Thesis Summary

Elliptic Quantum Algebra $U_{q,p}(\widehat{\mathfrak{g}})$,

Dynamical Quantum $Z-$algebra and Higher Level Representation

(楕円量子代数 $U_{q,\,p}(\hat{\mathfrak{g}})$，ダイナミカル量子 Z 代数，および高レベル表現)

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In this thesis, we study the infinite dimensional representation of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ of untwisted affine Lie algebras $\widehat{\mathfrak{g}}$. In particular, we discuss the existence of the dynamical $\mathcal{Z}_k-$algebra structure of the level$-k$ $U_{q,p}(\widehat{\mathfrak{g}})-$modules and show the construction of the level$-k$ highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$ by using $\mathcal{Z}_k-$module. We also discuss that the irreducibility of the $\mathcal{Z}_k-$module leads to the irreducibility of the level$-k$ $U_{q,p}(\widehat{\mathfrak{g}})-$module. We give the level-1 standard representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for some types of $\widehat{\mathfrak{g}}$.

The elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ can be equipped with a Hope algebroid structure. We use the elliptic analogue of the Drinfeld coproduct and the level-1 standard realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ to construct the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. We also investigate the elliptic analogue of the condition of integrability of such representation and derive an elliptic analogue of the so called q-difference equation of certain vertex operators. We show the higher level realization of the quantum dynamical $Z-$algebra.

For $U_{q,p}(C_l^{(1)})$, we present a different type of elliptic bosons A_m^j from the (co-)roots type elliptic bosons $\alpha_{j,n}(\alpha^\vee_{j,n})$. We give an explicit construction of the fundamental weight type elliptic bosons A_m^j, the orthogonal basis type $\mathcal{E}_m^{\pm j}$, the elliptic currents $k_{\pm j}(z)$ and calculate several commutation relations among them.

In chapter 2, we review the basic notations and concepts of affine Lie algebra $\widehat{\mathfrak{g}}$, quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ and elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$.

In the first part, we recall the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. Namely, we consider the polynomial loop algebra associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ and perform two extensions of the loop algebra. We summarize the main structures of constructed untwisted affine Lie algebras $\widehat{\mathfrak{g}}$.

The affine quantum group $U_q(\widehat{\mathfrak{g}})$ is exposed in the second part. We present two isomorphic realization of $U_q(\widehat{\mathfrak{g}})$ whose defining relations are written down in term of Chevally generators and Drinfeld's generators, respectively. We review the coalgebra structure of $U_q(\widehat{\mathfrak{g}})$. We investigate a category of the level$-k$ highest weight modules of $U_q(\widehat{\mathfrak{g}})$ in an analogous way to the classical affine Lie algebra. After that we present a quantum analogue of Lepowsky-Wilson's $Z-$algebra which related to the level$-k$ $U_q(\widehat{\mathfrak{g}})-$modules and the defining relations of this algebra. The induced $U_q(\widehat{\mathfrak{g}})-$modules are constructed by using the Z_k-modules. We show that the Z_k-modules

determines the irreducibility of the resulted induced $U_q(\widehat{\mathfrak{g}})-$modules. Finally, we give the level-1 irreducible $U_q(\widehat{\mathfrak{g}})-$modules for some types of untwisted affine Lie algebras $\widehat{\mathfrak{g}}$.

In the last part, we expose a definition of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ as a topological algebra over the ring of formal power series in p. We introduce the field $\mathcal{M}_{H^*}$ of meromorphic functions on H^* the dual of H, a dynamical extension of the Cartan subalgebra. We introduce the level$-k$ representation of $U_{q,p}(\widehat{\mathfrak{g}})$ as an $H-$algebra homomorphism. A category of the level$-k$ $U_{q,p}(\widehat{\mathfrak{g}})-$modules is introduced.

The main results of the thesis are presented in four chapters.

In chapter 3, we discuss a quantum dynamical analogue of Lepowsky and Wilson's $Z-$algebra associated with the level$-k$ $U_{q,p}(\widehat{\mathfrak{g}})-$module. First, we define the Heisenberg subalgebra $U_{q,p}(\mathcal{H})$ of $U_{q,p}(\widehat{\mathfrak{g}})$ and introduce its level$-k$ module. Secondly, we introduce certain level$-k$ vertex operators in $U_{q,p}(\mathcal{H})$ and their commutation relations. After that we present a definition of the dynamical quantum analogue $\mathcal{Z}_{\mathcal{V}}$ of Lepowsky and Wilson's $Z-$ algebra associated with level$-k$ $U_{q,p}(\widehat{\mathfrak{g}})-$module $\mathcal{V}$. We then present the universal dynamical quantum $Z-$algebra $\mathcal{Z}_k$. We define a category of the level$-k$ $\mathcal{Z}_k-$modules.

In chapter 4, we study the generic level $-k$ representation of $U_{q,p}(\widehat{\mathfrak{g}})$ by the associated $\mathcal{Z}_k$ representation. We construct the induced $U_{q,p}(\widehat{\mathfrak{g}})-$module as a tensor product of the $\mathcal{Z}_k-$module and the $U_{q,p}(\mathcal{H})-$module. We show that the irreducibility of that module is governed by the $\mathcal{Z}_k-$module. For the level 1 ($k=1$), we present examples of the infinite dimensional irreducible representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$.

In chapter 5, we study the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and Its integrability. We recall the $H-$Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in the first part. Then we review the level-1 irreducible representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. In third part, using the elliptic analogue of the Drinfeld coproduct, we construct the level$-k+1$ realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as the tensor product of the level-1 modules. In the fourth part, we introduce vertex operators of the level$-k+1$ elliptic bosons and obtain the level$-k+1$ quantum dynamical $Z-$algebra. In the last part, we discuss the elliptic analogue of the integrable condition of the constructed level$-k+1$ $U_{q,p}(\widehat{\mathfrak{sl}}_2)-$modules and the $q-$difference equation of certain level$-k+1$ vertex operators.

In chapter 6, We give a definition of A_m^j for an arbitrary level c and construct the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ and the elliptic current $k_{\pm j}(z)$ of $U_{q,p}(C_l^{(1)})$. Then we drive various commutation relations among the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ as well as among the elliptic currents $k_{\pm j}(z)$.

In the last chapter we summarize the main results of this thesis and discuss some open problems.