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Relation	

ASYMPTOTIC PROPERTY OF DIVERGENT FORMAL SOLUTIONS IN LINEARIZATION OF SINGULAR VECTOR FIELD

MASAFUMI YOSHINO

ABSTRACT. We study asymptotic properties of divergent formal solutions appearing in the linearization problem of a sigular vector field without a Diophantine condition or an existence of additional first integrals. We will give an asymptotic meaning to divergent formal solutions constructed by a singular perturbative solution (cf. [6]).

1. INTRODUCTION

A linearizing transformation of a singular vector field satisfies a certain semilinear Fuchsian system of equations of several variables. (cf. (2.2)). The system has a formal power series solution under a general nonresonance condition, while formal solutions are divergent in general. (cf. [3] and Proposition 3.1 of [7].) The convergence of the series can be proved under a Diophantine condition or an existence of additional first integrals. In this paper we study equations of two independent variables, and we shall give an asymptotic meaning to a formal solution without any Diophantine condition or an existence of additional first integrals.

In [6], we constructed a singular perturbative solution with respect to a singular perturbative parameter ε by resumming a singular perturbative formal solution. If the so-called Poincaré condition and the nonresonance condition are verified, then by analytic continuation with respect to ε up to $\varepsilon = 1$ we obtain the classical Poincaré solution. In this paper we are interested in the case where the Poincaré condition or a Diophantine condition is not verified. By the same method as in [6] we can construct a singular perturbative solution and make an analytic continuation with respect to ε up to a sector with vertex at $\varepsilon = 1$ as well. On the other hand the analytic continuation of the resummed singular perturbative solution does not necessarily converge when $\varepsilon \to 1$.

Our goal in this paper is to show that the analytic continuation of the resummed singular perturbative solution is an asymptotic expansion of a certain analytic solution in a multi sector of the space variables uniformly with respect to ε in a sector with vertex at $\varepsilon = 1$. More precisely, we can show the assertion for equations with nonliear part satisfying certain support conditions (cf. (2.19) and (2.20)) for which a small

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denominator may appear. (See also [7].) We hope that our new approach to a linearization problem via an equation with singular perturbative parameter might be generalized to the case of general independent variables. We also remark that our proof does not use the so-called Newton method in constructing a solution, which make the proof simpler than the one based on the Newton method.

This paper is organized as follows. In section 2 we state our results. In section 3 we prepare a necessary lemma. In the last section we prove our main theorem.

2. Statement of results

Let $x = (x_1, x_2) \in \mathbb{C}^2$ be the variable in $\mathbb{C}^2$ and let $\mathbb{R}$ be the set of real numbers. For a 2-square constant matrix Λ, we denote by L_Λ the Lie derivative of the linear vector field $x\Lambda \cdot \partial_x$

$$L_\Lambda := [x\Lambda\, \partial_x, \cdot\,] = \langle x\Lambda, \partial_x\rangle - \Lambda, \tag{2.1}$$

where $\langle x\Lambda, \partial_x\rangle = \sum_{j=1}^{2} (x\Lambda)_j (\partial/\partial x_j)$, with $(x\Lambda)_j$ being the j-th component of $x\Lambda$. It is well known that the following system of equations is the linearizing equation of the singular vector field $x\Lambda \cdot \partial_x + R(x)\partial_x$

$$L_\Lambda u = R(x + u(x)), \tag{2.2}$$

where $u = {}^t(u_1, u_2)$ is an unknown vector function and the function

$$R(y) = {}^t(R_1(y), R_2(y)) \tag{2.3}$$

is holomorphic in some neighborhood of $y = 0$ in $\mathbb{C}^2$ such that $R(y) = O(|y|^2)$ when $|y| \to 0$. In order to study (2.2) we consider the following equation with the parameter ε

$$L^\varepsilon_\Lambda u \equiv \varepsilon\langle x\Lambda, \partial_x\rangle u - u\Lambda = R(x + u(x)), \tag{2.4}$$

then we let $\varepsilon \to 1$.

In the following we assume that Λ is a diagonal matrix whose components are given by 1 and $-\tau < 0$, where $\tau > 0$ is an irrational number. Hence we have

$$\langle x\Lambda, \partial_x\rangle = x_1\partial_1 - \tau x_2\partial_2. \tag{2.5}$$

We first construct a formal solution of (2.4) $u^W(x, \varepsilon)$ in a formal power series of ε

$$u^W(x, \varepsilon) = \sum_{\nu=0}^{\infty} \varepsilon^\nu u^W_\nu(x) = u^W_0(x) + \varepsilon u^W_1(x) + \cdots, \tag{2.6}$$

where the coefficients $u^W_\nu(x)$ $(\nu = 0, 1, \dots)$ are holomorphic vector functions of x in some open set independent of ν. We will substitute the expansion (2.6) into (2.4). We first note

$$\varepsilon\langle x\Lambda, \partial_x\rangle u^W - u^W\Lambda = \sum_{\nu=0}^{\infty} (\varepsilon\langle x\Lambda, \partial_x\rangle u^W_\nu(x) - u^W_\nu(x)\Lambda)\varepsilon^\nu, \tag{2.7}$$

$$\begin{aligned} R(x + u^W) &= R(x + u^W_0 + u^W_1\varepsilon + u^W_2\varepsilon^2 + \cdots) \\ &= R(x + u^W_0) + \varepsilon u^W_1 (\nabla R)(x + u^W_0) + O(\varepsilon^2). \end{aligned} \tag{2.8}$$

By comparing the coefficients of $\varepsilon^0 = 1$ and ε, we obtain

$$u_0^W(x)\Lambda + R(x + u_0^W) = 0. \tag{2.9}$$

$$\langle x\Lambda, \partial_x\rangle u_0^W = u_1^W\Lambda + u_1^W(\nabla R)(x + u_0^W). \tag{2.10}$$

Because Λ is invertible and $u_0^W(x) = (|x|^2)$ when $x \to 0$, we can determine u_0^W as a holomorphic vector function in some neighborhood of the origin $x = 0$ from (2.9). On the other hand, by noting that $\Lambda + (\nabla R)(x + u_0^W)$ is an invertible matrix in some neighborhood of the origin $x = 0$ by the assumption $R(x) = O(|x|^2)$, we can determine u_1^W as a holomorphic function in some neighborhood of the origin $x = 0$ from (2.10). In order to determine u_ν^W ($\nu \geq 2$) we compare the coefficients of ε^ν of both sides of (2.4). Namely, we differentiate (2.4) with respect to ε, ν times and we put $\varepsilon = 0$. Then, we obtain

$$\begin{aligned} \langle x\Lambda, \partial_x\rangle u_{\nu-1}^W &= u_\nu^W\Lambda + u_\nu^W(\nabla R)(x + u_0^W) \\ &+ \ (\text{terms consisting of } u_i^W,\ i \leq \nu - 1). \end{aligned} \tag{2.11}$$

Clearly we can determine u_ν^W as a holomorphic function in some neighborhood of the origin $x = 0$ from (2.11). Hence we can determine u^W. We note that u_ν^W's are holomorphic in some neighborhood of the origin independent of ν in view of the above argument.

By expanding $u_\nu^W(x)$ ($\nu = 0, 1, \ldots$) into the power series of x, $u_\nu^W(x) = \sum_\alpha u_{\nu,\alpha}^W x^\alpha$, and summing up with respect to ν, we obtain the formal expansion of $u^W(x, \varepsilon)$,

$$u^W(x, \varepsilon) = \sum_{\alpha \in \mathbf{Z}_+^2} u_\alpha^W(\varepsilon) x^\alpha \tag{2.12}$$

with u_α^W being the formal power series of ε. In [6] we proved that, if τ is irrational, then the formal series $u_\alpha^W(\varepsilon)$ converges in some neighborhood of $\varepsilon = 1$ independent of α such that $u^W(x, \varepsilon)$ coincides with a unique formal power series solution of (2.4), a classical Poincaré series. Hence we can construct the solution of (2.2) from $u^W(x, \varepsilon)$ by setting $\varepsilon = 1$ in the class of formal power series. Note that we do not use any Diophantine condition in the argument.

In order to give an analytical meaning to this argument, we begin with the resummation of $u^W(x, \varepsilon)$ when ε is in some sector. We define $\tilde{u}^W(x, \varepsilon) = u^W(x, \varepsilon) - u_0^W(x)$. Then the (formal) Borel transform of $\tilde{u}^W$ is defined by

$$B(\tilde{u}^W)(x, \zeta) := \sum_{\nu=1}^\infty u_\nu^W(x) \frac{\zeta^{\nu-1}}{(\nu - 1)!}. \tag{2.13}$$

Because $u_\nu^W(x)$ is holomorphic in some neighborhood of the origin $x = 0$ independent of ν, the expansion $u_\nu^W(x) = \sum_\alpha u_{\nu,\alpha}^W x^\alpha$ converges in a common neighborhood of the origin independent of ν. By substituting the expansion into (2.13) we obtain

$$B(\tilde{u}^W)(x, \zeta) = \sum_{\nu=1}^\infty \sum_\alpha u_{\nu,\alpha}^W x^\alpha \frac{\zeta^{\nu-1}}{(\nu - 1)!}. \tag{2.14}$$

Let us assume that the right-hand side of (2.14) absolutely converges in some neighborhood of $(x,\zeta)=(0,0)$. (For the rigorous proof of this fact we refer [6].) Then, by changing the order of the summation we obtain

$$B(\tilde{u}^W)(x,\zeta)=\sum_{\alpha}\sum_{\nu=1}^{\infty}u^W_{\nu,\alpha}\frac{\zeta^{\nu-1}}{(\nu-1)!}x^{\alpha}. \tag{2.15}$$

We define the Laplace transform $\tilde{U}^W(x,\varepsilon)$ of $B(\tilde{u}^W)(x,\zeta)$ by

$$\tilde{U}^W(x,\varepsilon):=\sum_{\alpha}L\left(\sum_{\nu=1}^{\infty}u^W_{\nu,\alpha}\frac{\zeta^{\nu-1}}{(\nu-1)!}\right)x^{\alpha}, \tag{2.16}$$

where the operator L is given by

$$Lf(\varepsilon)=\int_0^{\infty}e^{-\zeta/\varepsilon}f(\zeta)d\zeta.$$

Here we assume an appropriate growth condition on $f(\zeta)$. We define

$$U^W(x,\varepsilon):=\tilde{U}^W(x,\varepsilon)+u_0^W(x).$$

If we recall that the Borel transform is the inverse of the Laplace transform, $U^W(x,\varepsilon)$ gives a holomorphic function of ε in a sectorial domain with the asymptotic expansion $u^W(x,\varepsilon)$. We call $U^W(x,\varepsilon)$ a resummation of a singular perturbative solution u^W. For the direction ξ, $(0\le\xi<2\pi)$ and the opening $\theta>0$ we define the sector $S_{\xi,\theta}$ by

$$S_{\xi,\theta}=\left\{\varepsilon\in\mathbb{C};\ |\arg\varepsilon-\xi|<\frac{\theta}{2},\ \varepsilon\neq 0\right\}. \tag{2.17}$$

The following theorem was proved in [6]. (cf. Theorem 2 of [6].)

Theorem 1. *There exist a direction ξ, an opening $\theta>0$ and a neighborhood Ω_0 of the origin $x=0$ such that $U^W(x,\varepsilon)$ is holomorphic in $(x,\varepsilon)\in\Omega_0\times S_{\xi,\theta}$ and satisfies (2.4). The formal solution $u^W(x,\varepsilon)$, (2.12) is an asymptotic expansion of $U^W(x,\varepsilon)$ in $\Omega_0\times S_{\xi,\theta}$ with respect to $\varepsilon\in S_{\xi,\theta}$.*

We note that one can take ξ any direction such that $\xi\neq 0,\pi$. Suppose $\tau<0$, namely the Poincaré condition is verified. By Theorem 4 of [6] $U^W(x,\varepsilon)$ can be analytically continued with respect to ε up to $\varepsilon=1$ when x is in some neighborhood of the origin independent of ε.

We now consider the case $\tau>0$, irrational. By Theorem 4 of [6] $U^W(x,\varepsilon)$ can be analytically continued with respect to ε up to a neighborhood of $\varepsilon=1$ such that $\operatorname{Im}\varepsilon>0$ (or $\operatorname{Im}\varepsilon<0$) when x is in some neighborhood of the origin which may depend on ε. By well known results on the divergence of the linearizing transformation in the case of non Diophantine case we cannot expect the convergence of $u^W(x,\varepsilon)|_{\varepsilon=1}$ as a formal power series of x. (cf. [7]). In the following we study the asymptotic meaning of the series.

Let η_1 and η_2 be such that $\eta_1>0$, $0<\eta_2<\pi/2$ and $\eta_1+\eta_2/\tau<\pi/2$. Let $S_1\subset\mathbb{C}$ and $S_2\subset\mathbb{C}$ be sectors with the openings η_1 and η_2, respectively, namely $S_j:=\{x_j\in\mathbb{C};|\arg x_j|<\eta_j/2\}$, $(j=1,2)$. For $0<\rho\le 1$ we define $S_{j,\rho}:=S_j\cap\{|x_j|<\rho\}$. Let

θ, $0 < \theta < \pi$ be given. We denote by $\mathcal{C}_{\pm,\theta}$ the cone with vertex at $\varepsilon = 1$ with opening θ

$$\mathcal{C}_{\pm,\theta} := \{\varepsilon \in \mathbb{C}; |\arg(\varepsilon - 1) \mp \pi/2| < \theta/2\}. \tag{2.18}$$

We define $\mathcal{C}_{\pm,\theta,\rho} = \mathcal{C}_{\pm,\theta} \cap \{|\varepsilon| < \rho\}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ we set $|\alpha| = \alpha_1 + \alpha_2$.

We assume that $R(x)$ is holomorphic in some neighborhood of the origin with the Taylor expansion given either by

$$R(x) = \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2, \alpha_1-\tau\alpha_2<-2\tau} R_\alpha x^\alpha, \tag{2.19}$$

or

$$R(x) = \sum_{\alpha=(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2, \alpha_1-\tau\alpha_2>2\tau} R_\alpha x^\alpha. \tag{2.20}$$

Then our main result in this paper is the following

Theorem 2. *Suppose that either (2.19) or (2.20) is satisfied. Let $0 < \theta < \pi$. Then there exists $\rho > 0$ such that (2.4) has a solution $u_\pm(x, \varepsilon)$ being holomorphic in $S_{1,\rho} \times S_{2,\rho} \times \mathcal{C}_{\pm,\theta,\rho}$ such that, for every $\varepsilon \in \mathcal{C}_{\pm,\theta,\rho}$ and $\nu = 0, 1, 2, \ldots$*

$$u_\pm(x, \varepsilon) - \sum_{|\alpha|\le\nu} u_\alpha^W(\varepsilon) x^\alpha = O(|x|^{\nu+1}), \quad \textit{when } x \to 0,\ x \in S_{1,\rho} \times S_{2,\rho}. \tag{2.21}$$

Remark 1. *If $\tau < 0$, namely the Poincaré condition is verified, then we may take $u_\pm(x, \varepsilon)$ in Theorem 2 as an analytic continuation of $U^W(x, \varepsilon)$ up to $\varepsilon = 1$. (cf. [6]). Theorem 2 assures the existence of a similar function in the case $\tau > 0$. We expect that our argument here also works for a resonant case after appropriate modifications, which is left for a future problem.*

3. Preliminary lemma

In this section we prove the solvability of (2.4) modulo flat functions. We define

$$S_\rho := S_1 \times S_2 \cap \{(x_1, x_2) \in \mathbb{C}^2; |x_1||x_2|^{1/\tau} < \rho, |x_2| < \rho\}. \tag{3.1}$$

For every $n \ge 1$ we choose the smallest positive integer k_n such that $n - \tau k_n < 0$. Namely, k_n is determined by the relation $-\tau < n - \tau k_n < 0$. We set $\alpha_n = (n, k_n)$. Let $U^W(x, \varepsilon) = \sum_{\alpha\in\mathbb{Z}_+^2} u_\alpha^W(\varepsilon) x^\alpha$ be the one given in Theorem 1. Then we have

Lemma 3. *Suppose that (2.19) is satisfied. Let $0 < \theta < \pi$. Then there exist $\rho > 0$ and a function $V(x, \varepsilon)$ being holomorphic in $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ and continuous up to the boundary such that, for every n ($n = 0, 1, 2 \ldots$) there exists $\tilde{g}_n(x, \varepsilon)$ holomorphic in $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ and continuous up to the boudary such that, for every $\varepsilon \in \mathcal{C}_{\pm,\theta,\rho}$*

$$R(x + V) - L_\Lambda^\varepsilon V = x^{\alpha_n} \tilde{g}_n(x, \varepsilon),\ x \in S_\rho, \tag{3.2}$$

$$V(x, \varepsilon) - \sum_{|\alpha|\le n} u_\alpha^W(\varepsilon) x^\alpha = O(|x|^{n+1}), \quad \textit{when } x \to 0,\ x \in S_\rho. \tag{3.3}$$

Moreover there exist infinitely many α_{n_ν} ($\nu = 1, 2, \ldots$) and $\theta', 0 < \theta' < 1$ independent of α_{n_ν} such that $x_2^{-1-\theta'} \tilde{g}_n(x, \varepsilon)$ is holomorphic and bounded in $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$.

Remark 2. *If (2.20) is satisfied, then we interchange the roles of x_1 and x_2. Then Lemma 3 holds true. The proof is the same as that of Lemma 3.*

Proof of Lemma 3. We divide the proof into 12 steps.

Step 1. For the sake of simplicity we denote $\mathcal{C}_{\pm,\theta}$ and $\mathcal{C}_{\pm,\theta,\rho}$ by $\mathcal{C}$ and $\mathcal{C}_\rho$, respectively. We will look for $U \equiv U(x,\varepsilon)$ in the form

$$U = a_0 + b_0 + \sum_{j=1}^{\infty} x^{\alpha_j}(a_j + b_j), \tag{3.4}$$

with $a_j \equiv a_j(x_1,\varepsilon)$ and $b_j \equiv b_j(x_2,\varepsilon)$ being holomorphic and bounded in $S_1 \times \mathcal{C}_\rho$ and $S_{2,\rho} \times \mathcal{C}_\rho$, respectively, and

$$a_0 = O(x_1^2),\ b_j = O(x_2^2), \quad j = 0,1,2,\dots, \tag{3.5}$$

such that the functions U_{n-1} $(n \geq 1)$

$$U_{n-1} := a_0 + b_0 + \sum_{j=1}^{n-1} x^{\alpha_j}(a_j + b_j) \tag{3.6}$$

satisfy

$$\mathcal{R}_{n-1} := L_\Lambda^\varepsilon U_{n-1} - R(x + U_{n-1}) = x^{\alpha_n}\tilde{\mathcal{R}}_{n-1}(x,\varepsilon) \tag{3.7}$$

for some $\tilde{\mathcal{R}}_{n-1}(x,\varepsilon)$ being holomorphic in $S_\rho \times \mathcal{C}_\rho$ and continuous up to the boundary such that $\tilde{\mathcal{R}}_{n-1} = O(x_2^2)$ as $x_2 \to 0$.

Step 2. We will construct a_j and b_j in (3.4) formally. We first rewrite $U^W(x,\varepsilon) = \sum_{\alpha \in \mathbf{Z}_+^2} u_\alpha^W(\varepsilon)x^\alpha$ in the following form

$$U^W = \tilde{a}_0(x_1,\varepsilon) + \tilde{b}_0(x_2,\varepsilon) + \sum_{n=1}^{\infty} x^{\alpha_n}(\tilde{a}_n(x_1,\varepsilon) + \tilde{b}_n(x_2,\varepsilon)), \tag{3.8}$$

where the formal power series $\tilde{a}_n(x_1,\varepsilon)$ and $\tilde{b}_n(x_2,\varepsilon)$ $(n = 0,1,\dots)$ satisfy

$$\tilde{a}_0(x_1,\varepsilon) = O(x_1^2),\ \tilde{b}_n(x_2,\varepsilon) = O(x_2^2), \qquad n = 0,1,2,\dots \tag{3.9}$$

We consider the case $\tau > 1$. We note $k_j \leq j$ for every j. We determine $\tilde{a}_0(x_1,\varepsilon)$ and $\tilde{b}_0(x_2,\varepsilon)$ as the Taylor series in U^W consisting of powers of x_1 and x_2 only, respectively. By subtracting $\tilde{a}_0 + \tilde{b}_0$ from U^W we see that the resultant term is divisable by x_1x_2. Hence we can choose terms which are divisable by x^{α_1}. By determing $\tilde{a}_1$ and $\tilde{b}_1$ similarly as $\tilde{a}_0$ and $\tilde{b}_0$ we subtract $x^{\alpha_1}(\tilde{a}_1 + \tilde{b}_1)$ again and see that the remaing term is divisable by $x^{\alpha_1}x_1x_2$. Hence it is divisable by x^{α_2}. We repeat the argument and we can rearrange the series U^W in the above form. We note that because we may have $k_1 = k_2 = \cdots = k_\ell$ for some $\ell > 1$, the expression is not unique in general.

Next we consider the case $0 < \tau < 1$. Because we have $k_j > j$ for some j the situation is different from the case $\tau > 1$. We first show that the support of the Taylor expansion of U^W is contained in the convex cone $\Gamma_0 := \{(\alpha_1,\alpha_2) \in \mathbb{R}^2; \alpha_j \geq 0, \alpha_1 - \tau\alpha_2 < -2\tau\}$. To see this, we recall that U^W is the formal power series solution of (2.4) such that $U^W = O(|x|^2)$. On the other hand, by (2.19) the term of degree 2 in the expansion of R vanishes. Because L^ε_Λ preserves monomials, it follows that terms with degree 2 in U^W also vanishes. Next the terms of degree 3 in R is given by the constant times x_2^3. Indeed, by the support condition on R, $\alpha_1 - \tau\alpha_2 < -2\tau$ we have $\alpha_2 \geq 3$ if $\alpha_1 \geq 1$. Because L^ε_Λ preserves monomials, it follows that terms with degree 3 in U^W has the weight $\alpha_1 - \tau\alpha_2 < -2\tau$. Let us suppose that the assertion holds for every term x^α in U^W up to $|\alpha| \leq \nu$. Consider the monomial x^β, $\beta = (\beta_1, \beta_2)$, $|\beta| = \nu + 1$ appearing from $R(x + u)$. We may consider $(x_1 + u_1)^k (x_2 + u_2)^m$ for $k + m \leq \nu + 1$ instead of $R(x + u)$ without loss of generality. In order to estimate the weight $\beta_1 - \tau\beta_2$ from the above for every x^β appearing from $(x_1 + u_1)^k (x_2 + u_2)^m$, it is sufficient to consider terms which contains x_1^k because the weight of terms appearing from $(x_1 + u_1)^k$ is less than or equals to k. As for the weight of terms appearing from $(x_2 + u_2)^m$ it is largest when x_2^m appears because the weight of every monomial in u_2 is strictly smaller than -2τ by inductive assumption. Because $k - \tau m < -2\tau$ by (2.19), we see that every monomial x^β, $|\beta| = \nu + 1$ appearing from $R(x + u)$ has the desired property. Hence the support of the Taylor expansion of U^W is contained in Γ_0.

In order to write U^W in the form (3.8) we determine $\tilde{a}_0$ and $\tilde{b}_0$ similarly as in the case $\tau > 1$. Subtracting $\tilde{a}_0 + \tilde{b}_0$ from U^W we see that the resultant term is divisable by $x_1 x_2$. Moreover, since k_1 satisfies that $-\tau < 1 - \tau k_1 < 0$, it follows that $m \geq k_1 + 2$ if $(1, m)$ is on the support of U^W. Hence the resultant term is divisable by $x^{\alpha_1} x_2^2$. We now determine a_1 and b_1 as in the case $\tau > 1$ and consider $U^W - \sum_{j=0}^1 x^{\alpha_j}(a_j + b_j)$, where $x^{\alpha_0} = 1$. It satisfies the same support condition as U^W. Hence we can proceed in the same way by noting that $m \geq k_n + 2$ if (n, m) is on the support of U^W. This proves that U^W can be expanded in (3.8).

Step 3. We will determine a_0 and b_0 such that

$$\mathcal{R}_0 := L^\varepsilon_\Lambda(a_0 + b_0) - R(x + a_0 + b_0) = x^{\alpha_1}\tilde{\mathcal{R}}_0(x, \varepsilon) \tag{3.10}$$

for some holomorphic function $\tilde{\mathcal{R}}_0(x, \varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ being continuous up to the boundary such that $\tilde{\mathcal{R}}_0 = O(x_2^2)$. Putting $x_2 = 0$ or $x_1 = 0$ in (3.10) we see that $w := a_0$ (resp. $w := b_0$), $w = (w_1, w_2)$ satisfies the system of equations

$$\varepsilon x_1 \partial_1 w_1 - w_1 = R_1(x_1 + w_1, w_2), \tag{3.11}$$

$$\varepsilon x_1 \partial_1 w_2 + \tau w_2 = R_2(x_1 + w_1, w_2), \tag{3.12}$$

respectively

$$-\varepsilon\tau x_2 \partial_2 w_1 - w_1 = R_1(w_1, x_2 + w_2), \tag{3.13}$$

$$-\varepsilon\tau x_2 \partial_2 w_2 + \tau w_2 = R_2(w_1, x_2 + w_2). \tag{3.14}$$

We note that $\tilde{a}_0$ (resp. $\tilde{b}_0$) is the formal solution of (3.11)–(3.12) (resp. (3.13)–(3.14)). We will show that $\tilde{a}_0 = 0$. By (2.19) we have $R(x_1, 0) \equiv 0$. It follows that the terms of order x_1^2 in $R(x_1 + w_1, w_2)$ appear from the terms of the form $(x_1 + w_1)w_2$ or w_2^2. By (3.9) these terms are $O(x_1^3)$. In order to see that the coefficients of x_1^2 of w_1 and w_2 vanish, we note that $\varepsilon\nu - 1 \neq 0$ and $\varepsilon\nu + \tau \neq 0$ for all integers $\nu \geq 2$ and $\varepsilon \in \mathcal{C}_{\pm,\theta}$ because $\operatorname{Im}\varepsilon \neq 0$. Hence, the coefficient of x_1^2 in $\tilde{a}_0$ vanishes. Next, the coefficients of x_1^3 in the right-hand sides of (3.11)–(3.12) vanish by the similar argument because $\tilde{a}_0 = O(x_1^3)$. Hence the coefficient of x_1^3 in $\tilde{a}_0$ vanish by (3.11) and (3.12). By induction we obtain $\tilde{a}_0 = 0$. By the condition $R_j(x_1, 0) \equiv 0$ $(j = 1, 2)$, we can put $a_0 = 0$.

We consider (3.13)–(3.14). By the nonresonance condition in proving $\tilde{a}_0 = 0$ and (3.5) we see that (3.13)–(3.14) has a unique formal power series solution $\tilde{b}_0 = (\tilde{w}_1, \tilde{w}_2)$. By the well-known Briot-Bouquet theorem, $\tilde{b}_0$ converges in some neighborhood of the origin. (cf. [2]). We set $b_0 := \tilde{b}_0$. By taking ρ sufficiently small we may assume that b_0 is holomorphic in $\{|x_2| < \rho\}$. We can easily see that b_0 is holomorphic with respect to ε in some neighborhood of $\varepsilon = 1$. By taking ρ sufficiently small we may asssume that b_0 is holomorphic in $\{|\varepsilon - 1| < \rho\}$.

We will estimate the remainder term $\tilde{\mathcal{R}}_0$ in (3.10). By (3.13) and (3.14) we have

$$L_\Lambda^\varepsilon b_0 = R((0, x_2) + b_0).$$

Hence, by setting $y_1 = (x_1, 0)$ and $y_2 = (0, x_2) + b_0(x_2, \varepsilon)$ and by recalling $R(0) = 0$ we have

$$\mathcal{R}_0 = -R(y_1 + y_2) + R(y_2) = -\int_0^1 x_1(\partial_{x_1} R)(t_1 y_1 + y_2) dt_1. \tag{3.15}$$

By (2.19), if $(1, m)$ is on the support of R, then we have $m \geq k_1 + 2$. Hence $\mathcal{R}_0$ satisfies $\mathcal{R}_0 = x^{\alpha_1} \tilde{\mathcal{R}}_0$ for some $\tilde{\mathcal{R}}_0$ being holomorphic and bounded when $x \in S_\rho$ and $\varepsilon \in \mathcal{C}_\rho$ and satisfying $\tilde{\mathcal{R}}_0 = O(x_2^2)$.

Step 4. We will determine a_1 and b_1. For $0 \leq t \leq 1$ we set

$$u_t = b_0(x_2, \varepsilon) + t x^{\alpha_1} (a_1(x_1, \varepsilon) + b_1(x_2, \varepsilon)), \tag{3.16}$$

and we determine a_1 and b_1 $(b_1 = O(x_2^2))$ such that

$$\begin{aligned} \mathcal{R}_1 &:= L_\Lambda^\varepsilon (b_0 + x^{\alpha_1}(a_1 + b_1)) - R(x + b_0 + x^{\alpha_1}(a_1 + b_1)) \\ &= L_\Lambda^\varepsilon (x^{\alpha_1}(a_1 + b_1)) + T_1 + \mathcal{R}_0 = x^{\alpha_2} \tilde{\mathcal{R}}_1(x, \varepsilon), \\ T_1 &:= R(x + u_0) - R(x + u_1), \end{aligned} \tag{3.17}$$

for some holomorphic function $\tilde{\mathcal{R}}_1(x, \varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ being continuous up to the boundary such that $\tilde{\mathcal{R}}_1 = O(x_2^2)$.

We first show

$$\mathcal{R}_0 = -x^{\alpha_1} \beta_1(x_2, \varepsilon) + x^{\alpha_2} \Omega(x, \varepsilon) \tag{3.18}$$

for some holomorphic functions $\beta_1(x_2, \varepsilon)$ and $\Omega(x, \varepsilon)$ in $S_\rho \times \mathcal{C}_\rho$ being continous up to the boundary. Indeed, by Taylor's formula the integrand in the right-hand side of (3.15) is written in

$$x_1(\partial_{x_1} R)(t_1 y_1 + y_2) = x_1(\partial_{x_1} R)(y_2) + \int_0^1 t_1 x_1^2 (\partial_{x_1}^2 R)(t_1 t_2 y_1 + y_2) dt_2.$$

Hence, by (3.15) we have

$$\mathcal{R}_0 = -x_1(\partial_{x_1}R)(y_2) - \int_0^1 dt_1 \int_0^1 t_1 x_1^2 (\partial_{x_1}^2 R)(t_1 t_2 y_1 + y_2) dt_2 \equiv -x^{\alpha_1}\beta_1(x_2,\varepsilon) + x^{\alpha_2}\Omega(x,\varepsilon). \tag{3.19}$$

By the support condition on R and (3.19) the function $\Omega(x,\varepsilon)$ is a bounded holomorphic function on $S_\rho \times \mathcal{C}_\rho$. Hence we obtain the desired decomposition of $\mathcal{R}_0$. We note that $\beta_1 = O(x_2^2)$ and $\Omega = O(x_2^2)$ by (2.19) and (3.19).

We consider T_1. By Taylor's formula we have

$$x^{-\alpha_1}T_1 = -\int_0^1 (a_1+b_1)\nabla R(x+u_t)dt. \tag{3.20}$$

We set

$$\Theta_1 := \nabla R(x_1,0), \qquad \Theta_2 := \nabla R((0,x_2)+b_0(x_2,\varepsilon)). \tag{3.21}$$

First we shall show that Θ_1 identically vanishes. Indeed, by (2.19) and $R(x) = O(|x|^2)$ we obtain $R(x) = O(x_2^3)$, from which we have the assertion. By letting $x_2 \to 0$ in (3.20) and by recalling $b_0(0,\varepsilon) \equiv b_1(0,\varepsilon) \equiv 0$ we see that the right-hand side of (3.20) tends to 0. Similarly, by letting $x_1 \to 0$ in (3.20) we obtain $-(b_1 + a_1(0,\varepsilon))\Theta_2$. Therefore we have

$$T_1 + x^{\alpha_1}\left((b_1 + a_1(0,\varepsilon))\Theta_2\right) = x^{\alpha_1}x_1x_2\tilde{T}_1(x,\varepsilon), \tag{3.22}$$

for some $\tilde{T}_1(x,\varepsilon)$ holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$. Indeed, $x^{-\alpha_1}$ times the left-hand side of (3.22) is divisable by x_1x_2 by definition.

In order to obtain the equations for a_1 and b_1, we note, for U given by (3.4),

$$(x_1\partial_1 - \tau x_2\partial_2)(U - b_0) = \sum x^{\alpha_n}(x_1\partial_1 - \tau x_2\partial_2 + n - \tau k_n)(a_n + b_n). \tag{3.23}$$

By (3.17), (3.18) and (3.22) we have

$$\begin{aligned}\mathcal{R}_1 &= x^{\alpha_1}(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)a_1 + x^{\alpha_1}(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1 - x^{\alpha_1}\beta_1(x_2,\varepsilon) \\ &- x^{\alpha_1}(b_1 + a_1(0,\varepsilon))\Theta_2 + x^{\alpha_1}x_1x_2\tilde{T}_1(x,\varepsilon) + x^{\alpha_2}\Omega(x,\varepsilon).\end{aligned} \tag{3.24}$$

Step 5. We will solve the equations for a_1 and b_1. By equating the coefficients of x^{α_1} in (3.24) which are functions of x_1 we obtain

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)a_1 = 0. \tag{3.25}$$

Clearly, $a_1 = \tilde{a}_1(x_1,\varepsilon) \equiv 0$ is the unique formal power series solution of (3.25) by assumption. Indeed, this follows from the assumption that $\operatorname{Im}\varepsilon \neq 0$. Hence we may set $a_1 = 0$.

As for b_1, we obtain

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1 = b_1\Theta_2 + \beta_1(x_2,\varepsilon). \tag{3.26}$$

Let $\tilde{b}_1(x_2,\varepsilon) = \sum_{n=2}^\infty \gamma_n^{(0)}(\varepsilon)x_2^n$ be the unique formal power series solution of (3.26). Clearly, $\gamma_n^{(0)}(\varepsilon)$ is holomorphic in $\mathcal{C}_\rho$ and continuous up to the boundary. We define

$\|\gamma_n^{(0)}\|$ as the maximum of $|\gamma_n^{(0)}(\varepsilon)|$ on the closure of $\mathcal{C}_\rho$. Let $0 < \delta < 1$ be a positive small number chosen later and define, for $x_2 \in S_{2,\rho}$,

$$b_1^{(0)} = \sum_{n=2}^{\infty} \gamma_n^{(0)}(\varepsilon)\phi_n(x_2)^2 x_2^n, \tag{3.27}$$

where

$$\phi_n(x_2) = \begin{cases} 1 - \exp\left(-\dfrac{\delta^n}{(\|\gamma_n^{(0)}\|+1)x_2(n-1)!}\right), & (\text{ if } \|\gamma_n^{(0)}\| \neq 0), \\ 1 & (\text{ if } \|\gamma_n^{(0)}\| = 0). \end{cases} \tag{3.28}$$

In order to show the convergence of (3.27) we recall the inequality

$$|1 - e^{-z}| < |z|, \qquad \operatorname{Re} z > 0. \tag{3.29}$$

Noting that $\operatorname{Re} x_2 > 0$ ($x_2 \in S_{2,\rho}$) and

$$\frac{\delta^n}{(\|\gamma_n^{(0)}\|+1)(n-1)!} \le 1,$$

we have that, for $x_2 \in S_{2,\rho}$, $\gamma_n^{(0)} \neq 0$ and $n \ge 2$

$$|\gamma_n^{(0)}(\varepsilon)||x_2^n||\phi_n(x_2)|^2 \le |\gamma_n^{(0)}(\varepsilon)||x_2^n| \left(\frac{\delta^n}{(\|\gamma_n^{(0)}\|+1)|x_2|(n-1)!}\right)^2 \le \frac{|x_2|^{n-2}\delta^n}{(n-1)!}. \tag{3.30}$$

Hence the series in (3.27) converges uniformly on $S_{2,\rho} \times \mathcal{C}_\rho$, and the limit function is holomorphic in $(x, \varepsilon) \in S_{2,\rho} \times \mathcal{C}_\rho$ and bounded on its closure. Indeed, we have

$$\sum_{n \ge 2} |\gamma_n^{(0)}||x_2^n||\phi_n(x_2)|^2 \le \delta^2 \sum_{n\ge 2} \frac{|x_2|^{n-2}\delta^{n-2}}{(n-2)!} \le \delta^2 e^{\delta|x_2|}. \tag{3.31}$$

If $x_2 \in S_{2,\rho}$ and $\delta > 0$ is sufficiently small, then the right-hand side term can be made arbitrarily small. One can easily show that (cf. [1], p.68) $\tilde{b}_1$ is the asymptotic expansion of $b_1^{(0)}$ when $x_2 \to 0$, $x_2 \in S_{2,\rho}$. Moreover we can easily see that $b_1^{(0)}$ solves (3.26) asymptotically. Namely we have, for every $n = 0, 1, 2, \ldots$, there exist $R_n^{(0)}(x_2, \varepsilon)$ being holomorphic and bounded in $S_{2,\rho} \times \mathcal{C}_\rho$ such that, for every $\varepsilon \in \mathcal{C}_\rho$

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)b_1^{(0)} - b_1^{(0)}\Theta_2 - \beta_1 = x_2^n R_n^{(0)}(x_2, \varepsilon), \quad x_2 \in S_{2,\rho},\ x_2 \to 0. \tag{3.32}$$

Step 6. For a holomorphic and bounded (vector) function $v = v(x_2, \varepsilon)$ in $S_{2,\rho} \times \mathcal{C}_\rho$, we define the norm of v by

$$\|v\| := \sup_{x_2 \in S_{2,\rho},\, \varepsilon \in \mathcal{C}_\rho} |v(x_2, \varepsilon)|. \tag{3.33}$$

We similarly define the norm of a (vector) function $v = v(x_1, \varepsilon)$ on $S_1 \times \mathcal{C}_\rho$.

In order to solve (3.26) in $S_{2,\rho}$ we define the approximate sequence $w^{(\nu)} = (w_1^{(\nu)}, w_2^{(\nu)})$ ($\nu = 0, 1, 2, \dots$) by $w^{(0)} = b_1^{(0)}$ and

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(1)} = \beta_1 + w^{(0)}\Theta_2 - (L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(0)}, \tag{3.34}$$

$$(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)w^{(\nu)} = w^{(\nu-1)}\Theta_2, \nu = 2, 3, \dots \tag{3.35}$$

If we can show the uniform convergence $b_1 := w^{(0)} + w^{(1)} + \cdots$ on $S_{2,\rho} \times \mathcal{C}_\rho$, then b_1 is the desired holomorphic solution of (3.26) in $S_{2,\rho} \times \mathcal{C}_\rho$.

We will estimate $w^{(j)}$. In order to solve (3.34)–(3.35) we recall that for every g holomorphic and bounded in $S_{2,\rho}$ with all derivatives vanishing at the origin and a complex number $\lambda \neq 0$, the solution of the equation $(x_2\partial_2 - \lambda)u = g$ is given by

$$u = (x_2\partial_2 - \lambda)^{-1}g = \int_{-\infty}^0 e^{-\lambda t} g(e^t x_2)dt, \tag{3.36}$$

where the integral converges by the assumption on g if $\operatorname{Re}\lambda \geq 0$. It follows that $w_1^{(1)}$ is well defined, holomorphic and bounded in $S_{2,\rho}$.

We shall prove that there exist constants $\eta_0 > 0$ and $0 < r_0 < 1$ such that

$$\|w_k^{(\nu)}\| \leq \eta_0 r_0^\nu, \quad k = 1, 2;\ \nu = 0, 1, \dots, \tag{3.37}$$

where $\eta_0 > 0$ can be chosen arbitrarily small if we take $\delta > 0$ sufficiently small. Clearly, if we can prove (3.37), then the limit $w_k := w_k^{(0)} + w_k^{(1)} + \cdots$ ($k = 1, 2$) exists on $S_{2,\rho} \times \mathcal{C}_\rho$ and $b_1 := (w_1, w_2)$ gives the desired solution. We will estimate $w^{(1)}$ by (3.34) and (3.32). For simplicity, let us denote the right-hand side of (3.34) by h_0. We take n in (3.32) sufficiently large so that $(L_\Lambda^\varepsilon + \varepsilon - \varepsilon\tau k_1)^{-1}h_0$ is well defined. In view of the formula (3.36) we see that the norm of $w^{(1)}$ can be made arbitrarily small on $S_{2,\rho} \times \mathcal{C}_\rho$ by taking ρ sufficiently small because there appears a power x_2^n.

As for the estimate of $w^{(\nu)}$, we can recursively estimate the term in view of the recurrence relation (3.35) and the smallness of Θ_2. Indeed, Θ_2 vanishes up to order 2 by the assumption $R(x) = O(x_2^3)$.

Next we will show that $\tilde{b}_1$ is the asymptotic expansion of $b_1 := \sum_{\nu=0}^\infty w^{(\nu)}$. Because $\tilde{b}_1$ is the asymptotic expansion of $b_1^{(0)}$ we will show that $\sum_{\nu=1}^\infty w^{(\nu)} \sim 0$ when $x \to 0$. In order to show this, let $\ell \geq 2$ be a given integer and consider the sum $\sum_{\nu=1}^\infty \tilde{w}^{(\nu)}$, where $\tilde{w}^{(\nu)} = x_2^{-\ell} w^{(\nu)}$. If we can show the uniform convergence of $\sum_{\nu=1}^\infty \tilde{w}^{(\nu)}$ on $S_{2,\rho}$, then we see that $b_1 - b_1^{(0)}$ vanishes up to order ℓ when $x_2 \to 0$. Because $\ell \geq 2$ is arbitrary, this proves that the asymptotic expansion of b_1 is equal to $b_1^{(0)}$.

We define $\tilde{g}(z) := z^{-\ell} g(z)$. Then, we get from (3.36)

$$\tilde{u}(x_2) := x_2^{-\ell} u(x_2) = \int_{-\infty}^0 e^{-\lambda t + \ell t} \tilde{g}(e^t x_2)dt. \tag{3.38}$$

We note that $e^{-\lambda t + \ell t}$ is integrable if ℓ is sufficiently large. Hence we can estimate $\tilde{u}$ in terms of $\tilde{g}$. By (3.34) we can estimate $\tilde{w}^{(1)}$ in terms of the right-hand side of (3.34). By (3.35) we can similarly estimate $\tilde{w}^{(\nu)}$ in terms of $\tilde{g}$ with $g = w^{(\nu-1)}\Theta_2$. Because $\tilde{g}(z) = z^{-\ell} w^{(\nu-1)}\Theta_2 = \tilde{w}^{(\nu-1)}\Theta_2$, this proves the uniform convergence of $\sum_{\nu=1}^\infty \tilde{w}^{(\nu)}$.

Step 7. We will show (3.17) for some $\mathcal{R}_1 = O(x_2^2)$. We want to prove

$$T_1 + x^{\alpha_1} b_1 \Theta_2 = x^{\alpha_2} \tilde{T}_1, \tag{3.39}$$

for some holomorphic and bounded function $\tilde{T}_1(x,\varepsilon)$ on $S_\rho \times \mathcal{C}_\rho$ such that $\tilde{T}_1 = O(x_2^2)$. If we can prove this, then (3.18), (3.26) and (3.39) imply (3.17) for $\tilde{\mathcal{R}}_1 = \tilde{T}_1 + \Omega$.

We first show that $\alpha_j + \alpha_1 \geq \alpha_{j+1}$ for every $j \geq 1$. Indeed, by definition we have $-\tau < j - \tau k_j < 0$ for every j. Hence, by adding the inequalities for $j = j$ and $j = 1$ we obtain $-2\tau < j + 1 - \tau(k_j + k_1) < 0$. By the minimality of k_{j+1} we have $k_j + k_1 \geq k_{j+1}$.

In order to show (3.39) we first note, by (3.20) and $a_1 = 0$

$$-x^{-\alpha_1} T_1 - \Theta_2 b_1 = \int b_1 \left(\nabla R(x + b_0 + t x^{\alpha_1} b_1) - \Theta_2\right) dt. \tag{3.40}$$

By the definition of R and Θ_2 we can easily see that $\nabla R(x + b_0 + t x^{\alpha_1} b_1) - \Theta_2$ is divisable by x^{α_1} with the quotient being holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$. In view of (3.40) and $2\alpha_1 \geq \alpha_2$ we have (3.39).

We easily see that the support of T_1 is contained in the set $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2; \alpha_1 - \tau\alpha_2 < -2\tau\}$ in view of (3.40). It follows that $\tilde{T}_1 = O(x_2^2)$. Because the support of $\mathcal{R}_0$ is contained in the set $\{(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2; \alpha_1 - \tau\alpha_2 < -2\tau\}$, the same assertion holds for that of $\mathcal{R}_1$.

Step 8. We will determine a_2 and b_2. We set $u_1 = b_0 + x^{\alpha_1} b_1$, and we determine $a_2(x_1, \varepsilon)$ and $b_2(x_2, \varepsilon)$ $(b_2(0, \varepsilon) \equiv 0)$ such that

$$\mathcal{R}_2 := L_\Lambda^\varepsilon(x^{\alpha_2}(a_2 + b_2)) + T_2 + \mathcal{R}_1 = x^{\alpha_3} \tilde{\mathcal{R}}_2(x), \tag{3.41}$$

where $\tilde{\mathcal{R}}_2(x) = O(x_2^2)$ and

$$T_2 := -R(x + u_1 + x^{\alpha_2}(a_2 + b_2)) + R(x + u_1), \tag{3.42}$$

and $\mathcal{R}_1$ is given by (3.17) with $a_1 = 0$. In the following we do not indicate the dependency with respect to ε explicitly if there is no fear of confusion. We will show that

$$T_2 = -x^{\alpha_2} b_2(x_2)\Theta_2 + x^{\alpha_2} x_1 x_2 \tilde{T}_2 \tag{3.43}$$

for some bounded holomorphic function $\tilde{T}_2$ in $S_\rho \times \mathcal{C}_\rho$. Indeed, by Taylor's formula and by similar calculations in proving (3.22) we can easily see that the term of order $O(x^{\alpha_2})$ in T_2 is given by $-x^{\alpha_2}(b_2(x_2) + a_2(0))\Theta_2$. Moreover, we have

$$T_2 x^{-\alpha_2} + (b_2(x_2) + a_2(0))\Theta_2 = O(x_1 x_2)$$

in view of the definition of the remainder term.

Next, let $\mathcal{R}_1 = x^{\alpha_2}(\tilde{T}_1(x) + \Omega(x))$ be given by (3.17). Because the term $L_\Lambda^\varepsilon(x^{\alpha_2}(a_2 + b_2))$ cancels with the corresponding terms in $T_2 + \mathcal{R}_1$ of order $O(x^{\alpha_2})$, we look for the decomposition

$$\mathcal{R}_1 = -x^{\alpha_2}(\gamma_2(x_1, \varepsilon) + \beta_2(x_2, \varepsilon)) + x^{\alpha_3}\Omega_1(x, \varepsilon) \tag{3.44}$$

for some $\gamma_2(x_1, \varepsilon)$ and $\beta_2(x_2, \varepsilon)$, $\beta_2 = O(x_2^2)$ holomorphic in S_1 and $S_{2,\rho}$, respectively, and Ω_1 being holomorphic in $S_\rho \times \mathcal{C}_\rho$. In order to compute γ_2 and β_2 we restrict

$\tilde{T}_1(x,\varepsilon)+\Omega(x,\varepsilon)$ to $x_2=0$ or $x_1=0$. By the definition of $\Omega(x,\varepsilon)$ in (3.19) and the assumption (2.19) we have $\Omega(x_1,0,\varepsilon)\equiv 0$. Next, by (3.40) and $b_1(0)=0$ we have $\tilde{T}_1(x_1,0,\varepsilon)\equiv 0$. Hence we have $\gamma_2=0$. By defining

$$\beta_2(x_2,\varepsilon)=-\tilde{T}_1(0,x_2,\varepsilon)-\Omega(0,x_2,\varepsilon)$$

we will show (3.44). In view of (3.40) and (2.19) we see that $\tilde{T}_1(x,\varepsilon)-\tilde{T}_1(0,x_2,\varepsilon)$ is divisable by x^{α_1}. By $\alpha_2+\alpha_1\ge\alpha_3$ we see that $x^{\alpha_2}(\tilde{T}_1(x,\varepsilon)-\tilde{T}_1(0,x_2,\varepsilon))$is divisable by x^{α_3}. On the other hand, by (3.19) and (2.19) $x^{\alpha_2}(\Omega(x,\varepsilon)-\Omega(0,x_2,\varepsilon))$ is divisable by x^{α_3}. We also note that $\beta_2(x_2,\varepsilon)=O(x_2^2)$.

Therefore we will determine a_2 and b_2 by the equations

$$(L_\Lambda^\varepsilon+2\varepsilon-\varepsilon\tau k_2)a_2 = 0, \tag{3.45}$$

$$(L_\Lambda^\varepsilon+2\varepsilon-\varepsilon\tau k_2)b_2 = (b_2+a_2(0))\Theta_2+\beta_2. \tag{3.46}$$

We easily see that $\tilde{a}_2=0$ and we can take $a_2=0$. Because (3.46) has the formal power series solution $\tilde{b}_2$, we define $b_2^{(0)}$ by the formula similar to (3.27). Then $b_2^{(0)}$ has an asymptotic expansion $\tilde{b}_2$. We note that the modulus of $b_2^{(0)}$ can be taken arbitrarily small in a neighborhood of the origin by taking δ in (3.27) sufficiently small. In order to solve (3.46) we construct the approximate sequence $w^{(\nu)}$ $(\nu\ge 1)$ by the relations like (3.34) and (3.35). We can easily see that $b_2:=w^{(0)}+w^{(1)}+\cdots$ converges in $S_{2,\rho}\times\mathcal{C}_\rho$ and gives a holomorphic solution of (3.46) with asymptotic expansion $\tilde{b}_2$. We can show that

$$T_2=-x^{\alpha_2}b_2\Theta_2+x^{\alpha_3}\tilde{T}_2 \tag{3.47}$$

for a possibly different holomorphic function $\tilde{T}_2=\tilde{T}_2(x,\varepsilon)$ in $S_\rho\times\mathcal{C}_\rho$ such that $\tilde{T}_2=O(x_2^2)$. We can also prove that the support of T_2 is contained in the set $\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2;\alpha_1-\tau\alpha_2<-2\tau\}$. Indeed, these facts follow from the support conditions of R and b_2 by applying Taylor's formula in integral form to (3.42).

Step 9. We will determine a_n and b_n. Suppose that we have determined $a_j=0$ and b_j, $b_j=O(x_2^2)$ as holomorphic and bounded functions on $S_{2,\rho}\times\mathcal{C}_\rho$ for all $j\le n-1$ satisfying (3.7) up to n such that the support of $\mathcal{R}_{n-1}$ is contained in $\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2;\alpha_1-\tau\alpha_2<-2\tau\}$. We will determine $a_n(x_1,\varepsilon)$ (resp. $b_n(x_2,\varepsilon)$) such that

$$U_n:=U_{n-1}+x^{\alpha_n}(a_n(x_1,\varepsilon)+b_n(x_2,\varepsilon))$$

satisfies (3.7) with n replaced by $n+1$. Let x_1 and x_2 be so small that $R(x+U_n)$ is well defined. First we consider

$$\begin{aligned}\mathcal{R}_n &:= L_\Lambda^\varepsilon U_n-R(x+U_n)=L_\Lambda^\varepsilon U_{n-1}-R(x+U_{n-1})\\ &+ L_\Lambda^\varepsilon(x^{\alpha_n}(a_n+b_n))+R(x+U_{n-1})-R(x+U_n)\\ &= \mathcal{R}_{n-1}+L_\Lambda^\varepsilon(x^{\alpha_n}(a_n+b_n))+T_n,\end{aligned} \tag{3.48}$$

where $T_n=R(x+U_{n-1})-R(x+U_n)$.

We want to write

$$\mathcal{R}_{n-1}=x^{\alpha_n}\tilde{\mathcal{R}}_{n-1}(x,\varepsilon)=-x^{\alpha_n}(\gamma_n(x_1,\varepsilon)+\beta_n(x_2,\varepsilon))+x^{\alpha_{n+1}}\Omega_n(x,\varepsilon). \tag{3.49}$$

Indeed, by an approriate choice of β_n and γ_n we have $\tilde{\mathcal{R}}_{n-1}(x,\varepsilon)+\beta_n+\gamma_n = O(x_1x_2)$. By the support property of $\mathcal{R}_{n-1}$ we may define $\gamma_n = 0$. Moreover, by (2.19), we have $\beta_n = O(x_2^2)$. We will show that the $O(x_1x_2x^{\alpha_n})$ term in $\mathcal{R}_{n-1}$ is $O(x^{\alpha_{n+1}})$. This is clear when $\tau > 1$ because $k_{n+1} = k_n$ or $k_{n+1} = k_n + 1$. On the other hand, if $0 < \tau < 1$, then, in view of the support property of $\mathcal{R}_{n-1}$ the $O(x_1x_2x^{\alpha_n})$ term in $\mathcal{R}_{n-1}$ is $O(x^{\alpha_{n+1}})$ and, consequently, $O(x_2^2x^{\alpha_{n+1}})$ by the same condition.

In order to obtain the equations for a_n and b_n we note

$$
\begin{aligned}
T_n &= -x^{\alpha_n}\int_0^1 (a_n+b_n)\nabla R(x+U_{n-1}+tx^{\alpha_n}(a_n+b_n))dt \\
&= x^{\alpha_n}(b_n+a_n(0,\varepsilon))\Theta_2 + O(x_1x_2x^{\alpha_n}).
\end{aligned} \tag{3.50}
$$

Therefore, by dividing (3.7) with n replaced by $n+1$ by x^{α_n} and by setting $x_2 = 0$ we obtain, in view of (3.23) and (3.48)

$$
(L_\Lambda^\varepsilon + n\varepsilon - k_n\tau\varepsilon)a_n = 0. \tag{3.51}
$$

As in the previous case, the formal solution $\tilde{a}_n$ of (3.51) vanishes and we may define $a_n = 0$. Next we consider the equation for b_n. We divide (3.7) with n replaced by $n+1$ by x^{α_n}. Then, by setting $x_1 = 0$ we obtain

$$
(L_\Lambda^\varepsilon + n\varepsilon - k_n\tau\varepsilon)b_n = b_n\Theta_2 + \beta_n(x_2,\varepsilon). \tag{3.52}
$$

By the same argument as for b_1 we can determine b_n as a bounded holomorphic function on $S_{2,\rho}\times\mathcal{C}_\rho$ such that $b_n = O(x_2^2)$. Therefore we can determine the formal solution U in (3.4).

We can see from (3.48) and the inductive assumption for $\mathcal{R}_{n-1}$ that the support of $\mathcal{R}_n$ is contained in $\{(\alpha_1,\alpha_2)\in\mathbb{Z}_+^2; \alpha_1-\tau\alpha_2 < -2\tau\}$, because the support of T_n is contained in the same set. In order to prove (3.7), note that $O(x_1x_2x^{\alpha_n})$ terms in (3.50) are, indeed, $O(x_2^2x^{\alpha_{n+1}})$, which can be shown by the support condition on R.

Step 10. We make the Borel-Ritt type argument to the formal series (3.4). By definition we have $\alpha_n = (n,k_n)$, $-\tau < n-\tau k_n < 0$. Hence we have $\lim_{n\to\infty}\alpha_n/n = (1,\tau^{-1})$. By the definition of S_ρ we can show that there exists $N\ge 1$ such that for any $n\ge N$ we have $\operatorname{Re} x^{\alpha_n/n} > 0$ on S_ρ. Indeed, by setting $x_j = r_je^{i\theta_j}$ for $j=1,2$ with $0<r_1<\infty$, $0<r_2\le\rho$, $|\theta_j|\le\eta_j$, we obtain

$$
x^{\alpha_n/n} = r_1r_2^{k_n/n}\exp(i(\theta_1+\theta_2k_n/n)).
$$

By the assumption $\eta_1+\eta_2/\tau < \pi/2$ and the relation $k_n/n\to\tau^{-1}$, we see that there exists $N>0$ such that for $n\ge N$, we have $|\theta_1+\theta_2k_n/n| < \pi/2$. This shows the assertion.

Suppose $\delta > 0$. Then we define

$$
\gamma_n := \max_{x,\varepsilon}\{\|x_2^{-1}b_n\|+1, \|x_2^{-1}\varepsilon(-\tau x_2\partial_{x_2}+n-\tau k_n)b_n)\|\}, \tag{3.53}
$$

and define $V(x,\varepsilon)$ on $S_\rho\times\mathcal{C}_\rho$ by

$$
V(x,\varepsilon) = \sum_{n=0}^\infty b_n(x_2,\varepsilon)\varphi_n(x)^2x^{\alpha_n}, \tag{3.54}
$$

where $\varphi_n = 1$ for $0 \leq n < N$, and for $n \geq N$

$$\varphi_n(x) = 1 - \exp\left(-\frac{\delta^n}{\gamma_n x^{\alpha_n/n}(n-1)!}\right). \tag{3.55}$$

In order to show that $V(x,\varepsilon)$ is holomorphic in $S_\rho \times \mathcal{C}_\rho$ we make a similar argument as for (3.27). Let $\operatorname{Re} x^{\alpha_n/n} > 0$ on S_ρ. Then we have, for $n \geq N$

$$\begin{aligned} |b_n||\varphi_n|^2|x^{\alpha_n}| &\leq \|x_2^{-1}b_n\||x_2x^{\alpha_n}|\left(\frac{\delta^n}{\gamma_n|x^{\alpha_n/n}|(n-1)!}\right)^2 \\ &\leq \delta^{2n}|x_2x^{\alpha_n(1-2/n)}|((n-1)!)^{-2}. \end{aligned} \tag{3.56}$$

Because $\alpha_n(1-2/n) = (n-2)(1, \frac{k_n}{n})$, we see that the sum

$$\sum \delta^{2n}|x_2x^{\alpha_n(1-2/n)}|((n-1)!)^{-2}$$

convergers on $S_\rho \times \mathcal{C}_\rho$. Hence the series (3.54) converges on $S_\rho \times \mathcal{C}_\rho$.

Step 11. We will show (3.3). Take any positive integer $n \geq N$ and write

$$\begin{aligned} V(x,\varepsilon) &= \sum_{j=0}^{n} x^{\alpha_j}b_j(x_2,\varepsilon) + \sum_{j=0}^{n} x^{\alpha_j}b_j(x_2,\varepsilon)(\varphi_j(x)^2 - 1) \\ &+ \sum_{j=n+1}^{\infty} x^{\alpha_j}b_j(x_2,\varepsilon)\varphi_j(x)^2 \equiv V_1 + V_2 + V_3. \end{aligned} \tag{3.57}$$

First, we show that $V_2 = O(x_2^2x^{\alpha_{n+1}})$ when $x \to 0$, $x \in S_\rho$. Indeed, for $j \geq N$ we have

$$\varphi_j(x) - 1 = -\exp\left(-\frac{\delta^j}{\gamma_j x^{\alpha_j/j}(j-1)!}\right).$$

For every $\nu \geq 1$ the right-hand side is $O(|x^{\alpha_j\nu/j}|)$ on S_ρ when $x \to 0$. Hence, by taking ν sufficiently large, it is divisable by $x^{\alpha_{n+1}}$ with the quotient bounded holomorphic in S_ρ. Because $b_j = O(x_2^2)$, we have $V_2 = O(x_2^2x^{\alpha_{n+1}})$. Next we will show that V_3 is divisable by $x^{\alpha_{n+1}}$ with the quotient being bounded holomorphic in $S_\rho \times \mathcal{C}_\rho$. Because $\alpha_j \geq \alpha_{n+1}$ for every $j \geq n+1$ and $b_j = (x_2^2)$ it is sufficient to prove that $|x_2x^{\alpha_j-\alpha_{n+1}}| < \rho^{j-n-1}$ on S_ρ for every $j > n+1$.

Indeed, by definition we have $j - \tau k_j < 0$ and $-\tau < n+1-\tau k_{n+1} < 0$. It follows that

$$k_j - k_{n+1} > \tau^{-1}(j-n-1) - 1.$$

Therefore, since $|x_2| < 1$ and $|x_1||x_2|^{1/\tau} < \rho$ on S_ρ, we have

$$\begin{aligned} |x_2x^{\alpha_j-\alpha_{n+1}}| &= |x_1|^{j-n-1}|x_2|^{k_j-k_{n+1}+1} \leq |x_1|^{j-n-1}|x_2|^{\tau^{-1}(j-n-1)} \\ &\leq (|x_1||x_2|^{1/\tau})^{j-n-1} \leq \rho^{j-n-1}. \end{aligned}$$

Therefore we have (3.3).

Step 12. We will prove (3.2). We set $g = R(x+V) - L^{\varepsilon}_{\Lambda}V$, where $R(x+V)$ is well defined for sufficiently small $\delta > 0$ in view of the definition of V. We write V in the form (3.57) and for x sufficiently small we write

$$g = R(x+W) - L^{\varepsilon}_{\Lambda}W + R(x+W+V_3) - R(x+W) - L^{\varepsilon}_{\Lambda}V_3, \tag{3.58}$$

where $V = W + V_3$ and $W := V_1 + V_2$.

We want to show that $L^{\varepsilon}_{\Lambda}V_3 = x^{\alpha_{n+1}}A_1(x,\varepsilon)$ for some holomorphic and bounded function $A_1 = O(x_2)$ on $S_\rho \times \mathcal{C}_\rho$. Indeed, if a derivation in $L^{\varepsilon}_{\Lambda}$ is applied to $\varphi_j(x)^2$, then, by the same argument as for the convergence of V, we see that the resultant series is convergent and divisable by x_2^2. We also note that every term in the series has the factor x^{α_j} with $\alpha_j \geq \alpha_{n+1}$. If $L^{\varepsilon}_{\Lambda}$ is applied to the term $x^{\alpha_j}b_j(x_2,\varepsilon)$ in $x^{\alpha_j}b_j(x_2,\varepsilon)\varphi_j(x)^2$, then we have

$$L^{\varepsilon}_{\Lambda}(x^{\alpha_j}b_j) = x^{\alpha_j}\left(\varepsilon(-\tau x_2\partial_{x_2} + j - \tau k_j) - \Lambda\right)b_j(x_2). \tag{3.59}$$

In view of (3.53) and the proof of the convergence of $V(x,\varepsilon)$ we see that the sum of terms in the right-hand side (3.59) converges and bounded when $S_\rho \times \mathcal{C}_\rho$.

In view of the estimate of V_3, we can see that A_1 is divisable by x_2. it is also easy to see that if $0 < \theta' < 1$ satisfies $-(1-\theta')\tau < n+1-\tau k_{n+1} < 0$, then we have $|x_2|^{1-\theta'}|x^{\alpha_j - \alpha_{n+1}}| \leq \rho^{j-n-1}$. In fact, for every $0 < \theta' < 1$ there exist infinitely many k_n such that $-(1-\theta')\tau < n+1-\tau k_{n+1} < 0$. For those n's we have $A_1 = O(|x_2|^{1+\theta'})$.

Next, by Taylor's formula we have

$$R(x+W+V_3) - R(x+W) = \int_0^1 V_3 \cdot \nabla R(x+W+tV_3)dt.$$

It follows that $R(x+W+V_3) - R(x+W) = x^{\alpha_{n+1}}A_2(x,\varepsilon)$ for some holomorphic and bounded function A_2 in $S_\rho \times \mathcal{C}_\rho$. In view of the estimate of V_3 and $\nabla R(x+W+tV_3) = O(x_2)$ we see that $A_2 = O(x_2^2)$.

We consider

$$R(x+W) - L^{\varepsilon}_{\Lambda}W = R(x+W) - L^{\varepsilon}_{\Lambda}V_1 - L^{\varepsilon}_{\Lambda}V_2.$$

It is easy to see that $L^{\varepsilon}_{\Lambda}V_2 = x^{\alpha_{n+1}}A_3(x,\varepsilon)$ for some holomorphic and bounded function A_3 in $S_\rho \times \mathcal{C}_\rho$ such that $A_3 = O(x_2^2)$. Indeed, the functions $\varphi_j(x)^2 - 1$ in V_2 and $L^{\varepsilon}_{\Lambda}(\varphi_j(x)^2 - 1)$ can be divisable by an arbitrary power of $x^{\alpha_j/j} = x_1 x_2^{d_j/j}$ such that the quotinent is holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$. Because $d_j/j > \tau^{-1}$, we see that it is $O(x_2^2 x^{\alpha_{n+1}})$.

We take $\rho' \leq \rho$ sufficiently small such that for every x with $|x_1||x_2|^{1/\tau} < \rho'$ and $|x_2| < \rho$ the values $x + V_1$, $x + V_1 + V_2$ are in the domain of R. Then we have $R(x+V_1+V_2) - R(x+V_1) = \int_0^1 V_2 \cdot \nabla R(x+V_1+tV_2)dt$. Clearly, the right-hand side function can be written in $x^{\alpha_{n+1}}A_4(x,\varepsilon)$ for some holomorphic and bounded function A_4 in $S_\rho \times \mathcal{C}_\rho$ with $|x_1||x_2|^{1/\tau} < \rho'$ such that $A_4 = O(x_2^2)$. Now we have

$$R(x+W) - L^{\varepsilon}_{\Lambda}W = R(x+V_1+V_2) - R(x+V_1) + R(x+V_1) - L^{\varepsilon}_{\Lambda}V_1 - L^{\varepsilon}_{\Lambda}V_2.$$

By the definition of V_1 we see that $R(x+V_1) - L^{\varepsilon}_{\Lambda}V_1 = x^{\alpha_{n+1}}A_5(x,\varepsilon)$ for some holomorphic and bounded function A_5 in $x \in S_\rho$, $|x_1||x_2|^{1/\tau} < \rho'$ such that $A_5 =$

$O(x_2^2)$. It follows that $F(x) := x^{-\alpha_{n+1}}(R(x+W) - L_\Lambda^\varepsilon W)$ is holomorphic and bounded in S_ρ such that $|x_1||x_2|^{1/\tau} < \rho'$. Because $R(x+W) - L_\Lambda^\varepsilon W$ is holomorphic in S_ρ and $x^{\alpha_{n+1}}$ does not vanish in $\rho' \le |x_1||x_2|^{1/\tau} \le \rho$, we see that $F(x)$ is also holomorphic in S_ρ. In order to prove the boundedness of $F(x)$ in S_ρ, we will show the boundedness of $F(x)$ when $\rho' \le |x_1||x_2|^{1/\tau} \le \rho$. We may assume, without loss of generality, that $0 < |x_2| < 1$. We note

$$|x^{\alpha_{n+1}}| = (|x_1||x_2|^{1/\tau}|x_2|^{\frac{k_{n+1}}{n+1} - \frac{1}{\tau}})^{n+1} \ge (\rho')^{n+1}|x_2|^{\left(\frac{k_{n+1}}{n+1} - \frac{1}{\tau}\right)(n+1)}.$$

Because

$$\frac{k_{n+1}}{n+1} - \frac{1}{\tau} < \frac{1}{n+1}$$

it follows that

$$|x_2|^{\left(\frac{k_{n+1}}{n+1} - \frac{1}{\tau}\right)(n+1)} > |x_2|.$$

On the other hand, we have $R(x+W) - L_\Lambda^\varepsilon W = O(x_2)$. This proves that $F(x)$ is bounded when $\rho' \le |x_1||x_2|^{1/\tau} \le \rho$. Because n is arbitrary, we have proved (3.2). This completes the proof of the lemma.

4. Proof of Theorem

Proof of Theorem 2. We prove Theorem 2 in case (2.19) is satisfied. We can similarly argue in case (2.20) is verified by changing the roles of x_1 and x_2. Let V be given by Lemma 3. Let $\alpha_N = (N, k_N)$ satisfy that $x_2^{-1-\theta'}\tilde{g}_N(x,\varepsilon)$ is holomorphic and bounded in $S_\rho \times \mathcal{C}_{\pm,\theta,\rho}$ as in Lemma 3. In order to solve (2.4), set $u(x) = v(x) + V(x)$ and consider

$$L_\Lambda^\varepsilon v = R(x+V+v) - L_\Lambda^\varepsilon V = R(x+V+v) - R(x+V) + g, \tag{4.1}$$

where $g := R(x+V) - L_\Lambda^\varepsilon V$.

Let $\rho > 0$ and $N \ge 1$ be an integer. For a holomorphic and bounded (vector) function

$$h = (h_1, h_2) = x^{\alpha_N}\tilde{h}(x,\varepsilon) = x^{\alpha_N}(\tilde{h}_1, \tilde{h}_2)$$

in $S_\rho \times \mathcal{C}_\rho$ with $\tilde{h}(x,\varepsilon)$ being holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$, we define the norm of h by

$$\|h\|_N := \sup_{x\in S_\rho, \varepsilon\in\mathcal{C}_\rho}(|x^{-\alpha_N}h_1(x,\varepsilon)| + |x^{-\alpha_N}x_2^{-1-\theta'/2}h_2(x,\varepsilon)|). \tag{4.2}$$

Let X_N be the set of functions h being holomorphic and bounded in $S_\rho \times \mathcal{C}_\rho$ such that $\|h\|_N < \infty$. Clearly, X_N is the Banach space with the norm (4.2). We choose a sequence $\alpha_N = (N, k_N)$, $N = N_\nu (\nu = 1, 2, \dots)$ such that for every pair α_N and α_ℓ in the sequence with $N > \ell$ we have

$$d_N - \frac{N}{\tau} \ge d_\ell - \frac{\ell}{\tau}.$$

Because $q - p/\tau$ is dense on $\mathbb{R}$ if p and q run in $\mathbb{Z}$, we can choose $\{\alpha_N\}$ satisfying the condition. We shall show that X_N is continuously embedded into X_ℓ. Indeed, for every $h = x^{\alpha_N}\tilde{h}_N \in X_N$ we have

$$x^{\alpha_N}\tilde{h}_N = x^{\alpha_\ell}x^{\alpha_N-\alpha_\ell}\tilde{h}_N = x^{\alpha_\ell}(x_1x_2^{1/\tau})^{N-\ell}x_2^{d_N-d_\ell-(N-\ell)/\tau}\tilde{h}_N.$$

Because $d_N - d_\ell - (N-\ell)/\tau > 0$ by assumption, we see that there exists $C > 0$ such that $\|h\|_\ell \le C\|h\|_N$. This proves the assertion.

For $\|h\|_N < \infty$ we define

$$v := -\frac{1}{\varepsilon}\int_0^\infty e^{-\Lambda t/\varepsilon}h(e^{t\Lambda}x,\varepsilon)dt. \tag{4.3}$$

Because $|x_1^N e^{Nt}x_2^{k_N}e^{-k_N\tau t}| = |x_1^N x_2^{k_N}e^{t(N-\tau k_N)}| \le |x_1^N x_2^{k_N}|$ for all $t \ge 0$, we see that $h(e^{t\Lambda}x,\varepsilon)$ in the integrand is bounded if $x \in S_\rho$, $\varepsilon \in \mathcal{C}_\rho$, $t \ge 0$. In order to show that the integral (4.3) converges we may consider the second component. In the integrand the following factor appears:

$$e^{t\tau/\varepsilon}e^{-(1+\theta'/2)\tau t}, \quad t \ge 0.$$

Therefore, if ε is sufficiently close to 1, then the integral converges. We easily see that v is the solution of the equation $L_\Lambda^\varepsilon v = h$, namely $v = (L_\Lambda^\varepsilon)^{-1}h$, where $(L_\Lambda^\varepsilon)^{-1}$ has the expression (4.3). Moreover, we have $\|v\|_N < \infty$.

We want to define the approximate sequence $\{v^{(k)}\}$ by

$$\begin{aligned} v^{(0)} &:= (L_\Lambda^\varepsilon)^{-1}g, \quad v^{(1)} := (L_\Lambda^\varepsilon)^{-1}(R(x+V+v^{(0)}) - R(x+V)), \\ v^{(k)} &:= (L_\Lambda^\varepsilon)^{-1}(R(x+V+v^{(0)}+\cdots+v^{(k-1)}) \\ &\qquad -R(x+V+v^{(0)}+\cdots+v^{(k-2)})), \quad k = 2,3,\ldots \end{aligned} \tag{4.4}$$

It is easy to see that if $v := \sum_{k=0}^\infty v^{(k)}$ converges, then v solves (4.1). In order to see that $v^{(k)}$'s are well defined, we note, from the definition of V in Lemma 3 and (2.19) that $g(x,\varepsilon) = x^{\alpha_N}\tilde{g}(x,\varepsilon)$ for some bounded holomorphic function $\tilde{g}$ in $S_\rho \times \mathcal{C}_\rho$ such that $\tilde{g} = O(x_2^{1+\theta'})$. Especially we have $\|g\|_N < \infty$. Hence $v^{(0)} \in X_N$. In order to estimate $v^{(0)}$ we obtain, in view of (4.3) and (4.4)

$$\begin{aligned} \|v^{(0)}\|_N &\le \sup\frac{1}{|\varepsilon|}\int_0^\infty e^{t(N-\tau k_N)}\left(|e^{-t/\varepsilon}\tilde{g}_1(e^{t\Lambda}x,\varepsilon)|\right. \\ &+ \left.|e^{t\tau/\varepsilon-\tau t-\theta'\tau t/2}(x_2^{-1-\theta'/2}\tilde{g}_2)(e^{t\Lambda}x,\varepsilon)|\right)dt \le C\rho^{\theta'/2}\|g\|_N, \end{aligned} \tag{4.5}$$

for some constant $C > 0$ independent of N since $N - \tau k_N < 0$ and $|x_2| < \rho$. Indeed, there appears $(x_2e^{-t\tau})^{\theta'/2}$ from $\tilde{g}_1(e^{t\Lambda}x,\varepsilon)$ and $(x_2^{-1-\theta'/2}\tilde{g}_2)(e^{t\Lambda}x,\varepsilon)$.

By (4.5) the function $R(x+V+v^{(0)}) - R(x+V)$ is well defined if $\delta > 0$ and $\rho > 0$ are sufficiently small, and it is divisable by x_2^2. Hence v_1 is well defined. Moreover we have

$$\|v_1\|_N \leq C\left\|\int_0^1 v^{(0)}\cdot\nabla R(\cdot+V+tv^{(0)})dt\right\|_N \leq C\int_0^1 \|v^{(0)}\|_N |||R|||dt = C\|v^{(0)}\|_N|||R|||, \tag{4.6}$$

where $|||R||| = \sup_x \|\nabla R(x)\|$. Take $\rho > 0$ so that $C|||R||| < 1/2$. Then we have that $\|v^{(1)}\|_N \leq \|v^{(0)}\|_N 2^{-1}$. Hence $v^{(2)}$ is well defined and it has the same property as $v^{(1)}$ if $v^{(0)}$ is sufficiently small. Moreover we have the estimate $\|v^{(2)}\|_N \leq \|v^{(0)}\|_N 2^{-2}$. In the same way, we can determine $v^{(k)}$ as bounded holomorphic functions in $S_\rho \times \mathcal{C}_\rho$ such that $\|v^{(k)}\|_N \leq \|v^{(0)}\|_N 2^{-k}$, $(k = 1, 2, \ldots)$. This proves that the limit $v := \sum_{k=0}^{\infty} v^{(k)}$ exists in $S_\rho \times \mathcal{C}_\rho$ in $\|\cdot\|_N$-norm. By the definition of the norm we have $v(x) = O(x^{\alpha_N})$ as $x \to 0$. The limit function v is independent of N because X_N is continuously embedded into X_ℓ for every $N > \ell$. Because there exist infinitely many N, this proves Theorem 2.

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