AN ORIGIN OF PRESCRIPTIONS FOR OUR MATHEMATICAL REASONING

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To build a supplementary theory from which we can derive a practical way of fostering inquiring minds in mathematics, this paper proposes a theoretical perspective that is compatible with existing ideas in mathematics education (radical constructivism, social constructivism, APOS theory, David Tall's framework, the framework of embodied cognition, new materialist ontologies). We focus on the fact that descriptive and prescriptive statements can be treated simultaneously, and consider both descriptive and instantiated models in our minds. This indicates that descriptive statements in mathematics come from our descriptions of models, and prescriptive statements come from the instantiatedness of the instantiated models and non-existence of counterexample. As a practical suggestion from the proposed perspective, we point out that careful communication is needed so that students do not recognize the refutation of their arguments as a denial of their way of mathematical thinking.

Key words: Inquiring minds, Prescriptive perspective, Mathematical reasoning

Introduction

Some undergraduate students seem to have only inadequately *inquiring minds* in mathematics, though inquiring minds are vital in continuing to study advanced mathematics. For example, a mathematician interviewed by Weber (2012) says, "I find students read a proof like they would read a newspaper and it's impossible to understand proofs that way" (p. 475). This comment implies that some students tend to accept proofs even before reading them, and as a result tend not to obtain the new insights that they would acquire through reading them. This is regarded as a "lack of inquiring minds" in this paper. In addition, these students also seem to uncritically accept most mathematical statements provided by their teachers in mathematics lectures. Following the distinction between a *mathematical attitude* and *an attitude toward mathematics* (Freudenthal, 1981, pp. 142–143), they lack mathematical attitudes, though they may have attitudes toward mathematics.

This lack of inquiring minds or mathematical attitudes may also be conceptualized as a lack of "mathematical integrity," a quality that involves commitment to mathematical truth (DeBellis & Goldin, 2006). To be specific, the phenomenon results from a lack of the unconscious belief that the discoverability of new mathematical results or the rediscoverability of already known mathematical results is open to everyone. For students without this *discoverability belief*, reading proofs or participating in mathematics lectures is not a process of (re)discovering mathematical results, but may instead be just a matter of encountering claims dependent on historical contingency, temporary human discourse, or authority. Some such students fail to make sense of mathematical statements, while others try to construct meanings such that the statements make sense to them. They do not check the validity of the statements, because they think that the statements are always correct if they only make sense. These students have difficulty in continuing to study mathematics. Therefore, in order to obtain practical implications from these cases to support successful mathematics learning, we need to identify the origins of discoverability beliefs and understand how they influence students.

For this purpose, it is not enough to explain the origin of one's discoverability belief as one's successful experience in discovering some mathematical results by oneself. As an example of such a successful experience, we may take a kind of sudden insight within problem solving, known as an *AHA! experience* (cf. Liljedahl, 2005). Discoverability beliefs also seem to depend on experiences of such subjective feelings. However, this explanation does not clarify the reasons why some students feel as if they have discovered something mathematical and others in the same lecture room do not. We may perhaps ascribe discoverability beliefs to uncontrollable subjective factors, but such a theoretical perspective is not useful for educational practice. For example, in terms of AHA! experiences, Liljedahl (2005) pointed out that “the environment for such an experience can be orchestrated, but the experience itself cannot” (p. 232). This implies that in order to get practical implications for the establishment of an adequate learning environment, we need to identify controllable objective factors that increase one’s probability of a successful experience or decrease one’s probability of an unsuccessful experience in (re)discovering some mathematical result.

One possible approach to this problem is the *epistemological* one. Although new mathematical findings may sometimes depend on empirical evidence, establishing the validity of mathematical statements does not need empirical support in many cases. What is needed to establish mathematical truth is usually just mathematical reasoning. Thus, one’s feeling about discovery mainly depends on one’s own process of establishing mathematical knowledge. An origin of discoverability beliefs can be supposed to consist in such an epistemological process of human mathematical reasoning if mathematical truth does not depend on arbitrary human judgments.

Several epistemological approaches to the process of establishing mathematical knowledge exist in mathematics education research, such as radical constructivism (Thompson, 2000; von Glasersfeld, 1995), social constructivism (Ernest, 1991, 1998), APOS theory (Dubinsky & McDonald, 2002), the three worlds of mathematics (Tall, 2004, 2008, 2011), embodied cognition (Lakoff & Núñez, 2000), and new materialist ontologies (de Freitas & Sinclair, 2013). However, none of these explain how the discoverability belief, or whatever its counterpart is in each theory, arises. (Of course, they do provide explanations for broader educational phenomena, and their scant attention to discoverability belief is thus forgivable, because each theoretical perspective has its own purpose.)

Thus, in order to obtain practical implications to support successful mathematics learning, we need a new supplementary theoretical perspective. As Cobb (2007) argued, “we should view the various co-existing perspectives as sources of ideas to be adapted to our purposes”; therefore, if the existing paradigms do not provide a direct solution, we must build a supplementary perspective integrating useful pieces of existing theoretical knowledge for a certain educational purpose. This paper attempts to build such a supplementary theory from which we can derive practical implications for the establishment of a learning environment where students can eventually acquire inquiring minds.

Sufficient Conditions of the Supplementary Theory

It is important to declare what conditions of the supplementary theory will be sufficient before trying to establish that supplementary theory. This will provide us with the needed constraints on the establishing process. In this regard, we make five assumptions in this paper.

The first assumption is that the objective factors determine uniquely the set of viable subjective knowledge. In this paper, we must identify controllable objective factors related to discoverability beliefs in the subjective processes of students’ mathematical reasoning. Some readers may feel that this attempt is paradoxical because of the attempt to find objective factors in subjective processes. However, this paradox disappears if we specify that we are using the term “objective” to describe something from the observer’s (e.g., the teacher’s or the researcher’s) perspective, and the term “subjective” to describe something from the learner’s perspective. In radical constructivist theory, subjective knowledge is to an objective problem what a key is to a lock (von Glasersfeld & Cobb, 1984). Although no single particular key can be uniquely determined by the particular lock, the *set* of usable keys is physically uniquely determined by the lock. Similarly, although valid subjective knowledge appropriate to solving an objective problem cannot be uniquely determined, we can assume that some set of viable subjective knowledge is uniquely determined by the problem. We thus establish the possibility of identifying influential objective factors for the viability of subjective knowledge, and will try to build a theoretical framework to capture such factors.

The second assumption is that as a result of students' use of learning strategies, their cognitive development follows David Tall's theory (Tall, 2008) when they construct new mathematical concepts, even when reading proofs or participating in lectures. The theory partially incorporates APOS Theory (Dubinsky & McDonald, 2002) and conceptual metaphor theory (Lakoff & Núñez, 2000). It explains students' cognitive transition from the earliest pre-school mathematics to graduate mathematics. However, it mainly explains successful development (outcomes), and is not directly suggestive for affective aspects such as attitudes or beliefs. The framework necessary for our purpose will be one which explains how some students autonomously begin to use successful learning strategies, resulting in the kind of cognitive development described by Tall's theory, when reading proofs or participating in lectures. This explanation will elaborate an origin for discoverability beliefs.

The third assumption is that we can compare the degrees of freedom of the solutions to a certain objective problem with those of an equivalent problem. Following the first assumption, for any problem for any student, the set of viable subjective knowledge for solving the problem is unique, but we cannot predict which knowledge in the set will actually survive or vie for survival, because many accidental factors influence the student. On the other hand, even if two problems are objectively equivalent, it is not necessarily warranted that two problems are subjectively similar to each other. If this third assumption holds, we can choose one among the equivalent problems, which will increases the probability that the intended knowledge actually vies. This paper will build a theory satisfying the third assumption.

The fourth assumption is that the patterns of mathematical reasoning are common among students but that their consequences can differ because accidental factors cause students to arbitrarily arrange the patterns in their reasoning processes. This assumption is, albeit indirectly, supported by the existing research. For example, Nesher (1987) indicates that "most [misconceptions] are overgeneralizations of previously learned, limited knowledge which is now wrongly applied" (p. 37). Even unsuccessful students with misconceptions, as well as successful students, have some mathematical attitude toward generalization of their subjective knowledge. Another example is from the research on concept images. According to Tall and Vinner (1981), students have their own subjective images of each concept; some students successfully use concept images (Pinto & Tall, 2002) and others use them unsuccessfully (Tall & Vinner, 1981; Vinner & Dreyfus, 1989). However, both successful and unsuccessful mathematical thinking have some aspects in common. As a radical extrapolation from this fact, we make the forth assumption: if some pattern of mathematical reasoning is appropriately modeled, the process of loss of discoverability beliefs related to it can be explained in terms of the following four steps. First, all students use common patterns of mathematical reasoning in early learning. Second, however, due to accidental factors, some students fail to learn mathematics, in spite of the fact that their learning strategies are the same as those of successful students. Third, unsuccessful students mistakenly perceive that the reason why they failed is because they are using inadequate learning strategies. Finally, as a result, they eventually lose their discoverability beliefs and do not come to use successful learning strategies. Therefore, our theoretical framework must provide an appropriate model of human mathematical reasoning. One of the practical goals of the framework will be to help students correctly recognize the validity of their initial learning strategies, because it is difficult to completely remove accidental factors.

The final assumption is that successful experience of mathematical discovery depends mainly on mathematical reasoning, though some types of mathematical discovery may depend on physical evidence. If this assumption does not hold, we will not be able to understand why the validity of mathematical knowledge depends mainly on reasoning. On the other hand, this assumption implies that mathematical reasoning must have some prescriptive aspects. One example of such is that if the propositions $P \to Q$ and P are true, then the proposition Q should be true. If students do not perceive this prescriptive proposition from their mathematical reasoning, then they will not have experienced mathematical discovery.

On the basis of these five assumptions, the main research task of this paper is to model the common patterns of human mathematical reasoning. The model must satisfy the following four sufficient conditions. First, it must identify the factors influencing the degrees of freedom of the solutions to a problem. Second, the model must explain how higher degrees of freedom tend to produce more accidental factors. Third, the model shows that the mechanisms of both successful and unsuccessful reasoning are the same except for the tendency to accept the influence of accidental factors. Finally, the model explains that a certain type of arrangement of reasoning patterns causes students to feel the presence of prescriptiveness in the knowledge at stake.

In the following section, we will discuss the dual aspects of mathematical reasoning: *prescription* and *description*. Through the elaboration of both aspects, we will eventually succeed in modeling a mathematical reasoning that can satisfy the above conditions.

Duality of Prescription and Description

Ernest (1998) pointed out the limitations of *prescriptive* accounts of mathematics:

> Absolutist philosophies of mathematics such as logicism, formalism, and intuitionism attempt to provide *prescriptive* accounts of the nature of mathematics. Such accounts are programmatic, legislating how mathematics should be understood, rather than providing accurately *descriptive* accounts of the nature of mathematics. Thus they are failing to account for mathematics as it is, in the hope of fulfilling their vision of how it should be. (pp. 50-51, italics in the original)

Thus, Ernest's (1998) social constructivism takes a descriptive stance. It provides no account of which way of doing mathematics is correct, but rather describes how people do mathematics. Other existing research perspectives for mathematics education also take descriptive stances. They provide no account of which method of understanding mathematics is correct, but merely explain how students do mathematics. However, the preceding discussion is based on the following implicit assumption: we must exclusively choose prescriptive or descriptive philosophies. Both the prescriptive statement "X should be Y" and the descriptive statement "X is Y" can be simultaneously correct.

For example, consider a group $(G,*)$. Suppose that G is a set, and that $*$ is a binary operation on G. The group axioms are as follows: (i) For all a, b in G, $a * b$ is also in G. (ii) For all a, b and c in G, $(a * b) * c = a * (b * c)$. (iii) There exists an element e in G such that, for every element a in G, the equation $a * e = e * a = a$ holds. (iv) For each a in G, there exists an element b in G such that $a * b = b * a = e$, where e is the element defined in axiom (iii). From these axioms, we can derive the statement that the element e postulated in (iii) is unique, and we will say that e postulated in (iii) should be unique if someone argues that there are many elements postulated in (iii). In this case, both statements (involving "is" and "should be") appear correct. This is explained by distinguishing between *in* and *out of* the axiomatic system. The statement that the element e postulated in (iii) is unique is a description of components *in* the system. The statement that the element e postulated in (iii) should be unique (or, more strictly, the statement that we should argue that e postulated in (iii) is unique) is a prescription for us who are *out of* the system. It is important that the element e (or the entity in the system) is not itself bound by the rules of logic, but that all thinking subjects who are out of the system and agree on the group axioms have an obligation to obey some logical inference rules.

In general, a descriptive statement in an axiomatic system and the corresponding prescriptive statement out of the system can be simultaneously correct, because we can always distinguish between in and out of the given system. It is, therefore, an unjustifiable assumption that we cannot simultaneously consider both prescription and description. If we have the ability to self-reflect, and to distinguish between the outside of an axiomatic system and the overall framework that contains the inside and the outside of the system, then prescriptive statements and descriptive statements are dual properties of the overall framework. In addition, it is also important that humans out of the system are prescribed, and the entities in the system are described at the same time.

Origin of Prescription

If our reasoning always followed the rules of formal logic, the discoverability belief would be justified by the independence between these rules and human minds. In general, it is difficult to describe the actual practices of mathematics only by formal logic (e.g., Fallis, 2003). Thus, we argue that the schemata of descriptions actually prescribe human reasoning.

The schema of descriptions is, for example, the format of implication statements "$P \rightarrow Q$." We do not assume that it pre-existed the modus ponens. Rather, we argue that modus ponens pre-existed the schema "$P \rightarrow Q$," and that the schema was invented to describe a situation where one may infer Q after knowing that P is true. Given the propositions P and $P \rightarrow Q$, we usually deduce proposition Q for any propositions P and Q. This does not imply the validity of modus ponens, but implies that there can be a situation where one may infer Q after knowing that P is true. Similarly, the rule of conjecture elimination (inferring P from $P \wedge Q$) pre-existed the schema "$P \wedge Q$," and the rule of universal instantiation (inferring $A(a)$ for any element a from $\forall x\, A(x)$) pre-existed the schema "$\forall x\, A(x)$." In general, an inference rule pre-existed its related schema. Thus, what one should infer depends on how one describes a given situation, and not on formal logic.

From this perspective, it is necessary to identify what determines a valid description of the situation. Next, we shift to the question of how descriptive statements arise.

Origin of Description

In mathematics, some descriptive statements are contained within the axioms of the system under consideration, but even in advanced mathematics, we do not always think in completely formalized systems. We propose that, instead, descriptive statements originate from *models* in our minds. In the present paper, the term *model* has a dual meaning. In this regard, Mason's (1989) idea is highly suggestive. According to Mason (1989), mathematical abstraction is described as "a delicate shift of attention from seeing an expression *as* an expression of generality, to seeing the expression *as* an object or property" (p. 2, italics in the original). Using the idea of "a shift of attention," we will show the dual meaning of "model."

One meaning is "something that a copy can be based on because it is an … example of its type" ("Model," n.d.-a). We call this an *instantiated model*. For example, the set of all integers, together with the operation $+$, is an instantiated model of a group in our minds, because it is a typical example of a group. With this in our minds, we can easily understand any example of a group by analogy. We can also show that the set of all integers with the operation $+$ is an instantiated model satisfying the group axioms. Similarly, because the experience of typicality can depend on subjective experiences, any example of a group can be an instantiated model. As it has not only the essential features of a group, but also non-essential ones, it has more information than a group as an abstract object without any non-essential features of a group. In general, an instantiated model satisfies a certain set of axioms, and carries more information than an abstract object without any properties which the axioms do not imply. A set of axioms do not have to be commonly accepted. Arbitrary logical expressions may be axioms. If a set of axioms is consistent, there exists at least one instantiated model for them.

Another meaning of the term "model" is "something that represents another thing … as a simple description that can be used in calculations" ("Model," n.d.-b). We call this a *descriptive model*. For example, a line in mathematics may be regarded as a descriptive model of a physical line, such as that made by a pencil, in our minds. A line in mathematics is defined by focusing attention on only some of the features of a physical line. It is a result of neglecting uninteresting features that. While a physical line does have width, we usually require in mathematics that a line have no width. In general, a descriptive model is created by focusing attention on only some of the features of other descriptive models or physical objects. Such a temporal creation is then refined with certain provisos (e.g., "it has no width"). The provisos prevent us from focusing attention on uninteresting features of the source descriptive models or objects.

Most relevant here is the relativity between instantiatedness and descriptiveness. That is, when we focus attention on some essential features of an instantiated model, the abstract object constrained by the logical expressions of those features is a descriptive model of the instantiated model. When we create a new object by adding some extra features to an abstract object that is a descriptive model, the new object is an instantiated model of the descriptive model. In other words, any model in our minds can always be both instantiated and descriptive. Any model other than a physical object is an instantiated model of more abstract models or objects, and it is simultaneously a descriptive model of more concrete models or objects. The relativity between instantiatedness and descriptiveness allows us to dispense with the distinction between the terms "model" and "object." In this sense, both terms may be used interchangeably, because every model can become an object of thought, and vice versa.

By using the term "model," one of the predominant origins of descriptive statements in mathematics can be explained as descriptions of models in our minds. We will provide two examples: the fundamental theorem of cyclic groups, and the construction of an equilateral triangle. Let us explain their possible models, for example, in the author's mind.

The fundamental theorem of cyclic groups: The theorem states that every subgroup of a cyclic group is cyclic. Let $\langle g \rangle$ be a cyclic group generated by g. Following the definition of a cyclic group, $\langle g \rangle$ simply consists of $\cdots, g^{-2}, g^{-1}, e, g, g^2, \cdots$; there is no other element in $\langle g \rangle$. If a subgroup of $\langle g \rangle$ has n different elements, they can be represented by $g^{k_1}, g^{k_2}, \cdots, g^{k_n}$. From the group axioms, the subgroup contains $g^{\mathrm{GCD}(k_1, k_2, \cdots, k_n)}$, and $g^{\mathrm{GCD}(k_1, k_2, \cdots, k_n)}$ generates all elements in the subgroup. Thus, the theorem seems to be true.

This way of creating descriptions of models in our minds implies various prescriptions. For example, when someone says that $\langle g \rangle$ might not contain e, the author should argue that $\langle g \rangle$ always contains e because $\langle g \rangle$ is an example of a group. As another example, when someone points out that the order of a subgroup of $\langle g \rangle$ is not always finite, the author should recognize that an example of a subgroup of $\langle g \rangle$ in his mind is too specific.

The construction of an equilateral triangle on a given line segment: Let AB be the given line segment. Draw a semicircle with center A and radius AB. Again, draw a semicircle with center B and radius BA on the same side as the first semicircle. Let C be the point of intersection of the semicircles. Then, the triangle ABC is equilateral. This is because the semicircles centered at A and B have radii of equal length, and all three segments AB, BC, and CA are the length of their radii. Thus, the construction seems to be valid.

There are also various prescriptions in this case. For example, when someone says that the three edges AB, BC, and CA are not always equal, the author should argue that they are always equal, for the following reason. The point C is regarded as our instantiated model of the points on the semicircles A and B; the pairs CA, AB and AB, BC are regarded as our instantiated models of equivalent radii, and the lengths of AB, BC, and CA are regarded as our instantiated models of the transitivity rule. As another example, if someone points out that the author's consideration depends on the belief that the two semicircles always intersect with each other, he should recognize that his consideration depends on a visual representation.

Generally speaking, descriptive statements of some mathematical objects are created by accessing their models in human minds, and then describing these models. Given an axiomatic system (that is, a descriptive model), one creates an instantiated model of the given descriptive model in mind. Creating a descriptive statement in the system is creating a descriptive model of the current model in mind. There are two types of creation. One creates a description of a common property among all the instantiated models of the given descriptive model. The other creates a description of a property satisfied by only a particular instantiated model of the given descriptive model. If one mistakenly argues something based on the latter type, and someone points this out, then one should recognize the mistake (for example, that an example of a subgroup of $\langle g \rangle$ is too specific, or the consideration of an equilateral triangle depends on a visual representation). Descriptive statements in mathematics, therefore, can come from descriptions of models in our minds, and prescriptive statements can come from the instantiatedness of the instantiated models and non-existence of counterexamples. From this perspective, the reason why proofs and refutations (Lakatos, 1976) occur in the history of mathematics might be because humans (including mathematicians) sometimes create a description of a property satisfied by only a particular instantiated model of the given descriptive model.

Conclusion

In order to obtain the practical implications to support successful mathematics learning, especially with regard to discoverability, the author attempts to build a model of mathematical reasoning from a new theoretical perspective, presupposing the presence of mental models in the human mind. This paper asserts that strictly two types of mathematical reasoning exist, involving either the creation of instantiated or descriptive models from the mind's present model.

This proposed model of mathematical reasoning satisfies the four conditions presented in the second section. First, a factor influencing the degree of freedom in solving a problem lacks sufficient constraints to ignore non-essential features. For example, student overgeneralization of certain mathematical topics can be attributed to the creation of descriptive models focusing on their non-essential features. In other cases, student misjudgment might be attributable to an overly specific mathematical concept image caused by the creation of instantiated models with additional non-essential features.

Second, if a learning environment permits students to focus on non-essential features, the probability of invalid mathematical reasoning will increase. The third vital property for successful reasoning entails focusing solely on the essential features that educators wish to teach; in contrast, unsuccessful reasoning is typified by a focus on non-essential features.

Finally, students who focus entirely on essential features will feel a sense of prescriptiveness. If a descriptive model of a common property among all instantiated models of a current mental model is created, prescriptiveness arises from subjective non-falsifiability. Discoverability beliefs originate from the repeated exposure of non-existent counterexamples.

As a practical suggestion from the proposed perspective, we point out that students might lose the discoverability belief if they recognize the refutation of their argument as a denial of their way of mathematical thinking. What the refutation actually denies might not be their attitude toward creating an instantiated model of the given descriptive model, but only the particular instantiated model contingently created at that time. If creating an instantiated model and describing it is an essential process of mathematics, a chain of reasoning means a chain of creating instantiated models or descriptive models of the already-created models. Then, many chains of reasoning are not deductive. If a student seems to mistakenly make a non-deductive chain of reasoning, the teacher should carefully communicate with the student, and try to recognize which chain would make such a conclusion. Otherwise, proofs and refutations do not work well as a social construction of mathematical knowledge in classrooms, and intersubjectivity cannot be established. In particular, it seems to be important for the teacher to pay attention not only to the student's conclusion but also to their attitude toward developing new findings in order to foster inquiring minds in mathematics. This teacher's attention can be one of controllable objective factors that increase one's probability of a successful experience or decrease one's probability of an unsuccessful experience in (re)discovering some mathematical result.

There are at least two limitations of the proposed perspective. First, it is still not clear whether it is completely compatible with each existing research perspective. Second, the above practical suggestion is still based on assumptions whose validity is not always warranted (for example, whether reasoning always means creating models). The suggestion describes only a possible situation in classrooms. Further development of our theoretical framework in this regard provides an avenue for future research.

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