A NEW FRAMEWORK BASED ON THE METHODOLOGY OF SCIENTIFIC RESEARCH PROGRAMS FOR DESCRIBING THE QUALITY OF MATHEMATICAL ACTIVITIES

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This paper proposes a new framework for describing the quality of mathematical activities under radical constructivism. It is based on Lakatos' philosophy of science, instead of his philosophy of mathematics. We focus on a structural similarity between mathematical problem-solving activities and scientific research programs. While Lakatos' philosophy of mathematics is only a model of a progressive activity, the new framework can distinguish between progressive and degenerative activities. To show its usefulness, we provide a sample analysis. Based on the analysis, we hypothesize that the zig-zag process of solving a mathematical problem is driven by a hard core: A set of one's unrevised assumptions that one would like to continue to maintain. The necessity of further research with the proposed framework is suggested.

INTRODUCTION

Lakatos' (1976) logic of mathematical discovery (LMD), known as proofs and refutations, is one of the most cited philosophies in mathematics education research. It characterized mathematics as an informal repeated process of conjecturing, proving, and refuting. Based on the LMD, several scholars have advocated a fallibilistic nature of learning mathematics (e.g., Confrey, 1991; Ernest, 1998; Lampert, 1990). The application range of the LMD is wide: From problem-solving at the elementary school level (Lampert, 1990) to theorem reinvention at the undergraduate level (Larsen & Zandieh, 2007). However, as Sriraman and Mousoulides (2014) point out, "[t]he didactic possibilities of Lakatos' thought experiment abound but not much is present in the mathematics education literature in terms of teaching experiments that try to replicate the 'ideal' classroom conceptualized by Lakatos" (p. 513).

The rare replications of the LMD style in classrooms stem from the gap between naïve and sophisticated mathematical activities. Although the LMD suggests a fallibilistic nature of learning mathematics, disagreements about a conjecture do not always contribute to mathematical development in a classroom. Note that the LMD originates from sophisticated activities among professionals, not among novices. We need more empirical data on the relationship between naïve and sophisticated activities.

This paper proposes an alternative theoretical framework for describing mathematical activities. The proposed framework is based not on Lakatos' (1976) philosophy of mathematics, but his philosophy of science (1978): The methodology of scientific research programs (MSRP). The LMD is useful for describing relatively sophisticated

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activities (e.g., Larsen & Zandieh, 2007), but the proposed framework based on the MSRP will be able to describe both naïve and sophisticated mathematical activities and will provide a descriptive framework for contrasting the two.

This paper consists of the following sections: (1) an overview of the MSRP, (2) an overview of radical constructivism (RC) proposed by von Glasersfeld (1995), (3) the proposal of a new theoretical framework, and (4) a sample analysis. Through the analysis, we will argue the usefulness of the proposed framework for describing mathematical activities.

LAKATOS' PHILOSOPHY OF SCIENCE

The scientific research program (SRP) is a series of activities with the same paradigm carried out by scientists. An SRP contains a hard core and protective belts. The hard core is a set of theoretical assumptions and the protective belts are auxiliary hypotheses, and any scientific claim in the SRP is based on both. If a counterexample of the claim is observed, either parts of the core or some of the belts are false. Thus, scientists, like pseudo-scientists, do not have to give up their own hard core and can protect it by revising some of the belts. This process is called a problem shift. In principle, the assumptions in the hard core can be arbitrarily selected. Lakatos (1978) abstracted this methodology from the history of science.

Although Yuxin (1990) pointed out the similarity between the LMD and the MSRP, there is a significant difference between them: The spirit of the LMD is "antidemarcationist," while that of the MSRP is "demarcationist" (Ernest, 1998, p. 111). That is, Lakatos provided a distinction between good and bad scientific activities: Science must predict the next empirical evidence. If an SRP predicts the next empirical evidence, its problem shift is called progressive; if not, it is called degenerative. In principle, the LMD cannot require mathematicians to completely give up a mathematical research program because the LMD is related to informal mathematics and not pseudo-mathematics. On the other hand, MSRP requires scientists to completely give up an SRP if it cannot predict the next empirical evidence.

RADICAL CONSTRUCTIVISM

RC is a philosophy which begins from "the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience" (von Glasersfeld, 1995, p. 1). This assumption leads to the possibility that even if an observed behavior looks irrational from the observer's perspective, it is rational from the behaver's own perspective. Therefore, any learner's behavior should be interpreted as at least *locally rational* from his or her own perspective at that moment (Confrey, 1991; Uegatani & Koyama, 2015).

For our purpose, we introduce two key concepts in RC: viability and action scheme. The concept of *viability* is: Pieces of knowledge are viable "if they fit the purposive or descriptive contexts in which [learners] use them" (von Glasersfeld, 1995, p. 14).

Action schemes (AS) consist of the following three parts: "1 Recognition of a certain situation; 2 a specific activity associated with that situation; and 3 the expectation that the activity produces a certain previously experienced result" (von Glasersfeld, 1995, p. 65).

Let us consider the example of an AS and its viability. Suppose a learner needs to solve an equation $x^2 = 3179$ (1. Situation). His or her next activity will be to test the divisibility of 3179 by 2, 3, 5, and 7 (2. Activity) with an expectation that 3179 is divisible by a certain number (3. Expectation). Using the AS means testing the consistency between the expected and the actual results of the activity. If the results are consistent, they will become more viable. If not, they will become less viable or be revised.

An AS can be revised in certain ways. Importantly, when a learner senses inconsistency, he or she cannot uniquely determine what causes it. In the above example, since 3179 is divisible by neither 2, 3, 5, nor 7, the learner may sense inconsistency. Then, he or she can arbitrarily suspect at least either the suitability of divisibility testing in the situation or the sufficiency of testing integers from 2 to 7. If the learner chooses the former, he or she may solve the inconsistency by considering the activity not suitable for the situation. If he or she chooses the latter, he or she may solve the inconsistency by considering that the activity should test divisibility by 11. The AS can be arbitrarily revised as long as the inconsistency is solved (Uegatani & Koyama, 2015). "The viability of concepts […] is not measured by their practical value, but by their non-contradictory fit into the largest possible conceptual network" (von Glasersfeld, 1995, p. 68).

A NEW THEORETICAL FRAMEWORK

There is a structural similarity between an AS and an SRP. The concept of viability corresponds to that of progressiveness. An AS has the following three features. (AS-a) If the AS predicts the next expected result, it remains viable and if not, becomes less viable. (AS-b) Even if the AS is viable at one moment, there may not be any consistency between the expected and the actual results in the next moment. (AS-c) When dealing with an inconsistency, the AS can be arbitrarily revised whether it becomes more or less viable. Similarly, an SRP has the following three features. (SRP-a) If the SRP predicts the next empirical data, it remains progressive, and if not, becomes degenerative. (SRP-b) Even if the SRP is progressive in one moment, there may not be any consistency between the predicted and the actual data in the next moment. (SRP-c) When dealing with an inconsistency, the SRP can be arbitrarily revised, whether it becomes progressive or degenerative (though a degenerative SRP is not qualified as science). Thus, in the analogy with an SRP, when we observe a revision of an AS, we will be able to identify the elements corresponding to "protective belts" and "a hard core." In this context, *protective belts* can be defined as pieces of knowledge used by the learner to predict a result, but recognized as inappropriately used; a *hard core* can be defined as a set of unrevised assumptions the learner would like to continue to maintain. We propose the MSRP based framework, which focuses especially on the hard core of a mathematical activity. The advantage of the new framework compared to the LMD based framework is that it enables us to describe the quality of mathematical activities as progressive or degenerative, for example, to understand the variation between progressive and degenerative activities.

SAMPLE ANALYSIS

To show the usefulness of the framework, we provide a sample analysis.

Background of a sample

The sample episode was videotaped in a part of the first author's mathematics lesson. This is a transcript of 11th grade students' group work. The group members (all names are pseudonyms) were Mr. Ham (leader), Ms. Uts (subleader), Mr. Ike (recorder), Mr. Tak (calculator), and Ms. Hor (presenter). Although each member was given his or her role to enhance the group discussion, the roles were often forgotten because of the heated discussion. The given task was identifying more digits of 2^{54} than other groups. The following episode is a vignette taken while performing the task.

Episode in a group work

Ike had already predicted the need for a logarithm before the task was presented:

6 Ike: Maybe, we are to refer to the table of common logarithms.

7 Ham: Really? … Like enough.

The reason why they predicted the need for a logarithm seems to be that they had learned to use the table of common logarithms in the last class. After the task was presented, Ham immediately decided to use common logarithms.

8 Ham: OK, take the common logarithm. The common logarithm of 2.

17 Ike: OK, well, 0.3010 (Referred to the table of common logarithms).

18 Ham: Calculate 54 times 0.3010. (Said to the student with the calculator, Tak)

22 Tak: (using a pocket calculator) 16.254.

On the other hand, Hor, who observed the boys' approach in silence, suddenly started to calculate 2^{54} by paper and pencil with Uts, but independent of the boys:

25 Hor: [Inaudible] … let's calculate 2^{54}. (Said to Uts, and started to calculate)

26 Uts: Oh….

27 Ham: So, is the value between 16th and 17th powers of 10?

28 Ike: Yes, yes.

Ham and Ike continued their approach without paying attention to the girls:

29 Ham: Ah …, so, then …., 16 digits …, Uh ….

30 Ike: So, after that, so, taking the logarithm of it, 16. …, 16.254. So, try to find a value as near as possible to 16.254 repeatedly. Maybe, we should take the antilogarithm of the value.

However, Tak alone started trying to directly calculate 2^{54} with a pocket calculator after observing the girls' approach. Hor noticed that, stopped calculating, and tried to communicate with the boys:

41	Hor:	Hey, how many digits can the pocket calculator use? (Said to Tak)
42	Tak:	Um …, 1, 2, 3, 4, 5, 6, 7, 8.
43	Ham:	So, how can we calculate $10^{16.3}$, for example? (Said to Ike)
44	Hor:	No way. (Said to Tak)
45	Ike:	We can do it if we can calculate $10^{0.3}$. (Said to Ham)
47	Ham:	Wow! Oh! That's true!

Immediately after hearing Ham's exclamation, Hor asked Ham:

48	Hor:	What did you say? What of 10? (Said to Ham)
49	Ham:	So, so, decompose …, in case of 16.3th power …, $10^{0.3}$…, and what is 10^{16}?
50	Ike:	Ah, so, let's use the table of the common logarithms. If you find the value whose common logarithm is 0.2 in the table, ….

However, Ham and Ike were absorbed in their thinking and perhaps unintentionally neglected Hor. Then, Hor gave up her communication with them.

After that, Hor and Uts continued to calculate by paper and pencil together. Ike began to seek the next promising step alone, and Tak proposed his opinion:

58	Ham:	Ike might solve alone ….
60	Tak:	Let's calculate 2^{54} in a step-by-step fashion!
61	Ike:	(Laugh) I don't recommend it.
62	All:	(Laugh)
63	Uts:	But, now she is calculating (Pointing to Hor)
64	Hor:	Without thinking difficult math, ah …, simply 1024^5 times 16.
65	Ike:	Do you have enough courage to calculate it?
66	Hor:	Yes, let's calculate it.
67	Uts:	Now, she has already been calculating.
68	Hor:	Yes, now I am calculating.

Despite this communication, Ike and Ham ignored Hor's approach. Tak began trying to directly calculate 2^{54} independent of the other members of the team.

Although Ike had directed Ham in solving the task until that time, Ike's original plan started becoming unstable. Consequently, they began supporting each other.

71	Ike:	The direct reference (to the table of the common logarithms) might be better. So, the target is $10^{16.254}$ …, 0.254, 254, (searching the nearest value of 0.254 in the table of common logarithms) … about 1.8?
72	Ham:	No, (the common logarithm of) 1.79 is nearer (to 0.254).
73	Ike:	1.79 …, so, oh, what can we do next?

74 Ham: Multiplying 10^{16} … (Writing down "17900000000000000"). So, "nearly equal" is not clear. In this case, "equal to or more than" is suitable, isn't it? "More than," isn't it?

75 Ike: But, any further inquiry is impossible because of (the precision of) our table of common logarithms.

76 Ham: Umm, in that case, is it better to calculate this (pointing to the common logarithm of 1.8 in the table). 1.8 means multiplying 2.54? No, it doesn't.

77 Ike: It means …. (Writing down $1.79 \times 10^{16} < 2^{54} < 1.80 \times 10^{16}$). (Note: Their judgment was mathematically incorrect because their consideration to a margin of error is not proper.)

Finally, after identifying some digits of 2^{54}, their discussion became deadlocked:

89 Ike: Now, what can we do next? There is no cue (for raising the precision)

90 Ham: Improving is impossible by using our table of common logarithms, isn't it?

91 Ike: Now what can we do?

Then, Ike noticed Hor and Uts's progress:

93 Ike: … Oh, you all have been really calculating by paper and pencil!

94 Hor: Really, we are still calculating.

95 Ike: Really?

96 Hor: If our calculation is finished (Hor and Uts had already finished calculating 2^{40} and 2^{14}), then we will finish all.

99 Ike: Oh, what can we do? What can we do? (Laughing and looking around)

Discussion

From the beginning of the episode, Ike and Ham seemed to share the same AS. Although they often found inconsistencies between the expected and actual results of their activities (e.g., #29, #49, #73, and #75), they immediately tried to change the interpretations of either their situations, or their activities in order to eliminate their sensed inconsistency (e.g., #30, #50, #74, and #76). Therefore, we can say that the pieces of knowledge that formed the rejected interpretations were the protective belts, while the unrevised assumptions that using logarithms is a better approach were the hard core, and that using logarithm seemed to be a policy rather than a conclusion. Although Yuxin (1990) argued that the term "hard core" in the MSRP corresponded to the term "main conclusion" in the LMD, this correspondence were not observed in the activity. In addition, when the inconsistency made Ike anxious that they could not find the next promising step, he tried to communicate with other members (#61, #65, and #93). Since Ike and Ham's approach could not predict the next expected result, it became degenerative (e.g., #89 and #99).

On the other hand, Hor seemed to have a different AS than Ike and Ham. As the inconsistency made her anxious that her direct calculation might not end in time, she tried to communicate with other members twice. The first time she tried to use Tak's pocket calculator (#41), and the second time she tried to get inspiration from Ike and Ham's approach (#48). However, since she was not inspired, she finally continued to calculate by paper and pencil. She seemed to adhere to her approach not only because she was in rivalry with Ike and Ham but also because she believed the rationality of her approach (#64). Thus, we can say the unrevised assumptions that direct calculation is a better approach were the hard core.

Although Uts and Tak calculated directly, they rarely contributed to the group activity. Because their cores were not hard enough, that is, because they were not confident enough of the validity of their cores, they seemed to follow Hor's lead.

In the above description with the MSRP based framework, we can observe the role of the hard cores in problem-solving activities. We hypothesize that the zig-zag process, the repeated process of confronting and eliminating inconsistencies, is driven by a hard core. Because of their hard cores, Ike, Ham, and Hor could take the initiative in problem-solving, at least temporarily. On the other hand, since Uts and Tak's cores were not hard enough, they could only follow Hor's lead. In addition, when the confidence in their cores was shaken, Ike and Hor tried to communicate with others in the group, and follow them. This suggests that taking the initiative in problem-solving in group work is related to the hardness of the core. In fact, while both Ike and Ham and Hor's ASs were progressive, they were incommensurable, and students did not need to communicate with the others in the group.

Implication for practice in mathematics education

Of course, a progressive series of the revised ASs offers no more guarantee of success in problem-solving than a progressive SRP offers of approaching the truth. However, if our hypothesis is valid, we can say that the existence of a single hard core is a necessary condition for a progressive mathematical activity. The participants can discuss and support each other if they share the same core like Ike and Ham, while they cannot effectively communicate with each other because of the incommensurability of their own hard cores like Ike and Hor. Thus, if the teacher intends to enhance his or her students' progressive mathematical activity, he or she must support them in constructing an appropriate shared hard core.

The origin of the hard core is not necessarily mathematical. For example, Ike and Ham created their core by predicting the contents of that day's lesson. On the other hand, Hor created her own core because she felt that Ike and Ham's approach was too complicated. Although the three students were in almost the same social or cultural settings, their hard cores were different. This means that RC cannot claim that the social or cultural settings themselves have an influence on one's core (cf. Lerman, 1996); it must state that depending on subjective interpretations of the social or cultural settings, different hard cores can be created even in the same settings.

Even if a core is created from the other participants' cores, it might be too weak to maintain like Uts and Tak. On the other hand, too strong hard cores will make the AS degenerative. If one wants to keep a progressive mathematical activity, then one may sometimes need to give up one's hard core and create a new core. Further empirical research is needed to explore what helps learners create their own core.

CONCLUSION

In this paper, we proposed the MSRP based framework as an alternative to the LMD based framework for describing mathematical activities. It enables us to describe the quality of mathematical activities as progressive or degenerative. We provided one sample analysis to show the usefulness of the framework. Based on the analysis, we hypothesized that the zig-zag process of solving a mathematical problem is driven by a hard core. For this reason, two persons with different hard cores are incommensurable. Although progressiveness does not always offer a guarantee of success in problem-solving, a hard core seems to be needed for progressive mathematical activities. The sample analysis empirically supports the validity of our hypothesis, even though that is not the purpose of this paper. We need further empirical research with the proposed framework to explore what helps learners create their own cores and have an appropriate shared core.

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