学位論文要旨

Degeneration of Fermat hypersurfaces in positive characteristic

（正標数の体上定義されたフェルマー多様体の退化）

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We work over an algebraically closed field k of positive characteristic p. Let q be a power of p. Let n be a positive integer. We denote by $M_{n+1}(k)$ the set of square matrices of size $n+1$ with coefficients in k. For a nonzero matrix $A = (a_{ij})_{0\le i,j\le n} \in M_{n+1}(k)$, we denote by X_A the hypersurface of degree $q+1$ defined by the equation

$$\sum a_{ij} x_i x_j^q = 0$$

in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \dots, x_n)$. When the rank of matrix A is $n+1$, the hypersurface X_A is projectively isomorphic to the Fermat hypersurface of degree $q+1$. The Fermat hypersurface has been a subject of numerous papers. It has many interesting properties, such as supersingularity, or unirationality. Moreover, the hypersurface X_A associated with the matrix A with coefficients a_{ij} in the finite field $\mathbb{F}_{q^2}$, which is called a Hermitian variety, has also been studied for many applications, such as coding theory. Therefore it is important to extended these studies to degenerate cases.

In the case where characteristic $p \neq 2$, the hypersurface defined by the quadratic form $\sum a_{ij} x_i x_j = 0$ is projectively isomorphic to the hypersurface defined by $x_0^2 + \cdots + x_{r-1}^2 = 0$, where r is the rank of $A = (a_{ij})$. This result has been extended the case of characteristic 2 by Dolgachev. Therefore we have a question what is the normal form of the hypersurfaces defined by a form $\sum a_{ij} x_i x_j^q = 0$. When A satisfies ${}^t A = A^{(q)}$ and hence this form is the Hermitian form over $\mathbb{F}_q$, the hypersurface X_A is projectively isomorphic over $\mathbb{F}_{q^2}$ to

$$x_0^{q+1} + \cdots + x_{r-1}^{q+1} = 0,$$

where r is the rank of A.

In this thesis, we classify the hypersurfaces X_A associated with the matrices A of rank n *over an algebraically closed field*, and determine the automorphism group.

We define I_s to be the $s \times s$ identity matrix, and E_r to be the $r \times r$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

In particular, $E_1 = (0)$ and E_0 is the 0×0 matrix. Our main theorem is as follows:

Theorem 0.1. *Let $A = (a_{ij})_{0\le i,j\le n}$ be a nonzero matrix in $M_{n+1}(k)$, and let X_A be the hypersurface of degree $q+1$ defined by $\sum a_{ij} x_i x_j^q = 0$ in the projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0, x_1, \dots, x_n)$. Suppose that the rank of A is n. Then* *the hypersurface X_A is projectively isomorphic to one of the hypersurfaces X_s associated with the matrices*

$$W_s = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s+1} \end{array} \right),$$

where $0 \le s \le n$. Moreover, if $s \ne s'$, then X_s and $X_{s'}$ are not projectively isomorphic.

We also determine the automorphism group

$$\mathrm{Aut}(X_s) = \{g \in PGL_{n+1}(k) \mid g(X_s) = X_s\},$$

of the hypersurface X_s for each s. For $M \in GL_{n+1}(k)$, we denote by $[M] \in PGL_{n+1}(k)$ the image of M by the natural projection.

Theorem 0.2. *Let X_s be the hypersurface associated with the matrix W_s in the projective space $\mathbb{P}^n$. The projective automorphism group $\mathrm{Aut}(X_s)$ with $s \le n-2$ is the group consisting of $[M]$ with*

$$M = \left(\begin{array}{c|c|c} T & {}^t\mathbf{a} & 0 \\ \hline 0 & d & 0 \\ \hline \mathbf{c} & e & 1 \end{array} \right),$$

where $T \in GL_{n-1}(k)$, $\mathbf{a}, \mathbf{c}$ are row vectors of dimension $n-1$, and $d, e \in k$, and they satisfy the following conditions:

(i) $[T] \in \mathrm{Aut}(X_s^{n-2})$, ${}^tTW_s'T^{(q)} = \delta W_s'$, $\delta = \delta^q \ne 0$, *where X_s^{n-2} is the hypersurface defined in $\mathbb{P}^{n-2}$ by the matrix*

$$W_s' = \left(\begin{array}{c|c} I_s & \\ \hline & E_{n-s-1} \end{array} \right)$$

(ii) $d = \delta$,

(iii) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot T^{(q)} = \delta(0, \cdots, 0, 1)$,

(iv) ${}^tTW_s' \cdot {}^t\mathbf{a}^{(q)} + {}^t\mathbf{c}d^q = 0$,

(v) $[\mathbf{a}W_s' + d(0, \cdots, 0, 1)] \cdot {}^t\mathbf{a}^{(q)} + ed^q = 0$.

Moreover, we have

$$\mathrm{Aut}(X_n) = \left\{ \left[\begin{array}{c|c} T_n & \\ \hline \mathbf{u} & 1 \end{array} \right] \;\middle|\; \begin{array}{l} {}^tT_nT_n^{(q)} = \lambda I_n, T_n \in GL_n(k),\ \lambda \ne 0, \\ \mathbf{u} \textit{ is a row vector of dimension } n \end{array} \right\},$$

and

$$\mathrm{Aut}(X_{n-1}) = \left\{ \left[\begin{array}{c|c|c} T_{n-1} & & \\ \hline & \beta & \\ \hline & & 1 \end{array} \right] \;\middle|\; \begin{array}{l} {}^tT_{n-1}T_{n-1}^{(q)} = \beta^q I_{n-1}, \\ T_{n-1} \in GL_{n-1}(k),\ 0 \ne \beta \in k \end{array} \right\}$$

We investigate the plane curve X_A associated with the matrix A of rank ≤ 2 in the projective plane $\mathbb{P}^2$, and recover Homma's unpublished work.