for.

Thus, this institute has a good capacity to promote the school curriculum to its students. KHU is one of the teacher training institutions in rural area of Mongolia. This institution trains secondary school teachers; therefore, students in this institution are expected to have reasonable knowledge of teaching and learning.

Teachers' MKT as of the institutions

Teachers' MKT is clustered by the pre-service teacher training institutions that teachers graduated from.

Figure 9. Teachers' MKT versus PRESET institutions

By Figure 9, it is seen that, in overall, teachers' KCT does not have significant difference for all the pre-service teacher training institutions. However, other sub-domains have slight differences depending upon the institution.

Except KCT, teachers who are trained in MSUE have reasonably better KCS; and it is due to a fact that MSUE students study more teaching and learning than students in other PRESET institutions. By the MSUE curricula for 2016-2019 academic years, there are 39 compulsory and 20 optional subjects. Out of total 59 subjects, 22 subjects are strongly related to teaching and learning of mathematics, and 17 subjects in mathematics discipline. The remaining are subjects in fundamentals of higher education, physics and astronomy.

Meantime, as for teachers from NUM, their CCK and KCC look better than others. Their CCK is comparatively better. This result can be interpreted referencing from who enroll in the NUM and what curriculum the NUM teach to students who is majoring in secondary mathematics teacher qualification.

By the entrance exam statistics, the NUM is usually first of high school graduates whose achievement in mathematics is higher. The NUM is traditionally perceived as the prestigious higher education institution in Mongolia, thus, students and parents tend to choose this institution as a first choice in entrance process. It means that students in the NUM have better subject matter knowledge of school mathematics. Other hand, by the curricula of the NUM, out of total 48 subjects, 12 of them related to teaching mathematics; and 27 subjects related to mathematics discipline. In addition, professors in NUM are leading curriculum developes of secondary school mathematics. Therefore, NUM educators are very much knowledgeable in curriculum development. It could be a factor why NUM teachers have better KCC.

As for KHU, teachers are comparatively less knowledgeable in sub-domains except KCT. There could be 2 reasons. One is that these institutions are 4th or 5th choice of students who would like to pursue higher education. Second, the KHU curriculum has 47 subjects; and 10 subjects are related to teaching mathematics. 20 subjects are in mathematics discipline. One of the features of this institution is that it emphasizes psychology subjects. The KHU teaches 5 psychology subjects; and it is not the feature of other 2 institutions.

In order to dig into why KCT is dominantly well in Mongolian teachers, a group of educators (5) from MSUE and NUM is interviewed. These educators teach subjects like advanced geometry, mathematics didactics and functional analysis to prospective teachers. All of them also lead prospective teachers' teaching practice at schools. Some of them are involved in secondary mathematics curriculum and textbook development. They have 8-23 teaching experience in pre-service teacher training institutions. Protocol of the interview is summarized in Figure 10.

Figure 10. Protocol of interview with teacher educators

Researcher: *What do you think about school mathematics education in Mongolia?*

Profs: Well, many things have been changed since 1990. We used to emphasize academic mathematics in school, yet, nowadays, it has changed to focus on practical mathematics. Many perspectives and ideas are imported in school mathematics education from overseas. This import makes Mongolian secondary (primary as well) school teachers to abandon conceptual connections among the mathematics concepts and to separately teach mathematics concepts. In school mathematics, all content is supposed to be taught by forming mathematical concepts which has hierarchical structure. Mean, mathematics content in previous grade are a foundation for content at next grade. This structure helps students to develop mathematical concept formation. For example, trapezoid area concept is built on areas of triangle and rectangle.

Researcher: *What about geometry education in schools of Mongolia?*

Prof: Geometry is indeed very important subject for developing students' spatial thinking, mathematics thinking as well. Nowadays, geometry content is increased in secondary school mathematics. However, current geometry curriculum emphasize pragmatic geometry which causes complications in school teachers.

Researcher: *What is a pragmatic geometry?*

Prof: School students can create geometrical knowledge by hands-on or practical activities like drawing, folding papers, etc. Our school teachers are also taught how to make students to create knowledge in geometry teaching. However, it is very superficial. School teachers do not conceptually understand it.
Researcher: *In your training institutions, what subjects are related to geometry and mathematics teaching and learning in secondary school?*
Prof: We teach 3 subjects - mathematics didactics including teaching practice, general pedagogy and psychology. These 3 subjects prepare our students to acquire knowledge and skills related to mathematics teaching.
Researcher: *What is a focus of mathematics didactics course in your institutions?*
Prof: First to say, we try to base on what students learned in a school. We have a subject called *Mathematics Didactics*. By this subject, we teach how to teach mathematics to school children in general. Please note "in general". For example, how we teach mathematics based on secondary school children previous knowledge, how to motivate them in mathematics, how to facilitate them to create new knowledge in mathematics teaching. Our secondary school teachers mainly rely on textbooks.
Researcher: *Do you teach your students how children learn mathematics or geometry?*
Prof: Oh no, we do not teach these specific things. You know, teachers know it when they actually work with children. Thus, it is not necessary to be taught. Why we teach general didactics is that every individual teacher has own methodology, so, we can teach general didactic to help them to develop their own methodology for mathematics teaching. You know, from experience, it is well known fact that if there is one good teacher at particular schools, other teachers learn a lot from this teacher. So, I think, this is a matter of schools.
Researcher: *What do you mean "good teacher"?*
Prof: Good teacher is a teacher who knows mathematics or its teaching better. If there is a good methodology teacher in a school, other peer teachers are good at mathematics teaching. If there is a teachers who is awarded in mathematics Olympiads, other peer teachers know mathematics subject matter better than other school teachers. School by school, teachers are different in general.
Researcher: *Do you teach like how students learn, I mean, how secondary school students learn mathematics, like what mistakes or difficulties are in learning a specific mathematics concepts, why they do mistakes?*
Prof: No, we do not teach these things, we teach very general things related to teaching as I said before. People learn when they do or act. But we teach mathematics theories to students.
Researcher: *Theories? If so, do you teach things like why a certain algorithm works for a particular problem, if a particular solution are mathematically correct?*
Prof: Well, we believe that these things can be learned when they become school teachers. Maybe, it is a one of the reasons why our school teachers emphasize teaching algorithms and procedures. Probably, because of that, they do not teach conceptual knowledge to students. Our children know usually algorithms and procedures.
Researcher: *Why they emphasize algorithms and procedures?*
Prof: Well, teachers teach to students in grades 6 to 12. During grades 6-10, teachers emphasize the algorithms. At senior grades, they teach children how to perform well in tests. They focus on correct answers or solutions. This tendency is similar to junior grade teaching. They do not teach why a particular algorithm works for the problem. Maybe, our school teachers do not know it. Our children also do not know beyond the correct solutions and answers. Even students who entered our school have the same knowledge. They know algorithms and procedures in mathematics. Because of that, our school initiated more subjects related to school mathematics including geometry.
Researcher: *I would like to ask about concept image and concept definition in geometry. What do you think about it?*
Prof: As I said before, concept image and concept definition was a way how to construct concept formation in geometry. It is very important for developing geometrical thinking in children. I suppose, school teachers do not pay attention on building concept image in children. Instead, they are likely to teach formal concept definitions. This is maybe a consequence of teaching academic mathematics to our students.
Researcher: *Are Mongolian teachers aware of it?*
Prof: Not sure, our teachers usually teach what is written in the curriculum and mathematics textbook.
Researcher: *What about geometry curriculum?*

Prof: Geometry is one of the domains in mathematics curriculum. Prospective teachers face the curriculum once they become school teachers. School teachers may know all lists of contents, yet, they are not aware of why a particular content is there. Teachers are not aware of vertical and broader span of mathematics content in the curriculum.
Researcher: *What do you teach in general pedagogy?*
Prof: We do not teach pedagogy. Pedagogy teachers usually teach cognitive theories in teaching.

This interview precisely expresses that KCT is a focus of pre-service teacher training in Mongolia, as well as, they appreciate significance of content knowledge for school teachers. KCS is left to teachers as they gain this knowledge of how students learn specific mathematics content through actual teaching experience in schools.

Moreover, to see how pre-service teacher training contributes teacher's MKT, it is necessary to see MKT of prospective teachers in MSUE because majority (68.4%) of sample teachers is from this institution.

MSUE Prospective teachers' MKT

In total, 38 prospective teachers are involved in the study. Most of them are females; and they are aged 17 - 23. The sample prospective teachers are selected from all course years at the training institutions. Their demographic data is described in Table 40.

Table 40. Demographic data of prospective teachers

Gender (%)	Age (%)	Course year (%)
Female – 80.7% Male – 19.3%	17 and 18 – 22.58% 19 – 32.26% 20 – 12.9% 21 – 22.58% 22 and 23 - 9.68%	Year 1 – 29.03% Year 2 – 32.26% Year 3 – 9.68% Year 4 – 29.03%

Prospective teachers are given the same questionnaire as teachers in the study. Their CCK, SCK, KCS, KCT and KCC is illustrated in PivotTable of Excel (Table 40) using average estimation of marks for their responses to each sub-domain items.

Table 41. Prospective teachers' MKTCI

Question	Expected response	MKT sub-domain	Teachers' responses	Response %
Q1	A or D	CCKCI	A or D	68.1%
			B, C	31.9%
Q2	T	SCKCI	T	31.03%
			F	62.06%
			U	6.94%
Q3	T	SCKCI	T	29.03%

			F	70.97%
Q4	T	SCKCI	T	46%
			F	43%
			U	11%
Q5	F	SCKCI	T	40%
			F	56%
			U	4%
Q6	C	KCSCI	A	5%
			B	0%
			C	18%
			D	40%
			E	37.%
Q7	A	KCTCI	A	42.1%
			B	42.1%
			C	5.8%
Q8	Interpretation to why both representation 1 or 2 (A, B) are important	*Reason of why Representation 1&2 are chosen:* - It will be easier for students to understand as it shows other geometrical shapes including various triangles - It enables students to observe how properties of a triangle are different from other shapes - To recognize triangles from the shapes - It represents that every three-angled shapes cannot be triangles - It contains appropriate shapes to show attributes of triangle - More shapes are included so it is effective to compare triangles with other shapes		
Q9	B	KCCCI	A	19%
			B	28.6%
			C, D, F	71.2%
Q11	Interpretation to what learning objectives could be set up the given exercise for students - KCCCI	*When I use this practical exercise for teaching, I would set up the following learning objectives*: - An objective could be as students to create knowledge about the symmetry - An objective of this practical exercise is to enable students to construct the symmetry, and create triangles using hands-on materials - The objective could be to construct symmetrical triangles along the given point using the tools - Students like interesting problems - It is appropriate to extend students' understanding because it demands more knowledge about triangle from students		

By Table 41, 68.1% of the prospective teachers responded to CCKCI question, correctly. Majority of them know the quadrilaterals whose 2 diagonals are both lines of symmetry. Prospective teachers' responses to SCKCI questions reveal that they do not know symmetrical properties when lines of the symmetry cut through sides of the polygon. Yet, they knew how the symmetry line divides the area of the shape and symmetrical properties of the rectangle. They responded 2 questions correctly and 2 questions incorrectly for

SCKCI. Most of the prospective teachers (82%) incorrectly responded to KCSCI questions. They did not know students' misconception about the meaning of the interior angles of non-convex polygon. Results of KCTCI question indicate that the same number of prospective teachers have chosen representation 1 (42.1%) and representation 2 (42.1%). They have explained why the particular representation is chosen. In contrast to school teachers, prospective teachers tend to emphasize attributes of a triangle in choosing the representation. This maybe implication that, for the teaching, prospective teachers are likely to give significance on developing mental images of the shapes in students. As for KCCCI, prospective teachers are not familiar with the curriculum. However, their assumption to set the learning objectives looks a quite interesting as they are likely to promote geometrical construction.

Table 42. Prospective teachers' MKTCD

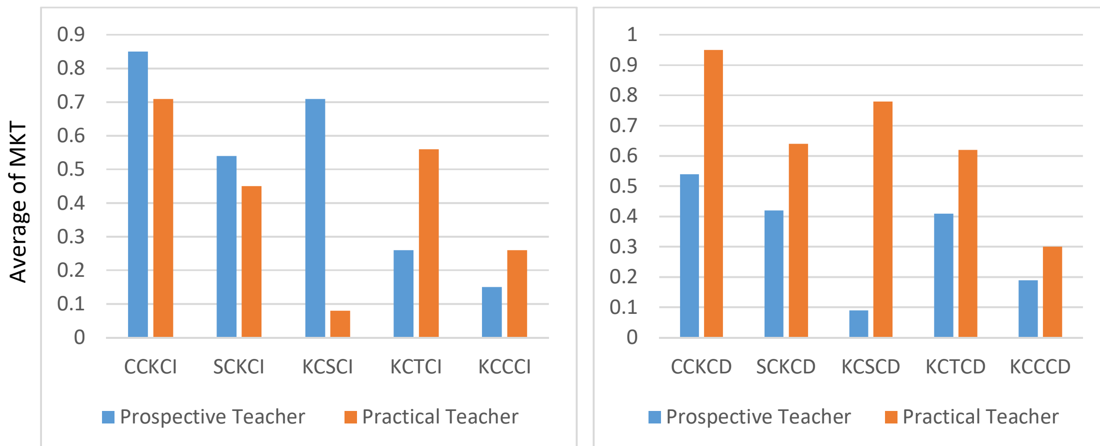
Item	Expected response	MKT sub-domain	Teachers' responses	Response %
Q12	F	CCKCD	A	42%
			B, G	3%
			C, D, E	53%
			F	-
Q13	A	SCKCD	A	17.8%
			B, C	57.1%
			D	14.3%
			G	-
			F	10.8%
Q14	Open	KCSCD	*The complicating idea for given statement of a student would be related*: - If I were this teacher, I would explain chord angle and circumference angles using other representations - I would have more descriptions about what this student talked - I would consider this statement as one of the practical examples - I would ask this student to explain what he said using geometrical tools such as ruler, protractor to other students - There is the same chord so the angles must be the same - "the opening" - It intersects the common chord - Inscribed angles take up the same length chord is not correct	

Q15	D	KCTCD	A	29.03%
			B	29.03%
			C	41.94%
			D	-
Q16	Interpretations to what advantage Representation C has	Because *REPRESENTATION C* enables to: - In order to make student to be understood geometry knowledge, first of all, example shapes are critical, so, I choose the first representation - There are sufficient examples to explain the triangle concept. For example, scalene, right, triangles are there. - This representation includes all types of triangles – classification of triangles - This representation precisely differentiate triangles form other shapes - I think it enables students to understand that angles of triangles can be varied - In order to understand attributes of triangle shape, students need to compare with other shapes - Attributes of triangles are emphasized - Triangle is a shape with three angles. This understanding must be given - The shapes are effective (1) to discuss about attributes of triangle, (2) to consider triangle area, (3) to recognize differences among the shapes		
Q17	B	KCCCD	A	33.33%
			B	66.67%
			C, D, E, F	-
Q18	Interpretations to why the given definition is appropriate Grade 7	*A reason why it can be used for Grade 7 is that*: - Definition is understandable for grade 7 children - This definition is simpler because grade 7 students have not yet studied AA' notation - Definition A is difficult, - AA' notation may cause confusion for students. Definition B is more precise and accurate		

Table 42 shows results of the prospective teachers' MKTCD. A result of CCKCD implies that a half of (53%) the prospective teachers did not know the formal definition for the rectangle. A term formal means that it is written in the curriculum and textbook. The same as CCKCD, 57.1% of the prospective teachers incorrectly answered to SCKCD question which is about inclusive and exclusive definitions for the quadrilaterals. Most prospective teachers could not answer to KCSCD question. As a result of KCTCD question, 41.94% of the prospective teachers have chosen Representation C to help students to improve their understanding of the formal definition for the triangle. They interpreted why they have chosen this representation is that it includes almost all different triangles. This is an indication that they may have focused on the triangle classification. Interesting result in

MKTCD is that 66.67% of the prospective teachers responded to KCCCD question correctly. However, their interpretation to why the given definition is appropriate to Grade 7 students was not accurate. A main point in their interpretation was about how AA' notation is used. Comparison of practical and prospective teachers' MKTCI and MKTCD may present clearer picture about the results.

Figure 11. Comparison of practical and prospective teachers' MKTCI and MKTCD



By Figure 11, practical and prospective teachers' CCKCI and SCKCI have the same tendency. However, prospective teachers are more knowledgeable in CCKCI, SCKCI and KCSCI. As for KCSCI, this result is probably due to a fact that the prospective teachers have some fresh memory of being school students, thus, they still remember what was difficult or misunderstood in school geometry. In contrary, school teachers are more knowledgeable in KCTCI and KCCCI compared to prospective teachers, and it is not surprising result.

As for MKTCD, prospective teachers know less than school teachers. Their results in CCKCD, SCKCD, KCSCD, KCTCD and KCCCD are lower than school teachers.

In order to dig deeper about prospective teachers' MKT, it is wise to understand how prospective teachers' MKT is likely to progress through years in the pre-service training institution, in particular, MSUE (Figure 12).

Figure 12. MSUE Prospective teachers’ MKT versus study years

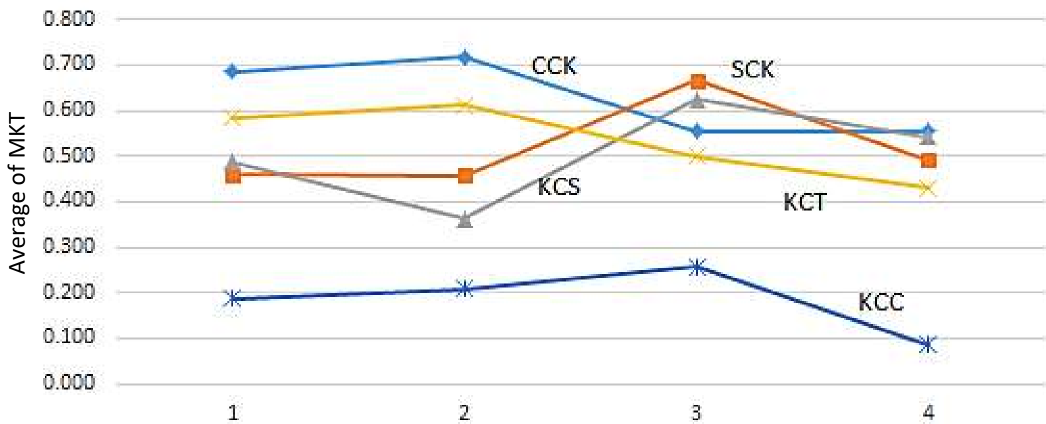


Figure 12 presents that prospective teachers’ CCK decreases as years go. Whereas, 3rd year prospective teachers’ SCK is the best, nevertheless, at 4th year, it decreased. Prospective teachers’ KCS is better at 3rd year, and 4th year prospective teachers’ did not have the same level KCS as 3rd year. KCT decreases as years go. In general, KCC is the knowledge that teachers know least, and it is not wondering because pre-service teacher training does not train them in the school curriculum.

Contribution of pre-service teacher training to teachers’ MKT

In order to answer to the research question about the contribution of pre-service teacher training to school teachers’ MKTCI and MKTCD, a correlation between practical and prospective teachers’ MKT is investigated as for MSUE using SPSS.

Estimation of the correlation resulted weak to moderate relationships; yet, a few of them are statistically significant. The following Table 43 presents correlations between practical and prospective teachers’ MKTCI and MKTCD.

Table 43. Correlation between practical and prospective teachers’ MKTCI and MKTCD

		Prospective teacher									
		CCKCI	SCKCI	KCSCI	KCTCI	KCCCI	CCKCD	SCKCD	KCSCD	KCTCD	KCCCD
School	CCKCI	.237	-.064	-.226	-.041	.211	-.064	.201	.193	.051	.003
	SCKCI	-.195	-.049	.166	-.383*	.022	-.371*	-.126	.010	-.054	-.123
	KCSCI	-.061	.099	-.097	-.038	.255	-.255	-.120	-192	-.096	.041
	KCTCI	.376*	.087	-.227	-.042	.188	.161	.082	.131	-.071	.061

	KCCCI	.061	.321	.139	-.115	.323	.192	.368*	-.082	-.205	-.333
	CCKCD	-.310	.046	.197	.098	.199	.156	.021	.100	.051	.003
	SCKCD	.110	.008	-.162	-.088	.233	-.285	.133	-.109	.237	.221
	KCSCD	-.234	.246	.-001	-.171	.115	-.069	-.272	.024	.052	-.207
	KCTCD	.048	.048	-.287	.101	.013	.383*	-.192	.265	.327	.257
	KCCCD	.185	.237	-.169	-.078	.172	.203	.157	-.094	-.430*	-.106

Note: * Correlation is significant at the .05 level

Table 43 shows the relationships between prospective and school teachers' MKTCI and MKTCD. Majority of the relationships are not statistically significant; however, there are few significant relationships. Prospective teachers' CCKCD is negatively correlated (*r=-.371, p<.05*) to school teachers' SCKCI, and positively related (*r=.383, p<.05*) to school teachers' KCTCD. Moreover, prospective teachers' SCKCD is positively related (*r=.368, p<.05*) to school teachers' KCCCI. Interestingly, prospective teachers' KCTCD is negatively related (*r=-.430, p<.05*) to school teachers' KCCCD.

The above statistically significant relationships presents that there could have an association between what teachers know in pre-service teacher education and what teachers know in practice. In order to dig into this association, the linear regression analysis is applied (Table 44).

In addition, school teachers' MKTCI and MKTCD is investigated the subdomain by subdomain to reveal how prospective teachers' MKTCI and MKTCD contributes to their knowledge. In total, 10 regression models are estimated; and only one model is estimated for school teachers' KCTCD as statistically fit.

Table 44. Model summary of the linear regression analysis as of teachers'KCTCD

Model	R	R square	Adjusted R Square	Std.Error		
1	.591[a]	.349	.224	.148		
		Sum of Squares	df	Mean Square	F	Sig.
	Regression	.305	5	.061	2.790	.038[b]
	Residual	.569	26	.022		
	Total	.875	31			

Note: a,b. Predictors: (Constant), Prospective teachers' KCCCD, CCKCD, KCSCD, SCKCD, KCSCD, KCTCD

Table 45. Correlations between teachers' KCTCD and prospective teachers' MKTCD

Model		Unstandardized B	Coefficients Std.Error	Standardized Coefficients Beta	t	Sig.
1	(Constant)	.109	.133		.818	.421

	CCKCD	.319	.129	.419	2.471	.020
	SCKCD	-.051	.093	-.096	-.542	.592
	KCSCD	.073	.204	.063	.358	.723
	KCTCD	.328	.191	.307	1.720	.097
	KCCCD	.258	.206	.207	1.253	.221

By results of the regression analysis in Tables 44 and 45, prospective teachers' KCCCD, CCKCD, KCSCD, SCKCD, KCSCD and KCTCD contribute to school teachers' KCTCD; however, the contributions of the subdomains are varied. By the standardized beta coefficients, contribution of prospective teachers' CCKCD is positive and significant ($\beta=.419,\ p<.05$), SCKCD is negative, but not significant ($\beta=-.096,\ p>.05$), KCSCD, KCTCD and KCCCD are positive, but not significant ($\beta=.063,\ \beta=.307,\ \beta=.207,\ p>.05$). It practically implicate that recruiting students with generally good common knowledge of mathematics can significantly contribute school teachers' KCTCD.

CHAPTER SIX. CONCLUSIONS AND RECOMMENDATIONS

This research aims (1) to identify Mongolian secondary school teachers' mathematical knowledge for teaching with a particular reference to geometry taking into consideration of teacher belief and context where teachers are trained and work, and (2) to provide implications to the teachers to have more profound mathematical knowledge for teaching geometry.

In order to achieve the aims, the following questions need to be answered:

1. What is Mongolian secondary school teachers' MKT geometry?
2. How do these teachers believe about the nature of geometry and its teaching and learning?
3. How are teacher beliefs associated with their MKT geometry?
4. How does school context situate secondary school teacher MKT geometry?
5. How is pre-service teacher education context in Mongolia likely to contribute to teachers' MKT geometry?

6.1 Conclusions

Research conclusions are drawn in conformity with the research questions.

Research question 1: Teacher MKT geometry - Applying CICD theory for the plane shape. Based on results and findings in teachers' mathematical knowledge for teaching concept image and concept definition of the shapes (MKTCI and MKTCD) (Table 24), in overall, Mongolian secondary school mathematics teachers' mathematical knowledge for teaching geometry is characterized by their knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT), and knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, determining learning objectives for a particular activity (KCC). However, some inconsistencies in their Knowledge that is used in wide variety of settings, not unique to teaching - common with how it is used in many other professions or occupations that also use mathematics (CCK), Knowledge that is unique to teaching and allows teachers to engage in particular teaching tasks and knowing if the given statements or solutions are mathematically true or not, why the solving method works (SCK) and knowledge of students and mathematics content - familiarity with, and anticipation of, students' conception and misconceptions about a particular mathematics content and causes of these misconceptions (KCS) depending upon the concept image or formal definitions of the plane shapes.

Characteristics of common content knowledge (CCK) of geometry is identified as proper knowledge of quadrilateral images involving symmetry (CCKCI) and limited common knowledge that rectangle is formally defined as a parallelogram (CCKCD). Meanwhile, teachers' specialized content knowledge (SCK) of geometry is featured as proper knowledge of images of polygons with particular symmetrical properties that is not commonly discussed and knowing if the given statements about the polygon images are mathematically true or not. It also includes knowledge of critical attributes of the polygon (SCKCI) and lack of knowledge choosing mathematically correct formal definition of the concept of rectangle and recognizing what is involved (excluded or included classifications of shapes) in the various definitions. It also includes knowledge of structure of (necessary and sufficient condition) a formal definition of the shape concept (SCKCD). This knowledge is limited by the formal definitions do not pay attention on structure of (necessary and sufficient condition) the concept definition of the shapes. Teachers' knowledge of content and student (KCS) in geometry is captured as limited knowledge of students' common misconception related to quadrilateral images including causes of students' misconception on inner angles of quadrilaterals (KCSCI) and proper knowledge related to students and concept definition including knowing what is confusing in their ideas related to the formal definition of inscribed angles and students' incomplete interpretation of this definition (KCSCD).

Teachers have about students' common misconception related to quadrilateral images and causes of students' misconception on inner angles of quadrilaterals. In contrary, teachers have proper knowledge related to students and concept definition. Moreover, teacher' knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) of geometry are limited for both concept image and formal definitions of the shapes. They have limited knowledge about the choice of representation for teaching triangle concept images. To make a choice of the representation, they picked the most appropriate representation, yet, focus of the choice was on the classification of the shapes, therefore, their interpretation of instructional advantages and disadvantages of the chosen representation did not consider examples and non-examples of a triangle that highlight critical attributes of the shape for developing students' images of triangle concept. Their knowledge about representing the concept definition for a triangle to students is limited by prototype examples of the shapes. In addition, teachers know what geometry topic should be taught to particular grade students, nevertheless, they do not know why this topic is appropriate to particular grade students; hence, what learning objectives could be set for students' learning.

Research question 2: Teacher Beliefs. By the research findings in Tables 27, it can be concluded that Mongolian secondary school mathematics teachers tend to hold Platonist view of belief about the nature of school and discipline geometry. Teachers believe that school geometry at each grade is interconnected; then, students learn certain level of geometry at each grade and can apply it in practice. These teachers believe that school geometry is a part of a body of hierarchical interconnected knowledge of understanding of which forms the basis on which some will learn higher level mathematics. This finding could be explained that Beswick conceptualization of distinctive belief about nature of school and discipline mathematics is theoretically built, no concrete evidences supported it.

Moreover, by the research findings summarized in Table 29, Mongolian secondary school mathematics teachers tend to hold learner-focused belief about the geometry learning which enables students to figure out own ways to solve a problem and to explore different ways without teachers' direct help. This is revealed through moderate relationship between belief about the nature of geometry and geometry learning. This conclusion is a quite controversial because by Ernest (1989), teachers who hold Platonist belief tend to perceive a teacher as an explainer and the learning as the reception of knowledge. Platonist view can lead to the teacher's insistence on there being a single 'correct' method for solving each problem. However, it is not surprising finding. By Beswick (2011), there are apparent inconsistencies among teachers' beliefs about mathematics and its teaching and learning. Teachers' belief about the nature of geometry is more cognitively oriented, whilst, belief about the geometry learning is more likely to be shaped up through the experiences in a particular classroom and school context. As it is noted in Ernest (1989), this inconsistencies could be resulted from the institutionalized curriculum embodied in adopted texts, the system of assessment, and so on.

Research question 3: Association of beliefs to MKT. The research results in Table 31 and attached findings, it is concluded that teacher' belief is negatively related to their specialized content knowledge that is unique to teaching and allows teachers to engage in particular teaching tasks and knowing if the given statements or solutions are mathematically true or not, why the solving method works (SCK) and knowledge of students and mathematics content - familiarity with, and anticipation of, students' conception and misconceptions about a particular mathematics content and causes of these misconceptions (KCS). Teachers who hold stronger Problem solving and Instrumentalist view of belief tend to have less knowledge that is unique to teaching and allows teachers to engage in particular teaching tasks and knowing if the given statements or solutions are mathematically true or not, why the solving method works (SCK). These teachers perceive the school geometry as students' motivation and basic skills. Moreover, teachers who hold stronger Problem solving and Platonist view of belief have also weaker knowledge that is unique to teaching and allows teachers to engage in particular teaching tasks and knowing if the given statements or solutions are mathematically true or not, why the solving method works (SCK). These teachers believe the geometry as students' motivation and a body of hierarchical interconnected knowledge understanding of which forms the basis on which some learn higher level geometry content. Teachers, who hold stronger Platonist and Problem solving view of belief, tend to have weak knowledge of students and mathematics content - familiarity with, and anticipation of, students' conception and misconceptions about a particular mathematics content and causes of these misconceptions (KCS). These teachers believe the school geometry as part of a body of hierarchical interconnected knowledge understanding of which enables the gifted few eventually to be mathematically creative; and students' motivation is significant.

Research question 4: How school context situates teachers' MKT. The research found that the school context is likely to situate teachers' mathematical knowledge for teaching (MKT) geometry through individual and collaborative reflections. When teachers observe how students image and define the shapes during the teaching, they gain more knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, determining learning objectives for a particular activity (KCC). In contrary, as teachers read more about the representations and students' thinking, they are likely to confuse in knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT). Moreover, when teachers reflect by listening to other teachers about effectiveness of representations and students' misconceptions, their knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT) faces challenge. This means that the listening to peer teachers without contributing to the discussion does not promote teachers' knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT). It is identified that teachers' discussion with peers on how to develop alternative learning activities to tackle with student difficulty or misconceptions in geometry helps teachers to have better KCS.

Research question 5: Contribution of pre-service teacher training to teachers' MKT. The research identified that pre-service teacher training is likely to contribute to school teachers' knowledge of selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation including knowledge of how to use examples and non-examples in the representation to define the triangle concept (KCTCD) through the recruitment of students with good knowledge of the formal concept definition of the shapes (CCKCD) based on results of the simple correlation and linear regression in Tables 44 and 45. As prospective teachers have better knowledge of the formal concept definition of the shapes (CCKCD), school teachers are likely to be more knowledgeable in selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation including knowledge of how to use examples and non-examples in the representation to define the triangle concept (KCTCD). Moreover, if pre-service teacher training recruits students with better knowledge of the formal concept definition of the shapes (CCKCD) and enables students to grow their knowledge related to students and concept definition including knowing what is confusing in their ideas related to the formal definition of inscribed angles and students' incomplete interpretation of this definition (KCSCD), knowledge for selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation taking into consideration of knowledge of how to use examples and non-examples in the representation to define the triangle concept (KCTCD) and knowledge of curriculum content of the concept definition for quadrilateral symmetry, at what grade level students are typically taught the formal definition of symmetry and students' familiarity (previous and after knowledge related to definition) with the definitions (KCCCD) during the training, school teachers tend to have better knowledge for selecting the most appropriate representation to illustrate triangle concept definition and reasons beyond the chosen representation taking into consideration of knowledge of how to use examples and non-examples in the representation to define the triangle concept (KCTCD). In particular, prospective teachers' mathematical knowledge for teaching (MKT) concept definition of the shapes is likely to contribute to school teachers' knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT) concept definition of the shapes. At the end, in Mongolia, teachers' mathematical knowledge for teaching (MKT) geometry can be characterized by limited knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations (KCT) and knowledge of articulating the strands of the curriculum, knowing students' prior and after knowledge in the curriculum, determining learning objectives for a particular activity (KCC). These teachers hold Platonist view of belief about the nature of geometry, and learner-focused view of belief about the geometry learning. Teachers' mathematical knowledge for teaching (MKT) geometry is situated in school context through professional community activities such as Mathematics Olympiads and Lesson study, as well as, teachers' individual and collaborative reflections. Teachers' knowledge for representing the concept definition for a triangle to students (KCTCD) is contributed by pre-service teacher training.

6.2 Recommendations

Therefore, it is recommended that school curriculum need to signify concept image well and include exclusive and inclusive concept definitions for the plane shape. More assignments and tasks need to be developed in school textbooks taking into consideration of essential and non-essential properties of the shapes that enable students to evoke proper image of the shapes and avoid the conflicts in the learning. Vinner (1983) claims that in order to handle concepts, one needs a concept image and not a concept definition. Concept definitions will remain inactive or even be forgotten. In thinking, almost always the concept image will be evoked. Thus, teachers need to be aware of raising the significance of the concept image to tackle with the tension between concept image and definition for the shape.

It is highly recommended to the teachers that they need to develop their mathematical knowledge for teaching geometry. In particular, they need to develop knowledge related to teaching and mathematics content that include choosing the appropriate representation, knowing advantages and disadvantages of the representations focusing on advantages and disadvantages of the chosen representations, and consideration of examples and non-examples in the representations. In addition, they need to be aware of that the curriculum is not about a list of topics or content, more, it is about knowing beyond the content, and reflecting it in students' learning. Conceptual connection in the content needs to be emphasized.

It is also recommended to schools to promote professional community activities, and teachers' reflections those focus on students' learning. In particular, at schools, teachers need to be provided opportunity to discuss about what student common errors and misconceptions are related to specific geometry content, most essential ideas in specific geometry content, what most or least difficult part of specific geometry content and concepts are, and how students develop concept image and definitions.

At pre-service teacher education institution, their curriculum need to emphasize how to develop prospective teachers' mathematical knowledge for teaching geometry, in particular, their knowledge related to how students learn concept images and definitions in geometry, knowledge for representing the concept definition for a triangle to school students, what advantages and disadvantages can be discussed and knowledge of curriculum content of the concept definition for quadrilateral symmetry, at what grade level students are typically taught the formal definition of symmetry and students' familiarity (previous and after knowledge related to definition) with the definitions during the training, school teachers tend to have, at least, better knowledge for representing the concept definition for a triangle to students.

6.3 Issues for future studies

Through the research, it is known that more systematic research needs to be done in how school context situate teachers' mathematical knowledge for teaching geometry. In particular, in-depth research in what school teachers usually communicate about during formal professional development activities using more comprehensive interview method. For example, how they deliver Mathematics Olympiads activities, how they run Lesson study at a school must be deeply researched. In addition, more "knowledgeable" teachers in a particular school needs to be identified and how this teacher influence other teachers' mathematical knowledge for teaching geometry seems very critical. Yet, it should be operationally defined who would be "knowledgeable" teacher.

Teacher educators' belief about the nature of geometry, teaching and learning, and attitude toward prospective teachers are also likely to influence the contextual aspect of teachers' mathematical knowledge for teaching geometry. Thorough investigation is needed in this direction.

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APPENDIX

Teacher questionnaires

The questionnaire is designed to collect data on current situation of teacher education in Mongolia; it does not have any intentions to treat your authority and reputation related to your job. Your responses will be purely used for research purposes; and will be kept behind. THANK YOU VERY MUCH FOR YOU SUPPORT!

Part ONE. DEMOGRAPHIC INFORMATION

Please answer the following.

Q11. Age: ___ Q12. Gender: ___ Q13. Number of years in teaching mathematics: ____

Q14. Please indicate your highest level of education:

__ College, diploma __ University, B.A degree

__ Graduate school, Master degree __ Graduate school, PhD degree

Q15. Please indicate the pre-service teacher school you graduated from:

__ Mongolian State University of Education

__ National University of Mongolia

__ Khovd University

__ Arkhangai, or Dornod Teacher College

__ Gurvan-Erdene Teacher College

Q16. Please answer when you graduated from the above school (YEAR): ____________

Q17. Please mark ONE (A, B, C, or D) which is the most professional collaborative activity among your school teachers, and check ONE (*often, occasionally, rarely* and *never*) in each activity:

	Often 4	Occasionally 3	Rarely 2	Never 1
Q171. Mathematics teaching methodology unit assembly	☐	☐	☐	☐
Q172. Mathematics Olympiads	☐	☐	☐	☐
Q173. Discussion with my friend teachers	☐	☐	☐	☐
Q174. Lesson Study	☐	☐	☐	☐
Q175. Pilot curriculum and textbook team meetings	☐	☐	☐	☐
Q176. Open lesson	☐	☐	☐	☐

Part TWO. SCHOOL CONTEXT: Situated and distribute nature of MKT

1. Teacher individual reflection

Please check ONE (*often, occasionally, rarely* and *never*) in each statement on to what extent do you reflect the following issues.

	R11-R19	Often	Occasionally	Rarely	Never
1.	From various references excluding the curriculum and textbooks, I read about how to accurately represent a subject matter to students and unusual solution methods in problems related to these subject matters	☐	☐	☐	☐
2.	From various references excluding the curriculum and textbooks, I read about how students thinking of the subject matters	☐	☐	☐	☐
3.	From various references excluding the curriculum and textbooks, I read about what is the most or least effective representations to develop student understanding of the subject matters	☐	☐	☐	☐
4.	I observe how other teachers tackle with student unusual methods and errors at geometry lessons	☐	☐	☐	☐
5.	I observe how other teachers represent the shapes to students at geometry lessons KCS	☐	☐	☐	☐
6.	I observe what common errors of my students likely to repeat during my teaching	☐	☐	☐	☐
7.	I observe how students develop the image and definition for the shapes during the teaching	☐	☐	☐	☐
8.	I listen to other teachers while they discuss about effectiveness of their chosen representations of the subject matters without saying my ideas	☐	☐	☐	☐
9.	I listen to other teachers while they discuss about student common errors, misconceptions related to the subject matter without saying my ideas	☐	☐	☐	☐

2. Teacher collaborative reflection

Please mark ONE (*often, occasionally, rarely* and *never*) to what extent do you discuss about the issues in the most common professional activity that you marked in the beginning of this questionnaire.

At the end, please add 3 more issues and check ONE (*often, occasionally, rarely* and *never*) in each.

R21-R26	Often	Occasionally	Rarely	Never
1. We discuss about how to cover intended content of a topic within teaching hour	☐	☐	☐	☐
2. We discuss what are possible representations of the subject matter, which is the most appropriate and why	☐	☐	☐	☐
3. We discuss what are student common errors and misconceptions related to specific content of geometry	☐	☐	☐	☐
4. We discuss how to develop alternative learning activities to tackle with student difficulty or misconceptions in geometry lessons	☐	☐	☐	☐
5. We discuss what is the most or least difficult part of specific content for teaching geometry	☐	☐	☐	☐
6. We discuss what the most essential subject matter is in the geometry content	☐	☐	☐	☐
7.	☐	☐	☐	☐
8.	☐	☐	☐	☐
9.	☐	☐	☐	☐

Part THREE. TEACHER MKT

1. Concept Image

Q1. Ms. Tsetseg found the following problem in the textbook she was using:

What do you call <u>quadrilaterals</u> whose two diagonals are both lines of symmetry?

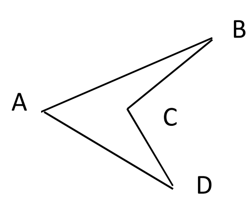
Which of the following is the correct answer for this problem? (Select ONE answer).

A. Squares; B. Rectangles; C. Parallelograms; D. Rhombi; E. Trapezoids

Q2-5. In a lesson on symmetry, Ms. Bayasgalan asked his class to generate polygons with at least one line of symmetry and to make observations about symmetric polygons. For each of the following claims, decide whether or not it is mathematically true. (Select TRUE, FALSE, or I AM NOT SURE for each).

	TRUE **T**	FALSE **F**	I'M NOT SURE **U**
Q2. If a line of symmetry cuts through a side then it makes a right angle with that side			
Q3. If a line of symmetry passes through a vertex, then it bisects the angle at that vertex			
Q4. The areas on each side of the line of symmetry are equal			
Q5. If a quadrilateral has exactly two lines of symmetry, then it must be a rectangle			

Q6. Ms. Ariunaa's students know that the sum of the angles in a triangle is 180^0. She states that the sum of the angles in a quadrilateral is 360^0 and illustrates this with three examples – a rectangle, a parallelogram, and an irregular quadrilateral. She then asks the class to check other examples. Bayar, a student, raises his hand and says that he has a counterexample. When Ms. Ariunaa asks him to show it to the class, he draws the figure below:

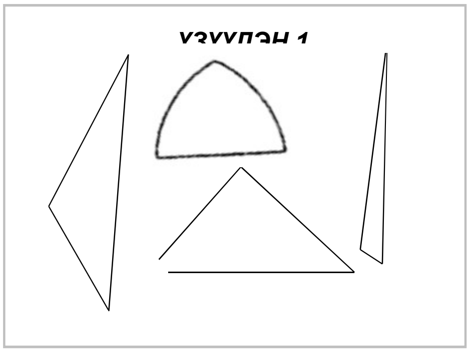


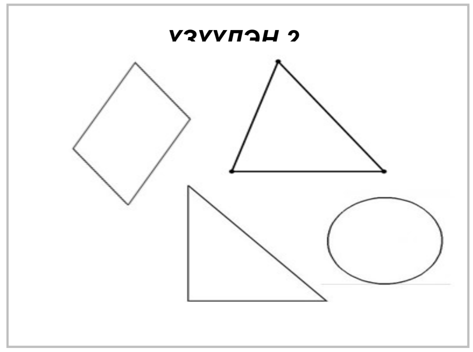
Bayar argues that angle A is about a right angle, angle C is only slightly larger, and angles B and D are very small, so the sum A+B+C+D cannot be the same as four right angles. Which of the following is the most reasonable appraisal of this situation? (select ONE answer)

A. The angle sum formula applies only to convex quadrilaterals;
B. Bayar's argument is not convincing because it is based on inexact estimates;
C. Bayar does not seem to understand the meaning of interior angles in the case of non-convex polygons;
D. Bayar does not understand what a counterexample is;
E. The figure Bayar drew is not a quadrilaterals;

Q7-8. At professional development workshop, teachers are given assignment to develop representations to teach the concept image of triangle to students. They have developed the following three different representations (example and non-example set) on the topic.

Q7. Please answer which representation would you choose (Tick as ✓)?





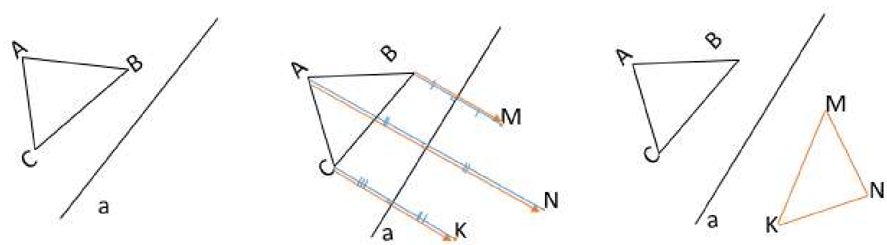
A. ___ Representation 1
B. ___Representation 2
C. ___Both are equally important
D. ___I am not sure

Q8. Please write up all advantages and disadvantages for each representation in the following Table.

	Advantage	Disadvantage
Representation 1		
Representation 2		

Q9-Q11. In school mathematics textbook, there are several practical exercises and one of them is given as follow:

<u>*Practical exercise*</u>



1. *Draw line "a" and ABC triangle as first figure.*
2. *Construct symmetrical points of A, B, and C along the line "a"; and present the symmetrical points as , N, M and K respectively.*
3. *Connect M, N and K points by line segments.*
4. *What if ABC triangle is folded as "a" line, do ABC and MNK triangles overlap?*

Please answer what grade would you use this practical exercise and what would be learning goals about a triangle shape for this practical exercise? Please write on the following space.

Q9. I would use this practical exercise for grade (*Choose one of the following responses*):

A. Grade 8; B. Grade 7; C. Grade 9; D. Grade 6; E. Any grade; F. I am not sure

Q10. Why is it appropriate with this grade . . . .
. . . .

Q11. If you use this practical exercise for teaching triangle concept, what learning objectives would you set up? . . . .
. . . .

2. Concept Definition

Q12. Which of the following definitions could be a definition for a rectangle? (Circle up ONES that can be a definition for a rectangle)

A. A rectangle has four right angles, four straight sides, two equal diagonals and two equal and parallel opposite sides;
B. A rectangle is one of the geometrical shapes;
C. A rectangle is a quadrangle, which has four right angles;
D. A rectangle is a quadrangle, which has four right angles and which adjacent sides are different lengths;
E. A rectangle is a quadrangle, whose opposite sides are parallel;
F. A rectangle is a parallelogram, which has one right angle;
G. A rectangle looks like a stretched square;

Q13. Ms. Ankhaa is preparing to teach a lesson on quadrilaterals. She sees that her textbook uses a different definition of trapezoid from the one that was in her college math method book.

Her teacher' edition defines a trapezoid as a quadrilaterals with exactly one pair of parallel sides. (Definition I)
Her college math methods book defines a trapezoid as a quadrilateral with at least one pair of parallel sides. (Definition II)

Ms. Ankhaa thinks the choice of definition might affect how one classifies shapes. Which of the following is true? (Mark the correct ONE as ✓)

		Tick
A.	A rectangle is a trapezoid according to Definition II but not according to Definition I	
B.	A rectangle is a trapezoid according to Definition I but not according to Definition II	
C.	A rectangle is a trapezoid by both definitions	
D.	All quadrilaterals are trapezoid according to Definition II	
E.	The definitions are really the same	
F.	I am not sure	

Q14. Mr.Bat was teaching about inscribed angles to students; and he introduced a conjecture that "inscribed angles that share a common chord have the same measure" is true. One of his students explained as follow:

Student: *I noticed the two angles take up the same amount of the circumference so their openings must be the same.*

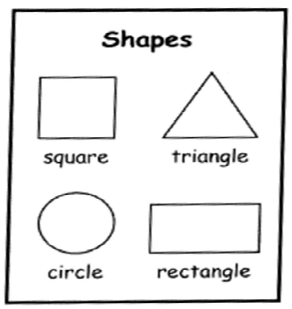
Please write up what would be a complicating idea for this student statement. Complicating idea would be :

. . . .
. . . .
. . . .

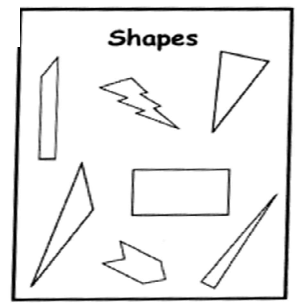
Q15-16. Ms. Bayarmaa wants her students to understand the structure of definition for triangle, and improve their understanding. To help them, she wants to give them some shapes using the representations.

She goes to the store to look for a visual aid to help with this lesson. Which of the following is <u>most</u> likely to help students improve their definition? (Circle ONE answer.) Please explain your choice.

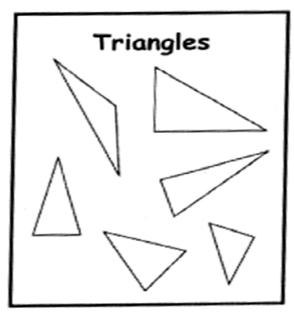
A



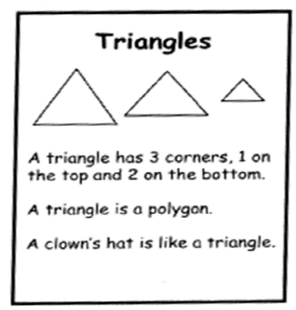
D



C



B

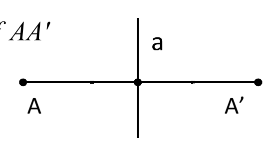


Please explain why you did choose the representation . . . .

Q17-Q18. In mathematics textbooks, two different definitions for different grades are given as follow:

<u>Definition A</u>: *If line "a" crosses through the midpoint of AB segment; and this line is perpendicular with the segment, points A and B will be the symmetrical along the line "a". Line "a" is called as a mirror line of the symmetry.*

<u>Definition B</u>: *If the line "a" crosses through the midpoint of AA' segment, and perpendicular with the segment, AA' is symmetrical point along the line "a". All points on this line are symmetrical along itself.*



Which of the definition for grade 7 and why do you think so? Please state the grade in first column and give a reason in next column of Table.

Q17. Which definition is taught to grade 7 students?

A. Definition B **B**. Definition A **C**. I am not sure

Q18. Why do you think your selected definition is appropriate with grade 7?

. . . .

. . . .

Part FOUR. TEACHER BELIEF

1. Combined Belief about the nature of the school and discipline geometry and teaching
To what extent do you agree or disagree with the following beliefs about the nature of the school mathematics? Check ONE in each row.

	1	2	3	4	5	6
B11-B19	Strongly disagree	Disagree	Slightly disagree	Slightly agree	Agree	Strongly agree
1. School geometry is a set of basic skills that needed to solve everyday life problems	☐	☐	☐	☐	☐	☐
2. School geometry compromises from basic skills that is needed for later grade or higher mathematics	☐	☐	☐	☐	☐	☐
3. School geometry is just for basic skills, so it is impossible to have creative ideas	☐	☐	☐	☐	☐	☐
4. Geometry at each grade is interconnected, then, students learn certain level of geometry at each grade and can apply it in practice	☐	☐	☐	☐	☐	☐
5. School geometry is interconnected through grades; prior grade geometry is a basis of next grade mathematics	☐	☐	☐	☐	☐	☐
6. Creative ideas can be constructed at each grade of school geometry, however, the most creative one will be at high school geometry by only gifted students	☐	☐	☐	☐	☐	☐
7. Solving school geometry problems with basic skills is a process; however, without motivation, it is not possible	☐	☐	☐	☐	☐	☐
8. Geometry at each grade can be possessed through processes, however, with motivation is a crucial	☐	☐	☐	☐	☐	☐
9. Many ideas can be came through creative processes that embedded in school geometry, so, students appreciate it	☐	☐	☐	☐	☐	☐

2. Belief about learning geometry

From your perspective, to what extent do you agree or disagree with the following beliefs about mathematics teaching and learning? Check ONE in each row. (Adapted and modified f*rom TEDS-M Study, 2008*)

B21-B214	Strongly disagree	Disagree	Slightly disagree	Slightly agree	Agree	Strongly agree
1. The best way to do well in geometry is to memorize all the formulas	☐	☐	☐	☐	☐	☐
2. Pupils need to be taught exact procedures for solving geometrical problems	☐	☐	☐	☐	☐	☐
3. It does not really matter if you understand a geometrical problem, if you can get the right answer	☐	☐	☐	☐	☐	☐
4. To be good in geometry you must be able to solve problem quickly	☐	☐	☐	☐	☐	☐
5. Pupils learn geometry best by attending to the teacher's explanations	☐	☐	☐	☐	☐	☐
6. When pupils are working geometry problems, more emphasis should be put on getting correct answer than on the process followed	☐	☐	☐	☐	☐	☐
7. In addition to getting a right answer in geometry, it is important to understand why the answer is correct	☐	☐	☐	☐	☐	☐
8. Teachers should allow pupils to figure out their own ways to solve geometrical problems	☐	☐	☐	☐	☐	☐
9. Non-standard procedures should be discouraged because then can interfere with learning the correct procedure	☐	☐	☐	☐	☐	☐
10. Hand-on geometry experiences aren't worth the time and expense	☐	☐	☐	☐	☐	☐
11. Time used to investigate why a solution to geometrical problem works is time well spent	☐	☐	☐	☐	☐	☐
12. Pupils can figure out a way to solve geometrical problems without a teacher's help	☐	☐	☐	☐	☐	☐
13. Teachers should encourage pupils to find their own solutions to geometrical problems even if they are inefficient	☐	☐	☐	☐	☐	☐
14. It is helpful for pupils to discuss different ways to solve particular problems	☐	☐	☐	☐	☐	☐