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Generating functions of Box and Ball System

(箱玉系の母関数)

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目次

1．主論文

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GENERATING FUNCTIONS OF BOX AND BALL SYSTEM

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Abstract. Generating functions of Box and Ball System (BBS) are defined and studied. When the number of balls is finite, we show that the generating function is a rational function. When there are infinitely many balls, we conjecture that the generating function is rational if and only if the BBS is semi-periodic. We prove the conjecture in a special case. We also study the generating function of the BBS with a limited cart, including semi-periodic cases.

1. Introduction

In 1990, Takahashi-Satsuma [6] introduced a discrete soliton system called box and ball system(BBS). This is a kind of cellular automaton obtained by the ultra-discrete Lotka-Voltera equation [2]. A state of the BBS consists of an infinite sequence of boxes, and each box can accommodate one ball at most. BBS has been studied and generalized from variety of aspects; ultra-discretization of soliton equation [8], crystal base [1][3], inverse scattering method [4], and so on.

In this paper, we define a generating function of BBS, and ask if they are rational functions.

In the classical BBS with finite balls, the generating function is always rational, which essentially follows from the result of Takahashi-Satsuma [6] and Tokihiro-Nagai-Satsuma [7]. By writing the generating function as a rational function, we can describe the whole behavior of the BBS by finite words. The rationality of the generating function reflects the fact that BBS is an integrable system. In spite of its highly nonlinear behavior, the rationality of the generating function implies the predictability of the system. We extend the rationality result to a limited cart case. We also consider the BBS with infinite balls. A necessary condition for a rationality is semi-periodicity, which is a generalization of periodic BBS [9].

We conjecture that this is actually sufficient condition also, and prove rationality in a special case.

2. Generating Functions of BBS

The goal of this section is to define the generating function $F_{\mathrm{B}}(z,t)$ of a BBS B, and prove that $F_{\mathrm{B}}(z,t)$ is a rational function when B is a classical BBS. We start by quickly reviewing their definition.

Definition 2.1. A state of a BBS *is a one dimensional array of boxes as picture below, with some boxes filled by balls bounded on the left side, but unbounded on the right side. Each box contains at most one ball. We denote a vacant box by 0 and a filled box by 1.*
So,

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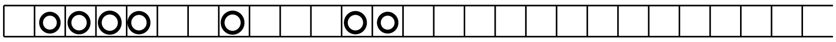


is denoted by

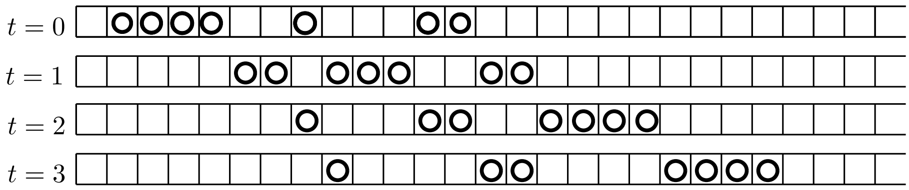
$$(0111100100011000000000000000\cdots).$$

The time evolution rule from time step t to $t+1$ is described as follows. Assume that the state of the BBS at time t is $(a_0^t, a_1^t, a_2^t, \cdots)$, then a_i^{t+1} is defined inductively on i as

$$a_i^{t+1} = \begin{cases} 1 & (\textit{if } a_i^t = 0 \textit{ and } \sum_{j=0}^{i-1} a_j^t > \sum_{j=0}^{i-1} a_j^{t+1}) \\ 0 & (\textit{otherwise}) \end{cases}$$

(see [6]*).*

To give an alternative explanation, one can imagine a carrier, moving from the left to the right. On the way at each box, the carrier moves the balls according to the following rules.
(i) If there is a ball in the box, the carrier picks up the ball into his cart (and move on to the next box).
(ii) If there is no ball in the box and the carrier has at least one ball, then he puts a ball into the empty box (and move on to the next box).
(iii) If there is no ball in the box and the carrier has no ball, then he does nothing (and move on to the next box).
When the carrier finishes all the boxes, one time-evolution step finishes.
For example, the time-evolution of the example in Definition 2.1 proceeds like



Remark 2.2. *First, we explain the movement of the balls in the time evolution from time $t = 0$ to $t = 1$. Let the most left box be the 0-th box. Then, from the 1st box to the 4-th box, a carrier picks up the balls in the boxes and puts into the cart by rule (i). From 5-th box to 6-th box which are empty, he puts the balls into the empty boxes by rule (ii). In 7-th box, he picks up the ball, and he puts into the balls from 8-the box to 10-the box. In 11-th and 12-th box, he picks up the balls and puts the balls in 13-th and 14-th box. From 15-th box, there is no ball in the box and he has no ball in the cart, then he does nothing by rule (iii) and one time-evolution finishes. The time-evolution after $t = 1$ proceeds same as the above.*

From $t = 0$ to $t = 3$, clusters of the balls (we call the cluster soliton) collides and the size of the clusters change, but the number of the clusters does not change.

From $t = 3$ to $t = 4$ (to be precise, after $t = 3$), the clusters of the balls have no collides and move in proportion to the size of the clusters of the balls.

Definition 2.3. *We say that BBS is* finite *if the initial state has only a finite number of balls.*

Definition 2.4. *Let $B = B_0$ be the initial state of BBS, and $B_j = (a_0^j, a_1^j, a_2^j, \cdots)$ be the state of BBS at time j. Then we define its* generating function *$F_B(z, t)$ to be the following.*

$$F_{\mathrm{B}}(z,t) := \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_i^j z^i t^j.$$

We also define $f_{B_j}(z) = \sum_{i=0}^{\infty} a_i^j z^i$. *Hence,* $F_{\mathrm{B}}(z,t) = \sum_{j=0}^{\infty} f_{B_j}(z)t^j$. *Here, we let the leftmost box be the 0th box, and the initial time be t=0.*

The following is the main theorem of this section.

Theorem 2.5. *The generating function of a finite BBS is a rational function of* z *and* t.

Essentially, Theorem 2.5 is a corollary to Theorem 2.8, the result by Takahashi-Satsuma and Tokihiro-Nagai-Satsuma. Let us introduce the necessary notation.

Definition 2.6. *When* B *is a state of a BBS, we define* $N \in \mathbb{Z}_{\geq 0}, E_0 \in \mathbb{Z}_{\geq 0}, Q_i, E_i \in \mathbb{Z}_{>0} \quad (i = 1, 2, \cdots, N)$ *by*

$$B = (\overbrace{0\cdots0}^{E_0}\overbrace{1\cdots1}^{Q_1}\overbrace{0\cdots0}^{E_1}\cdots\overbrace{0\cdots0}^{E_{N-1}}\overbrace{1\cdots1}^{Q_N}00\cdots).$$

We also write as E_i^t *and* Q_i^t *when we would like to specify the time* t. *Notice that the data* $(E_0^t, Q_1^t, E_1^t, \cdots, E_{N\ 1}^t, Q_N^t)$ *recover the state of BBS. We define a soliton to be one of* Q_i *consecutive boxes filled with balls (namely* Q_i *consecutive* 1*'s), and we define* Q_i *to be the size of the soliton.*
When B_t *is*

$$B_t = (\overbrace{0\cdots0}^{E_0^t}\overbrace{1\cdots1}^{Q_1^t}\overbrace{0\cdots0}^{E_1^t}\cdots\overbrace{0\cdots0}^{E_{N-1}^t}\overbrace{1\cdots1}^{Q_N^t}00\cdots),$$

there are N *solitons at time* t, *with sizes* $Q_1^t, Q_2^t, \cdots, Q_N^t$ *from the left to the right.*

Definition 2.7. *In a BBS with the data* $(E_0^t, Q_1^t, E_1^t, \cdots, E_{N\ 1}^t, Q_N^t)$*, we say that* the collision occurs at time t *if and only if* $Q_i^t > E_i^t$ *for some* $i \in \{1, 2, \cdots, N\quad 1\}$. *Namely the collision occurs in the BBS if the carrier has at least one ball at the beginning of some soliton.*

The following Theorem is due to Takahashi-Satsuma [6], proved carefully in [7 Theorem 1].

Theorem 2.8. *For any finite BBS, there exists some time* T *such that for any* $t \geq T$, *writing the state of BBS as* $(E_0^t, Q_1^t, E_1^t, \cdots, E_{N\ 1}^t, Q_N^t)$*, the conditions*

$$\begin{cases}(1) Q_1^T \leq Q_2^T \leq \cdots \leq Q_N^T \\ (2) Q_1^T \leq E_1^T, \cdots, Q_{N\ 1}^T \leq E_{N\ 1}^T\end{cases}$$

hold. □

Proof. (of Theorem 2.5)
By Theorem 2.8, there exists time T, and balls are divided into solitons in non-decreasing order and have no collision at $t \geq T$.
The number of the balls which belong to each soliton from the left are $Q_1^T, Q_2^T, \cdots, Q_N^T$ $(Q_1^T \leq Q_2^T \leq \cdots \leq Q_N^T)$ at $t = T$, and these data do not change at $t \geq T$.
At $t = T$,

$$\begin{aligned} f_{B_T}(z) &= z^{S_1^T}(1+z+\cdots+z^{Q_1^T\ \ 1})+z^{S_2^T}(1+z+\cdots+z^{Q_2^T\ \ 1}) \\ &\quad +\cdots+z^{S_N^T}(1+z+\cdots+z^{Q_N^T\ \ 1}) \\ &= \frac{1\quad z^{Q_1^T}}{1\quad z}\cdot z^{S_1^T}+\frac{1\quad z^{Q_2^T}}{1\quad z}\cdot z^{S_2^T}+\cdots+\frac{1\quad z^{Q_N^T}}{1\quad z}\cdot z^{S_N^T} \end{aligned}$$

where $S_i^T := E_0^T + (\sum_{k=1}^{i\ \ 1} E_k^T + Q_k^T)$.
Each soliton has no collision at $t \geq T$, hence we have

$$f_{B_{T+k}}(z) = \frac{1\quad z^{Q_1^T}}{1\quad z}\cdot z^{S_1^T}\cdot z^{k\cdot Q_1^T}+\frac{1\quad z^{Q_2^T}}{1\quad z}\cdot z^{S_2^T}\cdot z^{k\cdot Q_2^T}+\cdots+\frac{1\quad z^{Q_N^T}}{1\quad z}\cdot z^{S_N^T}\cdot z^{k\cdot Q_N^T}.$$

Therefore,

$$\sum_{j=T}^{\infty} f_{B_j}(z)\cdot t^j = \sum_{i=1}^{N}\frac{1\quad z^{Q_i^T}}{1\quad z}\cdot\frac{z^{S_i^T}\cdot t^T}{1\quad z^{Q_i^T}\cdot t}.$$

Hence this is a rational function.
Consequently,

$$F_{\mathrm{B}}(z,t) = \left(\sum_{j=0}^{T\ \ 1} f_{B_j}(z)\cdot t^j\right)+\left(\sum_{j=T}^{\infty} f_{B_j}(z)\cdot t^j\right)$$

is a sum of a polynomial and a rational function, hence is a rational function. □

3. Generating Functions of infinite-BBS

Definition 3.1. *We say that a BBS is* infinite *if the initial state has an infinite number of balls.*

In this section, we consider the rationality of the generating function $F_{\mathrm{B}}(z,t)$ of an infinite BBS for which the rationality of $F_{\mathrm{B}}(z,0)$ is a necessary condition. We need the following definition to test the rationality of $F_{\mathrm{B}}(z,t)$.

Definition 3.2. *A state of BBS $B_t = (a_0^t, a_1^t, a_2^t, \cdots)$ is* semi-periodic *at time t if there exist $S \geq 1$ and $k > 0$ such that $a_i^t = a_{i+k}^t$ for all $i \geq S$. A BBS is called semi-periodic if it is semi-periodic at any time t.*

In Proposition 3.3, we will show that $F_{\mathrm{B}}(z,0)$ is rational if and only if the BBS is semi-periodic at time $t=0$. In Proposition 3.5, we will show that the BBS is semi-periodic at time $t=0$, then it is semi-periodic at any time t. Moreover, when the BBS is semi-periodic, we will show that its periodic part behaves exactly like the classical periodic BBS (see [9]), hence our semi-periodic BBS is a generalization of the classical periodic BBS.

Proposition 3.3. *When $B_t = (a_0^t, a_1^t, a_2^t, \cdots)$ is a state of BBS at time t, $F_{\mathrm{B}}(z,0) = \sum_{t=0}^{\infty} a_i^0 z^i$ is a rational function if and only if B_0 is semi-periodic.*

Acknowledgment I thank Prof. Nobuyoshi Takahashi for teaching me the following proof.

Proof. We will write a_i instead of a_i^0.

We start by showing necessary condition. By Gauss Lemma, we can write $F_{\mathrm{B}}(z,0) \cdot P(z) = Q(z)$ with $Q(z), P(z) \in \mathbb{Z}[z]$. Writing $P(z) = \sum_{n=0}^{m} p_n \cdot z^n$, we have $p_0 \cdot a_n + p_1 \cdot a_{n-1} + \cdots + p_m \cdot a_{n-m} = 0$ for $n \gg 0$. Hence, a_n is determined by $(a_{n-1}, \cdots, a_{n-m})$ which has only 2^m possibilities. Therefore, it is periodic.

Next, we prove sufficient condition. Assume that there exist $S \gg 0$, $k > 0$ such that $a_{i+k} = a_i$ if $S \le i$.
Then, we get

$$\begin{aligned}
\sum_{i=0}^{\infty} a_i \cdot z^i &= (a_0 \cdot z^0 + \cdots + a_{S-1} \cdot z^{S-1}) + (a_S \cdot z^S + \cdots + a_{S+k-1} \cdot z^{S+k-1}) \\
&\quad + (a_{S+k} \cdot z^{S+k} + \cdots + a_{S+2k-1} \cdot z^{S+2k-1}) + \cdots \\
&= (a_0 + \cdots + a_{S-1} \cdot z^{S-1}) + a_S \cdot z^S (1 + z^k + z^{2k} + \cdots) \\
&\quad + \cdots + a_{S+k-1} \cdot z^{S+k-1}(1 + z^k + z^{2k} + \cdots) \\
&= (a_0 \cdot z^0 + \cdots + a_{S-1} \cdot z^{S-1}) + \frac{a^S \cdot z^S + \cdots + a^{S+k-1} \cdot z^{S+k-1}}{1 - z^k}.
\end{aligned}$$

□

Proposition 3.4. *If a state of BBS at time $t = 0$ is semi-periodic with the period N, then it stays to be periodic at any time $t > 0$ with the same period N. Moreover, the periodic pattern at time t is determined by the pattern at time $t = 0$, and is independent of the non-periodic part.*

Proof. It is enough to show that the BBS is semi-periodic at time $t + 1$ assuming that it is semi-periodic at time t.

Let $i_1(t)$ be the index of the box where the periodic part starts at time t, and N be the length of the period. Then $a_q^t = a_{q+N}^t$ for all $q \ge i_1(t)$, and M be the number of the balls in each period. We write $i_k(t) := i_1(t) + (k-1)N$, the index of the box where the k-th period starts at time t. Let $p_k(t)$ be the number of the balls the carrier has at the $i_k(t)$-th box at time t.

Let $(b_1, b_2, \cdots, b_N)$ be the state of the periodic part. We define the function ϕ by setting $\phi(\ell)$ to be the number of the balls that the carrier has at the end of periodic part when he has ℓ balls at the beginning of the periodic part. We separate the cases depending on $N < 2M, N = 2M$, and $N > 2M$.
(a)The case $N < 2M$:
We have $\phi(\ell) > \ell$ when $N < 2M$, because the carrier has to pick up all M balls, and there are only empty $N - M (< M)$ boxes. Hence, he picks up more balls than he drops.
Moreover, if $\ell \ge N - M$, the state of periodic part after at time $t + 1$ will be $(1 - b_1, 1 - b_2, \cdots, 1 - b_N)$ since the carrier drops balls to all empty boxes and picks up all balls when he passed the periodic part.

Therefore, if $N < 2M$ at time t, the state of BBS at time $t + 1$ is also periodic from $i_k(t)$-th box where $k \gg 1$, with the period $(1 - b_1, 1 - b_2, \cdots, 1 - b_N)$. In particular, we have $N > 2M$ at time $t + 1$.

(b)The case $N > 2M$:
Assume that $\phi(0) = p$. That means, if the carrier enters the periodic part with 0 balls, he picks up M balls and drops $(M - p)$ balls out of $N - M$ empty boxes, hence he passed by $(N - M) - (M - p) = N - 2M + p$ empty boxes without balls.

(b.i)The case $\ell \leq N \quad 2M + p$:
If the carrier has ℓ balls at the beginning of the periodic part with $\ell \leq N \quad 2M+p$, he drops ℓ extra balls out of $N \quad 2M + p$ boxes where the carrier who has no balls at the beginning (we call the carrier "$\ell = 0$ carrier") passes by, then his status (no balls) is the exactly same as the $\ell = 0$ carrier, and the result is $\phi(\ell) = \phi(0) = p$. Thus, $\phi(\ell) = p$.
(b.ii)The case $\ell > N \quad 2M + p(> p)$:
If the carrier has more than $N \quad 2M + p$ balls, then he drops balls at all $N \quad M$ empty boxes. He picks up M balls, so at the end of the period, his balls is reduced by $N \quad 2M$, hence $\phi(\ell) = \ell \quad (N \quad 2M)$.

Therefore, if $N > 2M$ at time t, the state of BBS at time $t + 1$ is also periodic from $i_k(t)$-th box where $k \gg 1$, then $p_k(t) = p = \phi(p)$. We have the same period N with the same number of balls M.

(c)The case $N = 2M$:
Assume that $\phi(0) = p$. Then. there are empty $N \quad M = M$ boxes at time t in each period, and the carrier picks up all M balls and drops $(M \quad p)$ balls in time evilution when the carrier enters the period part with 0 balls, hence he passed by $M \quad (M \quad p) = p$ empty boxes without balls.
(c.i)The case $\ell \leq p$:
Similarly to the case (b.i), we have $\phi(\ell) = p$.
(c.ii)The case $\ell > p$:
Similarly to the case (b.ii), the carrier drops balls to all empty M boxes, hence $\phi(\ell) = \ell + M \quad M = \ell$.

Therefore if $N = 2M$ at time t, the state of BBS at time $t+1$ is also semi-periodic from $i_2(t)$-th box, and the state of the periodic part is $(1 \quad b_1, 1 \quad b_2, \cdots, 1 \quad b_N)$.

Therefore, when $N \geq 2M$, the number of the balls M in each period and the length of the periodic part N are unchanged, and the pattern of periodic part at time $t + 1$ only depends only on the pattern at time t.

The classical periodic BBS is built up so that $\phi(\ell) = \ell$, hence behaves exactly in the same way as our periodic part.

□

Conjecture 3.5. *We conjecture that the generating function $F(z, t)$ is rational for all semi-periodic BBS.*

In Theorem 3.9, we will prove the conjecture in some special case. Rationality in some other cases, $\ell \quad (\ell + 1)$ BBS which is a semi-perioic BBS with the period $\overbrace{1 \cdots 1}^{\ell}\overbrace{0 \cdots 0}^{\ell+1}$, will appear elsewhere.

Definition 3.6. *We define ℓ-ℓ BBS to be a semi-periodic BBS with the period*

$$a_{s+i} = \begin{cases} 1 & (0 \leq i < \ell) \\ 0 & (\ell \leq i < 2\ell) \end{cases}$$

for some $s \gg 0$, of length 2ℓ at time $t = 0$. Hence the period looks like $\overbrace{11 \cdots 1}^{\ell}\overbrace{00 \cdots 0}^{\ell}$. We denote the initial state of BBS as below.

$$B = (\overbrace{0\cdots0}^{E_0^0}\overbrace{1\cdots1}^{Q_1^0}\overbrace{0\cdots0}^{E_1^0}\cdots\overbrace{1\cdots1}^{Q_M^0}\overbrace{0\cdots0}^{E_M^0}\overbrace{\overbrace{11\cdots1}^{\ell}\overbrace{00\cdots0}^{\ell}\cdots}^{\infty}).$$

Define L_t to be

$$L_t = (\sum_{i=0}^{M} Q_i^0 + E_i^0) + t \cdot \ell.$$

Then, at any time t, the state of ℓ-ℓ BBS, starting from the L_t-th box, is periodic with the period $\overbrace{11\cdots1}^{\ell}\overbrace{00\cdots0}^{\ell}$. Also, define M_t to be the number of the solitons in the non-periodic part at time t, and N_t to be the number of the balls in the non-periodic part at time t.

Definition 3.7. *We say that in a semi-periodic BBS,* P-collision occurs *at time t in getting B_{t+1} from B_t if the carrier has at least one ball at the beginning of the periodic part.*

Lemma 3.8.
(1) *If no P-collision occurs at time t, $N_t = N_{t+1}$.*
(2) *If no P-collision occurs at time t, time evolutions in the non-periodic part and the periodic part are independent.*
(3) *If a P-collision occurs at time t, $N_{t+1} < N_t$.*

Proof.
(1) The carrier passes through L_t with no balls, so the number of balls in the non-periodic part does not change.
(2) The carrier passes through L_t with no balls, so no interaction between the non-periodic part and the periodic part occurs.
(3) As the P-collision occurs, the carrier carries d (> 0) balls at L_t-th box which balls disappear to the infinity, hence $N_{t+1} = N_t \quad d$. □

Theorem 3.9. *In the case of ℓ-ℓ BBS, $F(z,t)$ is a rational function of z and t.*

Proof.
Since the balls in the non-periodic part are finite, the number of the P-collision is also finite. Let $T' \quad 1$ be the time of the last P-collision, then time evolutions in the non-periodic part and the periodic part at $t \geq T'$ are independent.

By Theorem 2.8, there exists time T such that the collisions of solitions in non-periodic part are over. Then, the state of the ℓ-ℓ semi-periodic BBS looks like the following at $t \geq T$.

$$\begin{cases} Q_1^t \leq Q_2^t \leq \cdots \leq Q_{M_{T'}}^t \leq \ell \\ M_{T'} = M_{T'+1} = \cdots \\ Q_i^t \leq E_i^t \quad (i = 1, 2, \cdots, M_{T'}) \end{cases}$$

$$B = (\overbrace{0\cdots0}^{E_0^t}\overbrace{1\cdots1}^{Q_1^t}\overbrace{0\cdots0}^{E_1^t}\cdots\overbrace{1\cdots1}^{Q_{M_{T'}}^t}\overbrace{0\cdots0}^{E_{M_{T'}}^t}\overbrace{\overbrace{11\cdots1}^{\ell}\overbrace{00\cdots0}^{\ell}\cdots}^{\infty})$$

Then,

$F(z,t) = \sum_{j=0}^{T\ 1} f_{B_j} t^j$
$+\sum_{i=1}^{M_{T'}} \frac{z^{S_i^T}(1+\cdots+z^{Q_i^T-1})\cdot t^T}{1\ \ z^{Q_i^T}\cdot t} + \frac{z^{S_{M_{T'}}^T+Q_{M_{T'}}^T+E_{M_{T'}}^T}(1+\cdots+z^{\ell-1})\cdot t^T}{(1\ \ z^{2\ell})(1\ \ z^{\ell}\cdot t)}$
where $S_i^T := E_0^T + (\sum_{k=1}^{i\ 1} E_k^T + Q_k^T)$. □

4. Generating Functions of BBS with a limited cart

4.1. Finite BBS with a limited cart.

BBS with a limited cart is another type of BBS proposed by Takahashi-Matsukidaira [5]. In this section, we consider a finite BBS with a carrier who has a limited cart. Assume that the carrier can carry at most k (> 0) balls.

Definition 4.1. *We call this k as the* capacity *of the BBS.*

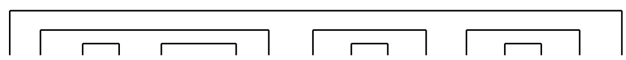
The time evolution rule from time step t to $t+1$ is described as follows. Assume that the state of the BBS with a limited cart at time t is $(a_0, a_1, a_2, \cdots)$. Then, a_i is defined inductively on i as

$$a_i^{t+1} = \min(1\quad a_i^t, \sum_{j=0}^{i\ 1} a_j^t \quad \sum_{j=0}^{i\ 1} a_j^{t+1}) + \max(0, \sum_{j=0}^{i} a_j^t \quad \sum_{j=0}^{i\ 1} a_j^{t+1} \quad k)$$

(see [5]).

The time evolution above can be reinterpreted as the following rule from time t to $t+1$ in terms of the cart and the carrier, moving from the left to the right, as in Definition 2.1.
(1) If there is a ball in the box, the carrier picks up the ball into his cart, unless the cart has k balls. If the cart has k balls, then he does nothing. (He moves on to the next box.)
$(2), (3)$ If there is no ball in the box, the carrier behaves like in the classical BBS, as described in Definition 2.1.
When the carrier finishes all the boxes, one time-evolution step finishes.

Definition 4.2. *In a BBS, we define the*marking *as follows: when the carrier drops a ball into a box, we assume that he always drops the newest ball, namely he behaves like a last-in first-out stack memory in computer program. Then each carried ball moves from one box to another, and we mark the move by connecting the two boxes by lines as below. The following example is the case of $k = 3$.*

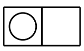


1 1 1 0 1 1 0 0 1 1 0 0 1 1 0 0 0 0 0 $\cdots$

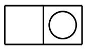
When the ball is on the i-th waiting list at most in the cart, then we say that the corresponding marking as depth i. *We denote the number of depth i markings at time t by $p_i(t)$.*

For each depth i marking, either it has a unique depth $i+1$ ball waiting behind it, or it does not have such a ball. It determines a surjection from some of the depth i balls to all of the depth $i+1$ balls, hence $p_i(t) \geq p_{i+1}(t)$. Therefore we obtain a non-increasing sequence of positive integers $p_1(t) \geq p_2(t) \geq \cdots \geq p_k(t) \geq 0$.

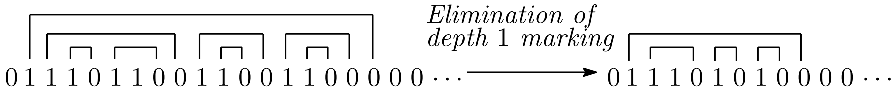
Definition 4.3. *We define the* 10 *pair to be two boxes which look like*



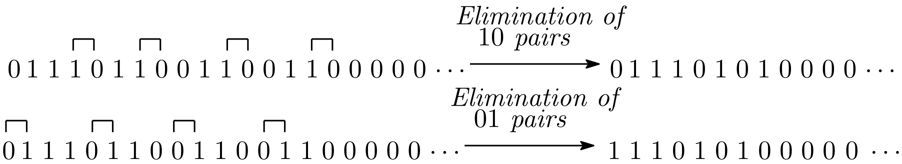
(*so using the notation of Definition 2.1, it is 10*). *We also define* 01 *pair similarly:*



Definition 4.4. *Let B be a BBS with the capacity k, we mark B, and we define the* elimination of depth 1 markings *of B to be the BBS with the capacity k 1, with all balls and boxes of the markings of depth* 1 *(in and out) eliminated.*
For example,

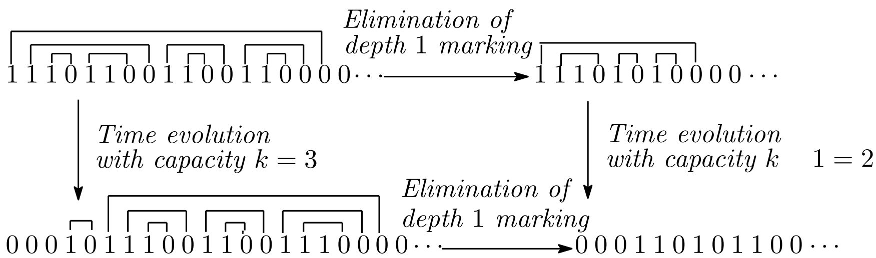


Similarly, we also define the elimination of 10 (or 01) pairs *of B to be the BBS with the capacity k 1, with all* 10 *(or* 01*) pairs eliminated.*



Note that the movement of the balls is not affected by the elimination of depth 1 markings, and that all the depths of the markings are reduced by 1.

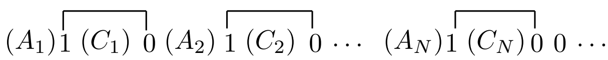
Proposition 4.5. *The elimination of depth* 1 *markings and the time evolution commute modulo numberings of the boxes. To be precise, if you eliminate the depth* 1 *markings and let the time evolve by* 1 *with the capacity k 1, the state of the BBS is the same as when you let the time evolve by* 1 *with the capacity k, eliminate the depth* 1 *markings and eliminate the box numbered* 0 *(so, all boxes will have* 1 *less numbers).*



Proof. Assume that the state of BBS is given at time t is as follows:

$$B = (\overbrace{0\cdots 0}^{E_0^t}\overbrace{1\cdots 1}^{Q_1^t}\overbrace{0\cdots 0}^{E_1^t}\overbrace{1\cdots 1}^{Q_2^t}\cdots\overbrace{0\cdots 0}^{E_{N-1}^t}\overbrace{1\cdots 1}^{Q_N^t}\overbrace{00\cdots}^{E_N^t=\infty})$$

Notice that the left most empty boxes of $E_1^t, E_2^t, \cdots, E_N^t$ are marked with depth 1, connected with some balled box in $Q_1^t, \cdots, Q_N^t$, hence the depth 1 markings look as follows:



$$\text{where} \begin{cases} A_i : \overbrace{0\cdots 0}^{\alpha_i}\overbrace{1\cdots 1}^{\beta_i} & (\alpha_i \geq 0, \beta_i \geq 0) \\ C_i : \overbrace{1\cdots 1}^{\gamma_i} & (\gamma_i \geq 0). \end{cases}$$

We denote the state of BBS at time $t+1$ as below.

$$(B_1)0(C_1)1(B_2)0(C_2)1\cdots(B_N)0(C_N)1(B_{N+1})00\cdots$$

where B_i: $\overbrace{1\cdots 1}^{\delta_i}\overbrace{0\cdots 0}^{\epsilon_i}$ $\quad(\delta_i \geq 0, \delta_1 = 0, \epsilon_i \geq 0)$.

Remark 4.6.
(1) *One can observe that the in-boxes of depth* 1 *marking at time* t *are exactly the* 0*-boxes of the* 10 *pairs at time* t*. The out-boxes of depth* 1 *markings at time* t *are exactly the* 0*-boxes of the* 01 *pairs at time* $t+1$*.*
(2) *Each* C_i *consists of* $1'$*s, and the carrier does not move these balls at time* t*, because his cart is full.*
(3) *Thus the elimination of depth* 1 *markings has exactly the same effect as elimination of* 10 *pairs.*

We prove that μ_r, the number of balls in $k \quad 1$ limited cart at the end of $(A_1)(C_1)(A_2)(C_2)\cdots(A_r)$, equals to ν_r, the number of balls in k limited cart at the end of $(A_1)1(C_1)0(A_2)1(C_2)0\cdots(A_r)1(C_r)0$, by induction on r.
When $r = 0$, $\mu_0 = \nu_0 = 0$. Hence, the equality holds.
Assume that $\mu_{r \ 1} = \nu_{r \ 1}$. Along the boxes (A_r), we claim that the number of the balls in the cart is less than $k \quad 1$ (hence the same behavior for the capacities $k \quad 1$ and k).

As $A_r = \overbrace{0\cdots 0}^{\alpha_r}\overbrace{1\cdots 1}^{\beta_r}$, first the carrier drops $\min(\alpha_r, \mu_{r \ 1})$ balls, and start picking the balls.

If (C_r) is not empty, then in the capacity k case, the ball before (C_r) is the k-th ball in the cart. This k-th ball is not included in (A_r), so the carrier has exactly $k \quad 1$ balls (in both capacities) at the end of (A_r). In this case, $\mu_r = k \quad 1 = \nu_r$.

If (C_r) is empty, then in the capacity k case, even with the extra ball after (A_r), he is within the k-th limit. Hence, in (A_r), his balls are less than or equal to $k \quad 1$. Thus, both carriers have $\max(\mu_{r \ 1} \quad \alpha_r, 0) + \beta_r \leq k \quad 1$ balls at the end of (A_r). Hence $\mu_r = \max(\mu_{r \ 1} \quad \alpha_r, 0) + \beta_r = \nu_r$.

Thus one can observe that after the time evolution with the capacity $k \quad 1$, the BBS looks like

$$(B_1)(C_1)(B_2)(C_2)\cdots(B_N)(C_N)(B_{N+1})00\cdots.$$

By Remark 4.6 (1), this is exactly the result of 01-elimination of the original BBS with capacity k at time $t+1$, which is same as 10-elimination together with the elimination of the 0-th box. □

Proposition 4.7. *The number of depth* i *markings in the sequence is preserved, namely* $p_i(t) = p_i(t+1)$*.*

Proof. Remark that the number of 10 pairs and the number of 01 pairs are the number of the solitons, hence are equal. Also by Remark 4.6(1), the number of 10

pair at time t is same as the number of the depth 1 markings, which is same as the number of 01 pairs at time $t+1$.
Then,

$$\begin{aligned}
& \text{The number of solitons at time } t \\
= \; & \text{The number of 10 pairs at time } t \ (= p_1(t)) \\
= \; & \text{The number of 01 pairs at time } t+1 \\
= \; & \text{The number of solitons at time } t+1 \\
= \; & \text{The number of 10 pairs at time } t+1 = p_1(t+1).
\end{aligned}$$

Hence, $p_1(t)$ is preserved in time.

By Proposition 4.5, eliminating depth 1 markings commutes with the time evolution, modulo numbering, hence eliminating depth 1 markings ($i \quad 1$) times also commutes with the time evolution, which implies $p_i(t) = p_i(t+1)$. □

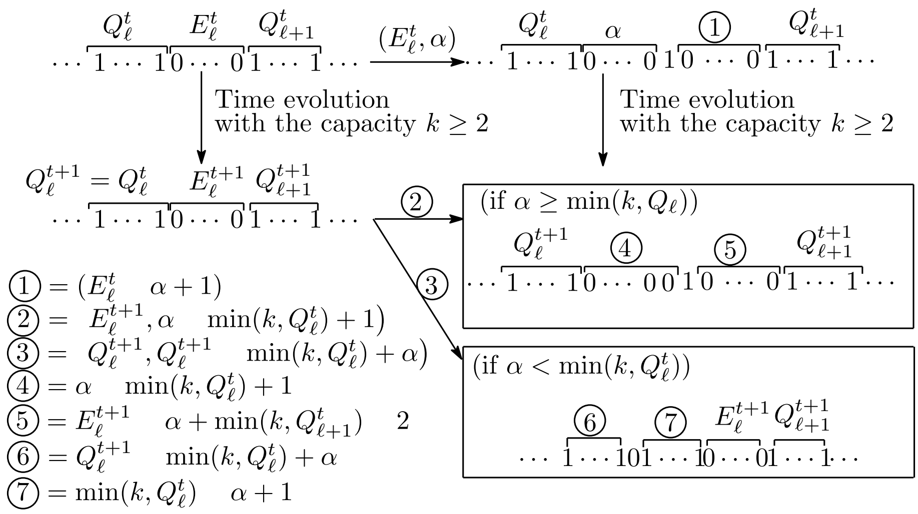
Similarly to the classical BBS, in Theorem 4.10 below, we will show that in the case of finite BBS with a limited cart, there exists finite time T such that the solitons are in correct order and no collision occurs at $t \geq T$.

Definition 4.8. *For a state of BBS $B_t = (E_0^t, Q_1^t, E_1^t, Q_2^t, \cdots)$, we define an operation (E_ℓ^t, α) with $0 < \alpha \leq E_\ell^t$, which inserts 10 pair into B_t by changing $E_\ell^t = \overbrace{0\cdots0}^{E_\ell^t}$ to $\overbrace{0\cdots0}^{\alpha}10\overbrace{0\cdots0}^{E_\ell^t\ \alpha}$. We also define an operation (Q_ℓ^t, α) with $0 < \alpha < Q_\ell^t$, which inserts 10 pair into B_t by changing $Q_\ell^t = \overbrace{1\cdots1}^{Q_\ell^t}$ to $\overbrace{1\cdots1}^{\alpha}10\overbrace{1\cdots1}^{Q_\ell^t\ \alpha}$.*

In the BBS B with the condition $(*)$ (with $s, t \geq T$), we say that $(E_\ell^s, \alpha) \leq (E_m^t, \beta)$ if $[\ell < m]$ or $[\ell = m$ and $\alpha < \beta]$, that $(Q_\ell^s, \alpha) \leq (Q_m^t, \beta)$ if $[\ell < m]$ or $[\ell = m$ and $\alpha < \beta]$, that $(E_\ell^s, \alpha) \leq (Q_m^t, \beta)$ if $\ell < m$, and that $(Q_\ell^s, \alpha) \leq (E_m^t, \beta)$ if $\ell \leq m$. In a word, $(*_\ell^s, \alpha) \leq (\sharp_m^t, \beta)$ when $(*_\ell^s, \alpha)$ is more left than $(\sharp_m^t, \beta)$ with $*, \sharp \in \{E, Q\}$.

Proposition 4.9. *Let C_k be the time-evolution operator of capacity k (i.e. $C_k(B_t) = B_{t+1}$). Then, for any $(*_\ell^t, \alpha)$, there exists $(\sharp_{\ell'}^{t+1}, \alpha')$ such that $C_k\left((*_\ell^t, \alpha)(B_t)\right) = (\sharp_{\ell'}^{t+1}, \alpha')\left(C_k(B_t)\right))$. Moreover, if the inserting operation is (Q_ℓ, α) or if the inserting operation is (E_ℓ, α) with $Q_\ell \geq 2$ after the evolution, we insert 10 pair more left to the original operation.*

Proof.
A figure below is an example of (E_ℓ, α).



The figure above shows that

$$C_k \;\; (E_\ell^t, \alpha)(B_t)\big) = \begin{cases} E_\ell^{t+1}, \alpha \quad \min(k, Q_\ell^t) + 1\big)\, (C_k(B_t)) & \text{if } (\alpha \geq \min(k, Q_\ell^t)) \\ Q_\ell^{t+1}, Q_\ell^{t+1} \quad \min(k, Q_\ell^t) + \alpha\big)\, (C_k(B_t)) & \text{(otherwise)}, \end{cases}$$

, which are more left to the original operation as far as $Q_\ell \geq 2$.
Similarly,

$$C_k \;\; (Q_\ell^t, \alpha)(B_t)\big) = \begin{cases} (Q_\ell^{t+1}, \alpha + 1 \quad k)\, (C_k(B_t)) & \text{if} (\alpha + 1 > k) \\ (E_{\ell \;\; 1}^{t+1}, E_{\ell \;\; 1}^{t+1} \quad \min(k \quad \alpha, Q_\ell \quad \alpha) + 1)\, (C_k(B_t)) & \text{(otherwise)}, \end{cases}$$

which are always more left to the original operation. Proposition 4.9 is proved. □

Theorem 4.10. *Assume that the number of the balls is finite in a BBS with M solitons and a limited cart. We denote the state of BBS at time t as below.*

$$B = (\overbrace{0 \cdots 0}^{E_0^t}\overbrace{1 \cdots 1}^{Q_1^t}\overbrace{0 \cdots 0}^{E_1^t} \cdots \overbrace{0 \cdots 0}^{E_{M-1}^t}\overbrace{1 \cdots 1}^{Q_M^t}\overbrace{000 \cdots}^{\infty}).$$

Then, there exists time T such that for $t \geq T$, we have

$$(*) \begin{cases} (1) \text{ There exists } r \text{ with } \; 0 \leq r \leq M, \text{ and } Q_1^t \leq Q_2^t \leq \cdots \leq Q_r^t < k \\ (2) Q_i^t \leq E_i^t \text{ if } \; 1 \leq i \leq r \\ (3) Q_i^t \geq k \text{ if } \; i > r \\ (4) k \leq E_i^t \text{ if } \; i > r \end{cases}$$

Hence, when $t \geq T$, each soliton has time evolution at constant speed and no collision occurs between them.

Proof. We proceed by induction on k.

When $k = 1$, the condition $(*)$ obviously holds for $T = 0$ with $r = 0$. In this case, each soliton proceeds at speed 1 with Q_i^t and $E_i^t (1 \leq r \leq M)$ unchanged and $E_i^t = E_0^T + (t \quad T)$.

Suppose that $(*)$ holds for $k \quad 1$. By Proposition 4.5, the elimination of depth 1 markings commutes with time evolution if we reduce the capacity by 1 after the elimination, modulo numbering the boxes (or ignoring E_0^t). The condition $(*)$ is not related to E_0^t, so we can ignore E_0^t.

By inductive assumption, there exists T such that when $t \geq T$, if we eliminate the depth 1 markings, the condition $(*)$ holds. If the elimination does not reduce the number of solitons, the Q_i^t's become $Q_i^t \quad 1$ and E_i^t's become $E_i^t \quad 1$ for $1 \leq i \leq M$ by Remark 4.6(3), so one can see that the condition $(*)$ holds before elimination.

Let us denote the state of original BBS at time T by B. We eliminate the depth 1 markings from B, and we denote the state by B'. By inductive assumption, we assume that B' satisfies the condition $(*)$.

We first treat the case when the number of the solitons on B' is 1 less than B. Assume $B' = (0\cdots0\overbrace{1\cdots1}^{q_1}\overbrace{0\cdots0}^{e_1}\cdots\overbrace{0\cdots0}^{e_{M-2}}\overbrace{1\cdots1}^{q_{M-1}}\overbrace{000\cdots}^{\infty=e_{M-1}})$, and we define $\bar{B}$ to be $\bar{B} = (0\cdots0\overbrace{1\cdots1}^{q_1+1}\overbrace{0\cdots0}^{e_1+1}\cdots\overbrace{0\cdots0}^{e_{M-2}+1}\ \overbrace{1\cdots1}^{q_{M-1}+1}\ \overbrace{000\cdots}^{\infty=e_{M-1}})$, namely placing back 10 pairs after all solitons of B'. Then B is obtained by placing back one 10 pair to $\bar{B}$, either in some $\overbrace{1\cdots1}^{q_i+1}$ or in some $\overbrace{0\cdots0}^{e_i+1}$.

Notice that if the 10 insertion point keeps moving to the left, like in Proposition 4.9, eventually we insert the 10 pair at (E_ℓ, α) with $Q_\ell = 1$ which means that at this point, the condition $(*)$ is satisfied. One can easily see that we can place back 10 pairs one by one, hence the induction on k completes. Now, Theorem 4.10 is proved. $\square$

Corollary 4.11. *The generating functions of finite BBS with a limited cart, $F(z,t)$, is a rational function of z and t.*

Essentially, Corollary 4.11 follows from Theorem 4.10.

Proof. By Theorem 4.10, there exists time T such that each soliton of finite BBS evolves at constant speed and no collision occurs between them. Namely, the state of BBS at time T satisfies the condition $(*)$ of Theorem 4.10.

$$B = (\overbrace{0\cdots0}^{E_0^T}\overbrace{1\cdots1}^{Q_1^T}\overbrace{0\cdots0}^{E_1^T}\cdots\overbrace{1\cdots1}^{Q_{r-1}^T}\overbrace{0\cdots0}^{E_{r-1}^T}\overbrace{1\cdots1}^{Q_r^T}\overbrace{0\cdots0}^{E_r^T}\cdots\overbrace{0\cdots0}^{E_{M-1}^T}\overbrace{1\cdots1}^{Q_M^T}\overbrace{00\cdots}^{\infty})$$

Then,

$$\begin{aligned} F(z,t) &= \sum_{j=0}^{T \quad 1} f_{B_j}(z)\cdot t^j \\ &\quad + \sum_{i=1}^{r \quad 1} \frac{z^{S_i^T}(1+z+\cdots z^{Q_i^T \quad 1})\cdot t^T}{1 \quad z^{Q_i^T}\cdot t} + \sum_{i=r}^{M} \frac{z^{S_i^T}(1+z+\cdots z^{Q_i^T \quad 1})\cdot t^T}{1 \quad z^k \cdot t} \end{aligned}$$

where $S_i^T := E_0^T + (\sum_{k=1}^{i \quad 1} E_k^T + Q_k^T)$. $\square$

4.2. **Infinite BBS with a limited cart.**

Now, consider the case of ℓ-ℓ BBS with a limited cart. Assume that there exists M solitions in the non-periodic part, and we denote the state of BBS as below.

$$B = (\overbrace{0\cdots0}^{E_0^t}\overbrace{1\cdots1}^{Q_1^t}\overbrace{0\cdots0}^{E_1^t}\cdots\overbrace{1\cdots1}^{Q_M^t}\overbrace{0\cdots0}^{E_M^t}\overbrace{\overbrace{11\cdots1}^{\ell}\overbrace{00\cdots0}^{\ell}\cdots}^{\infty})$$

Theorem 4.12. *In the case of ℓ-ℓ BBS with a limited cart, $F(z,t)$ is a rational function of z and t.*

Proof. When $\ell < k$, the proof is similar to Theorem 3.9. When $\ell \geq k$, if P-collision occurs, the balls are not moved away but remain in front of the periodic part, and we define the extended periodic part to be $\overbrace{1\cdots1}^{extended}\ \overbrace{\overbrace{1\cdots1}^{\ell}\overbrace{0\cdots0}^{\ell}\cdots}^{\infty}$. Then, the collision between the non-periodic part and the extended periodic part occurs only finitely many times, and the rest is similar. □

Conjecture 4.13. *The generating function of any semi-periodic BBS with a limited cart is a rational function of z and t.*

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