

Hiroshima University Doctoral Thesis

**Elliptic Quantum Algebra $U_{q,p}(\hat{\mathfrak{g}})$,
Dynamical Quantum Z -algebra and
Higher Level Representation**

(楕円量子代数 $U_{q,p}(\hat{\mathfrak{g}})$,
ダイナミカル量子 Z 代数, および高レベル表現)

2016

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(1) Elliptic Algebra $U_{q,p}(\hat{\mathfrak{g}})$ and Quantum Z -algebras

Rasha M. Farghly, Hitoshi Konno, and Kazuyuki Oshima

Algebra and Representation Theory, 18 (2015) 103–135.

(2) Higher Level Representation of the Elliptic Quantum Group
 $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its Integrability

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Hiroshima Mathematical Journal, in press.

Main Thesis

**Elliptic Quantum Algebra $U_{q,p}(\hat{\mathfrak{g}})$,
Dynamical Quantum Z -algebra and
Higher Level Representation**

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Abstract

Finding the infinite dimensional representations of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ has been very exciting research topic. We consider the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ as a topological algebra over the ring of formal power series in p . This dissertation deals on the one hand with the existence of the dynamical quantum \mathcal{Z}_k -algebra structure in the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module for general untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ [53]. We discuss the level- k irreducible highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$ in term of the dynamical quantum \mathcal{Z}_k -module and the module of level- k elliptic bosons. We show that the irreducible \mathcal{Z}_k -module guarantees the irreducibility of level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module. We present the level-1 irreducible highest weight representations of $U_{q,p}(\widehat{\mathfrak{g}})$, which we call the standard representations, for $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, B_l^{(1)}$.

On the other hand we discuss the construction of the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by taking an elliptic analogue of the Drinfeld coproduct of the level-1 standard representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [52]. We also study an elliptic analogue of the integrable condition of such representation and a "q-difference equation" of certain vertex operators obtained from the Drinfeld coproduct of the elliptic currents. We present the higher level realization of the dynamical quantum \mathcal{Z} -algebra.

For $U_{q,p}(C_n^{(1)})$ and at an arbitrary level c , we introduce an explicit construction of the elliptic bosons of the fundamental weight type A_m^j and the orthogonal basis type $\mathcal{E}_m^{\pm j}$, the elliptic currents $k_{\pm j}(z)$ and calculate useful commutation relations among them [53].

Acknowledgments

I am greatly indebted to Professor H. Konno, Tokyo University of Marine Science and Technology, for his outstanding guidance, great support, enthusiastic encouragement, invaluable advice, stimulating discussions, valuable suggestions, enlightening discussions, drawing our attention to this work and comprehensive, critical reading of the manuscript. I am very fortunate to have the opportunity to study under his supervision.

I also would like to express my thanks to my supervisor Professor M. Abe, Mathematics Department, Graduate School of Science, Hiroshima University, for his helpful advice and contributions to complete this study.

I would like to express my thanks to my co-advisor Professor Y. Agaoka, Mathematics Department, Graduate School of Science, Hiroshima University, for his kindly reviewing my doctoral thesis with invaluable comments and detailed suggestion.

I am indebted to all the Professors in Mathematics Department, Graduate School of Science, Hiroshima University, for their kind help and spontaneous cooperation and inspiration during my study in Hiroshima University.

Thanks are also given to the officer in Mathematics Department, Graduate School of Science, Hiroshima University, for their kind cooperation.

I am also gratefully acknowledged the Egyptian Government, Egypt for the financial support to my study in Japan.

I finally would like to express my deep thanks to all members of my family for giving me the opportunity to study abroad, enthusiastic encouragement throughout my study period.

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Chapter 1

Introduction

In this thesis, we expose the infinite dimensional representation of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ following the results published in papers [52, 53]. In particular, we discuss the dynamical \mathcal{Z}_k -algebra structure of the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules and show that the level- k highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$ can be realized as a tensor product of the quantum dynamical \mathcal{Z}_k -module and the module of the level- k elliptic bosons. We also discuss that the irreducibility of the \mathcal{Z}_k -module leads to the irreducibility of the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module. We give the level-1 standard representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, B_l^{(1)}$.

The elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ can be equipped with a Hope algebroid structure [39]. We use the elliptic analogue of the Drinfeld coproduct [43] and the level-1 standard realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [53] to construct the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [52]. We also investigate the elliptic analogue of the condition of integrability of such representation and derive an elliptic analogue of the so called q-difference equation of certain vertex operators. We show the higher level realization of the quantum dynamical Z -algebra.

For $U_{q,p}(C_l^{(1)})$, we give an explicit construction of the fundamental weight type elliptic bosons A_m^j , the orthogonal basis type $\mathcal{E}_m^{\pm j}$, the elliptic currents $k_{\pm j}(z)$ and calculate several commutation relations among them.

The manuscript of this dissertation is divided into 5 main chapters.

Chapter 2. We review the basic notations and concepts of affine Lie algebra $\widehat{\mathfrak{g}}$, quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ and elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$.

In the first part, we recall the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. Namely, we consider the polynomial loop algebra associated to a finite-dimensional simple Lie algebra \mathfrak{g} and perform two extensions of the loop algebra by the central element c and a derivation d . The constructed algebra $\widehat{\mathfrak{g}}$ is isomorphic to an untwisted affine Lie algebra. We summarize the main structures of constructed untwisted affine Lie algebras $\widehat{\mathfrak{g}}$: a root system decomposition, Chevalley generators and the triangular decomposition of $\widehat{\mathfrak{g}}$. We also give the definition of the generalized Cartan

matrix associated with the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$.

The affine quantum group $U_q(\widehat{\mathfrak{g}})$ is exposed in the second part. We present two realization of $U_q(\widehat{\mathfrak{g}})$ whose defining relations are written down in term of Chevally generators and Drinfeld's generators, respectively. We recall the isomorphism between the Chevally realization and the Drinfeld realization. We review the coalgebra structure of $U_q(\widehat{\mathfrak{g}})$ in term of the Chevally generators as well as in term of the Drinfeld currents, the generating functions of the Drinfeld generators. We investigate a category of the level- k highest weight modules of $U_q(\widehat{\mathfrak{g}})$ in an analogous way to the classical affine Lie algebra. After that we define a quantum analogue of Lepowsky-Wilson's Z -algebra which related to the level- k $U_q(\widehat{\mathfrak{g}})$ -modules. We present the defining relations of this algebra as well as the relations between its generators and those of $U_q(\widehat{\mathfrak{g}})$. To discuss the representation of the Z -algebra and $U_q(\widehat{\mathfrak{g}})$, we define the universal quantum Z -algebra Z_k , which is independent of the level- k $U_q(\widehat{\mathfrak{g}})$ -modules and consider a category of its level- k modules. The induced $U_q(\widehat{\mathfrak{g}})$ -modules are constructed by using the Z_k -modules. By using a functor and its reverse between the categories of Z_k -modules and $U_q(\widehat{\mathfrak{g}})$ -modules, we show that the Z_k -modules determines the irreducibility of the resulted induced $U_q(\widehat{\mathfrak{g}})$ -modules. Finally, we give the level-1 irreducible $U_q(\widehat{\mathfrak{g}})$ -modules for some types of untwisted affine Lie algebras.

In the last part, we expose a definition of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ [29, 39] as a topological algebra over the ring of formal power series in p . We introduce the field \mathcal{M}_{H^*} of meromorphic functions on H^* the dual of H , a dynamical extension of the Cartan subalgebra. We introduce the level- k representation of $U_{q,p}(\widehat{\mathfrak{g}})$ as an H -algebra homomorphism. This representation is called the dynamical representation due to \mathcal{M}_{H^*} . A category of the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules is introduced.

In the following four chapters we present the main results of the thesis.

Chapter 3. Dynamical Quantum Z -algebra of $U_{q,p}(\widehat{\mathfrak{g}})$. [53]

We discuss a quantum dynamical analogue of Lepowsky and Wilson's Z -algebra associated with the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module.

First, we define the Heisenberg subalgebra $U_{q,p}(\mathcal{H})$ of $U_{q,p}(\widehat{\mathfrak{g}})$ and introduce its level- k module. Secondly, we introduce certain level- k vertex operators in $U_{q,p}(\mathcal{H})$ and their commutation relations. After that we present a definition of the dynamical quantum analogue \mathcal{Z}_γ of Lepowsky and Wilson's Z -algebra associated with level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module \mathcal{V} . We then present the universal dynamical quantum Z -algebra \mathcal{Z}_k . We define a category of the level- k \mathcal{Z}_k -modules to prepare for construction of induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules in the next chapter.

Chapter 4. Representation theory of $U_{q,p}(\widehat{\mathfrak{g}})$. [53]

We study the generic level - k representation of $U_{q,p}(\widehat{\mathfrak{g}})$ by the associated \mathcal{Z}_k representation. We construct the induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module as a tensor product of the \mathcal{Z}_k -module and the

$U_{q,p}(\mathcal{H})$ -module. This becomes an infinite dimensional representation. We discuss its irreducibility. We show that the irreducibility is governed by the \mathcal{Z}_k -module. For the level-1 i.e. $k = 1$, we present examples of the infinite dimensional irreducible representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$.

Chapter 5. Higher level realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. [52]

We study the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the associated quantum dynamical Z -algebra realizing them explicitly. We also present the condition of the integrability of such representation.

We recall the H -Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in the first part. Then we review the level-1 irreducible representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. In third part, using the elliptic analogue of the Drinfeld coproduct, we construct the level- $k + 1$ realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ as the tensor product of the level-1 modules. In the fourth part, we introduce vertex operators of the level- $k + 1$ elliptic bosons and obtain the level- $k + 1$ quantum dynamical Z -algebra. In the last part, using the Z -algebra we discuss the elliptic analogue of the integrable condition [10, 13, 46] of the constructed level- $k + 1$ $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ -modules. We also show the elliptic analogue of the q -difference equation [10] of certain level- $k + 1$ vertex operators constructed from the elliptic currents.

Chapter 6. Elliptic bosons of $U_{q,p}(C_n^{(1)})$. [53]

In this chapter we present a different type of elliptic bosons A_m^j from the (co-)roots type elliptic bosons $\alpha_{j,n}(\alpha_{j,n}^\vee)$. We give a definition of A_m^j for an arbitrary level c and construct the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ and the elliptic current $k_{\pm j}(z)$ of $U_{q,p}(C_l^{(1)})$. Then we derive various commutation relations among the orthonormal basis type elliptic bosons $\mathcal{E}_m^{\pm j}$ as well as among the elliptic currents $k_{\pm j}(z)$.

In the last chapter we summarize the main results of this thesis and discuss some open problems.

Chapter 2

Preliminary

In this chapter we review some basic objects and facts which we will use in later chapters. In section 2.1, we review the untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. In particular, we present an explicit realization of $\widehat{\mathfrak{g}}$. In section 2.2, we give a definition of the quantum affine Lie algebra $U_q(\widehat{\mathfrak{g}})$ and discuss the quantum Z -algebra structure associated with the level- k representation of $U_q(\widehat{\mathfrak{g}})$. In section 2.3, we introduce the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ as a topological algebra and a concept of dynamical representation.

2.1 Untwisted affine Lie algebra $\widehat{\mathfrak{g}}$

There are wide classes of infinite dimensional Lie algebras called Kac-Moody algebras. Among them, a class of affine Lie algebras is most interesting because it has a rich mathematical structure in geometry, in arithmetics, in algebra, in representation theory and various applications in conformal field theory, in superstring theory, in soliton theory and in integrable modules. The affine Lie algebras are obtained by applying Serre's construction to a generalized Cartan matrix $A = (a_{ij}), i, j \in \{0\} \cup I, I = \{1, \dots, l\}$ of affine type. There exists also a geometrical construction by applying two extensions, namely a central extension and an extension by a derivation, of the polynomial loop algebra associated to a finite-dimensional simple Lie algebra \mathfrak{g} with a $l \times l$ Cartan matrix $(a_{ij}), i, j \in I$. An explicit construction of untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ as a central extension of loop algebra is presented in 2.1.1. We discuss the main structures of constructed untwisted affine Lie algebras $\widehat{\mathfrak{g}}$ such as a root system and Chevalley generators in 2.1.3 and 2.1.4 respectively. The triangular decomposition of $\widehat{\mathfrak{g}}$ was showed in 2.1.5. In 2.1.6, we give the definition of the generalized Cartan matrix associated with untwisted affine Lie algebra $\widehat{\mathfrak{g}}$.

2.1.1 Explicit construction of untwisted affine Lie algebras

Here we describe a realization of untwisted affine Lie algebra. Starting from a finite-dimensional simple Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathring{A})$, considering a loop algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ and performing extensions of $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ by the central element c and a derivation d , one obtains a Lie algebra $\widehat{\mathfrak{g}}$ isomorphic to an untwisted affine Lie algebra.

2.1.2 Central extensions of loop algebras

The extension of a Lie algebra is an enlargement of it by some others Lie algebras.

Definition 2.1.1. *Let \mathfrak{g} be a finite dimensional Lie algebra with bracket $[\cdot, \cdot]_{\mathfrak{g}}$. Then the polynomial loop algebra is defined by $\mathfrak{L}(\mathfrak{g}) = \{P(t) \otimes x \mid P(t) \in \mathbb{C}[t, t^{-1}], x \in \mathfrak{g}\}$ with the bracket*

$$[P(t) \otimes x, Q(t) \otimes y]_{\mathfrak{L}(\mathfrak{g})} = P(t)Q(t) \otimes [x, y]_{\mathfrak{g}},$$

where $\mathbb{C}[t, t^{-1}]$ is the infinite dimensional associative complex algebra of Laurent polynomials in the indeterminate variable t .

We can easily prove that the polynomial loop algebra $\mathfrak{L}(\mathfrak{g})$ of the finite dimensional simple Lie algebra \mathfrak{g} is not simple. Consider the set $I \in \mathfrak{L}(\mathfrak{g})$ is given by

$$I = \{(1+t)P(t) \otimes x \mid P(t) \in \mathbb{C}[t, t^{-1}], x \in \mathfrak{g}\},$$

we find $[I, \mathfrak{L}(\mathfrak{g})] \subset I$ which means I is a proper ideal in the polynomial loop algebra $\mathfrak{L}(\mathfrak{g})$.

Definition 2.1.2. *A symmetric associative bilinear form on $\mathfrak{L}(\mathfrak{g})$ is defined by*

$$(t^m \otimes x, t^n \otimes y) := \delta_{m+n,0} \mathbf{K}(x, y),$$

where \mathbf{K} is the Cartan Killing form on the simple Lie algebra \mathfrak{g} .

Definition 2.1.3. *A derivation \mathbf{d} on $\mathfrak{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ is defined by $\mathbf{d} = t \frac{d}{dt}$. \mathbf{d} satisfies*

$$(\mathbf{d}(t^m \otimes x), t^n \otimes y) = m\delta_{m+n,0} \mathbf{K}(x, y) = -n\delta_{m+n,0} \mathbf{K}(x, y) = -(t^m \otimes x, \mathbf{d}(t^n \otimes y)).$$

Definition 2.1.4. *The universal central extension of the loop algebra $\mathfrak{L}(\mathfrak{g})$ by a one dimensional abelian Lie algebra $\mathbb{C}\mathbf{c}$ is defined by*

$$\widetilde{\mathfrak{g}} = \mathfrak{L}(\mathfrak{g}) \oplus \mathbb{C}\mathbf{c} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}\mathbf{c} \tag{2.1.1}$$

with the bracket

$$[P(t) \otimes x \oplus \mu\mathbf{c}, Q(t) \otimes y \oplus \nu\mathbf{c}]_{\widetilde{\mathfrak{g}}} = P(t)Q(t) \otimes [x, y]_{\mathfrak{g}} + (dP(t) \otimes x, Q(t) \otimes y)\mathbf{c}, \tag{2.1.2}$$

where \mathbf{c} is the central element of $\mathfrak{L}(\mathfrak{g})$.

2.1.2 Central extensions of loop algebras

The Lie algebra $\tilde{\mathfrak{g}}$ in (2.1.1) has a finite dimension abelian subalgebra $\tilde{\mathfrak{h}}$ given by

$$\tilde{\mathfrak{h}} = (1 \otimes \mathfrak{h}) \oplus \mathbb{C}\mathbf{c},$$

where \mathfrak{h} is the maximal toral subalgebra (Cartan subalgebra) of \mathfrak{g} . Since $\dim \mathfrak{h} = l$, then we have $\dim \tilde{\mathfrak{h}} = l + 1$. To determine the adjoint action of $\tilde{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$, we will use the following definition.

Definition 2.1.5. *The dual contraction $\langle \cdot, \cdot \rangle: \tilde{\mathfrak{h}}^* \times \tilde{\mathfrak{h}} \rightarrow \mathbb{C}$ is defined by*

$$\langle \alpha, 1 \otimes h \rangle := \langle \alpha, h \rangle, \quad \langle \alpha, \mathbf{c} \rangle = 0, \quad (2.1.3)$$

where $\alpha \in \tilde{\mathfrak{h}}^*$, $\tilde{\mathfrak{h}}^*$ is the dual space of $\tilde{\mathfrak{h}}$.

Applying the adjoint action of $\tilde{\mathfrak{h}}$ on $\tilde{\mathfrak{g}}$, we obtain for $\tilde{h} = 1 \otimes h + \mu\mathbf{c} \in \tilde{\mathfrak{h}}$ and $t^m \otimes x \in \tilde{\mathfrak{g}}$

$$\text{ad}_{\tilde{h}}(t^m \otimes x) = t^m \otimes [h, x]_{\mathfrak{g}}.$$

For $x \in \mathfrak{g}_{\alpha}$ ($\alpha \in \Delta \subset \mathcal{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, $\mathfrak{g}_{\alpha} = \{x \in \mathfrak{g} \mid \text{ad}(h)x = \alpha(h)x \forall h \in \mathfrak{h}\}$), we find

$$\begin{aligned} [\tilde{h}, t^n \otimes x_{\alpha}]_{\tilde{\mathfrak{g}}} &= \langle \alpha, h \rangle t^n \otimes x_{\alpha} \\ &= \langle \alpha, \tilde{h} \rangle t^n \otimes x_{\alpha}. \end{aligned}$$

While for $x = h' \in \mathfrak{h}$ yields

$$[\tilde{h}, t^n \otimes h']_{\tilde{\mathfrak{g}}} = 0.$$

It is noted that $\text{ad}_{\tilde{\mathfrak{h}}}$ does not distinguish between the powers of t . Applying the adjoint action of \mathbf{d} on $x_{\alpha} \otimes t^n$, one has

$$[\mathbf{d}, x_{\alpha} \otimes t^n] = nt^n \otimes x_{\alpha}.$$

This means \mathbf{d} distinguishes between the power n of t . To obtain the analogue of the scaling element, we will perform on $\tilde{\mathfrak{g}}$ an extension by the derivation $\mathbf{d} = t \frac{d}{dt}$. We will extend Definition 2.1.3 of \mathbf{d} on \mathfrak{g} to $\tilde{\mathfrak{g}}$ by supposing the additional condition

$$\mathbf{d}(\mathbf{c}) = 0.$$

Definition 2.1.6. *The extension of $\tilde{\mathfrak{g}}$ by a derivation \mathbf{d} is defined by*

$$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}\mathbf{d} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}(\mathbb{A}) \oplus \mathbb{C}\mathbf{c} \oplus \mathbb{C}\mathbf{d},$$

as a vector space with a bracket defined by

$$\begin{aligned} [t^m \otimes x + \mu\mathbf{c} + \nu\mathbf{d}, t^n \otimes y + \mu'\mathbf{c} + \nu'\mathbf{d}] &:= t^{m+n} \otimes [x, y]_{\mathfrak{g}} + m\delta_{m+n,0}K(x, y)\mathbf{c} \\ &\quad + \nu nt^n \otimes y - \nu' mt^m \otimes x, \end{aligned} \quad (2.1.4)$$

and

$$[c, \hat{\mathfrak{g}}] = 0, \quad [d, t^n \otimes x] = n(t^n \otimes x), \quad [c, d] = 0.$$

Then $\hat{\mathfrak{g}}$ is a Lie algebra called *untwisted affine lie algebra*.

2.1.3 Root space decomposition of $\widehat{\mathfrak{g}}$

Definition 2.1.7. The abelian subalgebra $\widehat{\mathfrak{h}}$ of $\widehat{\mathfrak{g}}$ is defined by

$$\widehat{\mathfrak{h}} = \widetilde{\mathfrak{h}} \bigoplus \mathbb{C}\mathbf{d} = (1 \otimes \mathfrak{h}) \bigoplus \mathbb{C}\mathbf{c} \bigoplus \mathbb{C}\mathbf{d},$$

with dimension $\dim \widehat{\mathfrak{h}} = \dim \widetilde{\mathfrak{h}} + 1 = l + 2$.

Definition 2.1.8. For $x_n = t^n \otimes x, y_m = t^m \otimes y \in \widehat{\mathfrak{g}}$. The non-degenerate associative symmetric bilinear form $\mathfrak{B} : \widehat{\mathfrak{g}} \times \widehat{\mathfrak{g}} \rightarrow \mathbb{C}$, called the generalized Cartan-Killing form, is defined by

$$\begin{aligned} \mathfrak{B}(x_n, y_m) &= \delta_{n+m,0} \mathbf{K}(x, y), \\ \mathfrak{B}(x_n, c) &= \mathfrak{B}(x_n, d) = 0, \quad \mathfrak{B}(c, c) = \mathfrak{B}(d, d) = 0, \\ \mathfrak{B}(c, d) &= 1 \end{aligned}$$

and by requiring \mathfrak{B} to be bilinear and symmetric.

Definition 2.1.9. Define $\nu : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^*$ by

$$\langle \nu(\widehat{h}), \widehat{h}' \rangle = \mathfrak{B}(\widehat{h}, \widehat{h}'), \quad \nu(\widehat{h}) \in \widehat{\mathfrak{h}}^*, \widehat{h}, \widehat{h}' \in \widehat{\mathfrak{h}},$$

and $\delta := \nu(c), \Lambda_0 := \nu(d)$. Then we have the following pairings

$$\langle \alpha_i, \widehat{h}_j \rangle = \langle \alpha_i, h_j \rangle, \quad \langle \Lambda_0, c \rangle = 1 = \langle \delta, d \rangle \quad (2.1.5)$$

and the other pairings are zero.

2.1.3 Root space decomposition of $\widehat{\mathfrak{g}}$

Let $\Delta \subset \mathfrak{h}^*$ be the root system of the finite dimensional simple Lie algebra \mathfrak{g} and $\{\alpha_1, \dots, \alpha_l\}$ be the root basis. One of the main features of the finite dimensional simple Lie algebra \mathfrak{g} is the eigenvectors belonging to different eigenvalues are linearly independent. Then the finite dimensional simple Lie algebra has the root space decomposition structure $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \mathcal{Q}, \alpha \neq 0} \mathfrak{g}_\alpha$ with respect to \mathfrak{h} . Insert this expression of \mathfrak{g} into a definition of affine Lie algebra $\widehat{\mathfrak{g}}$, and rewrite it in the following form

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \bigoplus \left(\bigoplus_{n \in \mathbb{Z}_{n \neq 0}} t^n \otimes \mathfrak{h} \right) \bigoplus_{\alpha \in \Delta} \left(\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}_\alpha \right)$$

We define the root system of $\widehat{\mathfrak{g}}$ by considering the adjoint action of $\widehat{\mathfrak{h}}$ on $\widehat{\mathfrak{g}}$ as follows.

For $\widehat{h} = 1 \otimes h + \mu\mathbf{c} + \nu\mathbf{d} \in \widehat{\mathfrak{h}}, x_\alpha \otimes t^n \in \widehat{\mathfrak{g}}$ by using (2.1.4) and (2.1.5) one obtains

$$\begin{aligned} \text{ad}_{\widehat{h}}(t^n \otimes x_\alpha) &= [t^n \otimes x_\alpha, \widehat{h}] \\ &= t^n \otimes [h, x_\alpha] + \nu n t^n \otimes x_\alpha \\ &= (\langle \alpha, h \rangle + \nu n) t^n \otimes x_\alpha \\ &= (\langle \alpha, 1 \otimes h + \mu\mathbf{c} \rangle + \langle n\delta, \nu\mathbf{d} \rangle) t^n \otimes x_\alpha \\ &= \langle \alpha + n\delta, \widehat{h} \rangle t^n \otimes x_\alpha. \end{aligned} \quad (2.1.6)$$

2.1.3 Root space decomposition of $\widehat{\mathfrak{g}}$

For $x = h' \in \mathfrak{h}$ by using equations (2.1.4) and (2.1.5) we obtain

$$\begin{aligned} \text{ad}\hat{h}(t^n \otimes h') &= \nu n t^n \otimes h' \\ &= \langle n\delta, 1 \otimes h + \mu \mathbf{c} + \nu \mathbf{d} \rangle t^n \otimes h' \\ &= \langle n\delta, \hat{h} \rangle t^n \otimes h'. \end{aligned}$$

Hence, $t^n \otimes x_\alpha \in \widehat{\mathfrak{g}}$ belong to $\alpha + n\delta$ and $t^n \otimes h' \in \widehat{\mathfrak{g}}$ belong to the imaginary root $n\delta$. Applying the derivation \mathbf{d} on $t^n \otimes \mathfrak{g}_\alpha$ and $t^n \otimes \mathfrak{h}$, we get

$$\begin{aligned} \mathbf{d}(t^n \otimes \mathfrak{g}_\alpha) &= n(t^n \otimes \mathfrak{g}_\alpha) \\ &= \langle n\delta, \mathbf{d} \rangle (t^n \otimes \mathfrak{g}_\alpha), \\ &= \langle \alpha + n\delta, \mathbf{d} \rangle (t^n \otimes \mathfrak{g}_\alpha), \end{aligned}$$

$$\begin{aligned} \mathbf{d}(t^n \otimes \mathfrak{h}) &= n(t^n \otimes \mathfrak{h}) \\ &= \langle n\delta, \mathbf{d} \rangle (t^n \otimes \mathfrak{h}). \end{aligned}$$

Consequently, we can define the following root spaces in $\widehat{\mathfrak{g}}$

$$\widehat{\mathfrak{g}}_{\alpha+n\delta} = t^n \otimes \mathfrak{g}_\alpha, \quad \widehat{\mathfrak{g}}_{n\delta} = t^n \otimes \mathfrak{h}, \quad (2.1.7)$$

where

$$\dim \widehat{\mathfrak{g}}_{\alpha+n\delta} = \dim \mathfrak{g}_\alpha = 1, \quad \dim \widehat{\mathfrak{g}}_{n\delta} = \dim \mathfrak{h} = l. \quad (2.1.8)$$

We conclude that $\widehat{\mathfrak{g}}$ has a root space decomposition

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{h}} \bigoplus_{n \in \mathbb{Z}_{\neq 0}} \widehat{\mathfrak{g}}_{n\delta} \bigoplus_{\alpha \in \Delta, n \in \mathbb{Z}} \widehat{\mathfrak{g}}_{\alpha+n\delta}, \quad (2.1.9)$$

with root system

$$\widehat{\Delta} = \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z}_{\neq 0}\}.$$

Definition 2.1.10. *The Heisenberg Lie algebra \mathcal{H} is a subspace of $\widehat{\mathfrak{g}}$ defined by*

$$\mathcal{H} = \bigoplus_{n \in \mathbb{Z}_{\neq 0}} (\mathfrak{h} \otimes t^n) \oplus \mathbb{C}c,$$

such that for $h_n, h'_m \in \mathcal{H}$

$$[h_n, h'_m] = n\delta_{n+m,0} \mathfrak{B}(h, h')c.$$

2.1.4 Chevalley generators of $\widehat{\mathfrak{g}}$

Definition 2.1.11. *The set of Chevalley generators of the finite dimensional simple Lie algebra \mathfrak{g} is denoted by*

$$\{e_1, \dots, e_l, f_1, \dots, f_l, h_1, \dots, h_l\}.$$

They satisfy

$$\begin{aligned} [e_i, f_j] &= \delta_{ij} h_j, \\ [h_i, e_j] &= \langle \alpha_j, h_i \rangle e_j = a_{ij} e_j, \\ [h_i, f_j] &= \langle -\alpha_j, h_i \rangle f_j = -a_{ij} f_j \quad (i, j = 1, \dots, l). \end{aligned}$$

Where $\{\mathfrak{g} \otimes t^n\}_{n \in \mathbb{Z}}$ generates $\widehat{\mathfrak{g}}$ except d . It is obvious that $\{\mathfrak{g} \otimes 1\}$, $\{\mathfrak{g} \otimes t\}$, and $\{\mathfrak{g} \otimes t^{-1}\}$ generate $\{\mathfrak{g} \otimes t^n\}_{n \in \mathbb{Z}}$. By using the root space decomposition (2.1.9) and (2.1.7) for $\alpha = \pm \alpha_i, n = 0$ we get

$$\widehat{\mathfrak{g}}_{\alpha_i} = \mathbb{C} \widehat{e}_i, \quad \widehat{\mathfrak{g}}_{-\alpha_i} = \mathbb{C} \widehat{f}_i,$$

where we set

$$\widehat{e}_i := 1 \otimes e_i, \quad \widehat{f}_i := 1 \otimes f_i \quad \widehat{h}_i := 1 \otimes h_i \quad (i = 1, \dots, l). \quad (2.1.10)$$

Let $x_{\pm\theta}$ be an element in $\widehat{\mathfrak{g}}_{\pm\theta}$ with the highest root θ of \mathfrak{g} such that

$$\mathfrak{B}(x_\theta, x_{-\theta}) = 1, [x_\theta, x_{-\theta}] = h_\theta$$

Set

$$\widehat{e}_0 := t \otimes x_{-\theta}, \quad \widehat{f}_0 := t^{-1} \otimes x_\theta \quad (2.1.11)$$

$$\widehat{h}_0 := \frac{2}{(\theta | \theta)} (c - 1 \otimes h_\theta). \quad (2.1.12)$$

Then one can check that these generators satisfy the following commutation relations

$$\begin{aligned} [\widehat{e}_0, \widehat{f}_0] &= \widehat{h}_0, \\ [\widehat{e}_i, \widehat{f}_0] &= [\widehat{f}_0, \widehat{e}_i] = 0, \\ [\widehat{e}_i, \widehat{f}_i] &= \delta_{ij} \widehat{h}_j. \end{aligned}$$

If we apply the adjoint action of $\widehat{\mathfrak{h}}$ on \widehat{e}_0 and \widehat{f}_0 , we find that \widehat{e}_0 and \widehat{f}_0 belong to the root spaces $\widehat{\mathfrak{g}}_{\delta-\theta}$ and $\widehat{\mathfrak{g}}_{-(\delta-\theta)}$ respectively. Moreover if we set

$$\alpha_0 := \delta - \theta,$$

we obtain the following commutation relations of Chevalley generators

$$\begin{aligned} [\widehat{h}_i, \widehat{e}_j] &= \langle \alpha_j, \widehat{h}_i \rangle \widehat{e}_j \quad (i = 0, \dots, l), \\ [\widehat{h}_i, \widehat{f}_j] &= -\langle \alpha_j, \widehat{h}_i \rangle \widehat{f}_j \quad (i = 0, \dots, l), \\ [\widehat{e}_i, \widehat{f}_j] &= \delta_{ij} \widehat{h}_i \quad (i, j = 0, \dots, l). \end{aligned} \quad (2.1.13)$$

2.1.5 Triangular decomposition $\hat{L}(\mathfrak{g})$

Hence we obtain the Chevalley generators $\hat{e}_i, \hat{f}_i, \hat{h}_i$ ($i = 0, 1, \dots, l$) of $\hat{\mathfrak{g}}$ and the simple roots of the Lie algebra $\hat{\mathfrak{g}}$ is $\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_l\}$.

2.1.5 Triangular decomposition $\hat{L}(\mathfrak{g})$

From the root space decomposition, we can define a positive part and negative part of the root system by

$$\hat{\Delta}_{\pm} = \Delta_{\pm} \cup \{\alpha + n\delta \mid \alpha \in \Delta, n \in \mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}.$$

where Δ_+, Δ_- are the positive part and negative part of the root system of the finite dimensional simple Lie algebra \mathfrak{g} such that $\Delta_+ = -\Delta_-, \Delta_+ \cap \Delta_- = \phi, \Delta_+ \cup \Delta_- = \Delta$.

Define

$$\hat{N}_{\pm} := \bigoplus_{\pm\gamma \in \hat{\Delta}_{\pm}} \hat{\mathfrak{g}}_{\gamma}.$$

By using the definition of the Chevalley generators of $\hat{\mathfrak{g}}$ for positive and negative parts of roots we can prove \hat{N}_+ and \hat{N}_- are generated by $\{\hat{e}_0, \hat{e}_1, \dots, \hat{e}_k\}$ and $\{\hat{f}_0, \hat{f}_1, \dots, \hat{f}_k\}$ respectively. This means

$$\begin{aligned} \hat{N}_+ &= (t\mathbb{C}[t, t^{-1}] \otimes (N_- \oplus \mathfrak{h})) \oplus (\mathbb{C}[t, t^{-1}] \otimes N_+), \\ \hat{N}_- &= (t^{-1}\mathbb{C}[t, t^{-1}] \otimes (N_+ \oplus \mathfrak{h})) \oplus (\mathbb{C}[t, t^{-1}] \otimes N_-), \end{aligned}$$

where

$$\begin{aligned} N_+ &= \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{\alpha} = \sum_{i_1, \dots, i_l} c_{i_1 i_2 \dots i_l} \{e_{i_1}, [e_{i_1}, e_{i_2}], \dots, [e_{i_1}, e_{i_2}], e_{i_l}, \dots\}, \\ N_- &= \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_{\alpha} = \sum_{i_1, \dots, i_l} d_{i_1 i_2 \dots i_l} \{f_{i_1}, [f_{i_1}, f_{i_2}], \dots, [f_{i_1}, f_{i_2}], f_{i_l}, \dots\}. \end{aligned}$$

We can find the triangular decomposition in the following form

$$\hat{\mathfrak{g}} = \hat{N}_- \oplus \hat{\mathfrak{h}} \oplus \hat{N}_+$$

2.1.6 Generalized Cartan matrix corresponding to $\hat{\mathfrak{g}}$

Definition 2.1.12. The symmetric bilinear form (inner product) on $\hat{\mathfrak{h}}^*$ of $\hat{\mathfrak{g}}$ is defined by

$$(\alpha_i \mid \alpha_j) = \mathfrak{B}(\nu^{-1}(\alpha_i), \nu^{-1}(\alpha_j)) = \frac{1}{\epsilon_j} a_{ji} = b_{ji}, \quad (\delta \mid \Lambda_0) = 0,$$

where $B = (b_{ij})_{i,j=0}^l = B^T$, and $D = \text{diag}(\epsilon_0, \epsilon_1, \dots, \epsilon_l)$, ($0 < \epsilon_i = \frac{2}{2h^\vee(\alpha_i \mid \alpha_i)}$). $h^\vee := \frac{1}{(\theta \mid \theta)}$ is called the dual coxeter number.

Theorem 2.1.13. A matrix $A = (a_{ij})_{1 \leq i, j \leq l}$, $a_{ij} = \langle \alpha_j, \hat{h}_i \rangle = \frac{2(\alpha_j \mid \alpha_i)}{(\alpha_i \mid \alpha_i)}$ associated with $\hat{\mathfrak{g}}$ have the following properties

2.2 Affine quantum group $U_q(\widehat{\mathfrak{g}})$

1. $A_{ii} = 2 \quad (i = 0, 1, 2, \dots, l).$
2. $A_{ij} \in \{-1, -2, \dots\} \quad (i \neq j).$
3. $A_{ij} = 0 \iff A_{ji} = 0.$
4. A is indecomposable,

then A is called a generalized Cartan matrix. Moreover A satisfies

5. A is singular, $\det A = 0,$
6. $r = \text{rank } A = l, \text{ corank } A = 1,$
7. A is symmetrizable, means there is a diagonal matrix D and a symmetric matrix B such that $A = DB.$

We call A the generalized Cartan matrix of affine type.

2.2 Affine quantum group $U_q(\widehat{\mathfrak{g}})$

The quantum group was independently discovered by Drinfeld [14] and Jimbo [27,28]. The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is the quantum group associated to affine Lie algebra $\widehat{\mathfrak{g}}$. The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ was initially formulated using a q -deformed version of the commutation relations for the Chevalley generators of $\widehat{\mathfrak{g}}$. The Chevalley generators are the generators associated with the simple roots of $\widehat{\mathfrak{g}}$. Later on [16], Drinfeld proposed a new realization of $U_q(\widehat{\mathfrak{g}})$ in term of the elements $\{x_{i,m}^\pm, \psi_{i,n}, \phi_{i,-n} \mid (i \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{>0})\}$ called the Drinfeld generators (Drinfeld realization), which are associated with the affine root $\widehat{\Delta}$ of $U_q(\widehat{\mathfrak{g}})$. It is convenient to introduce the generating functions $x^\pm(z) = \sum_{m \in \mathbb{Z}} x_m^\pm z^{-m}, \widehat{\phi}(z) = \sum_{m \in \mathbb{Z}} \phi_m z^{-m}, \widehat{\psi}(z) = \sum_{m \in \mathbb{Z}} \psi_m z^{-m}$ called the Drinfeld currents. The Drinfeld's realization is a quantum analogue of the loop realization of affine Lie algebra $\widehat{\mathfrak{g}}$. Drinfeld [16] stated the exact isomorphism between the Chevalley generators realization and Drinfeld realization. The proof of his statement was studied for untwisted types in [2] and for some of lower rank cases in [9,48]. Their proof towards from the quantum group à La Drinfeld and Jimbo to the Drinfeld realization. The proof of the opposite direction was discussed for both twisted and untwisted cases in [31]. There is another realization of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ by means of the L -operator satisfying the RLL relation with the R -matrix, a solution of the quantum Yang- Baxter equation [20,54]. This is called the Faddeev- Reshetikhin-Semenov-Tian Shansky- Takhtajan (FRST) formulation. An explicit isomorphism between the FRST formulation and the Drinfeld's realization was exhibited in [11].

2.2.1 Definition

In this section, we review the definitions of affine quantum group $U_q(\widehat{\mathfrak{g}})$ and its infinite dimensional representation. Also we introduce the quantum Z -algebra structure associated with level- k $U_q(\widehat{\mathfrak{g}})$ -module as a quantum analogue of the classical Lepowsky-Wilson's Z -algebra.

In 2.2.1, we present two realization of the quantum affine algebra in terms of the Chevally generators and the Drinfeld generators, respectively. Also we write down the defining relations of the quantum affine algebra in term of the Drinfeld currents. In 2.2.3 and 2.2.4, we investigate the infinite dimensional representation of $U_q(\widehat{\mathfrak{g}})$ and introduce the quantum analogue of Lepowsky-Wilson's Z -algebra associated with the level- k $U_q(\widehat{\mathfrak{g}})$ -modules respectively. We provide the Serre relations [53] (2.2.69) which are not written in [31] explicitly. We give some examples of the level-1 infinite dimensional representations of $U_q(\widehat{\mathfrak{g}})$.

2.2.1 Definition

Let $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$, $q_i = q^{d_i}$. For any integer $n > 0$, we use the following notations

$$\begin{aligned}
 [n] &= \frac{q^n - q^{-n}}{q - q^{-1}}, & [n]_i &= \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, & [n]_j &= \frac{q^n - q^{-n}}{q_j - q_j^{-1}}, \\
 [n]_i! &= [n]_i [n-1]_i \cdots [1]_i, & \begin{bmatrix} m \\ n \end{bmatrix}_i &= \frac{[m]_i!}{[n]_i! [m-n]_i!}, \\
 (x; q)_\infty &= \prod_{n=0}^{\infty} (1 - xq^n), & (x; q, t)_\infty &= \prod_{n,m=0}^{\infty} (1 - xq^n t^m), \\
 \Theta_p(z) &= (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.
 \end{aligned}$$

Definition 2.2.1. [28] Quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ associated to a symmetrizable generalized Cartan matrix A is the unital associative algebra over \mathbb{C} with generators $\widehat{e}_i, \widehat{f}_i, q^{\pm \widehat{h}_i}, q^{\pm d}$ ($i \in$

$\{0\} \cup I)$ and the following relations

$$q^{\widehat{h}_i} q^{-\widehat{h}_i} = 1 = q^{-\widehat{h}_i} q^{\widehat{h}_i}, \quad (2.2.1)$$

$$q^{\widehat{h}_i} q^{\widehat{h}_j} = q^{\widehat{h}_j} q^{\widehat{h}_i}, \quad (2.2.2)$$

$$q^d q^{-d} = 1 = q^{-d} q^d, \quad (2.2.3)$$

$$q^d q^h q^{-d} = q^{-h} \quad (2.2.4)$$

$$q^d \widehat{e}_i q^{-d} = q^{\delta_{i,0}} \widehat{e}_i \quad q^d \widehat{f}_i q^{-d} = q^{\delta_{i,0}} \widehat{f}_i \quad (2.2.5)$$

$$q^{\widehat{h}_i} \widehat{e}_j q^{-\widehat{h}_i} = q^{a_{ij}} \widehat{e}_j, \quad q^{\widehat{h}_i} \widehat{f}_j q^{-\widehat{h}_i} = q^{-a_{ij}} \widehat{f}_j, \quad (2.2.6)$$

$$[\widehat{e}_j, \widehat{f}_j] = \delta_{ij} \frac{q_i^{\widehat{h}_i} - q_i^{-\widehat{h}_i}}{q_i - q_i^{-1}} \quad (2.2.7)$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (\widehat{e}_i)^r \widehat{e}_j (\widehat{e}_i)^{1-a_{ij}-r} = 0, \quad i \neq j \quad (2.2.8)$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} (\widehat{f}_i)^r \widehat{f}_j (\widehat{f}_i)^{1-a_{ij}-r} = 0, \quad i \neq j. \quad (2.2.9)$$

Theorem 2.2.2. $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra [35] with comultiplication Δ , counit ϵ and antipode a defined on generators by

$$\Delta(q^{\widehat{h}_i}) = q^{\widehat{h}_i} \otimes q^{\widehat{h}_i} \quad (2.2.10)$$

$$\Delta(\widehat{e}_i) = \widehat{e}_i \otimes q^{\widehat{h}_i} + 1 \otimes \widehat{e}_i \quad (2.2.11)$$

$$\Delta(\widehat{f}_i) = \widehat{f}_i \otimes 1 + q^{-\widehat{h}_i} \otimes \widehat{f}_i \quad (2.2.12)$$

$$\epsilon(q^{\widehat{h}_i}) = 1, \epsilon(\widehat{e}_i) = \epsilon(\widehat{f}_i) = 0, \quad (2.2.13)$$

$$a(q^{\widehat{h}_i}) = q^{-\widehat{h}_i}, \quad (2.2.14)$$

$$a(\widehat{e}_i) = -\widehat{e}_i q^{-\widehat{h}_i}, \quad (2.2.15)$$

$$a(\widehat{f}_i) = -q^{\widehat{h}_i} \widehat{f}_i. \quad (2.2.16)$$

Definition 2.2.3. [16] [The Drinfeld generators] The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization is a unital \mathbb{C} -algebra generated by $q^{\pm \widehat{h}_i}$ ($\widehat{h}_i \in \widehat{\mathfrak{h}}$), $a_{i,n}$, $x_{i,m}^{\pm}$ ($i \in I$, $n \in \mathbb{Z}_{\neq 0}$, $m \in \mathbb{Z}$) \bar{d} and the central element c .

2.2.1 Definition

The defining relations are as follows.

$$[q_i^{\pm \widehat{h}_i}, \bar{d}] = 0, \quad [\bar{d}, a_{i,n}] = na_{i,n}, \quad [\bar{d}, x_{i,n}^{\pm}] = nx_{i,n}^{\pm}, \quad (2.2.17)$$

$$[q_i^{\pm \widehat{h}_i}, a_{j,n}] = 0, \quad q_i^{\widehat{h}_i} x_{j,m}^{\pm} = q_i^{\pm a_{ij}} x_{j,m}^{\pm} q_i^{\widehat{h}_i}, \quad (2.2.18)$$

$$[a_{i,n}, a_{j,m}] = \frac{[b_{ij}n][cn]}{n} q^{-c|n|} \delta_{n+m,0}, \quad (2.2.19)$$

$$[a_{i,n}, x_{j,m}^+] = \frac{[b_{ij}n]}{n} q^{-c|n|} x_{j,m}^+, \quad (2.2.20)$$

$$[a_{i,n}, x_{j,m}^-] = -\frac{[b_{ij}n]}{n} x_{j,m}^-, \quad (2.2.21)$$

$$x_{i,m+1}^{\pm} x_{j,n}^{\pm} - q_i^{a_{ij}} x_{j,n}^{\pm} x_{i,m+1}^{\pm} = q_i^{a_{ij}} x_{i,m}^{\pm} x_{j,n+1}^{\pm} - x_{j,n+1}^{\pm} x_{i,m}^{\pm}, \quad (2.2.22)$$

$$[x_{i,m}^+, x_{j,n}^-] = \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(q^{\frac{c|m-n|}{2}} \psi_i(m+n) - q^{\frac{c|n-m|}{2}} \phi_i(m+n) \right), \quad (2.2.23)$$

$$\sum_{\sigma \in S_a} \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i x_{i,\sigma(1)}^{\pm} \cdots x_{i,\sigma(s)}^{\pm} x_{j,n}^{\pm} x_{i,\sigma(s+1)}^{\pm} \cdots x_{i,\sigma(a)}^{\pm} = 0, \quad (2.2.24)$$

$(i \neq j, a = 1 - a_{ij}),$

where $b_{ij} = d_i a_{ij}$.

Note we set $a_{i,n} = [d_i] q^{-\frac{c}{2}} a_{i,n}^{\vee}$ ($i \in I, n \in \mathbb{Z}_{\neq 0}$) be the simple root type Drinfeld bosons, where $a_{i,n}^{\vee}$ is the simple co-root type Drinfeld bosons in [31].

There is an isomorphism between Chevalley generators and Drinfeld generators given by [16, 31]

$$\widehat{e}_i \mapsto x_{i,0}^+, \quad \widehat{f}_i \mapsto x_{i,0}^-, \quad q^{\widehat{h}_i} \mapsto q^{\widehat{h}_i}, \quad (i = 1, \dots, l) \quad (2.2.25)$$

$$\widehat{e}_0 \mapsto [x_{i_{h-1},0}^-, [x_{i_{h-2},0}^-, [\dots, [x_{i_2,0}^-, x_{i_1,1}^-]_{q^{\epsilon_1}} \dots]_{q^{\epsilon_{h-1}}}]_{q^{\epsilon_{h-2}}} q^{\frac{c}{2}} q^{-\widehat{h}_1} \dots q^{-\widehat{h}_{h-1}} \quad (2.2.26)$$

$$\widehat{f}_0 \mapsto [x_{i_{h-1},0}^+, [x_{i_{h-2},0}^+, [\dots, [x_{i_2,0}^-, x_{i_1,1}^-]_{q^{\epsilon_1}} \dots]_{q^{\epsilon_{h-1}}}]_{q^{\epsilon_{h-2}}} \quad (2.2.27)$$

$$q^{\widehat{h}_0} \mapsto q^{\frac{c}{2}} q^{-\widehat{h}_1} \dots q^{-\widehat{h}_{h-1}}, \quad q^{\bar{d}} \mapsto q^{\bar{d}}. \quad (2.2.28)$$

where $\epsilon_{i_{h-2}} = (\alpha_{i_1} + \dots + \alpha_{i_{h-2}} | \alpha_{i_{h-1}}) \in \mathbb{Q}_{\leq 0}$ and h is the Coxeter number of the Lie algebra \mathfrak{g} . $h = l + 1$ for A_l , $= 2l$ for B_l, C_l , $= 2(l - 1)$ for D_l , $= 12$ for E_6 , $= 18$ for E_7 , $= 30$ for E_8 , $= 12$ for F_4 , $= 6$ for G_2 .

Extension of the comultiplication of Chevalley generators to Drinfeld's generators by using Drinfeld isomorphism [16] was studied in [7, 31] which give partial information and sufficient for the purpose of use, you can see [32, 33, 44].

The defining relations of $U_q(\widehat{\mathfrak{g}})$ in Definition 2.2.3 can be written in terms of Drinfeld currents

2.2.2 Co-algebra structure

$x^\pm(z), \psi(z), \phi(z)$ in a formal variable z [23]. We set

$$x_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^{-m}, \quad (2.2.29)$$

$$\psi_i(z) = q_i^{\widehat{h}_i} \exp \left((q - q^{-1}) \sum_{n>0} a_{i,n} z^{-n} \right), \quad (2.2.30)$$

$$\varphi_i(z) = q_i^{-\widehat{h}_i} \exp \left(-(q - q^{-1}) \sum_{n>0} a_{i,-n} z^n \right). \quad (2.2.31)$$

The defining relations are as follows.

$$[q_i^{\pm \widehat{h}_i}, \bar{d}] = 0, \quad [\bar{d}, a_{i,n}] = na_{i,n}, \quad [\bar{d}, x_{i,n}^\pm] = nx_{i,n}^\pm, \quad (2.2.32)$$

$$q_i^{\widehat{h}_i} x_j^\pm(z) = q_i^{\pm a_{ij}} x_j^\pm(z) q_i^{\widehat{h}_i}, \quad (2.2.33)$$

$$[q_i^{\pm \widehat{h}_i}, a_{j,n}] = 0, \quad (2.2.34)$$

$$[a_{i,n}, a_{j,m}] = \frac{[b_{ij}n][cn]}{n} q^{-c|n|} \delta_{n+m,0}, \quad (2.2.35)$$

$$[a_{i,n}, x_j^+(z)] = \frac{[b_{ij}n]}{n} q^{-c|n|} z^n x_j^+(z), \quad (2.2.36)$$

$$[a_{i,n}, x_j^-(z)] = -\frac{[b_{ij}n]}{n} z^n x_j^-(z), \quad (2.2.37)$$

$$(z - q^{\pm b_{ij}} w) x_i^\pm(z) x_j^\pm(w) = (q^{\pm b_{ij}} z - w) x_j^\pm(w) x_i^\pm(z), \quad (2.2.38)$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-k} \frac{z}{w}) \psi_i(q^{k/2} w) - \delta(q^k \frac{z}{w}) \varphi_i(q^{-k/2} w) \right), \quad (2.2.39)$$

$$\sum_{\sigma \in S_a} \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(s)}) x_j^\pm(w) x_i^\pm(z_{\sigma(s+1)}) \cdots x_i^\pm(z_{\sigma(a)}) = 0, \quad (2.2.40)$$

$(i \neq j, a = 1 - a_{ij}).$

2.2.2 Co-algebra structure

The co-algebra structure of $U_q(\widehat{\mathfrak{g}})$ is defined by the algebra homomorphism Δ : coproduct and ε : counite over \mathbb{C} such that they satisfy the axioms of coassociativity and counit. The formulas Δ, ε endow $U_q(\widehat{\mathfrak{g}})$ with a Hopf algebra structure [35] if they preserve the defining relation of $U_q(\widehat{\mathfrak{g}})$. Drinfeld proposed a coproduct formula based on the current formulation. This coproduct is called the Drinfeld coproduct [10].

Theorem 2.2.4. [15] $U_q(\mathfrak{g})$ has a Hopf algebra structure, which is given by the following

2.2.3 Infinite dimensional representation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$

formulas

$$\Delta(q^c) = q^c \otimes q^c, \quad (2.2.41)$$

$$\Delta(\psi_i(z)) = \psi_i(q^{\frac{c(2)}{2}} z) \otimes \psi_i^\pm(q^{-\frac{c(1)}{2}} z) \quad (2.2.42)$$

$$\Delta(\phi_i(z)) = \phi_i(q^{-\frac{c(2)}{2}} z) \otimes \phi_i(q^{\frac{c(1)}{2}} z) \quad (2.2.43)$$

$$\Delta(x^+(z)) = x^+(q^{-c(2)} z) \otimes \phi(q^{-\frac{c(2)}{2}} z) + 1 \otimes x^+(z) \quad (2.2.44)$$

$$\Delta(x^-(z)) = x^-(z) \otimes 1 + \psi(q^{-\frac{c(1)}{2}} z) \otimes x^-(zq^{-c(1)}) \quad (2.2.45)$$

$$\varepsilon(q^c) = 1, \varepsilon(\psi(z)) = \varepsilon(\phi(z)) = 1 \quad (2.2.46)$$

$$\varepsilon(x^+(z)) = \varepsilon(x^-(z)) = 0, \varepsilon(\alpha_n) = 0 \quad (2.2.47)$$

$$a(q^c) = q^{-c}, a(\psi(z)) = \psi(z)^{-1}, a(\phi(z)) = \phi(z)^{-1} \quad (2.2.48)$$

$$a(x^+(z)) = -\phi(zq^{\frac{c}{2}})^{-1} x^+(q^c z) \quad (2.2.49)$$

$$a(x^-(z)) = -x^-(q^c z) \psi(zq^{\frac{c}{2}})^{-1}. \quad (2.2.50)$$

This Δ is called the Drinfeld coproduct. The maps Δ, ε, a satisfy the following relations

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad (2.2.51)$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta \quad (2.2.52)$$

$$m \circ (id \otimes a) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \quad \forall x \in U_q(\mathfrak{g}) \quad (2.2.53)$$

$$m \circ (a \otimes id) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in U_q(\mathfrak{g}), \quad (2.2.54)$$

where $m : U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is the algebra multiplication.

2.2.3 Infinite dimensional representation of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$

Definition 2.2.5. For $k \in \mathbb{C}$, we say that a $U_q(\widehat{\mathfrak{g}})$ -module V has level k if c act as the scalar k on it.

Definition 2.2.6. For $\omega \in \mathbb{C}$, we set

$$V_\omega = \{v \in V \mid -d \cdot v = \omega v\}$$

and we call V_ω the space of elements homogeneous of degree ω . We also say that $X \in \text{End}V$ is homogeneous of degree $\omega \in \mathbb{C}$ if

$$[-\bar{d}, X] = \omega X$$

Definition 2.2.7. For $k \in \mathbb{C}$, a $U_q(\widehat{\mathfrak{g}})$ -module $V(\lambda)$ is called the level- k highest weight module with the highest weight λ , if there exists a vector $v \in V(\lambda)$ such that

$$\begin{aligned} \mathcal{V}(\lambda) &= U_q(\widehat{\mathfrak{g}}) \cdot v, & N_+ \cdot v &= 0, \\ c \cdot v &= kv, & q^h \cdot v &= q^{\langle \lambda, h \rangle} v. \end{aligned}$$

2.2.4 Quantum Z -algebras

We define the category C_k in the analogous way to the classical affine Lie algebra case [47].

Definition 2.2.8. For $k \in \mathbb{C}$, C_k is the full subcategory of the category of $U_q(\widehat{\mathfrak{g}})$ -modules consisting of those modules V such that

(i) V has level k

(ii) $V = \bigsqcup_{\omega \in \mathbb{C}} V_\omega$

(iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $V_{\omega+n} = 0$.

2.2.4 Quantum Z -algebras

The quantum analogue of Lepowsky- Wilson's Z -algebras and applications to the representations of $U_q(\widehat{\mathfrak{g}})$ were partially investigated in [4,5,30,31,39,50]. Here we review this quantum Z -algebra. We provide the Serre relations [53] (2.2.69) which are not written in [31] explicitly.

Definition 2.2.9. Let $U_q(\widehat{\mathfrak{h}})$ be the quantum Heisenberg algebra generated by $\{a_{i,n}, q^{\pm \frac{c}{2}}, n \in \mathbb{Z}_{\neq 0}\}$ with relations (2.2.35).

Let $U_q(\widehat{\mathfrak{h}}^+)$ (resp. $U_q(\widehat{\mathfrak{h}}^-)$) be the commutative subalgebras of $U_q(\widehat{\mathfrak{h}})$ generated by $\{a_{i,n}, q^{\pm \frac{c}{2}}, n \in \mathbb{Z}_{>0}\}$ (resp. $\{a_{i,-n}, q^{\pm \frac{c}{2}}, n \in \mathbb{Z}_{>0}\}$). By the quantum analogue of Poincare-Birkhoff-Witt theorem for $U_q(\widehat{\mathfrak{h}})$, we have

$$U_q(\widehat{\mathfrak{h}}) = U_q(\widehat{\mathfrak{h}}^+)U_q(\widehat{\mathfrak{h}}^-)$$

Let $\mathbb{C}1_k$ be the one-dimensional $U_q(\widehat{\mathfrak{h}}_+)$ -module generated by the vacuum vector 1_k defined by

$$q^{\pm \frac{c}{2}} \cdot 1_k = q^{\pm \frac{k}{2}} 1_k \quad a_{i,n} \cdot 1_k = 0 \quad (n > 0).$$

Then we have the induced $U_q(\widehat{\mathfrak{h}})$ -module

$$F_{a,k} = U_q(\widehat{\mathfrak{h}}) \otimes_{U_q(\widehat{\mathfrak{h}}_+)} \mathbb{C}1_k.$$

We identify $F_{a,k}$ with a polynomial ring $\mathbb{C}[a_{i,-m} \ (i \in I, m > 0)]$ by

$$\begin{aligned} q^{\pm \frac{c}{2}} \cdot u &= q^{\pm \frac{k}{2}} u, & a_{i,-n} \cdot u &= a_{i,-n} u, \\ a_{i,n} \cdot u &= \sum_j \frac{[a_{ij}n][kn]}{n} q^{-kn} \frac{\partial}{\partial a_{j,-n}} u \quad (n > 0) \end{aligned}$$

for $u \in \mathbb{C}[a_{i,-m} \ (i \in I, m > 0)]$. This means that $U_q(\widehat{\mathfrak{h}}^-)$ is a canonical $U_q(\widehat{\mathfrak{h}})$ -module.

Definition 2.2.10. Let $k \in \mathbb{C}$ and $(V, \bar{\pi}) \in C_k$. We call $\bar{\pi}U_q(\mathcal{H}) \subset \text{End}V$ the level- k Heisenberg algebra. We define the following vertex operators in $\text{End}V[[z, z^{-1}]]$ in $U_q(\widehat{\mathfrak{h}})$ -module.

$$E_{\pm}^{\pm}(a_j, z) = \exp \left(\mp \sum_{n \geq 1} \frac{\bar{\pi}(a_{j,-n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^n \right), \quad E_{\pm}^{\pm}(a_j, z) = \exp \left(\pm \sum_{n > 0} \frac{\bar{\pi}(a_{j,n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^{-n} \right).$$

These vertex operators $E_{-}^{\pm}(a_j, z), E_{+}^{\pm}(a_j, z)$ satisfy the following relations.

Proposition 2.2.11. [31]

$$E_{+}^{\pm}(a_i, z)E_{-}^{\pm}(a_j, w) = \frac{(q^{-b_{ij}+k\mp k}w/z; q^{2k})_{\infty}}{(q^{b_{ij}+k\mp k}w/z; q^{2k})_{\infty}} E_{-}^{\pm}(a_j, w)E_{+}^{\pm}(a_i, z), \quad (2.2.55)$$

$$E_{+}^{\pm}(a_i, z)E_{-}^{\mp}(a_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_{\infty}}{(q^{-b_{ij}+k}w/z; q^{2k})_{\infty}} E_{-}^{\mp}(a_j, w)E_{+}^{\pm}(a_i, z), \quad (2.2.56)$$

$$E_{-}^{\pm}(a_i, z)E_{-}^{\mp}(a_i, q^{\mp k}) = E_{+}^{\pm}(a_i, z)E_{+}^{\mp}(a_i, z^{\pm k}) = 1, \quad (2.2.57)$$

$$E_{-}^{\pm}(a_i, z)^{-1}E_{-}^{\mp}(a_i, zq^k) = q_i^{\widehat{h}}\phi(zq^k), \quad (2.2.58)$$

$$E_{+}^{\pm}(a_i, z)^{-1}E_{+}^{\mp}(a_i, zq^{-k}) = q_i^{-\widehat{h}}\psi(zq^{-k}). \quad (2.2.59)$$

$$E_{+}^{\pm}(a_i, z)x_j^{\mp}(w) = \frac{(q^{\pm b_{ij}+\frac{1\mp 1}{2}k}(w/z); q^{2k})_{\infty}}{(q^{\mp b_{ij}+\frac{1\mp 1}{2}k}(w/z); q^{2k})_{\infty}} x_j^{\mp}(w)E_{+}^{\pm}(a_i, z), \quad (2.2.60)$$

$$E_{-}^{\pm}(a_i, z)x_j^{\mp}(w) = \frac{(q^{\mp b_{ij}+\frac{1\mp 1}{2}k}(z/w); q^{2k})_{\infty}}{(q^{\pm b_{ij}+\frac{1\mp 1}{2}k}(z/w); q^{2k})_{\infty}} x_j^{\mp}(w)E_{-}^{\pm}(a_i, z), \quad (2.2.61)$$

$$E_{+}^{\pm}(a_i, z)x_j^{\mp}(w) = \frac{(q^{\mp b_{ij}+k+\frac{1\mp 1}{2}k}(w/z); q^{2k})_{\infty}}{(q^{\pm b_{ij}+k+\frac{1\mp 1}{2}k}(w/z); q^{2k})_{\infty}} x_j^{\mp}(w)E_{+}^{\pm}(a_i, z), \quad (2.2.62)$$

$$E_{-}^{\pm}(a_i, z)x_j^{\mp}(w) = \frac{(q^{\pm b_{ij}+k+\frac{1\mp 1}{2}k}(z/w); q^{2k})_{\infty}}{(q^{\mp b_{ij}+k+\frac{1\mp 1}{2}k}(z/w); q^{2k})_{\infty}} x_j^{\mp}(w)E_{-}^{\pm}(a_i, z). \quad (2.2.63)$$

Definition 2.2.12. The Z -operators associated with level- k $U_q(\widehat{\mathfrak{g}})$ -module V is defined by

$$Z_i^{\pm}(z; V) = E_{-}^{\pm}(a_i, z)x_i^{\pm}(z)E_{+}^{\pm}(a_i, z).$$

for $i \in I$. The coefficients $Z_{i,n}^{\pm}(V)$ of $Z_i^{\pm}(z; V) = \sum_{n \in \mathbb{Z}} Z_{i,n}^{\pm}(V)z^{-n}$ in z are well defined elements in $\text{End}_{\mathbb{C}}V$.

From the defining relations of $U_q(\widehat{\mathfrak{g}})$, we obtain the following relations of the quantum Z operators.

Theorem 2.2.13. [31, 53] The Z -operators $Z_i^{\pm}(z; V)$ satisfy the following relations.

$$[\bar{d}, Z_j^{\pm}(z; V)] = -z \frac{\partial}{\partial z} Z_j^{\pm}(z; V), \quad (2.2.64)$$

$$[a_{i,m}, Z_j^{\pm}(w; V)] = 0, \quad (2.2.65)$$

$$q^{\pm \widehat{h}_i} Z_j^{\mp}(z; V) = q^{\mp b_{ij}} Z_j^{\mp}(z; V) q^{\pm \widehat{h}_i}, \quad q^{\pm \widehat{h}_i} Z_j^{\mp}(z; V) = q^{\pm b_{ij}} Z_j^{\mp}(z; V) q^{\pm \widehat{h}_i}, \quad (2.2.66)$$

$$z \frac{(q^{-b_{ij}}w/z; q^{2k})_{\infty}}{(q^{b_{ij}+2k}w/z; q^{2k})_{\infty}} Z_i^{\pm}(z; V) Z_j^{\pm}(w; V) = -w \frac{(q^{-b_{ij}}z/w; q^{2k})_{\infty}}{(q^{b_{ij}+2k}z/w; q^{2k})_{\infty}} Z_j^{\pm}(w; V) Z_i^{\pm}(z; V), \quad (2.2.67)$$

$$\begin{aligned} & \frac{(q^{b_{ij}+k}w/z; q^{2k})_{\infty}}{(q^{-b_{ij}+k}w/z; q^{2k})_{\infty}} Z_i^{\mp}(z; V) Z_j^{\mp}(w; V) - \frac{(q^{b_{ij}+k}z/w; q^{2k})_{\infty}}{(q^{-b_{ij}+k}z/w; q^{2k})_{\infty}} Z_j^{\mp}(w; V) Z_i^{\mp}(z; V) \\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(q^{-\widehat{h}_i} \delta(q^{-k}z/w) - q^{\widehat{h}_i} \delta(q^kz/w) \right), \end{aligned} \quad (2.2.68)$$

2.2.5 The universal algebra Z_k

$$\begin{aligned}
& \sum_{\sigma \in S_a} \prod_{1 \leq m < l \leq a} \frac{(q^{2+k\mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_{\infty}}{(q^{-2+k\mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_{\infty}} \\
& \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix} \prod_{1 \leq i \leq s} \frac{(q^{-b_{ij}+k\mp k} w/z_{\sigma(i)}; q^{2k})_{\infty}}{(q^{b_{ij}+k\mp k} w/z_{\sigma(i)}; q^{2k})_{\infty}} \prod_{s+1 \leq i \leq a} \frac{(q^{-b_{ij}+k\mp k} z_{\sigma(i)}/w; q^{2k})_{\infty}}{(q^{b_{ij}+k\mp k} z_{\sigma(i)}/w; q^{2k})_{\infty}} \\
& \times Z_i^{\pm}(z_{\sigma(1)}; V) \cdots Z_i^{\pm}(z_{\sigma(s)}; V) Z_j^{\pm}(w; V) Z_i^{\pm}(z_{\sigma(s+1)}; V) \cdots Z_i^{\pm}(z_{\sigma(a)}; V) = 0 \\
& (i \neq j, a = 1 - a_{ij}). \tag{2.2.69}
\end{aligned}$$

Remark. This theorem is essentially due to Jing [31]. However, in [31] Serre relations are not written.

Definition 2.2.14. For $k \in \mathbb{C}^{\times}$ and $(V, \bar{\pi}) \in C_k$, we call the subalgebra of $\text{End}_{\mathbb{C}} V$ generated by $Z_{i,m}^{\pm}(V)$, $q_i^{\pm h_i}$ ($i \in I, m \in \mathbb{Z}$) and \bar{d} the quantum Z -algebra Z_V associated with $(V, \bar{\pi})$.

2.2.5 The universal algebra Z_k

Using the relations in Theorem 2.2.13, we define the universal quantum Z -algebra as follows.

Definition 2.2.15. Let $Z_{i,m}^{\pm}$ ($i \in I, m \in \mathbb{Z}$) be abstract symbols. We set $Z_i^{\pm}(z) = \sum_{m \in \mathbb{Z}} Z_{i,m}^{\pm} z^{-m}$. We define the universal quantum Z -algebra Z_k to be a topological algebra over $\mathbb{C}[[q^{2k}]]$ generated by $Z_{i,m}^{\pm}, q_i^{\pm h_i}$ ($i \in I, m \in \mathbb{Z}$), \bar{d} subject to the relations in Theorem 2.2.13.

Note that for $(V, \bar{\pi}) \in C_k$ we extend $\bar{\pi}$ to the map $\bar{\pi} : Z_k \rightarrow \text{End} V$ by $\bar{\pi}(Z_{i,m}^{\pm}) = Z_{i,m}^{\pm}(V)$. Then V is a Z_k -module by $\bar{\pi}$.

Definition 2.2.16. For $k \in \mathbb{C}^{\times}$, we denote by D_k the full subcategory of the category of Z_k -modules consisting of those modules $(W, \bar{\sigma})$ such that

- (i) W has level k .
- (ii) $W = \bigsqcup_{\omega \in \mathbb{C}} W_{\omega}$, where $W_{\omega} = \{w \in W \mid -\bar{\sigma}(\bar{d})w = \omega w\}$
- (iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $W_{\omega+n} = 0$.

Let us consider $(V, \bar{\pi}) \in C_k$. Following Lepowsky and Wilson [47], we define the vacuum space $\bar{\Omega}_V$ by

$$\bar{\Omega}_V = \{v \in V \mid \bar{\pi}(a_{i,n})v = 0 \quad \forall i \in I, n \in \mathbb{Z}_{>0}\}.$$

From Theorem 2.2.13, $\bar{\Omega}_V$ is stable under the action of Z_V . For a morphism $\bar{f} : V \rightarrow V'$ in C_k , we have

$$\bar{f}(\bar{\Omega}_V) \subset \bar{\Omega}_{V'}.$$

2.2.6 The functor $\bar{\Lambda}$

Proposition 2.2.17. *For $(V, \bar{\pi}) \in C_k$, there is a unique representation $\bar{\sigma}$ of Z_k on $\bar{\Omega}_V$ such that $(\bar{\Omega}_V, \bar{\sigma}) \in D_k$,*

$$\bar{\sigma}(q^{\pm \hat{h}_i}) = \bar{\pi}(q^{\pm \hat{h}_i}), \quad \bar{\sigma}(Z_{i,m}^{\pm}) = Z_{i,m}^{\pm}(V) \quad \forall i \in I, m \in \mathbb{Z}.$$

We hence define a functor $\bar{\Omega} : C_k \rightarrow D_k$ by

$$\bar{\Omega}(V, \bar{\pi}) = (\bar{\Omega}_V, \bar{\sigma}), \quad \bar{\Omega}(\bar{f}) = \bar{f}|_{\bar{\Omega}_V} : \bar{\Omega}_V \rightarrow \bar{\Omega}_{V'}.$$

2.2.6 The functor $\bar{\Lambda}$

We define a reverse functor $\bar{\Lambda} : D_k \rightarrow C_k$ as follows. Let $(W, \bar{\sigma}) \in D_k$ be a Z_k -module. We define $U_q(\widehat{\mathfrak{h}})$ -module $\text{Ind } W$ by requiring $a_{i,m} \cdot W = 0$ and

$$\text{Ind } W = U_q(\widehat{\mathfrak{h}}) \otimes_{U_q(\widehat{\mathfrak{h}}_+)} W.$$

Let $F_{\alpha,k}$ be the level- k Fock module defined in sec.2.2.4 We have a natural isomorphism $F_{\alpha,k} \otimes_{\mathbb{C}} W \cong \text{Ind } W$ by $(u \otimes 1_k) \otimes w \mapsto u \otimes w$ [47]. We thus identify the $U_q(\widehat{\mathfrak{h}})$ -module $\text{Ind } W$ with $F_{\alpha,k} \otimes_{\mathbb{C}} W$, with the action $\bar{\pi}$ of $U_q(\widehat{\mathfrak{h}})$

$$\bar{\pi}(c) = 1 \otimes c, \quad \bar{\pi}(q^{\pm \hat{h}_i}) = 1 \otimes \bar{\sigma}(q^{\pm \hat{h}_i}), \quad \bar{\pi}(a_{i,m}) = a_{i,m} \otimes 1.$$

For $(W, \bar{\sigma}) \in D_k$ and $\text{Ind } W = F_{\alpha,k} \otimes_{\mathbb{C}} W$, we define $x_j^{+'}(z), x_j^{-'}(z) \in \text{End } W[[z, z^{-1}]]$ by

$$\begin{aligned} x_j^{+'}(z) &= E^-(a_j, z)^{-1} E^+(a_j, z)^{-1} \otimes \bar{\sigma}(Z_j^+(z)), \\ x_j^{-'}(z) &= E^-(a'_j, z)^{-1} E^+(a'_j, z)^{-1} \otimes \bar{\sigma}(Z_j^-(z)). \end{aligned}$$

These are well-defined elements of $\text{End } W[[z, z^{-1}]]$. By a similar argument to the proof of Theorem 2.2.13 one can show that $x_j^{+'}(z)$ and $x_j^{-'}(z)$ satisfy the defining relations of $U_q(\widehat{\mathfrak{g}})$ with $c = k$. We hence extend $\bar{\pi} : U_q(\widehat{\mathfrak{h}}) \rightarrow \text{End } W$ to $\bar{\pi} : U_q(\widehat{\mathfrak{g}}) \rightarrow \text{End } W$ as follows

$$\begin{aligned} \bar{\pi}(x_j^+(z)) &= x_j^{+'}(z), \quad \bar{\pi}(x_j^-(z)) = x_j^{-'}(z), \\ \bar{\pi}(\bar{d}) &= \bar{d} \otimes 1 + 1 \otimes \bar{\sigma}(\bar{d}). \end{aligned}$$

By construction, the latter map is uniquely determined.

Proposition 2.2.18. *For $(W, \bar{\sigma}) \in D_k$, there is a unique level- k $U_q(\widehat{\mathfrak{g}})$ -module $(\text{Ind } W, \bar{\pi}) \in C_k$.*

We thus reach the following definition.

Definition 2.2.19. *We define a functor $\bar{\Lambda} : D_k \rightarrow C_k$ by*

$$(i) \quad \bar{\Lambda}(W, \bar{\sigma}) = (\text{Ind } W, \bar{\pi})$$

2.2.7 Level-1 representations

(ii) For a morphism $\bar{f} : W \rightarrow W'$ in D_k , define $\bar{\Lambda}(\bar{f}) : \text{Ind } W \rightarrow \text{Ind } W'$ to be the induced $U_q(\widehat{\mathfrak{h}})$ -module map. Then $\bar{\Lambda}(\bar{f})$ is a $U_q(\widehat{\mathfrak{g}})$ -module map.

We obtain the following theorem analogously to the case of the affine Lie algebras [47].

Theorem 2.2.20. For $k \in \mathbb{C}^\times$, the two categories C_k and D_k are equivalent by the functors $\bar{\Omega} : C_k \rightarrow D_k$ and $\bar{\Lambda} : D_k \rightarrow C_k$. In particular, the level- k $U_q(\widehat{\mathfrak{g}})$ -module $\text{Ind } W = F_{\alpha,k} \otimes_C W \in C_k$ is irreducible if and only if $W \in D_k$ is an irreducible Z_k -module.

2.2.7 Level-1 representations

We here give some examples of the level-1 irreducible induced representations of $U_q(\widehat{\mathfrak{g}})$ of types $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$.

The simply laced case ADE :

Let $\mathbb{C}[\mathcal{Q}]$ be the group algebra of the root lattice $\mathcal{Q} = \oplus_i \mathbb{Z}\alpha_i$ with the central extension:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i|\alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad (i, j \in I).$$

Let us consider the fundamental weight Λ_a of $\widehat{\mathfrak{g}}$ with $0 \leq a \leq l$ for $A_l^{(1)}$, $a = 0, 1, l-1, l$ for $D_l^{(1)}$, $a = 0, 1, 2$ for $E_6^{(1)}$, $a = 0, 1$ for $E_7^{(1)}$, $a = 0$ for $E_8^{(1)}$.

Theorem 2.2.21. [23, 31] An inequivalent set of the level-1 irreducible $Z_1(\widehat{\mathfrak{g}})$ -module is given by $W(\Lambda_a) = e^{\bar{\Lambda}_a} \mathbb{C}[\mathcal{Q}]$, on which the actions of $Z_j^\pm(z)$ are given by

$$Z_j^\pm(z) = e^{\pm\alpha_j} z^{\pm h_j + 1} \quad (2.2.70)$$

with

$$e^{\alpha_i} e^{\alpha_j} = e^{\alpha_i + \alpha_j}, \quad z^{\pm h_i} e^{\pm\alpha_j} e^{\bar{\Lambda}_a} = z^{\pm(\alpha_i^\vee, \alpha_j + \bar{\Lambda}_a)} e^{\pm\alpha_j} e^{\bar{\Lambda}_a} \quad (i, j \in I).$$

The $B_l^{(1)}$ case

We follow the work [45] and its quantum analogues [3, 33] with a slight modification in the Ramond sector according to [26]. We take $d_i = 1$ ($1 \leq i \leq l-1$) and $d_l = 1/2$. Let e^{α_i} ($i \in I$) be the generators of the group algebra $\mathbb{C}[\mathcal{Q}]$ with the following central extension.

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} e^{\alpha_j} e^{\alpha_i}$$

As before we regard h_i ($i \in I$) as an operator such that

$$z^{\pm h_i} e^{\alpha_j} = z^{\pm(\alpha_i^\vee, \alpha_j)} e^{\alpha_j} z^{\pm h_i}.$$

2.2.7 Level-1 representations

We also need the Neveu-Schwartz (NS) fermion $\{\Psi_n | n \in \mathbb{Z} + \frac{1}{2}\}$ and the Ramond (R) fermion $\{\Psi_n | n \in \mathbb{Z}\}$ satisfying the following anti-commutation relations.

$$\{\Psi_m, \Psi_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$$

with $\mathcal{N} = 1/(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-\frac{1}{2}}, \Psi_{-\frac{3}{2}} \cdots], \quad \tilde{\mathcal{F}}^R = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \dots]$$

and their submodules $\mathcal{F}_{even}^{NS,R}$ (reps. $\mathcal{F}_{odd}^{NS,R}$) generated by the even (reps. odd) number of Ψ_{-m} 's. One should note that for the R fermion $\Psi_0^2 = \mathcal{N}$ and $\{\Psi_m, \Psi_0\} = 0$ for $m \neq 0$. So we have two degenerate vacuum states 1 and $\Psi_0 1$. We hence consider the extended space

$$\hat{\mathcal{F}}^R = \tilde{\mathcal{F}}^R \otimes \mathbb{C}^2$$

and realize the R -fermions by

$$\hat{\Psi}_m = \Psi_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \hat{\Psi}_0 = \mathcal{N}^{\frac{1}{2}} (1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that $\{\hat{\Psi}_m, \hat{\Psi}_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$. We set

$$\mathcal{F}^R = \mathcal{F}_{even}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}_{odd}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of Ψ_m on \mathcal{F}^{NS} is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_m \cdot u = \{\Psi_m, u\} \quad (m \in \mathbb{Z}_{>0}),$$

where $u \in \mathcal{F}^{NS}$, whereas $\hat{\Psi}_m$ acts on \mathcal{F}^R as

$$\begin{aligned} \hat{\Psi}_{-m} \cdot u \otimes v &= \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), & \hat{\Psi}_0 \cdot u \otimes v &= u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v, \\ \hat{\Psi}_m \cdot u \otimes v &= \{\Psi_m, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \end{aligned}$$

where $u \in \tilde{\mathcal{F}}^R$, $v \in \mathbb{C}^2$.

Let us define the fermion fields $\Psi^{NS}(z)$ and $\Psi^R(z)$ by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_n z^{-n}, \quad \Psi^R(z) = \sum_{n \in \mathbb{Z}} \hat{\Psi}_n z^{-n}.$$

One can derive the following operator product expansions.

$$\Psi(z)\Psi(w) =: \Psi(z)\Psi(w) : + \langle \Psi(z)\Psi(w) \rangle,$$

where

$$\langle \Psi(z)\Psi(w) \rangle = \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \mathcal{N} \frac{(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases}$$

Then the quantum Z -algebra $Z_1(B_l^{(1)})$ is realized as follows [31].

$$\begin{aligned} Z_i^\pm(z) &= e^{\pm\alpha_i} z^{\pm h_i + 1} \quad (1 \leq i \leq l-1), \\ Z_l^\pm(z) &= \frac{1}{\mathcal{N}^{1/2}} \Psi(z) e^{\pm\alpha_l} z^{\pm d_l h_l + d_l}. \end{aligned}$$

There are three irreducible $Z_1(B_l^{(1)})$ -modules given by

$$\begin{aligned} W(\Lambda_0) &= \mathcal{F}_{\text{even}}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] \oplus \mathcal{F}_{\text{odd}}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1}, \\ W(\Lambda_1) &= \mathcal{F}_{\text{even}}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1} \oplus \mathcal{F}_{\text{odd}}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0], \\ W(\Lambda_l) &= \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}] e^{\bar{\Lambda}_l} \cong \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_l} \oplus \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0] e^{\bar{\Lambda}_1 + \bar{\Lambda}_l}, \end{aligned}$$

where \mathcal{Q}_0 denotes the sublattice of \mathcal{Q} generated by the long roots.

2.3 Elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$

The elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ is an elliptic analogue [39] [29] of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization [16]. In 1998, Konno [39] introduced the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_2)$ as an elliptic analogue of $U_q(\widehat{\mathfrak{sl}}_2)$ to formulate the fusion SOS models. Then Jimbo, Konno, Odaka, Shiraishi [29] gave a constrictive definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, they modified the Drinfeld currents of $U_q(\widehat{\mathfrak{sl}}_2)$ and got a set of elliptic currents. The defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [29, 39] are expressed in term of the elliptic currents and Jacobi elliptic theta function $\vartheta(u), z = q^{2u}$. Farghly, Konno and Oshima [53] introduced a p -adic topology to $U_{q,p}(\widehat{\mathfrak{g}})$ and made the defining relations in the elliptic currents well defined. We treat the relations as formal Laurent series, where coefficients are usually infinite series of the generators. The p -adic topology let them well defined by a completion.

$U_{q,p}(\widehat{\mathfrak{g}})$ has two known coalgebra structures [42, 43] as H -Hopf algebroid [17, 18, 22, 36]. In [42], the H -Hopf algebroid structure was defined in term of the coproduct of the L -operator of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, whereas in [43] another coproduct called the Drinfeld coproduct for the elliptic Drinfeld currents was defined for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following the work [29].

In this section we review the $U_{q,p}(\widehat{\mathfrak{g}})$ defined as a topological algebra with a p -adic topology. We introduce the space of meromorphic functions \mathcal{M}_{H^*} on the dual space H^* of a dynamical extended Cartan subalgebra H . We define the level- k highest weight module \mathcal{V} of $U_{q,p}(\widehat{\mathfrak{g}})$. This representation is called the dynamical representation of $U_{q,p}(\widehat{\mathfrak{g}})$ because of \mathcal{M}_{H^*} . A category of level- k modules of $U_{q,p}(\widehat{\mathfrak{g}})$ is defined.

2.3.1 Definition of $U_{q,p}(\widehat{\mathfrak{g}})$

In 2.3.1, we give a definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ as a topological algebra generated by the elliptic Drinfeld currents. 2.3.2 devoted to a review of the dynamical representation of $U_{q,p}(\widehat{\mathfrak{g}})$.

2.3.1 Definition of $U_{q,p}(\widehat{\mathfrak{g}})$

Let $\widehat{\mathfrak{g}}$ be an untwisted affine Lie algebra associated with the generalized Cartan matrix $A = (a_{ij})$ $i, j \in \{0\} \cup I$. We denote by $B = (b_{ij})$, $b_{ij} = d_i a_{ij}$ the symmetrization of A . We take $d_i = 1$ ($i \in I$) for the simply laced cases, $d_i = 1$ ($i \leq i \leq l-1$), $d_l = 1/2$ for $B_l^{(1)}$ and $d_i = 1$ ($i \leq i \leq l-1$), $d_l = 2$ for $C_l^{(1)}$. Let $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$ and set $q_i = q^{d_i}$. Let p be an indeterminate.

Let $\widehat{\mathfrak{h}} = \widetilde{\mathfrak{h}} \oplus \mathbb{C}d$, $\widetilde{\mathfrak{h}} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$, $\bar{\mathfrak{h}} = \bigoplus_{i \in I} \mathbb{C}h_i$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$. Define $\delta, \Lambda_0, \alpha_i$ ($i \in I$) $\in \widehat{\mathfrak{h}}^*$ by

$$\langle \alpha_i, h_j \rangle = a_{j,i}, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \quad (2.3.1)$$

the other pairings are 0. We also define $\bar{\Lambda}_i$ ($i \in I$) $\in \widehat{\mathfrak{h}}^*$ by

$$\langle \bar{\Lambda}_i, h_j \rangle = \delta_{i,j}.$$

We set $\bar{\mathfrak{h}}^* = \bigoplus_{i \in I} \mathbb{C}\bar{\Lambda}_i$, $\widetilde{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $\mathcal{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathcal{P} = \bigoplus_{i \in I} \mathbb{Z}\bar{\Lambda}_i$. Let $N = l + 1$ for $X_l = A_l$, $= l$ for B_l, C_l, D_l , $= 7$ for E_6 , $= 8$ for E_7, E_8 , $= 3$ for G_2 , $= 4$ for F_4 and consider the orthonormal basis $\{\xi_j$ ($1 \leq j \leq N$) $\}$ in \mathbb{R}^N with the inner product $(\xi_j, \xi_k) = \delta_{j,k}$. For A_l , we also set

$$\bar{\xi}_j = \xi_j - \frac{1}{l+1} \sum_{j=1}^{l+1} \xi_j. \quad (2.3.2)$$

We define $\epsilon_j = \bar{\xi}_j$ for A_l and $= \xi_j$ for other X_l . The simple roots α_j and the fundamental weights $\bar{\Lambda}_j$ ($1 \leq j \leq l$) can be expressed as a linear sum of ϵ_j [6, 34]. We follow Kac's conventions. We define $h_{\epsilon_j} \in \bar{\mathfrak{h}}$ ($j \in I$) by $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$ and $h_\alpha \in \bar{\mathfrak{h}}$ for $\alpha = \sum_j c_j \epsilon_j$, $c_j \in \mathbb{C}$ by $h_\alpha = \sum_j c_j h_{\epsilon_j}$. We regard $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$ as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k]. \quad (2.3.3)$$

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h^j = h_{\bar{\Lambda}_j}$.

We introduce another Heisenberg algebra generated by P_α and Q_β ($\alpha, \beta \in \bar{\mathfrak{h}}^*$) satisfying the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}], \quad (2.3.4)$$

$$[P_{\epsilon_j}, \alpha] = [Q_{\epsilon_j}, \alpha] = 0, \quad (2.3.5)$$

$$[P_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = [Q_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = 0 \quad (2.3.6)$$

2.3.1 Definition of $U_{q,p}(\widehat{\mathfrak{g}})$

where $P_\alpha = \sum_j c_j P_{\epsilon_j}$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_{\tilde{\mathfrak{h}}} = \bigoplus_{j \in I} \mathbb{C} P_{\epsilon_j}$, $Q_{\tilde{\mathfrak{h}}} = \bigoplus_{j \in I} \mathbb{C} Q_{\epsilon_j}$, $P_j = P_{\alpha_j^\vee}$, $P^j = P_{\bar{\alpha}_j}$ and $Q_j = Q_{\alpha_j}$, $Q^j = Q_{\bar{\alpha}_j}$. Here $\alpha_j^\vee = 2\alpha_j / (\alpha_j, \alpha_j)$.

For the abelian group $\mathcal{R}_Q = \sum_{j=1}^N \mathbb{Z} Q_{\alpha_j}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over \mathbb{C} of \mathcal{R}_Q . We denote by e^α the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $\alpha \in \mathcal{R}_Q$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us set $H = \tilde{\mathfrak{h}} \oplus P_{\tilde{\mathfrak{h}}} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\epsilon_j}) + \sum_j \mathbb{C} P_{\epsilon_j} + \mathbb{C}c$ and denote its dual space by $H^* = \tilde{\mathfrak{h}}^* \oplus Q_{\tilde{\mathfrak{h}}}$. We define the pairing by (2.3.1), $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$ and $\langle Q_\alpha, h_\beta \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$. We define $\mathbb{F} = \mathcal{M}_{H^*}$ to be the field of meromorphic functions on H^* . We regard a function of $P+h = \sum_j a_j (P_{\epsilon_j} + h_{\epsilon_j})$, $P = \sum_j b_j P_{\epsilon_j}$ and c , $\widehat{f} = f(P+h, P, c)$, as an element in \mathbb{F} by $\widehat{f}(\mu) = f(\langle \mu, P+h \rangle, \langle \mu, P \rangle, \langle \mu, c \rangle)$ for $\mu \in H^*$.

Definition 2.3.1. [53] An elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by \mathcal{M}_{H^*} , $e_{j,m}, f_{j,m}, \alpha_{j,n}^\vee, K_j^\pm$, ($j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}$), d and the central element c . We assume K_j^\pm are invertible and set

$$\begin{aligned} e_j(z) &= \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, & f_j(z) &= \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m}, \\ \psi_j^+(q^{-\frac{c}{2}} z) &= K_j^+ \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1-p^n} z^{-n} \right), \\ \psi_j^-(q^{\frac{c}{2}} z) &= K_j^- \exp \left(-(q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,-n}^\vee}{1-p^n} z^n \right) \exp \left((q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1-p^n} z^{-n} \right). \end{aligned}$$

Note that $\psi_j^\pm(z)$ are formal Laurent series in z , whose coefficients are well defined in the p -adic topology. We call $e_j(z), f_j(z), \psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows.

2.3.1 Definition of $U_{q,p}(\widehat{\mathfrak{g}})$

For $g(P), g(P+h) \in \mathcal{M}_{H^*}$,

$$g(P+h)e_j(z) = e_j(z)g(P+h), \quad g(P)e_j(z) = e_j(z)g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.3.7)$$

$$g(P+h)f_j(z) = f_j(z)g(P+h - \langle \alpha_j, P+h \rangle), \quad g(P)f_j(z) = f_j(z)g(P), \quad (2.3.8)$$

$$[g(P), \alpha_{i,m}^\vee] = [g(P+h), \alpha_{i,n}^\vee] = 0, \quad (2.3.9)$$

$$g(P)K_j^\pm = K_j^\pm g(P - \langle Q_{\alpha_j}, P \rangle), \quad (2.3.10)$$

$$g(P+h)K_j^\pm = K_j^\pm g(P+h - \langle Q_{\alpha_j}, P \rangle), \quad (2.3.11)$$

$$[d, g(P+h, P)] = 0, \quad (2.3.12)$$

$$[d, \alpha_{j,n}^\vee] = n\alpha_{j,n}^\vee, \quad [d, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [d, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z), \quad (2.3.13)$$

$$K_i^\pm e_j(z) = q_i^{\mp a_{ij}} e_j(z) K_i^\pm, \quad K_i^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm, \quad (2.3.14)$$

$$[\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{ij}m]_i [cm]_j}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (2.3.15)$$

$$[\alpha_{i,m}^\vee, e_j(z)] = \frac{[a_{ij}m]_i}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \quad (2.3.16)$$

$$[\alpha_{i,m}^\vee, f_j(z)] = -\frac{[a_{ij}m]_i}{m} z^m f_j(z), \quad (2.3.17)$$

$$z_1 \frac{(q^{b_{ij}} z_2 / z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_\infty} e_i(z_1) e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_\infty} e_j(z_2) e_i(z_1), \quad (2.3.18)$$

$$z_1 \frac{(q^{-b_{ij}} z_2 / z_1; p)_\infty}{(pq^{b_{ij}} z_2 / z_1; p)_\infty} f_i(z_1) f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1 / z_2; p)_\infty}{(pq^{b_{ij}} z_1 / z_2; p)_\infty} f_j(z_2) f_i(z_1), \quad (2.3.19)$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right), \quad (2.3.20)$$

$$\begin{aligned} & \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)} / z_{\sigma(m)}; p^*)_\infty}{(p^* q^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p^*)_\infty} \\ & \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq i \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(i)}; p^*)_\infty}{(p^* q^{-b_{ij}} w / z_{\sigma(i)}; p^*)_\infty} \prod_{s+1 \leq i \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(i)} / w; p^*)_\infty}{(p^* q^{-b_{ij}} z_{\sigma(i)} / w; p^*)_\infty} \\ & \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0, \end{aligned} \quad (2.3.21)$$

$$\begin{aligned} & \sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(pq^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p)_\infty}{(pq^2 z_{\sigma(k)} / z_{\sigma(m)}; p)_\infty} \\ & \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq i \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(i)}; p)_\infty}{(pq^{b_{ij}} w / z_{\sigma(i)}; p)_\infty} \prod_{s+1 \leq i \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(i)} / w; p)_\infty}{(pq^{b_{ij}} z_{\sigma(i)} / w; p)_\infty} \\ & \times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(s)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}), \end{aligned} \quad (2.3.22)$$

where $p^* = pq^{-2c}$ and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We also denote by $U'_{q,p}(\widehat{\mathfrak{g}})$ the subalgebra obtained by removing d .

2.3.2 Dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$

We treat the relations (2.3.13), (2.3.16)-(2.3.22) as formal Laurent series in z, w and z_j 's. In each term of (2.3.18)-(2.3.22), the expansion direction of the structure function given by a ratio of infinite products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of (2.3.18), $\frac{(q^{b_{ij}} z_2/z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2/z_1; p^*)_\infty}$ should be expanded in z_2/z_1 , whereas in the r.h.s $\frac{(q^{b_{ij}} z_1/z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1/z_2; p^*)_\infty}$ should be expanded in z_1/z_2 . In each term in (2.3.21), the coefficient function is expanded in $z_{\sigma(k)}/z_{\sigma(m)}$ ($m < k$), $w/z_{\sigma(i)}$ ($i \leq s$) and $z_{\sigma(i)}/w$ ($i \geq s+1$). All the coefficients in z_j 's are well defined in the p -adic topology.

Remark. In [29,38,39,41], assuming that q is a transcendental complex number satisfying $|q| < 1$, we wrote (2.3.18), (2.3.19) as

$$\begin{aligned} z_1 \Theta_{p^*}(q^{b_{ij}} z_2/z_1) e_i(z_1) e_j(z_2) &= -z_2 \Theta_{p^*}(q^{b_{ij}} z_1/z_2) e_j(z_2) e_i(z_1), \\ z_1 \Theta_p(q^{-b_{ij}} z_2/z_1) f_i(z_1) f_j(z_2) &= -z_2 \Theta_p(q^{-b_{ij}} z_1/z_2) f_j(z_2) f_i(z_1), \end{aligned}$$

in the sense of analytic continuation.

Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra associated with $\widehat{\mathfrak{g}}$ in the Drinfeld realization [16]. $U_{q,p}(\widehat{\mathfrak{g}})$ is a natural elliptic deformation of $U_q(\widehat{\mathfrak{g}})$ in the following sense.

Theorem 2.3.2.

$$U_{q,p}(\widehat{\mathfrak{g}})/pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathbb{F} \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{g}})) \sharp \mathbb{C}[\mathcal{R}_Q].$$

Here the smash product \sharp is defined as follows.

$$\begin{aligned} g(P, P+h)x \otimes e^\alpha \cdot f(P, P+h)y \otimes e^\beta \\ = g(P, P+h)f(P - \langle \alpha, P \rangle, P+h - \langle \alpha + \text{wt}(x), P+h \rangle)xy \otimes e^{\alpha+\beta} \end{aligned}$$

where $\text{wt}(x) \in \bar{\mathfrak{h}}^*$ s.t. $q^h x q^{-h} = q^{\langle \text{wt}(x), h \rangle} x$ for $x, y \in U_q(\widehat{\mathfrak{g}})$, $f(P), g(P) \in \mathbb{F}$, $e^\alpha, e^\beta \in \mathbb{C}[\mathcal{R}_Q]$.

Proof. At $p = 0$, the relations for $\alpha_{j,m}^\vee, e_j(z), f_j(z)$ (2.3.13)-(2.3.22) coincide with those for $a_{i,m}^\vee, x_j^+(z), x_j^-(z)$ (2.2.35)-(2.2.40) of $U_q(\widehat{\mathfrak{g}})$. Therefore from (2.3.7)-(2.3.11), one has the isomorphism

$$e_j(z) \mapsto x_j^+(z) e^{-Q_{\alpha_j}}, \quad f_j(z) \mapsto x_j^-(z), \quad K_j^\pm \mapsto q_j^{\mp h_j} e^{-Q_{\alpha_j}}, \quad \alpha_{j,m}^\vee \mapsto a_{j,m}^\vee \pmod{pU_{q,p}(\widehat{\mathfrak{g}})}.$$

2.3.2 Dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$

Let us consider \mathcal{V} a vector space over \mathbb{F} . \mathcal{V} which is H -diagonalizable, i.e.

$$\mathcal{V} = \bigoplus_{\lambda, \mu \in H^*} \mathcal{V}_{\lambda, \mu}, \quad \mathcal{V}_{\lambda, \mu} = \{v \in \mathcal{V} \mid q^{P+h} \cdot v = q^{\langle \lambda, P+h \rangle} v, \quad q^P \cdot v = q^{\langle \mu, P \rangle} v \quad \forall P+h, P \in H\}.$$

2.3.2 Dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$

Let us define the H -algebra $\mathcal{D}_{H,\mathcal{V}}$ of the \mathbb{C} -linear operators on \mathcal{V} by

$$\mathcal{D}_{H,\mathcal{V}} = \bigoplus_{\alpha,\beta \in H^*} (\mathcal{D}_{H,\mathcal{V}})_{\alpha\beta},$$

$$(\mathcal{D}_{H,\mathcal{V}})_{\alpha\beta} = \left\{ X \in \text{End}_{\mathbb{C}}\mathcal{V} \left| \begin{array}{l} f(P+h)X = Xf(P+h + \langle \alpha, P+h \rangle)X, \\ f(P)X = Xf(P + \langle \beta, P \rangle)X \\ f(P), f(P+h) \in \mathbb{F}, X \cdot \mathcal{V}_{\lambda,\mu} \subseteq \mathcal{V}_{\lambda+\alpha,\mu+\beta} \end{array} \right. \right\},$$

$$\mu_l^{\mathcal{D}_{H,\mathcal{V}}}(\widehat{f})v = f(\langle \lambda, P+h \rangle, p)v, \quad \mu_r^{\mathcal{D}_{H,\mathcal{V}}}(\widehat{f})v = f(\langle \mu, P \rangle, p^*)v, \quad \widehat{f} \in \mathcal{M}_{H^*}, v \in \mathcal{V}_{\lambda,\mu}.$$

Definition 2.3.3. We define a dynamical representation of $U_{q,p}(\widehat{\mathfrak{g}})$ on \mathcal{V} to be an H -algebra homomorphism $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H,\mathcal{V}}$. By the action π of $U_{q,p}(\widehat{\mathfrak{g}})$ we regard \mathcal{V} as a $U_{q,p}(\widehat{\mathfrak{g}})$ -module.

Definition 2.3.4. For $k \in \mathbb{C}$, we say that a $U_{q,p}(\widehat{\mathfrak{g}})$ -module has level k if c act as the scalar k on it.

Definition 2.3.5. For $\omega \in \mathbb{C}$, set

$$\mathcal{V}_{\omega} = \{v \in \mathcal{V} \mid -d \cdot v = \omega v\}$$

and we call \mathcal{V}_{ω} the space of elements homogeneous of degree ω . We also say that $X \in \mathcal{D}_{H,\mathcal{V}}$ is homogeneous of degree $\omega \in \mathbb{C}$ if

$$[-d, X] = \omega X$$

and denote by $(\mathcal{D}_{H,\mathcal{V}})_{\omega}$ the space of all endomorphisms homogeneous of degree ω .

Definition 2.3.6. Let $\mathcal{H}, \mathcal{N}_+, \mathcal{N}_-$ be the subalgebras of $U_{q,p}(\widehat{\mathfrak{g}})$ generated by c, d, K_i^{\pm} ($i \in I$), by $\alpha_{i,n}^{\vee}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$) $f_{i,n}$ ($i \in I, n \in \mathbb{Z}_{>0}$) and by $\alpha_{i,-n}^{\vee}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $f_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$), respectively.

Definition 2.3.7. For $k \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$ and $\mu \in H^*$, a (dynamical) $U_{q,p}(\widehat{\mathfrak{g}})$ -module $\mathcal{V}(\lambda, \mu)$ is called the level- k highest weight module with the highest weight (λ, μ) , if there exists a vector $v \in \mathcal{V}(\lambda, \mu)$ such that

$$\begin{aligned} \mathcal{V}(\lambda, \mu) &= U_{q,p}(\widehat{\mathfrak{g}}) \cdot v, & \mathcal{N}_+ \cdot v &= 0, \\ c \cdot v &= kv, & f(P) \cdot v &= f(\langle \mu, P \rangle)v, & f(P+h) \cdot v &= f(\langle \lambda, P+h \rangle)v. \end{aligned}$$

We define the category \mathfrak{C}_k in the analogous way to the classical affine Lie algebra case [47].

Definition 2.3.8. For $k \in \mathbb{C}$, \mathfrak{C}_k is the full subcategory of the category of $U_{q,p}(\widehat{\mathfrak{g}})$ -modules consisting of those modules \mathcal{V} such that

- (i) \mathcal{V} has level k

2.3.2 Dynamical representations of $U_{q,p}(\widehat{\mathfrak{g}})$

$$(ii) \mathcal{V} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{V}_\omega$$

(iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{V}_{\omega+n} = 0$.

Since $\pi\mathcal{N}_+ \subset \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{D}_{H,\mathcal{V}})_n$, any level- k highest weight $U_{q,p}(\widehat{\mathfrak{g}})$ -modules belong to \mathfrak{C}_k .

Chapter 3

Dynamical Quantum Z -algebra of

$$U_{q,p}(\widehat{\mathfrak{g}})$$

There is an enormous literature on the representation theory of the quantum affine algebras as well as the affine Lie algebras. Based on the Drinfeld realization, a construction of higher level (integral) representation of $U_q(\widehat{\mathfrak{g}})$ appeared in [5, 31] which dealt with the theory of Z -algebras.

The theory of Z -algebras was introduced by Lepowsky and Wilson [47] in their study of Rogers-Ramanujan identities in the level- k standard representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ in the principal picture. Also, Lepowsky and Primc [45, 46] introduced the Z -algebra in the explicit realizations of higher level representation by using the homogeneous picture. The same construction was done in [51] for the higher rank cases. In 1987, Gepner [25] studied the theory of parafermions in conformal field theory. The coset conformal field theory $\widehat{\mathfrak{sl}}_2 \otimes \widehat{\mathfrak{sl}}_2 \supset (\widehat{\mathfrak{sl}}_2)_{diag}$ is known to be realized in terms of the level- k free boson and the Z_k -parafermion. The Z -algebra is obtained by taking a quotient of an affine Lie algebra $\widehat{\mathfrak{g}}$ of level k by its Heisenberg subalgebra. The Z_k -parafermion is obtained by taking a further quotient of the Z -algebra by the group algebra $\mathbb{C}[\mathcal{Q}]$ of the root lattice $\mathcal{Q} = \oplus_{i \in I} \mathbb{Z}\alpha_i$ (the zero modes of the boson operators). The coset conformal field theory associated with the general untwisted affine Lie algebras $\widehat{\mathfrak{g}}$ is constructed [8] in term of the generalized parafermions. The theory of Z -algebras has been a tool to construct an explicit representation of the affine Lie algebras.

The quantum deformation of the Z -algebras and its applications to the representations of $U_q(\widehat{\mathfrak{g}})$ was partially investigated in in [30, 39, 50] for level k (the central element acts by k). A higher level realization of the quantum Z -algebra was discussed in [5, 31]. The structure of the Z -algebra associated with the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ was investigated in [39], however for the general case $U_{q,p}(\widehat{\mathfrak{g}})$ it had not been studied. This structure of Z -algebra will find applications in representations of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ and Z -algebra itself.

In [53], we investigate if this structure exist or not for general $U_{q,p}(\widehat{\mathfrak{g}})$. Then we found the

3.1 The Heisenberg algebra $U_{q,p}(\mathcal{H})$

dynamical analogue of the quantum Z -algebras. In the next chapter we construct the infinite dimensional representation of $U_{q,p}(\widehat{\mathfrak{g}})$ in term of the Z -algebra modules.

This chapter addresses the following. In section 3.1, we define the Heisenberg algebra of $U_{q,p}(\widehat{\mathfrak{g}})$ and it's defining relations. Also the level- k module of the Heisenberg algebra of $U_{q,p}(\widehat{\mathfrak{g}})$ is constructed. We introduce a quantum and dynamical analogue \mathcal{Z}_γ of Lepowsky-Wilson's Z -algebra associated with the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules in section 3.2. We also provide the Serre relations (3.2.19) explicitly. Then we define the universal dynamical algebra \mathcal{Z}_k . Finally we define a category of the level- k \mathcal{Z}_k -modules. These results are published in [53].

3.1 The Heisenberg algebra $U_{q,p}(\mathcal{H})$

3.1.1 Definition

Let $U_{q,p}(\mathcal{H})$ be the subalgebra of $U_{q,p}(\widehat{\mathfrak{g}})$ generated by $\alpha_{i,n}^\vee$ ($i \in I, n \in \mathbb{Z}_{\neq 0}$) and c . It is convenient to introduce the simple root type generators $\alpha_{j,m}$ and $\alpha'_{j,m}$ defined by $\alpha_{j,m} = [d_j]\alpha_{j,m}^\vee$ and $\alpha'_{j,m} = \frac{1-p^{*m}}{1-p^m}q^{km}\alpha_{j,m}$, ($j \in I, n \neq 0$). From (2.3.15), (2.3.16), (2.3.17), we have

$$[\alpha_{i,m}, \alpha_{j,n}] = \frac{[b_{ij}m][cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-km} \delta_{m+n,0}, \quad (3.1.1)$$

$$[\alpha'_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m][cm]}{m} \frac{1-p^{*m}}{1-p^m} q^{km} \delta_{m+n,0}, \quad (3.1.2)$$

$$[\alpha_{i,m}, \alpha'_{j,n}] = \frac{[b_{ij}m][cm]}{m} \delta_{m+n,0}, \quad (3.1.3)$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ij}m]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} z^m e_j(z), \quad (3.1.4)$$

$$[\alpha'_{i,m}, f_j(z)] = -\frac{[b_{ij}m]}{m} \frac{1-p^{*m}}{1-p^m} q^{cm} z^m f_j(z). \quad (3.1.5)$$

Let $U_{q,p}(\mathcal{H}_+)$ (reps. $U_{q,p}(\mathcal{H}_-)$) be the commutative subalgebras of $U_{q,p}(\mathcal{H})$ generated by $\{c, \alpha_{i,n} (i \in I, n \in \mathbb{Z}_{>0})\}$ (resp. $\{\alpha_{i,-n} (i \in I, n \in \mathbb{Z}_{>0})\}$). We have

$$U_{q,p}(\mathcal{H}) = U_{q,p}(\mathcal{H}_-)U_{q,p}(\mathcal{H}_+).$$

Let $\mathbb{C}1_k$ be the one-dimensional $U_{q,p}(\mathcal{H}_+)$ -module generated by the vacuum vector 1_k defined by

$$c \cdot 1_k = k1_k \quad \alpha_{i,n} \cdot 1_k = 0 \quad (n > 0).$$

Then we have the induced $U_{q,p}(\mathcal{H})$ -module

$$\mathcal{F}_{\alpha,k} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathbb{C}1_k.$$

We identify $\mathcal{F}_{\alpha,k}$ with a polynomial ring $\mathbb{C}[\alpha_{i,-m} \ (i \in I, m > 0)]$ by

$$\begin{aligned} c \cdot u &= ku, & \alpha_{i,-n} \cdot u &= \alpha_{i,-n}u, \\ \alpha_{i,n} \cdot u &= \sum_j \frac{[b_{ij}n][kn]}{n} \frac{1-p^n}{1-p^*} q^{-kn} \frac{\partial}{\partial \alpha_{j,-n}} u \quad (n > 0) \end{aligned}$$

for $u \in \mathbb{C}[\alpha_{i,-m} \ (i \in I, m > 0)]$.

3.2 Dynamical Quantum Z -algebra

In this section we introduce a quantum and dynamical analogue \mathcal{Z}_k of Lepowsky-Wilson's Z -algebra associated with the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules \mathcal{V} .

3.2.1 The dynamical quantum Z -algebra $Z_{\mathcal{V}}$

Let $k \in \mathbb{C}^\times$ and $(\mathcal{V}, \pi) \in \mathfrak{C}_k$. We call $\pi U_{q,p}(\mathcal{H}) \subset (D_{H,\mathcal{V}})_{00}$ the level- k Heisenberg algebra. We define the following vertex operators in $(D_{H,\mathcal{V}})_{00}[[z, z^{-1}]]$.

$$E^\pm(\alpha_j, z) = \exp\left(\pm \sum_{n>0} \frac{\pi(\alpha_{j,\pm n})}{[kn]} z^{\mp n}\right), \quad E^\pm(\alpha'_j, z) = \exp\left(\mp \sum_{n>0} \frac{\pi(\alpha'_{j,\pm n})}{[kn]} z^{\mp n}\right). \quad (3.2.1)$$

These satisfy the following relations.

Proposition 3.2.1.

$$E^+(\alpha_i, z)E^-(\alpha_j, w) = \frac{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p^*)_\infty}{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p^*)_\infty} E^-(\alpha_j, w)E^+(\alpha_i, z), \quad (3.2.2)$$

$$E^+(\alpha'_i, z)E^-(\alpha'_j, w) = \frac{(q^{-b_{ij}}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p)_\infty}{(q^{b_{ij}}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p)_\infty} E^-(\alpha'_j, w)E^+(\alpha'_i, z), \quad (3.2.3)$$

$$E^+(\alpha_i, z)E^-(\alpha'_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha'_j, w)E^+(\alpha_i, z), \quad (3.2.4)$$

$$E^+(\alpha'_i, z)E^-(\alpha_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha_j, w)E^+(\alpha'_i, z), \quad (3.2.5)$$

$$E^\pm(\alpha_i, z)e_j(w) = \frac{(q^{\pm b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p^*)_\infty}{(q^{\mp b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p^*)_\infty} e_j(w)E^\pm(\alpha_i, z), \quad (3.2.6)$$

$$E^\pm(\alpha'_i, z)f_j(w) = \frac{(q^{\pm b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p)_\infty}{(q^{\mp b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p)_\infty} f_j(w)E^\pm(\alpha'_i, z), \quad (3.2.7)$$

$$E^\pm(\alpha'_i, z)e_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} e_j(w)E^\pm(\alpha'_i, z), \quad (3.2.8)$$

$$E^\pm(\alpha_i, z)f_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} f_j(w)E^\pm(\alpha_i, z). \quad (3.2.9)$$

3.2.1 The dynamical quantum Z -algebra $Z_{\mathcal{V}}$

Definition 3.2.2. We define $\mathcal{Z}_j^{\pm}(z; \mathcal{V}) \in \mathcal{D}_{H, \mathcal{V}}[[z, z^{-1}]]$ by

$$\mathcal{Z}_j^+(z; \mathcal{V}) := E^-(\alpha_j, z)\pi(e_j(z))E^+(\alpha_j, z), \quad (3.2.10)$$

$$\mathcal{Z}_j^-(z; \mathcal{V}) := E^-(\alpha'_j, z)\pi(f_j(z))E^+(\alpha'_j, z), \quad (3.2.11)$$

for $j \in I$ and call them the dynamical quantum Z operators associated with $(\mathcal{V}, \pi) \in \mathfrak{C}_k$.

Note that due to the truncation property of the grading of $\mathcal{V} \in \mathfrak{C}_k$ w.r.t $-d$, $\mathcal{Z}_j^{\pm}(z; \mathcal{V})$ are well defined i.e. the coefficients $\mathcal{Z}_{j,n}^{\pm}(\mathcal{V})$ of $\mathcal{Z}_j^{\pm}(z; \mathcal{V}) = \sum_{n \in \mathbb{Z}} \mathcal{Z}_{j,n}^{\pm}(\mathcal{V})z^{-n}$ in z are well defined elements in $(\mathcal{D}_{H, \mathcal{V}})_n$ for all $n \in \mathbb{Z}$. For the sake of simplicity of the presentation, we often drop π to denote the elements in $\mathcal{D}_{H, \mathcal{V}}$.

From the defining relations of $U_{q,p}(\widehat{\mathfrak{g}})$, we obtain the following relations of the dynamical quantum Z operators.

Theorem 3.2.3.

$$g(P+h)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P+h), \quad g(P)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P - \langle Q_{\alpha_i}, P \rangle), \quad (3.2.12)$$

$$g(P+h)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P+h - \langle \alpha_i, P+h \rangle), \quad g(P)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P), \quad (3.2.13)$$

$$[d, \mathcal{Z}_j^{\pm}(z; \mathcal{V})] = -z \frac{\partial}{\partial z} \mathcal{Z}_j^{\pm}(z; \mathcal{V}), \quad (3.2.14)$$

$$[\alpha_{i,m}, \mathcal{Z}_j^{\pm}(w; \mathcal{V})] = 0, \quad (3.2.15)$$

$$K_i^{\pm} \mathcal{Z}_j^{\pm}(z; \mathcal{V}) = q^{\mp b_{ij}} \mathcal{Z}_j^{\pm}(z; \mathcal{V}) K_i^{\pm}, \quad K_i^{\pm} \mathcal{Z}_j^{\mp}(z; \mathcal{V}) = q^{\pm b_{ij}} \mathcal{Z}_j^{\mp}(z; \mathcal{V}) K_i^{\pm}, \quad (3.2.16)$$

$$z \frac{(q^{-b_{ij}} w/z; q^{2k})_{\infty}}{(q^{b_{ij}+2k} w/z; q^{2k})_{\infty}} \mathcal{Z}_i^{\pm}(z; \mathcal{V}) \mathcal{Z}_j^{\pm}(w; \mathcal{V}) = -w \frac{(q^{-b_{ij}} z/w; q^{2k})_{\infty}}{(q^{b_{ij}+2k} z/w; q^{2k})_{\infty}} \mathcal{Z}_j^{\pm}(w; \mathcal{V}) \mathcal{Z}_i^{\pm}(z; \mathcal{V}), \quad (3.2.17)$$

$$\begin{aligned} & \frac{(q^{b_{ij}+k} w/z; q^{2k})_{\infty}}{(q^{-b_{ij}+k} w/z; q^{2k})_{\infty}} \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^-(w; \mathcal{V}) - \frac{(q^{b_{ij}+k} z/w; q^{2k})_{\infty}}{(q^{-b_{ij}+k} z/w; q^{2k})_{\infty}} \mathcal{Z}_j^-(w; \mathcal{V}) \mathcal{Z}_i^+(z; \mathcal{V}) \\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(K_i^- \delta(q^{-k} z/w) - K_i^+ \delta(q^k z/w) \right), \end{aligned} \quad (3.2.18)$$

$$\begin{aligned} & \sum_{\sigma \in S_a} \prod_{1 \leq m < l \leq a} \frac{(q^{2+k \mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_{\infty}}{(q^{-2+k \mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_{\infty}} \\ & \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix} \prod_{i=1 \leq i \leq s} \frac{(q^{-b_{ij}+k \mp k} w/z_{\sigma(i)}; q^{2k})_{\infty}}{(q^{b_{ij}+k \mp k} w/z_{\sigma(i)}; q^{2k})_{\infty}} \prod_{s+1 \leq i \leq a} \frac{(q^{-b_{ij}+k \mp k} z_{\sigma(i)}/w; q^{2k})_{\infty}}{(q^{b_{ij}+k \mp k} z_{\sigma(i)}/w; q^{2k})_{\infty}} \\ & \times \mathcal{Z}_i^{\pm}(z_{\sigma(1)}; \mathcal{V}) \cdots \mathcal{Z}_i^{\pm}(z_{\sigma(s)}; \mathcal{V}) \mathcal{Z}_j^{\pm}(w; \mathcal{V}) \mathcal{Z}_i^{\pm}(z_{\sigma(s+1)}; \mathcal{V}) \cdots \mathcal{Z}_i^{\pm}(z_{\sigma(a)}; \mathcal{V}) = 0 \\ & (i \neq j, a = 1 - a_{ij}). \end{aligned} \quad (3.2.19)$$

Proof. See [53].

3.2.2 The universal algebra \mathcal{Z}_k

Definition 3.2.4. For $k \in \mathbb{C}^\times$ and $(\mathcal{V}, \pi) \in \mathfrak{C}_k$, we call the H -subalgebra of $\mathcal{D}_{H, \mathcal{V}}$ generated by $\mathcal{Z}_{i,m}^\pm(\mathcal{V})$, K_i^\pm ($i \in I, m \in \mathbb{Z}$), \mathcal{M}_{H^*} and d the dynamical quantum Z -algebra $\mathcal{Z}_\mathcal{V}$ associated with (\mathcal{V}, π) .

3.2.2 The universal algebra \mathcal{Z}_k

Using the relations in Theorem 3.2.3, we define the universal dynamical quantum Z -algebra as follows.

Definition 3.2.5. Let $\mathcal{Z}_{i,m}^\pm$ ($i \in I, m \in \mathbb{Z}$) be abstract symbols. We set $\mathcal{Z}_i^\pm(z) = \sum_{m \in \mathbb{Z}} \mathcal{Z}_{i,m}^\pm z^{-m}$. We define the universal dynamical quantum Z -algebra \mathcal{Z}_k to be a topological algebra over $\mathbb{F}[[q^{2k}]]$ generated by $\mathcal{Z}_{i,m}^\pm, K_i^\pm$ ($i \in I, m \in \mathbb{Z}$), d, \mathcal{M}_{H^*} subject to the relations obtained by replacing $\mathcal{Z}_i^\pm(z; \mathcal{V})$ by $\mathcal{Z}_i^\pm(z)$ in Theorem 3.2.3.

We treat the relations as formal Laurent series in z, w and z_j 's in a similar way to those of $U_{q,p}(\widehat{\mathfrak{g}})$ in sec.2.1. The defining relations are well-defined in the q^{2k} -adic topology.

Proposition 3.2.6. \mathcal{Z}_k is an H -algebra with the same μ_l, μ_r as in $U_{q,p}(\widehat{\mathfrak{g}})$.

Note that for $(\mathcal{V}, \pi) \in \mathfrak{C}_k$ we extend π to the map $\pi : \mathcal{Z}_k \rightarrow \mathcal{D}_{H, \mathcal{V}}$ by $\pi(\mathcal{Z}_{i,m}^\pm) = \mathcal{Z}_{i,m}^\pm(\mathcal{V})$. Then \mathcal{V} is a \mathcal{Z}_k -module by π .

3.2.3 The Induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules from the \mathcal{Z}_k -modules

We define a category \mathfrak{D}_k of the \mathcal{Z}_k -modules. Each representation of \mathcal{Z}_k in \mathfrak{D}_k turns out to be a dynamical analogue of the quantum Z -algebra derived by Jing [31] from the level- k representation in the $U_q(\widehat{\mathfrak{g}})$ counterpart of \mathfrak{D}_k . Those statements will be used in the next chapter to construct the level- k induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules.

Definition 3.2.7. For $k \in \mathbb{C}^\times$, we denote by \mathfrak{D}_k the full subcategory of the category of \mathcal{Z}_k -modules consisting of those modules (\mathcal{W}, σ) such that

- (i) \mathcal{W} has level k ,
- (ii) $\mathcal{W} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{W}_\omega$, where $\mathcal{W}_\omega = \{w \in \mathcal{W} \mid -\sigma(d)w = \omega w\}$,
- (iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{W}_{\omega+n} = 0$.

Let us consider $(\mathcal{V}, \pi) \in \mathfrak{C}_k$. Following Lepowsky and Wilson [47], we define the vacuum space $\Omega_\mathcal{V}$ by

$$\Omega_\mathcal{V} = \{v \in \mathcal{V} \mid \pi(\alpha_{i,n})v = 0 \quad \forall i \in I, n \in \mathbb{Z}_{>0}\}.$$

3.2.3 The Induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules from the \mathcal{Z}_k -modules

From Theorem 3.2.3, $\Omega_{\mathcal{V}}$ is stable under the action of $\mathcal{Z}_{\mathcal{V}}$ and commutes with $U_{q,p}(\mathcal{H})$. For a morphism $f : \mathcal{V} \rightarrow \mathcal{V}'$ in \mathfrak{C}_k , we have

$$f(\Omega_{\mathcal{V}}) \subset \Omega_{\mathcal{V}'}$$

Proposition 3.2.8. *For $(\mathcal{V}, \pi) \in \mathfrak{C}_k$, there is a unique representation σ of \mathcal{Z}_k on $\Omega_{\mathcal{V}}$ such that $(\Omega_{\mathcal{V}}, \sigma) \in \mathfrak{D}_k$,*

$$\sigma(K_i^{\pm}) = \pi(K_i^{\pm}), \quad \sigma(\mathcal{Z}_{i,m}^{\pm}) = \mathcal{Z}_{i,m}^{\pm}(\mathcal{V}) \quad \forall i \in I, m \in \mathbb{Z}.$$

We hence define a functor $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$ by

$$\Omega(\mathcal{V}, \pi) = (\Omega_{\mathcal{V}}, \sigma), \quad \Omega(f) = f|_{\Omega_{\mathcal{V}}} : \Omega_{\mathcal{V}} \rightarrow \Omega_{\mathcal{V}'}$$

Chapter 4

Representation theory of $U_{q,p}(\widehat{\mathfrak{g}})$

This chapter deals with the infinite dimensional representation of $U_{q,p}(\widehat{\mathfrak{g}})$. We construct the induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module as a tensor product of the \mathcal{Z}_k -module and the $U_{q,p}(H)$ -modules. We discuss the irreducibility of the resultant induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module, we find that $U_{q,p}(\widehat{\mathfrak{g}})$ -module is irreducible if and only if \mathcal{Z}_k -module is irreducible. Namely, the irreducibility of the $U_{q,p}(\widehat{\mathfrak{g}})$ -module is controlled by the \mathcal{Z}_k -module. For $k = 1$, we present some examples of the level-1 infinite dimensional irreducible representations of $U_{q,p}(\widehat{\mathfrak{g}})$. The level-1 representations will be used in the next chapter to find the higher level representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

This chapter is organized as follow. In the beginning, we construct the $U_{q,p}(\widehat{\mathfrak{g}})$ -module which is expressed as a tensor product of the dynamical quantum Z -algebra module and the Heisenberg algebra $U_{q,p}(\mathcal{H})$ -module. Then we obtain a realization of $U_{q,p}(\widehat{\mathfrak{g}})$ in forms of the Z -algebra and the Heisenberg algebra. Then we define the reverse functor from the level- k category of \mathcal{Z}_k -modules to the category of the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. We show how the irreducibility of the $U_{q,p}(\mathcal{H})$ -module depends on the irreducibility of the \mathcal{Z}_k -algebra module. In the end of this chapter, we list some level-1 irreducible representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$. These results are published in [53].

4.1 Infinite dimensional representation of $U_{q,p}(\widehat{\mathfrak{g}})$

In 4.1.1, \mathcal{Z}_k is realized in term of the quantum Z -algebra associated with level- k $U_q(\widehat{\mathfrak{g}})$ -module which we state in section 2.2.4. We also give the infinite dimensional representation of $U_{q,p}(\widehat{\mathfrak{g}})$.

4.1.1 The functor Λ

We define a reverse functor $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$ which is equivalent to Ω such that we can construct the $U_{q,p}(\widehat{\mathfrak{g}})$ -modules from \mathcal{Z}_k -modules and visa versa.

4.1.1 The functor Λ

Let $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$ be a \mathcal{Z}_k -module. We define $U_{q,p}(\mathcal{H})$ -module $\text{Ind } \mathcal{W}$ by requiring $\alpha_{i,m} \cdot \mathcal{W} = 0$ and

$$\text{Ind } \mathcal{W} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathcal{W}.$$

Let $\mathcal{F}_{\alpha,k}$ be the level- k Fock module defined in section 3.1. We have a natural isomorphism $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \cong \text{Ind } \mathcal{W}$ by $(u \otimes 1_k) \otimes w \mapsto u \otimes w$ [47]. We thus identify the $U_{q,p}(\mathcal{H})$ -module $\text{Ind } \mathcal{W}$ with $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$, with the action π of $U_{q,p}(\mathcal{H})$

$$\pi(c) = 1 \otimes c, \quad \pi(K_i^{\pm}) = 1 \otimes \sigma(K_i^{\pm}), \quad \pi(\alpha_{i,m}) = \alpha_{i,m} \otimes 1.$$

For $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$ and $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$, we define $e'_j(z), f'_j(z) \in \mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$ by

$$\begin{aligned} e'_j(z) &= E^-(\alpha_j, z)^{-1} E^+(\alpha_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^+(z)), \\ f'_j(z) &= E^-(\alpha'_j, z)^{-1} E^+(\alpha'_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^-(z)). \end{aligned}$$

These are well-defined elements of $\mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$. By a similar argument to the proof of Theorem 3.2.3 one can show that $e'_j(z)$ and $f'_j(z)$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{g}})$ with $c = k$. We hence extend $\pi : U_{q,p}(\mathcal{H}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$ to $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$ as a H -algebra homomorphism by

$$\begin{aligned} \pi(e_j(z)) &= e'_j(z), \quad \pi(f_j(z)) = f'_j(z), \\ \pi(d) &= d \otimes 1 + 1 \otimes \sigma(d). \end{aligned}$$

By construction, the latter map is uniquely determined.

Proposition 4.1.1. *For $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$, there is a unique level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module $(\text{Ind } \mathcal{W}, \pi) \in \mathfrak{C}_k$.*

We thus reach the following definition.

Definition 4.1.2. *We define a functor $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$ by*

$$(i) \quad \Lambda(\mathcal{W}, \sigma) = (\text{Ind } \mathcal{W}, \pi)$$

(ii) *For a morphism $f : \mathcal{W} \rightarrow \mathcal{W}'$ in \mathfrak{D}_k , define $\Lambda(f) : \text{Ind } \mathcal{W} \rightarrow \text{Ind } \mathcal{W}'$ to be the induced $U_{q,p}(\mathcal{H})$ -module map. Then $\Lambda(f)$ is a $U_{q,p}(\widehat{\mathfrak{g}})$ -module map.*

We obtain the following theorem analogously to the case of the affine Lie algebras [47].

Theorem 4.1.3. *For $k \in \mathbb{C}^{\times}$, the two categories \mathfrak{C}_k and \mathfrak{D}_k are equivalent by the functors $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$ and $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$. In particular, the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \in \mathfrak{C}_k$ is irreducible if and only if $\mathcal{W} \in \mathfrak{D}_k$ is an irreducible \mathcal{Z}_k -module.*

Comparing the defining relations of \mathcal{Z}_k with those of Z_k , we obtain the following isomorphism.

Proposition 4.1.4. *We have the isomorphism*

$$\mathcal{Z}_k \cong (\mathbb{F} \otimes_{\mathbb{C}} Z_k) \sharp \mathbb{C}[\mathcal{R}_Q]$$

as an H -algebra by

$$\begin{aligned} \mathcal{Z}_{j,m}^+ &\mapsto Z_{j,m}^+ e^{-Q_{\alpha_j}}, & \mathcal{Z}_{j,m}^- &\mapsto Z_{j,m}^-, \\ K_i^{\pm} &\mapsto q_i^{\mp h_i} e^{-Q_{\alpha_j}} \quad (i \in I, m \in \mathbb{Z}) & d &\mapsto \bar{d}, \end{aligned}$$

where $Z_{j,m}^{\pm}$ denotes the generators in Z_k (Definition 3.2.4).

Theorem 4.1.5. *For $(W, \bar{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, there is a dynamical representation σ of \mathcal{Z}_k on $\mathcal{W}_{H,Q}(\mu) := (\mathbb{F} \otimes_{\mathbb{C}} W) \otimes_{\mathbb{C}} e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ such that $(\mathcal{W}_{H,Q}(\mu), \sigma) \in \mathfrak{C}_k$ and*

$$\begin{aligned} \sigma(c) &= k, \\ \sigma(\mathcal{Z}_{j,m}^+) &= \bar{\sigma}(Z_{j,m}^+) \otimes e^{-Q_{\alpha_j}}, & \sigma(\mathcal{Z}_{j,m}^-) &= \bar{\sigma}(Z_{j,m}^-) \otimes 1, \\ \sigma(K_j^{\pm}) &= \bar{\sigma}(q_j^{\mp h_j}) \otimes e^{-Q_{\alpha_j}}, & \sigma(d) &= \bar{\sigma}(\bar{d}) \otimes 1 + 1 \otimes P_d, \end{aligned}$$

where P_d denotes a \mathbb{C} -linear operator on $1 \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ such that

$$[1 \otimes P_d, \sigma(\mathcal{Z}_{j,m}^{\pm})] = 0.$$

Proposition 4.1.6. *The representation $(\mathcal{W}_{H,Q}(\mu), \sigma)$ of \mathcal{Z}_k is irreducible if and only if W is an irreducible Z_k -module.*

From this and Theorem 4.1.3, we obtain:

Proposition 4.1.7. *For a Z_k -module $(W, \bar{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, let $(\mathcal{W}_{H,Q}(\mu), \sigma)$ be the \mathcal{Z}_k -module constructed in Theorem 4.1.5 and $\text{Ind } \mathcal{W}_{H,Q}(\mu) = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}_{H,Q}(\mu)$ be the level- k induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module given in Prop.4.1.1. Then $(\text{Ind } \mathcal{W}_{H,Q}(\mu), \pi)$ is irreducible if and only if $(W, \bar{\sigma})$ is irreducible.*

4.2 Level-1 representations

We here give some examples of the level-1 irreducible induced representations of $U_{q,p}(\widehat{\mathfrak{g}})$ of types $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$.

4.2.1 The simply laced case :

Let us consider Theorem 2.2.21, then for generic $\mu \in \mathfrak{h}^*$, we have from Theorem 4.1.5 a level-1 irreducible $\mathcal{Z}_1(\widehat{\mathfrak{g}})$ module $\mathcal{W}_{H,Q}(\Lambda_a, \mu) := (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ with the action given by

$$\mathcal{Z}_j^+(z) = Z_j^+(z) \otimes e^{-Q_{\alpha_j}}, \quad \mathcal{Z}_j^-(z) = Z_j^-(z) \otimes 1. \quad (4.2.1)$$

Then from Proposition 4.1.7 we obtain:

4.2.2 The $B_l^{(1)}$ case

Theorem 4.2.1. *A level-1 irreducible highest weight representations of $U_{q,p}(\hat{\mathfrak{g}})$ is given by $\mathcal{V}(\Lambda_a + \mu, \mu) := \text{Ind } \mathcal{W}_{H,Q}(\Lambda_a, \mu)$ with the highest weight $(\Lambda_a + \mu, \mu)$:*

$$\mathcal{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha,1} \otimes \mathcal{W}_{H,Q}(\Lambda_a, \mu) = \bigoplus_{\gamma, \kappa \in \mathcal{Q}} \mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu),$$

where

$$\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu) = \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes e^{\bar{\Lambda}_a + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}},$$

The highest weight vector is $1_1 \otimes e^{\bar{\Lambda}_a} \otimes e^{Q_{\bar{\mu}}}$. The derivation operator d is realized as

$$\begin{aligned} d &= -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2)(P + h)^j, \\ N^\alpha &= \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j, \end{aligned}$$

where $r, r^* \in \mathbb{C}^\times$, and A_m^j are the fundamental weight type elliptic bosons given in section 5.1 in [53].

One can easily calculate the character of $\mathcal{V}(\Lambda_a, \mu)$:

$$\begin{aligned} ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} &= \text{tr}_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c(W(\hat{\mathfrak{g}}))}{24}} = \sum_{\gamma, \kappa \in \mathcal{Q}} ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)}, \\ ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)} &= \frac{1}{\eta(q)^l} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \bar{\rho})|^2}. \end{aligned}$$

Here $c(W(\hat{\mathfrak{g}})) = l(1 - \frac{g(g+1)}{rr^*})$, and $\eta(q)$ denotes Dedekind's η -function given by

$$\eta(q) = q^{\frac{1}{24}}(q; q)_\infty.$$

4.2.2 The $B_l^{(1)}$ case

We consider the quantum Z -algebra $Z_1(B_l^{(1)})$ and thier level-1 irreducible $Z_1(B_l^{(1)})$ -modules which stated in subsection 2.2.7. For generic $\mu \in \mathfrak{h}^*$ and $a = 0, 1, l$, we set $\mathcal{W}_{H,Q}(\Lambda_a, \mu) = (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$. From Proposition 4.1.1 we have the following three level-1 irreducible $U_{q,p}(\hat{B}_l^{(1)})$ -modules with the highest weight $(\Lambda_a + \mu, \mu)$:

$$\begin{aligned} \mathcal{V}(\Lambda_a + \mu, \mu) &= \mathcal{F}_{\alpha,1} \otimes_{\mathbb{C}} \mathcal{W}_{H,Q}(\Lambda_a, \mu) \\ &= \bigoplus_{\gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}} \bigoplus_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta}} \mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu), \end{aligned}$$

4.2.2 The $B_l^{(1)}$ case

where

$$\begin{aligned}
\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}_{\text{even}}^{NS} \otimes e^\gamma) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\
\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}_{\text{odd}}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\
\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}_{\text{even}}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\
\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}_{\text{odd}}^{NS} \otimes e^\gamma) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\
\mathcal{F}_{\Lambda_l, \gamma, \kappa}(\Lambda_l, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}, \\
\mathcal{F}_{\Lambda_l - \alpha_l, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \bar{\Lambda}_1 + \gamma}) \otimes e^{Q_{\bar{\mu} + \kappa}}.
\end{aligned}$$

The highest weight vectors are given by $1 \otimes 1 \otimes 1 \otimes e^{Q_{\bar{\mu}}}$ for $\mathcal{V}(\Lambda_0 + \mu, \mu)$, $1 \otimes 1 \otimes e^{\bar{\Lambda}_1} \otimes e^{Q_{\bar{\mu}}}$ for $\mathcal{V}(\Lambda_1 + \mu, \mu)$ and $1 \otimes 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e^{\bar{\Lambda}_l} \otimes e^{Q_{\bar{\mu}}}$ for $\mathcal{V}(\Lambda_l + \mu, \mu)$, respectively.

It is also easy to calculate the characters of these modules:

$$ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} = \text{tr}_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c_W}{24}} = \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta \\ \gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)},$$

where $c_W = (l + \frac{1}{2}) \left(1 - \frac{2l(2l-1)}{rr^*}\right)$ is the central charge of the WB_l algebra by Fateev and Lukyanov [49], and the derivation operator d is realized as

$$\begin{aligned}
d &= -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha - N^\Psi + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2) (P + h)^j, \\
N^\alpha &= \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j, \quad N^\Psi = \sum_{m > 0} \frac{m(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}{q^m + q^{-m}} \Psi_{-m} \Psi_m
\end{aligned}$$

where $r, r^* \in \mathbb{C}^\times$, and A_m^j are the fundamental weight type elliptic bosons of the type B_l given in section 5.1 in [53], Ψ_m denotes Ψ_m on \mathcal{F}^{NS} and $\hat{\Psi}_m$ on \mathcal{F}^R . We obtain:

$$\begin{aligned}
ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} &= \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta \\ \gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)}, \\
ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu)} &= c_{\Lambda_0}^{\Lambda_0} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\
ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu)} &= c_{\Lambda_1}^{\Lambda_0} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\
ch_{\mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu)} &= c_{\Lambda_1}^{\Lambda_1} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_1 + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\
ch_{\mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu)} &= c_{\Lambda_0}^{\Lambda_1} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\
ch_{\mathcal{F}_{\Lambda_l, \gamma, \kappa}(\Lambda_l, \mu)} &= c_{\Lambda_l}^{\Lambda_l} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_l + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2}, \\
ch_{\mathcal{F}_{\Lambda_l - \alpha_l, \gamma, \kappa}(\Lambda_1, \mu)} &= c_{\Lambda_l - \alpha_l}^{\Lambda_1} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_l + \bar{\mu} + \kappa + \gamma + \bar{\rho})|^2},
\end{aligned}$$

4.2.2 The $B_l^{(1)}$ case

where

$$\begin{aligned}c_{\Lambda_0}^{\Lambda_0} &= c_{\Lambda_1}^{\Lambda_1} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left((-q^{\frac{1}{2}}; q)_\infty + (q^{\frac{1}{2}}; q)_\infty \right), \\c_{\Lambda_0}^{\Lambda_1} &= c_{\Lambda_1}^{\Lambda_0} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left((-q^{\frac{1}{2}}; q)_\infty - (q^{\frac{1}{2}}; q)_\infty \right), \\c_{\Lambda_l}^{\Lambda_l} &= c_{\Lambda_l - \alpha_l}^{\Lambda_l} = \frac{q^{\frac{1}{24}}}{2\eta(q)^l} (-q; q)_\infty.\end{aligned}$$

Chapter 5

Higher level realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

For the affine Lie algebra, the integrable representation was defined in [34] as the highest weight module on which the locally Chevaly generators locally nilpotent. The higher level representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ was studied in [46]. Proposition VI.5 in Ref. [45] showed that on that level- $k+1$ ($k \in \mathbb{Z}_{>0}$) standard $\widehat{\mathfrak{sl}}_2$ -module, the currents $x_{\pm\alpha}(z)$ are nilpotent operators, $x_{\pm\alpha}(z)^{k+2} = 0$.

The quantum analogue of this condition appeared in [10]. The authors in [10, 13] studied the higher level representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$, quantum Z -algebra and the quantum integrable condition by using the Drinfeld coproduct [10] for the Drinfeld realization of level-1 $U_q(\widehat{\mathfrak{sl}}_2)$ [16]. Also they showed that the vertex operators $(x^{\pm}(z))^k$ satisfy certain q -difference equations $(x^+(zq^2))^k = \Delta^k \phi^{-1}(zq^{\frac{m+1}{2}})(x^+(z))^k \Delta^k \psi(zq^{\frac{3(m+1)}{2}})$, $(x^-(zq^2))^k = \Delta^k \phi(zq^{-\frac{3(m+1)}{2}})(x^-(z))^k \Delta^k \psi^{-1}(zq^{-\frac{(m+1)}{2}})$, where $\phi(z)$ and $\psi(z)$ are the generating functions of the bosons a_{-n}, a_n ($n \in \mathbb{Z}_{>0}$) respectively.

For the elliptic algebra $U_q(\widehat{\mathfrak{sl}}_2)$, the integrable condition of that higher level representation had not been studied. These subjects are addressed in this chapter [52]. In [29], the authors introduced the elliptic analogue of the Drinfeld coproduct for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The level-1 standard representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$ were given in [53]. Using the elliptic Drinfeld coproduct [29, 43] and the standard level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [53], we construct the higher level realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. The level- $k+1$ elliptic currents are expressed as a multiply coproduct of the level-1 elliptic currents. In particular, we obtain the level- $k+1$ Heisenberg algebra, then we introduce the vertex operators $E_{(k)}^{\pm}(\alpha, z), E_{(k)}^{\pm}(\alpha', z)$ and define the level- $k+1$ quantum Z -operators from the level- $k+1$ elliptic currents. Also, we give an elliptic analogue of the quantum integrable condition for level- $k+1$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [52]. There we found that the products $\mathcal{E}_N(z) = \Delta^k(e(z)) \Delta^k(e(zq^2)) \cdots \Delta^k(e(zq^{2(N-1)}))$ and $\mathcal{F}_N(z) = \Delta^k(f(z)) \Delta^k(f(zq^{-2})) \cdots \Delta^k(f(zq^{-2(N-1)}))$ vanish at $N = k+2$. We also show that the products $\mathcal{E}_N(z)$ and $\mathcal{F}_N(z)$ at $N = k+1$ give certain vertex operators associated with the

5.1 Co-algebra structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

level- $k + 1$ module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

This chapter is organized as follows. In section 5.1, we define the H -Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and formulate it as an elliptic quantum group. We apply the argument of last chapter to introduce the level-1 highest weight realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in section 5.2. After that, we show a construction of the level- $k + 1$ realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using Drinfeld coproduct at level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Level- $k + 1$ realization of Heisenberg algebra $U_{q,p}(H)$ is found too. Furthermore, we give a realization of the level- $k + 1$ Z -algebra associated with level- $k + 1$ realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in section 5.4. We present the elliptic analogue of quantum integrable condition for any level- $k + 1$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in the last section where we present a kind of vertex operators of the level- $k + 1$ elliptic bosons. These results are published in [52].

5.1 Co-algebra structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Here we follow [41–43] to present the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using the Drinfeld coproduct of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [29]. A different Hopf algebroid structure was given in [42] by using a different co-product.

5.1.1 H -Hopf algebroid

Let \mathcal{A} be a complex associative algebra, \mathcal{H} be a finite dimensional commutative subalgebra of \mathcal{A} , and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on \mathcal{H}^* the dual space of \mathcal{H} .

Definition 5.1.1 (\mathcal{H} -algebra). *An \mathcal{H} -algebra is an associative algebra \mathcal{A} with 1, which is bi-graded over \mathcal{H}^* , $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{A}_{00}$ (the left and right moment maps), such that*

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathcal{A}_{\alpha\beta}, \quad \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Definition 5.1.2. *An \mathcal{H} -algebra homomorphism is an algebra homomorphism $\pi : A \rightarrow B$ between two \mathcal{H} -algebras A and B such that for $\alpha, \beta \in \mathcal{H}^*$*

$$\pi(A_{\alpha\beta}) \subseteq B_{\alpha\beta}, \quad \pi(\mu_l^A(\widehat{f})) = \mu_l^B(\widehat{f}), \quad \pi(\mu_r^A(\widehat{f})) = \mu_r^B(\widehat{f}).$$

The tensor product $A \widetilde{\otimes} B = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} (A \widetilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} (\bigoplus_{\gamma \in \mathcal{H}^*} (A_{\alpha\gamma} \otimes_{\mathcal{M}_{\mathcal{H}^*}} B_{\gamma\beta}))$ is again an \mathcal{H} -algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$. The tensor product $\otimes_{\mathcal{M}_{\mathcal{H}^*}}$ refers to the usual tensor product modulo the following rule:

$$\mu_r^A(\widehat{f})a \otimes b = a \otimes \mu_l^B(\widehat{f})b, \quad a \in A, b \in B, \widehat{f} \in \mathcal{M}_{\mathcal{H}^*}. \quad (5.1.1)$$

5.1.2 H -Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

The unit object \mathcal{D} in the category of \mathcal{H} -algebras is an algebra of automorphisms $\mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{M}_{\mathcal{H}^*}$

$$\mathcal{D} = \left\{ \sum_i \widehat{f}_i T_{\beta_i} \mid \widehat{f}_i \in \mathcal{M}_{\mathcal{H}^*}, \beta_i \in \mathcal{H}^* \right\} = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{D}_{\alpha\alpha} \quad (5.1.2)$$

where $\mathcal{D}_{\alpha\alpha} = \{\widehat{f}T_{-\alpha} \mid \widehat{f} \in \mathcal{M}_{\mathcal{H}^*}, \alpha \in \mathcal{H}^*\}$ and the moment maps $\mu_l^{\mathcal{D}}, \mu_r^{\mathcal{D}} : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{D}_{00}$ are defined by $\mu_l^{\mathcal{D}}(\widehat{f}) = \mu_r^{\mathcal{D}}(\widehat{f}) = \widehat{f}T_0$.

Definition 5.1.3. An \mathcal{H} -Hopf algebroid is an \mathcal{H} -algebra A equipped with two \mathcal{H} -algebra homomorphisms: coproduct $\Delta : A \rightarrow A \widetilde{\otimes} A$, counit $\varepsilon : A \rightarrow \mathcal{D}$ and a \mathbb{C} -linear map: antipode $a : A \rightarrow A$. Δ, ε, a satisfy the following

$$(\Delta \widetilde{\otimes} id) \circ \Delta = (id \widetilde{\otimes} \Delta) \circ \Delta \quad (5.1.3)$$

$$(\varepsilon \widetilde{\otimes} id) \circ \Delta = (id \widetilde{\otimes} \varepsilon) \circ \Delta \quad (5.1.4)$$

$$m \circ (id \widetilde{\otimes} a) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \quad \forall x \in A \quad (5.1.5)$$

$$m \circ (a \widetilde{\otimes} id) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in A_{\alpha\beta}. \quad (5.1.6)$$

$m : A \widetilde{\otimes} A \rightarrow A$ refers the multiplication and $\varepsilon(x)1 (x \in A)$ refers the action of the operator $\varepsilon(x)$ on the constant function $1 \in \mathcal{M}_{\mathcal{H}^*}$.

5.1.2 H -Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Proposition 5.1.4. $U = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H -algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha\beta},$$

$$U_{\alpha\beta} = \left\{ x \in U \mid q^{P+h} x q^{-(P+h)} = q^{\langle \alpha, P+h \rangle} x, \quad q^P x q^{-P} = q^{\langle \beta, P \rangle} x \quad \forall P+h, P \in H \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{00}$ defined by

$$\mu_l(\widehat{f}) = f(P+h, p) \in \mathbb{F}[[p]], \quad \mu_r(\widehat{f}) = f(P, p^*) \in \mathbb{F}[[p]].$$

The tensor product $U \widetilde{\otimes} U = \bigoplus_{\alpha, \beta \in H^*} (U \widetilde{\otimes} U)_{\alpha\beta}$ is an H^* bigraded algebra.

The H -algebra \mathcal{D} of the shift operators is

$$\mathcal{D} = \left\{ \sum_i \widehat{f}_i T_{\alpha_i} \mid \widehat{f}_i \in \mathcal{M}_{H^*}, \alpha_i \in H^* \right\}.$$

with the bigraded structure and the moments map as in Definition 5.1.1.

In [42], Konno defined the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the coproduct of L -operator. Here we define the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the Drinfeld coproduct [29, 43].

5.2 Level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Theorem 5.1.5. [43] *The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ has an elliptic analogue of the Drinfeld coproduct $\Delta : U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow U_{q,p}(\widehat{\mathfrak{sl}}_2) \widetilde{\otimes} U_{q,p}(\widehat{\mathfrak{sl}}_2)$, the counit $\varepsilon : U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow \mathcal{D}$ and the antipode $a : U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow U_{q,p}(\widehat{\mathfrak{sl}}_2)$*

$$\Delta(q^c) = q^c \widetilde{\otimes} q^c, \Delta(q^h) = q^h \widetilde{\otimes} q^h \quad (5.1.7)$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(q^{\pm \frac{c(2)}{2}} z) \widetilde{\otimes} \psi^\pm(q^{\mp \frac{c(1)}{2}} z) \quad (5.1.8)$$

$$\Delta(\mu_r(\widehat{f})) = 1 \widetilde{\otimes} \mu_r(\widehat{f}), \Delta(\mu_l(\widehat{f})) = \mu_l(\widehat{f}) \widetilde{\otimes} 1 \quad (5.1.9)$$

$$\Delta(e(z)) = e(q^{-c(2)} z) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z) \quad (5.1.10)$$

$$\Delta(f(z)) = f(z) \widetilde{\otimes} 1 + \psi^+(q^{-\frac{c(1)}{2}} z) \widetilde{\otimes} f(zq^{-c(1)}) \quad (5.1.11)$$

$$\varepsilon(q^c) = 1, \varepsilon(\psi^+(z)) = \varepsilon(\psi^-(z)) = 1 \quad (5.1.12)$$

$$\varepsilon(\mu_r(\widehat{f})) = \varepsilon(\mu_l(\widehat{f})) = \widehat{f}T_0 \quad (5.1.13)$$

$$\varepsilon(e(z)) = \varepsilon(f(z)) = 0, \varepsilon(\alpha_n) = 0 \quad (5.1.14)$$

$$a(q^c) = q^{-c}, a(\psi^\pm(z)) = \psi^\pm(z)^{-1} \quad (5.1.15)$$

$$a(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), a(\mu_l(\widehat{f})) = \mu_r(\widehat{f}) \quad (5.1.16)$$

$$a(e(z)) = -\psi^-(zq^{\frac{c}{2}})^{-1} e(q^c z) \quad (5.1.17)$$

$$a(f(z)) = -f(q^c z) \psi^+(zq^{\frac{c}{2}})^{-1}. \quad (5.1.18)$$

Namely, the maps Δ, ε are algebra homomorphism and a is an anti-algebra homomorphism satisfying the relations (5.1.3)-(5.1.6) in Definition 5.1.3. Therefore the H -algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with Δ, ε, a is an H -Hopf algebroid.

Proof. See [52].

We call the H -Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, \mathcal{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, a)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

5.2 Level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

We derive the induced level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ which we use along with the Drinfeld's coproducts Theorem 5.1.5 in section 5.1. In the next section we construct the irreducible level- $k+1$ representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Definition 5.2.1. *Define the fundamental weight $\Lambda_a (a = 0, 1) \in \mathfrak{h}^*$ by $\Lambda_1 = \bar{\Lambda}_1 + \Lambda_0$ such that*

$$\langle \Lambda_a, h \rangle = \delta_{a,1}, \quad \langle \Lambda_a, c \rangle = \delta_{a,0}.$$

From Definitions 5.2.1, 2.3.6, 2.3.7, (3.1.1)-(3.1.3) and (2.2.21) we have

5.2 Level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

Theorem 5.2.2. [53] For $a = 0, 1$. Define $\mathcal{V}(\Lambda_a + \mu, \mu) = \bigoplus_{\gamma, \kappa \in \mathbb{Q}} (\mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes e^{\Lambda_a + \gamma}) \otimes e^{Q\bar{\mu} + \kappa})$.

Let $\rho : U_{q,p}(\widehat{\mathfrak{sl}}_2) \longrightarrow \text{End}(\mathcal{V}(\Lambda_a + \mu, \mu))$ by

$$\begin{aligned} \rho(\psi^+(z)) &= q^{-h} e^{-2Q} \exp\left(- (q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_{-n})}{1 - p^n} (zq^{\frac{1}{2}})^n\right) \\ &\quad \times \exp\left((q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_n)}{1 - p^n} (zq^{\frac{1}{2}})^{-n}\right) \end{aligned} \quad (5.2.1)$$

$$\begin{aligned} \rho(\psi^-(z)) &= q^h e^{-2Q} \exp\left(- (q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_{-n})}{1 - p^n} (zq^{-\frac{1}{2}})^n\right) \\ &\quad \times \exp\left((q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_n)}{1 - p^n} (zq^{-\frac{1}{2}})^{-n}\right) \end{aligned} \quad (5.2.2)$$

$$\rho(e(z)) =: \exp\left(- \sum_{n \neq 0} \frac{\rho(\alpha_n)}{[n]} z^{-n}\right) : e^\alpha z^{h+1} \quad (5.2.3)$$

$$\rho(f(z)) =: \exp\left(\sum_{n \neq 0} \frac{\rho(\alpha'_n)}{[n]} z^{-n}\right) : e^{-\alpha} z^{-h+1}, \quad (5.2.4)$$

where $\mathcal{F}_{\alpha,1}$ is the polynomial ring $\mathbb{C}[\alpha_{-m} \ (m > 0)]$. For $u \in \mathbb{C}[\alpha_{-m} \ (m > 0)]$

$$\begin{aligned} \rho(c) \cdot u &= u, \quad \rho(\alpha_{-n}) \cdot u = \alpha_{-n} u, \\ \rho(\alpha_n) \cdot u &= \frac{[2n][n]}{n} \frac{1 - p^n}{1 - p^{*n}} q^{-n} \frac{\partial}{\partial \alpha_{-n}} u \quad (n > 0). \end{aligned}$$

Then $\mathcal{V}(\Lambda_a + \mu, \mu)$ is the level-1 irreducible highest weight module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the highest weight $(\Lambda_a + \mu, \mu)$ and the highest weight vector $v_0 = 1 \otimes 1 \otimes e^{\Lambda_a} \otimes e^Q$.

For convention, we will drop ρ to refer the elements in $\text{End}(\mathcal{V}(\Lambda_a + \mu, \mu))$. The following proposition states the OPE relations of the level-1 elliptic operators. These OPE relations are used later in proof of some lemmas.

Proposition 5.2.3. The level-1 elliptic operators satisfy the following relations

$$e(z)e(w) = \frac{(q^{-2}p^* \frac{w}{z}; p^*)_\infty (q^{-2} \frac{w}{z}; q^{2c})_\infty}{(q^2 p^* \frac{w}{z}; p^*)_\infty (q^2 \frac{w}{z}; q^{2c})_\infty} : e(z)e(w) : \quad (5.2.5)$$

$$\psi^-(z)e(w) = \frac{(q^{-2-\frac{c}{2}} \frac{w}{z}; pq^{-2c})_\infty}{(q^{2-\frac{c}{2}} \frac{w}{z}; pq^{-2c})_\infty} : \psi^-(z)e(w) : \quad (5.2.6)$$

$$f(z)f(w) = \frac{(q^{-2} \frac{w}{z}; q^{2c})_\infty (q^2 \frac{w}{z}; p)_\infty}{(q^2 \frac{w}{z}; q^{2c})_\infty (q^{-2} \frac{w}{z}; p)_\infty} : f(z)f(w) : \quad (5.2.7)$$

$$f(z)\psi^+(w) = \frac{(q^{-2+\frac{c}{2}} \frac{w}{z}; p)_\infty}{(q^{2+\frac{c}{2}} \frac{w}{z}; p)_\infty} : f(z)\psi^+(w) : \quad (5.2.8)$$

$$\psi^\pm(z)\psi^\pm(w) = \frac{(q^{-2} \frac{w}{z}; pq^{-2c})_\infty (q^2 \frac{w}{z}; p)_\infty}{(q^2 \frac{w}{z}; pq^{-2c})_\infty (q^{-2} \frac{w}{z}; p)_\infty} : \psi^\pm(z)\psi^\pm(w) :, \quad (5.2.9)$$

where c acts on $\mathcal{V}(\Lambda_a + \mu, \mu)$ by 1.

5.3 Level- $k + 1$ realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

For $k > 0, \lambda_i \in \mathfrak{h}^*, \mu^{(i)} \in H^*(i \in \{0, 1, \dots, k+1\})$. Let's consider a tensor product of $k + 1$ copies of the level-1 highest weight modules $\mathcal{V}(\Lambda_a + \mu, \mu)(a = 0, 1)$

$$\begin{aligned} \mathcal{V}_{k+1}(\lambda_i, \mu) &= \mathcal{V}(\Lambda_{a^{(1)}} + \mu^{(1)}, \mu^{(1)}) \otimes \cdots \otimes \mathcal{V}(\Lambda_{a^{(i)}} + \mu^{(i)}, \mu^{(i)}) \otimes \mathcal{V}(\Lambda_{a^{(i+1)}} + \mu^{(i+1)}, \mu^{(i+1)}) \\ &\quad \otimes \cdots \otimes \mathcal{V}(\Lambda_{a^{(k+1)}} + \mu^{(k+1)}, \mu^{(k+1)}), \end{aligned} \quad (5.3.1)$$

such that $a^{(1)}, \dots, a^{(k+1)} \in \{0, 1\}$ and take i of a 's as 0 and $k + 1 - i$ of a 's as 1. Denote by $v^{(k+1)} \in \mathcal{V}_{k+1}(\lambda, \mu)$ the tensor product of the highest weight vectors in the tensor factors in relation (5.3.1). For calculation of the highest weight of $v^{(k+1)}$, we refer to [52].

Theorem 5.3.1. *The space $\mathcal{V}_{k+1}(\lambda_i, \mu)$ is the level- $k + 1$ highest weight module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the highest weight*

$$(\lambda_i, \mu) = (i\Lambda_0 + (k + 1 - i)\Lambda_1 + \sum_{j=1}^{k+1} \mu^{(j)}, \sum_{j=1}^{k+1} \mu^{(j)})$$

by the action

$$\begin{aligned} \Delta^k(e(z)) &= \sum_{i=1}^{k+1} e^i(z), \\ e^i(z) &= 1 \otimes \cdots \otimes 1 \otimes e(zq^{-(c^{(i+1)} + \cdots + c^{(k+1)})}) \otimes \psi^-(zq^{-(\frac{c^{(i+1)}}{2} + c^{(i+2)} + \cdots + c^{(k+1)})}) \\ &\quad \otimes \psi^-(zq^{-(\frac{c^{(i+2)}}{2} + c^{(i+3)} + \cdots + c^{(k+1)})}) \otimes \cdots \otimes \psi^-(zq^{-\frac{c^{(k+1)}}{2}}), \end{aligned} \quad (5.3.2)$$

$$\begin{aligned} \Delta^k(f(z)) &= \sum_{i=1}^{k+1} f^i(z), \\ f^i(z) &= \psi^+(zq^{-\frac{c^{(1)}}{2}}) \otimes \psi^+(zq^{-(c^{(1)} + \frac{c^{(2)}}{2})}) \otimes \cdots \otimes \\ &\quad \psi^+(zq^{-(c^{(1)} + \cdots + c^{(i-2)} + \frac{c^{(i-1)}}{2})}) \otimes f(zq^{-(c^{(1)} + \cdots + c^{(i-2)} + c^{(i-1)})}) \otimes 1 \cdots \otimes 1, \end{aligned} \quad (5.3.3)$$

$$\begin{aligned} \Delta^k(\psi^\pm(z)) &= \psi^\pm(zq^{\pm \frac{c^{(2)} + c^{(3)} + \cdots + c^{(k+1)}}{2}}) \otimes \psi^\pm(zq^{\mp \frac{c^{(1)}}{2} \pm \frac{c^{(3)} + c^{(4)} + \cdots + c^{(k+1)}}{2}}) \\ &\quad \otimes \psi^\pm(zq^{\mp \frac{c^{(1)} + c^{(2)}}{2} \pm \frac{c^{(4)} + c^{(5)} + \cdots + c^{(k+1)}}{2}}) \otimes \cdots \otimes \psi^\pm(zq^{-\frac{c^{(1)} + \cdots + c^{(k)}}{2}}), \end{aligned} \quad (5.3.4)$$

where $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$ and $c^{(i)}$ acts on $\mathcal{V}(\Lambda_{a^{(i)}} + \mu^{(i)}, \mu^{(i)})$ as 1.

Proof. See [52].

We also obtain the comultiplication formula Δ^k of boson operator $\alpha_n (n \neq 0)$ from $\Delta^k(\psi^\pm(z))$.

Corollary 5.3.2. *For $k \geq 1, n \neq 0$. The boson operator is*

$$\begin{aligned} \Delta^k(\alpha_n) = & \alpha_n \otimes 1 \cdots 1 \otimes 1 + \frac{(1-p^n)q^{-c^{(1)}n}}{1-p^nq^{-2c^{(1)}n}} \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)})n}} \otimes 1 \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \cdots + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}} \otimes 1 \cdots \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \cdots + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}} \otimes 1 \cdots \otimes \alpha_n, \end{aligned} \quad (5.3.5)$$

where $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$.

Proof. See [52].

The operators $\Delta^k(\alpha_n)(n \neq 0)$ give a level- $k + 1$ realization of the Heisenberg algebra $U_{q,p}(\mathcal{H})$.

Proposition 5.3.3. *The operators $\Delta^k(\alpha_n)(n \neq 0)$ and $\Delta^k(c)$ satisfy*

$$[\Delta^k(\alpha_m), \Delta^k(\alpha_n)] = \frac{[2m][\Delta^k(c)m]}{m} \frac{1-p^m}{1-p^mq^{-2\Delta^k(c)m}} q^{-\Delta^k(c)m} \delta_{m+n,0}, \quad (5.3.6)$$

$$[\Delta^k(\alpha_m), \Delta^k(e(z))] = \frac{[2m]}{m} \frac{1-p^m}{1-p^mq^{-2\Delta^k(c)m}} q^{-\Delta^k(c)m} z^m \Delta^k(e(z)), \quad (5.3.7)$$

$$[\Delta^k(\alpha_m), \Delta^k(f(z))] = -\frac{[2m]}{m} \frac{1-p^mq^{-2\Delta^k(c)m}}{1-p^m} q^{\Delta^k(c)m} z^m \Delta^k(f(z)). \quad (5.3.8)$$

By means of an elliptic analogue of the Drinfeld coproduct, we have found the higher level module of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

5.4 Quantum Z -algebra structure of level- $k + 1$ realization

Here we give a realization of the level- $k + 1$ quantum Z -algebra. Let's introduce the vertex operators $E_{(k)}^\pm(\alpha, z)$ and $E_{(k)}^\pm(\alpha', z)$ in the following definition.

Definition 5.4.1. *By using the level- $k + 1$ elliptic bosons $\Delta^k(\alpha_n)(n \neq 0)$, we define the vertex operators*

$$E_{(k)}^\pm(\alpha, z) = \exp\left(\pm \sum_{n>0} \frac{\Delta^k(\alpha_{\pm n})}{[\Delta^k(c)n]} z^{\mp n}\right), \quad E_{(k)}^\pm(\alpha', z) = \exp\left(\mp \sum_{n>0} \frac{\Delta^k(\alpha'_{\pm n})}{[\Delta^k(c)n]} z^{\mp n}\right),$$

which are formal Laurent series in z with coefficient in $\text{End}\mathcal{V}_{k+1}(\lambda_i, \mu)$.

The following proposition is a consequence of the commutation relations (5.3.6)-(5.3.8) in Proposition 5.3.3 with $\Delta^k(c)$ acts as the scalar $k + 1$.

Proposition 5.4.1. $E_{(k)}^{\pm}(\alpha, z)$ and $E_{(k)}^{\pm}(\alpha', z)$ satisfy the following relations:

$$E_{(k)}^{+}(\alpha, z)E_{(k)}^{-}(\alpha, w) = \frac{(q^{-2+2(k+1)}w/z; q^{2(k+1)})_{\infty}(q^{-2}w/z; pq^{-2(k+1)})_{\infty}}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_{\infty}(q^2w/z; pq^{-2(k+1)})_{\infty}} E_{(k)}^{-}(\alpha, w)E_{(k)}^{+}(\alpha, z), \quad (5.4.1)$$

$$E_{(k)}^{+}(\alpha', z)E_{(k)}^{-}(\alpha', w) = \frac{(q^{-2}w/z; q^{2(k+1)})_{\infty}(q^2w/z; p)_{\infty}}{(q^2w/z; q^{2(k+1)})_{\infty}(q^{-2}w/z; p)_{\infty}} E_{(k)}^{-}(\alpha', w)E_{(k)}^{+}(\alpha', z), \quad (5.4.2)$$

$$E_{(k)}^{+}(\alpha, z)E_{(k)}^{-}(\alpha', w) = \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_{\infty}} E_{(k)}^{-}(\alpha', w)E_{(k)}^{+}(\alpha, z), \quad (5.4.3)$$

$$E_{(k)}^{+}(\alpha', z)E_{(k)}^{-}(\alpha, w) = \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_{\infty}} E_{(k)}^{-}(\alpha, w)E_{(k)}^{+}(\alpha', z), \quad (5.4.4)$$

$$E_{(k)}^{\pm}(\alpha, z) \Delta^k(e(w)) = \frac{(q^{\pm 2+2(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\pm 2}(w/z)^{\pm 1}; pq^{-2(k+1)})_{\infty}}{(q^{\mp 2+2(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\mp 2}(w/z)^{\pm 1}; pq^{-2(k+1)})_{\infty}} \times \Delta^k(e(w))E_{(k)}^{\pm}(\alpha, z), \quad (5.4.5)$$

$$E_{(k)}^{\pm}(\alpha', z) \Delta^k(f(w)) = \frac{(q^{\pm 2}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\pm 2}(w/z)^{\pm 1}; p)_{\infty}}{(q^{\mp 2}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\mp 2}(w/z)^{\pm 1}; p)_{\infty}} \Delta^k(f(w))E_{(k)}^{\pm}(\alpha', z), \quad (5.4.6)$$

$$E_{(k)}^{\pm}(\alpha', z) \Delta^k(e(w)) = \frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}} \Delta^k(e(w))E_{(k)}^{\pm}(\alpha', z), \quad (5.4.7)$$

$$E_{(k)}^{\pm}(\alpha, z) \Delta^k(f(w)) = \frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}} \Delta^k(f(w))E_{(k)}^{\pm}(\alpha, z). \quad (5.4.8)$$

Definition 5.4.2. [53] For $k \in \mathbb{Z}_{>0}$. We define the level- $k+1$ quantum Z -operators by

$$\Delta^k(e(z)) = E(k, \alpha, z)Z^{+}(z)$$

$$\Delta^k(f(z)) = E(k, \alpha', z)Z^{-}(z)$$

where

$$\begin{aligned} E(k, \alpha, z) &= E_{(k)}^{-}(-\alpha, z)E_{(k)}^{+}(\alpha, z) \\ &= \exp\left(\sum_{n>0} \frac{\Delta^k(\alpha_{-n})}{[\Delta^k(c)n]} z^n\right) \exp\left(-\sum_{n>0} \frac{\Delta^k(\alpha_n)}{[\Delta^k(c)n]} z^{-n}\right), \end{aligned} \quad (5.4.9)$$

$$\begin{aligned} E(k, \alpha', z) &= E_{(k)}^{-}(-\alpha', z)E_{(k)}^{+}(\alpha', z) \\ &= \exp\left(-\sum_{n>0} \frac{\Delta^k(\alpha'_{-n})}{[\Delta^k(c)n]} z^n\right) \exp\left(\sum_{n>0} \frac{\Delta^k(\alpha'_n)}{[\Delta^k(cn)]} z^{-n}\right), \end{aligned} \quad (5.4.10)$$

$$Z^{\pm}(z) = \sum_{i=1}^{k+1} Z_i^{\pm}(z). \quad (5.4.11)$$

Since $\Delta^k(e(z))$ and $\Delta^k(f(z))$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we find that $Z^{\pm}(z)$ satisfy the following relations [53]:

Theorem 5.4.2. [53]

$$g(P + h)Z^+(z) = Z^+(z)g(P + h), \quad g(P)Z^+(z) = Z^+(z)g(P - \langle Q, P \rangle), \quad (5.4.12)$$

$$g(P + h)Z^-(z) = Z^-(z)g(P + h - \langle \alpha, P + h \rangle), \quad g(P)Z^-(z) = Z^-(z)g(P), \quad (5.4.13)$$

$$[d, Z^\pm(z)] = -z \frac{\partial}{\partial z} Z^\pm(z), \quad (5.4.14)$$

$$[\Delta^k(\alpha_m), Z^\pm(w)] = 0, \quad (5.4.15)$$

$$\Delta^k(K^\pm)Z^+(z) = q^{\mp 2(k+1)}Z^+(z)\Delta^k(K^\pm), \quad \Delta^k(K^\pm)Z^-(z) = q^{\pm 2(k+1)}Z^-(z)\Delta^k(K^\pm), \quad (5.4.16)$$

$$z \frac{(q^{-2}w/z; q^{2(k+1)})_\infty}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_\infty} Z^\pm(z)Z^\pm(w) = -w \frac{(q^{-2}z/w; q^{2(k+1)})_\infty}{(q^{2+2(k+1)}z/w; q^{2(k+1)})_\infty} Z^\pm(w)Z^\pm(z), \quad (5.4.17)$$

$$\begin{aligned} & \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_\infty} Z^+(z)Z^-(w) - \frac{(q^{2+(k+1)}z/w; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}z/w; q^{2(k+1)})_\infty} Z^-(w)Z^+(z) \\ &= \frac{1}{q - q^{-1}} \left(\Delta^k(K^-)\delta(q^{-(k+1)}z/w) - \Delta^k(K^+)\delta(q^{(k+1)}z/w) \right). \end{aligned} \quad (5.4.18)$$

Proof. See [52].

From Definition 5.4.2, Theorem 5.3.1 and Theorem 5.2.2 with $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$, we express the level- $k + 1$ quantum Z -operators as follows

$$\begin{aligned} Z^+(z) &= \sum_{i=1}^{k+1} E_{(k)}^-(\alpha, z) \mathfrak{E}_i^-(\alpha, z) \mathfrak{E}_i^+(\alpha, z) E_{(k)}^+(\alpha, z) \\ &\quad \times (1 \otimes \cdots \otimes e^\alpha \otimes e^{-2Q} \otimes \cdots \otimes e^{-2Q}) \\ &\quad \times (1 \otimes \cdots \otimes z^h q^{-(c^{(i+1)} + \cdots + c^{(k+1)})h} \otimes q^h \otimes \cdots \otimes q^h) z q^{-(c^{(i+1)} + \cdots + c^{(k+1)})}, \\ Z^-(z) &= \sum_{i=1}^{k+1} E_{(k)}^-(\alpha', z) \mathfrak{F}_i^-(\alpha, z) \mathfrak{F}_i^+(\alpha, z) E_{(k)}^+(\alpha', z) \\ &\quad \times (e^{-2Q} \otimes \cdots \otimes e^{-2Q} \otimes e^{-\alpha} \otimes 1 \cdots \otimes 1) \\ &\quad \times (q^{-h} \otimes \cdots \otimes q^{-h} \otimes z^{-h} q^{(c^{(i+1)} + \cdots + c^{(k+1)})h} \otimes 1 \cdots \otimes 1) z q^{-(c^{(i+1)} + \cdots + c^{(k+1)})}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{E}_i^-(\alpha, z) &= \exp \left((q^{-1} - q) \sum_{n>0} \left\{ 1 \otimes \cdots \otimes \frac{\alpha_{-n} q^{c^{(i)}n}}{1 - q^{2c^{(i)}n}} q^{-(c^{(i+1)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\ &\quad \left. \left. + 1 \otimes \cdots \otimes \frac{p^n \alpha_{-n}}{1 - p^n} q^{-(c^{(i+2)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \frac{p^n \alpha_{-n}}{1 - p^n} \right\} z^n \right) \\ \mathfrak{E}_i^+(\alpha, z) &= \exp \left((q - q^{-1}) \sum_{n>0} \left\{ 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - q^{2c^{(i)}n}} q^{(c^{(i)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\ &\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - p^n} q^{-(c^{(i+2)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - p^n} \right\} z^{-n} \right) \end{aligned}$$

$$\begin{aligned}
 \mathfrak{F}_i^-(\alpha, z) &= \exp \left(-(q - q^{-1}) \sum_{n>0} \left\{ \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(1)}n}} q^{-c^{(1)}n} \otimes 1 \cdots \otimes 1 \right. \right. \\
 &\quad + 1 \otimes \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(2)}n}} q^{-(c^{(1)}+c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots \\
 &\quad + 1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(i-1)}n}} q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n} \otimes 1 \cdots \otimes 1 \\
 &\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - q^{2c^{(i)}n}} q^{-(c^{(1)}+\cdots+c^{(i-1)}n+c^{(i)}n} \otimes 1 \cdots \otimes 1 \right\} z^n \right) \\
 \mathfrak{F}_i^+(\alpha, z) &= \exp \left((q - q^{-1}) \sum_{n>0} \left\{ \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(1)}n}} q^{-c^{(1)}n} \otimes 1 \cdots \otimes 1 \right. \right. \\
 &\quad + 1 \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(2)}n}} q^{(c^{(1)}-c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots \\
 &\quad + 1 \otimes \cdots \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(i-1)}n}} q^{(c^{(1)}+\cdots+c^{(i-2)}-c^{(i-1)})n} \otimes 1 \cdots \otimes 1 \\
 &\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_n}{1 - q^{2c^{(i)}n}} q^{(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)}+c^{(i)})n} \otimes 1 \cdots \otimes 1 \right\} z^{-n} \right).
 \end{aligned}$$

5.5 Integrable condition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ -module

The integrable representation was defined in [34] as the locally nilpotancy of Chevally generators on the highest weight module of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. The quantum analogue of this condition appeared in [10]. In this section we will investigate the elliptic analogue of the integrability condition.

Theorem 5.5.1. *For $k \geq 1$. On the level- $k+1$ integrable module $\mathcal{V}_{k+1}(\lambda_i, \mu)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we obtain a quantum analogue of the condition of integrability (an elliptic analogue of the Wheel condition) as*

$$\mathcal{E}_{k+2}(z) = \Delta^k(e(z)) \Delta^k(e(zq^2)) \cdots \Delta^k(e(zq^{2(k+1)})) = 0 \quad (5.5.1)$$

$$\mathcal{F}_{k+2}(z) = \Delta^k(f(z)) \Delta^k(f(zq^{-2m})) \cdots \Delta^k(f(zq^{-2(k+1)})) = 0. \quad (5.5.2)$$

In order to show the proof of Theorem 5.5.1, we need the following OPE relations for $e^i(z)$ and $f^i(z)$ in the expansion of $\Delta^k(e(z))$ and $\Delta^k(f(z))$ respectively.

Lemma 5.5.2. *Set $i < j$. For $e^i(z)$*

$$e^i(z)e^j(w) = \frac{(q^{-2\frac{w}{z}}; pq^{-2\Delta^k(c)})_\infty}{(q^2\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^i(z)e^j(w) :, \quad (5.5.3)$$

$$e^j(z)e^j(w) = \frac{(q^{-2+2c^{(j)}\frac{w}{z}}; q^{2c^{(j)}})_\infty (q^{-2\frac{w}{z}}; pq^{-2\Delta^k(c)})_\infty}{(q^{2+2c^{(j)}\frac{w}{z}}; q^{2c^{(j)}})_\infty (q^2\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^j(z)e^j(w) :, \quad (5.5.4)$$

$$e^j(z)e^i(w) = \frac{(q^{-2-2\Delta^k(c)}p\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty}{(q^{2-2\Delta^k(c)}p\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^j(z)e^i(w) :. \quad (5.5.5)$$

For $f^i(z)$

$$f^i(z)f^j(w) = \frac{(q^2 \frac{w}{z}; p)_\infty}{(q^{-2} \frac{w}{z}; p)_\infty} : f^i(z)f^j(w) :, \quad (5.5.6)$$

$$f^j(z)f^j(w) = \frac{(q^2 \frac{w}{z}; p)_\infty}{(q^{-2} \frac{w}{z}; p)_\infty} \frac{(q^{-2} \frac{w}{z}; q^{2c(j)})_\infty}{(q^2 \frac{w}{z}; q^{2c(j)})_\infty} : f^j(z)f^j(w) :, \quad (5.5.7)$$

$$f^j(z)f^i(w) = \frac{(q^{-2} p \frac{w}{z}; p)_\infty}{(q^2 p \frac{w}{z}; p)_\infty} : f^j(z)f^i(w) :. \quad (5.5.8)$$

Proof. This follows from Proposition 5.2.3.

Proof. Let us show the proof of (5.5.1). From the comultiplication (5.3.2) in Theorem 5.3.1, we have the following product on $V_{k+1}(\lambda_i, \mu)$ for some positive integer N over all possible decompositions

$$\sum_{i_1, \dots, i_N \in \{1, \dots, k+1\}} e^{i_1}(z_{i_1})e^{i_2}(z_{i_2}) \cdots e^{i_N}(z_{i_N}),$$

where $c^{(i)} = 1$.

From the relations (5.5.3)-(5.5.5) in Lemma 5.5.2, one can show that for $z_{i_{j+1}}/z_{i_j} = q^2$ all terms in (5.5.9) are zero except for those with indices $i_1 > \cdots > i_{k+1}$. Suppose $N = k + 2$ and $z_{i_{j+1}}/z_{i_j} = q^2$, then for $m \neq n$ there is $i_m = i_n$. Thus we get the first condition of integrability. Similarly one can prove the $\mathfrak{F}_{k+2}(z)$ case.

Theorem 5.5.3. $\mathcal{E}_{k+1}(z)$ and $\mathcal{F}_{k+1}(z)$ give the following vertex operators

$$\begin{aligned} \mathcal{E}_{k+1}(z) &= \mathfrak{S}(p, q)_e : \exp \left(\sum_{n \neq 0} -\frac{\Delta^k(\alpha_n)}{[n]} q^{-kn} z^{-n} \right) : (1 \otimes K^- \otimes K^- \otimes \cdots \otimes K^-) \\ &\quad \times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{h+1} \otimes \cdots \otimes z^{h+1})(q^{kh} \otimes q^{(k-1)h} \otimes \cdots \otimes 1) q^{\frac{k(k+1)}{2}}, \end{aligned} \quad (5.5.9)$$

$$\begin{aligned} \mathcal{F}_{k+1}(z) &= \mathfrak{S}(p, q)_f : \exp \left(\sum_{n \neq 0} \frac{\Delta^k(\alpha'_n)}{[n]} q^{kn} z^{-n} \right) : (K^+ \otimes K^+ \otimes \cdots \otimes K^+ \otimes 1) \\ &\quad \times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{-h+1} \otimes \cdots \otimes z^{-h+1})(q^{(k+1)h} \otimes q^{(k)h} \otimes \cdots \otimes q^h) q^{-\frac{(k+1)(k+2)}{2}}, \end{aligned} \quad (5.5.10)$$

where

$$\begin{aligned}
 \mathfrak{S}(p, q)_e &= \frac{(q^{-2}pq^{-2(\Delta(c))}; pq^{-2(\Delta(c))})_\infty}{(q^2pq^{-2(\Delta(c))}; pq^{-2(\Delta(c))})_\infty} \prod_{j=1}^k \prod_{i=1}^j \frac{(q^{-2}pq^{-2(\Delta^{(j)}(c)+(2i-1))}; pq^{-2(\Delta^{(j)}(c))})_\infty}{(q^2pq^{-2(\Delta^{(j)}(c)+(2i-1))}; pq^{-2(\Delta^{(j)}(c))})_\infty} \\
 &\times \prod_{j \leq l=1}^{k-1} \prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty}{(q^2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty} \frac{(q^2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}, \\
 \mathfrak{S}(p, q)_f &= \prod_{j=0}^{k-1} \prod_{i=1}^{k-j} \frac{(pq^{2-2i-2(c^{(1)}+\dots+c^{(j)})}; pq^{-2(c^{(1)}+\dots+c^{(j)})})_\infty}{(pq^{-2-2i-2(c^{(1)}+\dots+c^{(j)})}; pq^{-2(c^{(1)}+\dots+c^{(j)})})_\infty} \\
 &\times \prod_{j \leq l=1}^{k-1} \prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty}{(q^2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty} \frac{(q^2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}
 \end{aligned}$$

Proof. See [52].

Proposition 5.5.4. *On $\mathcal{V}_{k+1}(\lambda_i, \mu)$, the vertex operators $\mathcal{E}_{k+1}(z)$ and $\mathcal{F}_{k+1}(z)$ satisfy the following q -difference equations*

$$\begin{aligned}
 \mathcal{E}_{k+1}(zq^2) &= \Delta^k(q^{h+1}) \exp \left((q - q^{-1}) \sum_{n>0} \Delta^k(\alpha_{-n})(q^{k+1}z)^n \right) \\
 &\times \mathcal{E}_{k+1}(z) \Delta^k(q^{h+1}) \exp \left(-(q - q^{-1}) \sum_{n>0} \Delta^k(\alpha_n)(q^{k+1}z)^{-n} \right), \quad (5.5.11)
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{F}_{k+1}(zq^2) &= \Delta^k(q^{-(h+1)}) \exp \left((q - q^{-1}) \sum_{n>0} \Delta^k(\alpha'_{-n})(q^{k+1}z)^n \right) \\
 &\times \mathcal{F}_{k+1}(z) \Delta^k(q^{-(h+1)}) \exp \left(-(q - q^{-1}) \sum_{n>0} \Delta^k(\alpha'_n)(q^{k+1}z)^{-n} \right). \quad (5.5.12)
 \end{aligned}$$

Chapter 6

Elliptic bosons of $U_{q,p}(C_n^{(1)})$

The elliptic bosons $\alpha_{j,m}(\alpha_{j,m}^V)$ are the (co-) roots type Heisenberg generators appears in $U_{q,p}(\widehat{\mathfrak{g}})$. There is another type of elliptic bosons, the fundamental weight type A_m^j appeared in [1, 21] for the level-1 ($c = 1$) $\widehat{\mathfrak{sl}}_N$ and in [24] for level-1 $\widehat{\mathfrak{g}}$. A_m^j are used to construct the derivation operators d of the level-1 irreducible highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$. Also, the elliptic bosons $\mathcal{E}_m^{\pm j}$ of type of the orthogonal basis can be constructed explicitly and in turn they are used to realize the L operators and the vertex operators.

In this chapter we review an explicit construction of the elliptic bosons of the fundamental weight type A_m^j and the orthogonal basis type $\mathcal{E}_m^{\pm j}$ for $\widehat{\mathfrak{g}} = C_n^{(1)}$ for arbitrary level c . The level-1 bosons A_m^j and $\mathcal{E}_m^{\pm j}$ are used to realize the derivation operator d which appear in subsection 4.2. The elliptic currents $k_{\pm j}(z)$ associated with the orthonormal basis type $\mathcal{E}_m^{\pm j}$ are used to redefine the elliptic current $\psi_j^{\pm}(z)$. The elliptic currents $k_{\pm j}(z)$ are expected to play an important role to construct an L -operator of $U_{q,p}(C_n^{(1)})$. In section 6.1, we give definitions of the arbitrary level c fundamental weight type A_m^j and the orthogonal basis type $\mathcal{E}_m^{\pm j}$ for $U_{q,p}(C_l^{(1)})$ and their commutation relations. In section 6.2 we find the elliptic currents $k_{\pm j}(z)$ and calculate commutation relations among $\mathcal{E}_m^{\pm j}, k_{\pm j}(z)$ and the generators of $U_{q,p}(C_l^{(1)})$. These results are published in [53].

6.1 Definition

Let us set $\eta = -tg/2$ ($t = (\text{long root})^2/2$), where $g = n + 1, t = 2, \eta = -(l + 1)$. Let $\alpha_{i,m}$ be the elliptic bosons of the simple root type as in sec.2.3.1. We define the fundamental weight type elliptic bosons A_m^j ($1 \leq j \leq l, m \in \mathbb{Z}_{\neq 0}$) by

$$[\alpha_{i,m}, A_n^j] = -\delta_{i,j} \delta_{m+n,0} \frac{[cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} \quad (1 \leq i, j \leq l). \quad (6.1.1)$$

6.1 Definition

Note that using the matrix $B(m) = ([b_{i,j}m])_{1 \leq i,j \leq l}$, we have [24]

$$A_m^j = \sum_{k=1}^l (B(m)^{-1})_{kj} \alpha_{k,m}.$$

Solving (6.1.1) we obtain

$$\begin{aligned} A_m^j &= C_m \left((q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} \right. \\ &\quad \left. + [jm] \sum_{k=j+1}^{l-1} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} + [jm] \alpha_{l,m} \right), \quad (1 \leq j \leq l-1), \\ A_m^l &= C_m \left(\sum_{k=1}^{l-1} [km] \alpha_{k,m} + \frac{[m]}{[2m]} [lm] \alpha_{l,m} \right). \end{aligned}$$

Here

$$C_m = \frac{[\eta m]}{[m]^2 [2\eta m]}.$$

We then divide A_m^j into two terms and define the elliptic bosons $\mathcal{E}_m^{\pm j}$ of the orthogonal basis type as follows.

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j} \tag{6.1.2}$$

$$\begin{aligned} \mathcal{E}_m^{\pm j} &= q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-1} [(\eta+k)m]_+ \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right) \\ &\quad (1 \leq j \leq l-1) \end{aligned} \tag{6.1.3}$$

$$A_m^l = \frac{1}{q^m + q^{-m}} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}), \tag{6.1.4}$$

$$\mathcal{E}_m^{\pm l} = q^{\pm lm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{l-1} [km] \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right). \tag{6.1.5}$$

Proposition 6.1.1.

$$\alpha_{j,m} = \pm [m]^2 (q - q^{-1}) (\mathcal{E}_m^{\pm j} - q^{\mp \eta m} \mathcal{E}_m^{\pm(j+1)}), \quad (1 \leq j \leq l-1) \tag{6.1.6}$$

$$\alpha_{l,m} = [m]^2 (q - q^{-1}) (q^m \mathcal{E}_m^{+l} - q^{-m} \mathcal{E}_m^{-l}). \tag{6.1.7}$$

Proposition 6.1.2. *The following relations hold.*

$$\mathcal{E}_m^{\pm 1} = \pm \frac{q^{\pm m}}{q^m - q^{-m}} A_m^1, \quad \mathcal{E}_m^{\pm j} = \pm \frac{1}{q^m - q^{-m}} (q^{\pm m} A_m^j - A_m^{j-1}), \tag{6.1.8}$$

where $2 \leq j \leq l-1$. In addition, we have

$$\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} \left((q^m + q^{-m}) q^{\pm m} A_m^l - A_m^{l-1} \right). \tag{6.1.9}$$

6.2 The elliptic currents $k_{\pm j}(z)$

Remark. The level-1 case i.e. $c = 1$, the $C_l^{(1)}$ type relation is different from those given in [24]. At least the formulas seem to be reversed. Our definitions and relation are valid for arbitrary level c .

Although the expressions of $\mathcal{E}_m^{\pm j}$ are complicated depending on the types of the affine Lie algebras, their commutation relations are rather universal:

Theorem 6.1.3. *For $1 \leq j, k \leq l$ the following commutation relations hold.*

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = \delta_{m+n,0} \frac{[cm][\eta m][2(\eta+1)m]}{m(q-q^{-1})^2[m]^3[2\eta m][(\eta+1)m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (6.1.10)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \mp \delta_{m+n,0} \frac{q^{\pm jm}[cm][\eta m]}{m[m]^3(q-q^{-1})^2[2\eta m]} \frac{[rm]}{[r^*m]} \left(q^{\pm(\eta+j)m}[m] \pm q^{\mp(j-1)m}[\eta m] \right), \quad (6.1.11)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \mp \operatorname{sgn}(k-j) \delta_{m+n,0} q^{\mp(\operatorname{sgn}(k-j)\eta+k-j)m} \frac{[cm][\eta m]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (6.1.12)$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = \mp \delta_{m+n,0} q^{\pm(\eta+j+k)m} \frac{[cm][\eta m]}{m(q-q^{-1})[m]^2[2\eta m]} \frac{1-p^m}{1-p^{*m}} q^{-cm}, \quad (6.1.13)$$

where

$$\operatorname{sgn}(l-j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}$$

Proof. Straightforward calculation using Proposition 6.1.2 and (6.1.1).

Proposition 6.1.4. *For $1 \leq i \leq l$, $1 \leq j \leq l-1$, the following commutation relations hold*

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm j}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1}) \quad (6.1.14)$$

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm l}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} q^{-cm} (q^{\mp m} (q^m + q^{-m}) \delta_{i,l} - \delta_{i,l-1}) \quad (6.1.15)$$

From (2.3.16) and (2.3.17) we also obtain the following relations.

Proposition 6.1.5. *For $1 \leq i \leq l-1$, $1 \leq j \leq l$,*

$$[\mathcal{E}_m^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} e_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}), \quad (6.1.16)$$

$$[\mathcal{E}_m^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}), \quad (6.1.17)$$

$$[\mathcal{E}_m^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1-p^m}{1-p^{*m}} e_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}), \quad (6.1.18)$$

$$[\mathcal{E}_m^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}). \quad (6.1.19)$$

6.2 The elliptic currents $k_{\pm j}(z)$

Let us set

$$\psi_j(z) =: \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1 - p^m} p^m z^{-m} \right\} : . \quad (6.2.1)$$

6.2 The elliptic currents $k_{\pm j}(z)$

Then the elliptic currents $\psi_j^{\pm}(z)$ in Definition 2.3.1 can be written as

$$\psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}}z) = K_j^- \psi_j(pq^{-c}z). \quad (6.2.2)$$

Let us introduce the new currents $k_{\pm j}(z)$ ($1 \leq j \leq l$) associated with $\mathcal{E}_m^{\pm j}$ by

$$k_{\pm j}(z) = : \exp \left\{ - \sum_{m \neq 0} \frac{[m]^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} :. \quad (6.2.3)$$

Then from Proposition 6.1.1 we have the following decompositions.

Proposition 6.2.1. *For $1 \leq j \leq l - 1$ we have*

$$\psi_j(z) = : k_{+j}(z) k_{+(j+1)}(qz)^{-1} :=: k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1}z) : \quad (6.2.4)$$

$$\psi_l(z) = : k_{+l}(q^{-1}z) k_{-l}(qz)^{-1} :. \quad (6.2.5)$$

Now let us introduce the functions $\tilde{\rho}^+(z)$, which appear associated with the elliptic dynamical \mathcal{R} -matrices [40]:

$$\tilde{\rho}^+(z) = \frac{\{\xi z\}^2 \{\xi^2 q^{-2} z\} \{q^2 z\} \{p \xi^2 / z\} \{p / z\} \{p \xi q^2 / z\} \{p \xi q^{-2} / z\}}{\{\xi^2 z\} \{z\} \{\xi q^2 z\} \{\xi q^{-2} z\} \{p \xi / z\}^2 \{p \xi^2 q^{-2} / z\} \{p q^2 / z\}}, \quad (6.2.6)$$

where $\xi = q^{-2\eta}$, $\{z\} = (z; p, \xi^2)_{\infty}$. The following Theorem indicates a deep relationship between $k_{\pm j}(z)$'s and elliptic dynamical \mathcal{R} -matrices.

Theorem 6.2.2.

$$\begin{aligned} k_{\pm j}(z_1) k_{\pm j}(z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} k_{\pm j}(z_2) k_{\pm j}(z_1), \quad (1 \leq j \leq l), \\ k_{+j}(q^j z_1) k_{+k}(q^l z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{+l}(q^l z_2) k_{+j}(q^j z_1) \quad (1 \leq j < k \leq l), \\ k_{-j}(q^{-j} z_1) k_{-k}(q^{-k} z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{-k}(q^{-k} z_2) k_{-j}(q^{-j} z_1) \quad (1 \leq k < j \leq l), \\ k_{+j}(q^j z_1) k_{-k}(q^{-k} \xi z_2) &= \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} k_{-k}(q^{-k} \xi z_2) k_{+j}(q^j z_1) \quad (j \neq k), \\ k_{+j}(q^j z_1) k_{-j}(q^{-j} \xi z_2) &= \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{2j-2} \xi^{-1} z) \Theta_p(q^{2j} \xi^{-1} z)}{\Theta_{p^*}(q^{2j} \xi^{-1} z) \Theta_p(q^{2j-2} \xi^{-1} z)} \frac{\Theta_{p^*}(q^{-2}z) \Theta_p(z)}{\Theta_{p^*}(z) \Theta_p(q^{-2}z)} \\ &\quad \times k_{-j}(q^{-j} \xi z_2) k_{+j}(q^j z_1), \end{aligned}$$

where $z = z_1/z_2$ and $\tilde{\rho}^{+*}(z) = \tilde{\rho}^+(z)|_{p \rightarrow p^*}$.

Proof. Straightforward calculation using Theorem 6.1.3.

In addition from Proposition 6.1.5, we obtain:

Proposition 6.2.3. *For $1 \leq i \leq l-1$*

$$\begin{aligned}
 k_{\pm j}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c\mp 2}z)}e_j(z_2)k_{\pm j}(z_1) & (1 \leq j \leq l), \\
 k_{\pm j}(z_1)e_{j-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{j-1}(z_2)k_{\pm j}(z_1) & (2 \leq j \leq l), \\
 k_{\pm j}(z_1)e_k(z_2) &= e_k(z_2)k_{\pm j}(z_1) & (k \neq j, j-1), \\
 \\
 k_{\pm j}(z_1)f_j(z_2) &= \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)}f_j(z_2)k_{\pm j}(z_1) & (1 \leq j \leq l), \\
 k_{\pm j}(z_1)f_{j-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{j-1}(z_2)k_{\pm j}(z_1) & (2 \leq j \leq l), \\
 k_{\pm j}(z_1)f_k(z_2) &= f_k(z_2)k_{\pm j}(z_1) & (k \neq j, j-1).
 \end{aligned}$$

In addition, we have

$$\begin{aligned}
 k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z)}{\Theta_{p^*}(q^{-c\mp 3}z)}e_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) & (j \neq l, l-1), \\
 k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_p(q^{\mp 3}z)}{\Theta_p(q^{\pm 1}z)}f_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) & (j \neq l, l-1).
 \end{aligned}$$

The elliptic bosons $\mathcal{E}_m^{\pm j}$ and their elliptic currents $k_{\pm j}(z)$ are useful to realize the L -operators and the vertex operators for $U_{q,p}(C_n^{(1)})$ as well as deformation of the W -algebras.

Chapter 7

Discussions

Elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ had been widely researched in mathematics and physics, whereas its Z -algebra structure had not yet been studied so much. In this dissertation we have investigated the Z -algebra structure and how we can use it to construct the irreducible $U_{q,p}(\widehat{\mathfrak{g}})$ -module. We also have studied the higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its integrable condition by using the elliptic Drinfeld coproduct and the level-1 representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Finally, we have showed an explicit construction of another type of elliptic bosons for $U_{q,p}(C_l^{(1)})$.

First result. We have found the dynamical quantum Z -algebra \mathcal{Z}_k associated with the generic level- k module of the elliptic quantum algebra $U_{q,p}(\widehat{\mathfrak{g}})$ for general untwisted affine Lie algebras $\widehat{\mathfrak{g}}$. This result is a quantum dynamical analogue of Lepowsky-Wilson's Z -algebra [47]. We have derived the defining relations among the generators of \mathcal{Z}_k as well as the relations between its generators and those of $U_{q,p}(\widehat{\mathfrak{g}})$. The dynamical quantum Z -algebra turns out to be a tool to construct the induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules of general level. We have discussed the irreducibility of $U_{q,p}(\widehat{\mathfrak{g}})$ -module is governed by the dynamical quantum Z -algebra. We have given the example of level-1 irreducible dynamical quantum Z -algebra modules for $A_l^{(1)}, B_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and constructed the corresponding $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. The first result was discussed in Chapter 3 and 4 and published in [53].

According to this result, it is natural to search the dynamical quantum Z -algebra associated to $U_{q,p}(\widehat{\mathfrak{g}}^{(r)})$ for twisted affine Lie algebras $\widehat{\mathfrak{g}}^{(r)}$. Find the representation of $U_{q,p}$ for twisted and untwisted affine Lie algebras. Another problem can be considered is finding the explicit basis of the irreducible dynamical quantum Z -algebra modules. Define the $U_{q,p}(\widehat{\mathfrak{g}}^{(r)})$ by the associated Z -algebra and construct the screening operators of deformed W -algebra.

Second result. We have constructed the higher level (integral [10,13,45,46]) representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by using the elliptic Drinfeld coproduct [43] and the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [53]. We have found the elliptic analogue of the condition of integrability of the constructed

representation. The realization of the dynamical quantum Z -algebra for level- $k + 1$ is presented. This result is presented in Chapter 5 and is published in [52].

A direct open problems: what are the higher level representations of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}_N$ and in sequence for other types of twisted and untwisted affine Lie algebras?, find the Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{g}}^{(r)})$ for twisted affine Lie algebras $\widehat{\mathfrak{g}}^{(r)}$ and compute the correlation functions of the vertex operators by using the condition of integrability.

Third result. For arbitrary level, we have defined the fundamental basis type elliptic boson A_m^j [1, 21, 24] for $U_{q,p}(C_n^{(1)})$. We have derived the associated orthonormal basis type $\mathcal{E}_m^{\pm j}$ and the elliptic currents $k_{\pm j}(z)$. Several commutation relations are calculated among A_m^j , $\mathcal{E}_m^{\pm j}$, $k_{\pm j}(z)$, and the generators of $U_{q,p}(C_n^{(1)})$. The level-1 A_m^j is used in defining the derivation operator d of the highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$ [53]. $\mathcal{E}_m^{\pm j}$ and $k_{\pm j}(z)$ are useful in realization the L -operators, vertex operators and deformed W -algebra associated with $U_{q,p}(\widehat{\mathfrak{g}})$. This result is written in Chapter 6 and published in [53].

By those resulted structures, redefine the defining relations of $U_{q,p}(C_n^{(1)})$ in the sense of analytic continuation by using the elliptic currents $k_{\pm j}(z)$ to construct the half currents of $U_{q,p}(C_n^{(1)})$ and the L -operators. Define the H -Hopf algebroid structure in term of the L -operators and find the vertex operators as an intertwining operators of the $U_{q,p}(C_n^{(1)})$ -modules of generic level. Find the free field realization of the possible levels of the vertex operators. According to my knowledge, the fundamental basis type elliptic boson A_m^j , the orthonormal basis $\mathcal{E}_m^{\pm j}$ and the elliptic currents $k_{\pm j}(z)$ for twisted elliptic quantum algebra have not been studied yet.

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Articles

- (1) Elliptic Algebra $U_{q,p}(\hat{\mathfrak{g}})$ and Quantum Z -algebras
Rasha M. Farghly, Hitoshi Konno, and Kazuyuki Oshima
Algebra and Representation Theory, 18 (2015) 103–135.
- (2) Higher Level Representation of the Elliptic Quantum Group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$
and its Integrability
Rasha Mohamed Farghly
Hiroshima Mathematical Journal, in press.

Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{g}})$ and Quantum Z-algebras

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Received: 1 July 2013 / Accepted: 15 April 2014 / Published online: 3 June 2014
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Abstract A new definition of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ associated with an untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ is given as a topological algebra over the ring of formal power series in p . We also introduce a quantum dynamical analogue of Lepowsky-Wilson's Z-algebras. The Z-algebra governs the irreducibility of the infinite dimensional $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. Some level-1 examples indicate a direct connection of the irreducible $U_{q,p}(\widehat{\mathfrak{g}})$ -modules to those of the W-algebras associated with the coset $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}} \supset (\widehat{\mathfrak{g}})_{diag}$ with level $(r - g - 1, 1)$ (g : the dual Coxeter number), which includes Fateev-Lukyanov's WB_l -algebra.

Keywords Quantum group · Affine Lie algebra · Virasoro algebra · W-algebra · Z-algebra

Mathematics Subject Classification (2010) 17B37 · 20G42 · 81R10 · 81R50

1 Introduction

The algebra $U_{q,p}(\widehat{\mathfrak{g}})$ is an elliptic analogue [1, 2] of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization [3]. There are two types of the elliptic quantum groups, the vertex type and the face type [4, 5]. Deriving the L -operators [2, 6, 7] and introducing the

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Hopf-algebroid structure [8–10] $U_{q,p}(\widehat{\mathfrak{g}})$ is now recognized as a face type elliptic quantum group.

Originally $U_{q,p}(\widehat{\mathfrak{g}})$ with $p = q^{2r}$ was derived for $\widehat{\mathfrak{sl}}_2 = \widehat{\mathfrak{sl}}(2, \mathbb{C})$ [1] as a deformation of the screening currents of the coset conformal field theory (CFT) $\widehat{\mathfrak{sl}}_2 \oplus \widehat{\mathfrak{sl}}_2 \supset (\widehat{\mathfrak{sl}}_2)_{diag}$ with level $(r - k - 2, k)$ [11–16] instead of considering a deformation of $U_q(\widehat{\mathfrak{sl}}_2)$ itself. Such coset CFT is known to be realized in terms of the level- k free boson and the \mathbb{Z}_k -parafermion [17], or the Z -algebra [18] associated with the level- k standard representation of $\widehat{\mathfrak{sl}}_2^1$. It was then crucial in [1] to realize that the level- k boson should be deformed both q - and elliptically [19] whereas the \mathbb{Z}_k -parafermion gets only a q -deformation to obtain consistent relations for the generators in $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

In [2], a realization of $U_{q,p}(\widehat{\mathfrak{g}})$ for general untwisted affine Lie algebra $\widehat{\mathfrak{g}}$ was given by modifying the Drinfeld realization of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. However its structure associated with the quantum Z -algebras has not yet been discussed so far. The purpose of this paper is to address this subject. The general theory of the Z -algebra was studied by Lepowsky and Wilson [18] and by Gepner [20] in the representation theory of affine Lie algebras and in CFT, respectively. Its quantum deformation and application to the representations of $U_q(\widehat{\mathfrak{g}})$ was partially investigated in [1, 21–24]. A construction of the coset CFT associated with the general $\widehat{\mathfrak{g}}$ was also given [25] in terms of the generalized parafermions. We extend these studies to the elliptic algebras $U_{q,p}(\widehat{\mathfrak{g}})$. In particular, we define a dynamical analogue \mathcal{Z}_k of the quantum Z -algebras and show that the level- k highest weight representations of $U_{q,p}(\widehat{\mathfrak{g}})$ are realized in terms of \mathcal{Z}_k and the level- k elliptic bosons. It is then shown that the irreducibility of the infinite dimensional $U_{q,p}(\widehat{\mathfrak{g}})$ -modules is governed by the \mathcal{Z}_k -modules as in the affine Lie algebra cases [18].

On the other hand, it was conjectured [1, 2] that the $U_{q,p}(\widehat{\mathfrak{g}})$ provides an algebra of the screening currents of the deformation of the W -algebras associated with the coset $\widehat{\mathfrak{g}} \oplus \widehat{\mathfrak{g}} \supset (\widehat{\mathfrak{g}})_{diag}$ with level $(r - g - 1, 1)$. For the simply-laced $\widehat{\mathfrak{g}}$, such deformed W -algebras have been realized in [26–28], and in particular for the $\widehat{\mathfrak{sl}}_N$ case the conjecture has been established by an explicit comparison of the free field realizations [7, 26, 27, 29]. However for the non-simply laced $\widehat{\mathfrak{g}}$, deformation of the coset type W -algebras has not yet been studied at all. One should note that the coset type W -algebras associated with the non-simply laced $\widehat{\mathfrak{g}}$ are different from those obtained by the quantum Hamiltonian reduction. See for example [30]. We investigate this issue further by giving an explicit realization of the level-1 irreducible highest weight representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. We show that at least for $A_l^{(1)}$ and $D_l^{(1)}$ the level-1 elliptic currents $e_j(z)$ and $f_j(z)$ coincide with the screening currents of the deformed W -algebras obtained in [26–28]. We also show that the irreducible representations of $U_{q,p}(\widehat{\mathfrak{g}})$ is naturally decomposed into a direct sum of the irreducible W -algebras of the coset type for $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, D_l^{(1)}$. This suggests in particular an existence of a deformation of Fateev-Lukyanov's WB_l -algebra [31] as the commutant of the screening operators provided by the level-1 elliptic currents $e_j(z)$ and $f_j(z)$ of $U_{q,p}(B_l^{(1)})$.

It is also worth to mention that the coset type W -algebras describe a critical behavior of the face type elliptic solvable lattice models [32, 33]. Correspondingly the $U_{q,p}(\widehat{\mathfrak{g}})$ provides an algebraic framework to formulate the lattice model itself in the spirit of Jimbo and Miwa [34]. This has been established for $\widehat{\mathfrak{sl}}_N$ in [1, 2, 7, 10, 35, 36] by constructing the L -operator

¹The difference between Z -algebra and Parafermion is whether one adds zero-modes of the bosons to it or not.

and introducing the Hopf algebroid structure. In order to construct the L -operator of $U_{q,p}(\widehat{\mathfrak{g}})$ and also to get a realization of a generating function of the deformation of the W -algebras, it is crucial to introduce new types of elliptic bosons, which we call the fundamental weight type A_m^j and the orthonormal basis type $\mathcal{E}_m^{\pm j}$ distinguishing from the usual ones $\alpha_{j,m}$ ($\alpha_{j,m}^\vee$) corresponding to the simple (co-)root and appearing as generators of $U_{q,p}(\widehat{\mathfrak{g}})$. An idea of such bosons has already appeared in [26–28]. We give an explicit construction of them for $\widehat{\mathfrak{g}} = A_n^{(1)}, B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$. As a check we calculate the commutation relations among $\mathcal{E}_m^{\pm j}$ as well as among the elliptic currents $k_{\pm j}(z)$, the generating functions of $\mathcal{E}_m^{\pm j}$, and show that they have a universal form. See Theorem 5.3 and 5.7.

This paper is organized as follows. In section 2, we define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ as a topological algebra generated by the elliptic Drinfeld generators. This is a new definition of $U_{q,p}(\widehat{\mathfrak{g}})$ given independently of $U_q(\widehat{\mathfrak{g}})$ unlike the previous one in Appendix A in [2]. In section 3, we define a quantum dynamical analogue $\mathcal{Z}_\mathcal{V}$ of Lepowsky and Wilson’s Z -algebra associated with the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module \mathcal{V} and its universal counterpart \mathcal{Z}_k . The irreducibility of the level- k highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$ is shown to be governed by the \mathcal{Z}_k -module. In section 4, we give a simple realization of \mathcal{Z}_k in terms of the quantum (non-dynamical) Z -algebra associated with the level- k $U_q(\widehat{\mathfrak{g}})$ -module and define a standard representation of $U_{q,p}(\widehat{\mathfrak{g}})$. We provide some level-1 examples of the standard representations and discuss their relation to the deformation of the W -algebras. In section 5, we give a construction of the new elliptic bosons of the fundamental weight type and the orthonormal basis type and derive various commutation relations.

2 Elliptic Algebra $U_{q,p}(\widehat{\mathfrak{g}})$

2.1 Definition

Let $\widehat{\mathfrak{g}} = X_l^{(1)}$ be an untwisted affine Lie algebra associated with the generalized Cartan matrix $A = (a_{ij})$ $i, j \in \{0\} \cup I$, $I = \{1, \dots, l\}$. We denote by $B = (b_{ij})$, $b_{ij} = d_i a_{ij}$ the symmetrization of A . We take $d_i = 1$ ($i \in I$) for the simply laced cases, $d_i = 1$ ($1 \leq i \leq l - 1$), $d_l = 1/2$ for $B_l^{(1)}$ and $d_i = 1$ ($1 \leq i \leq l - 1$), $d_l = 2$ for $C_l^{(1)}$. Let $q = e^{\hbar} \in \mathbb{C}[[\hbar]]$ and set $q_i = q^{d_i}$. Let p be an indeterminate.

Let $\mathfrak{h} = \mathfrak{h} \oplus \mathbb{C}d$, $\bar{\mathfrak{h}} = \bar{\mathfrak{h}} \oplus \mathbb{C}c$, $\tilde{\mathfrak{h}} = \bigoplus_{i \in I} \mathbb{C}h_i$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$. Define $\delta, \Lambda_0, \alpha_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$\langle \alpha_i, h_j \rangle = a_{j,i}, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \tag{2.1}$$

the other pairings are 0. We also define $\bar{\Lambda}_i$ ($i \in I$) $\in \mathfrak{h}^*$ by

$$\langle \bar{\Lambda}_i, h_j \rangle = \delta_{i,j}.$$

We set $\bar{\mathfrak{h}}^* = \bigoplus_{i \in I} \mathbb{C}\bar{\Lambda}_i$, $\tilde{\mathfrak{h}}^* = \bar{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0$, $\mathcal{Q} = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\mathcal{P} = \bigoplus_{i \in I} \mathbb{Z}\bar{\Lambda}_i$. Let $N = l + 1$ for $X_l = A_l$, $= l$ for B_l, C_l, D_l , $= 7$ for E_6 , $= 8$ for E_7, E_8 , $= 3$ for G_2 , $= 4$ for F_4 and consider the orthonormal basis $\{\xi_j \mid (1 \leq j \leq N)\}$ in \mathbb{R}^N with the inner product $(\xi_j, \xi_k) = \delta_{j,k}$. For A_l , we also set

$$\bar{\xi}_j = \xi_j - \frac{1}{l+1} \sum_{j=1}^{l+1} \xi_j. \tag{2.2}$$

We define $\epsilon_j = \bar{\xi}_j$ for A_l and $= \xi_j$ for other X_l . The simple roots α_j and the fundamental weights $\bar{\Lambda}_j$ ($1 \leq j \leq l$) can be expressed as a linear sum of ϵ_j [37, 38]. We follow

Kac’s conventions. We define $h_{\epsilon_j} \in \bar{\mathfrak{h}}$ ($j \in I$) by $\langle \epsilon_i, h_{\epsilon_j} \rangle = (\epsilon_i, \epsilon_j)$ and $h_\alpha \in \bar{\mathfrak{h}}$ for $\alpha = \sum_j c_j \epsilon_j$, $c_j \in \mathbb{C}$ by $h_\alpha = \sum_j c_j h_{\epsilon_j}$. We regard $\bar{\mathfrak{h}} \oplus \bar{\mathfrak{h}}^*$ as the Heisenberg algebra by

$$[h_{\epsilon_j}, \epsilon_k] = (\epsilon_j, \epsilon_k), \quad [h_{\epsilon_j}, h_{\epsilon_k}] = 0 = [\epsilon_j, \epsilon_k]. \tag{2.3}$$

In particular, we have $[h_j, \alpha_k] = a_{jk}$. We also set $h^j = h_{\bar{\Lambda}_j}$.

In order to treat the dynamical shifts in the face type elliptic algebra systematically, we introduce another Heisenberg algebra generated by P_α and Q_β ($\alpha, \beta \in \bar{\mathfrak{h}}^*$) satisfying the commutation relations

$$[P_{\epsilon_j}, Q_{\epsilon_k}] = (\epsilon_j, \epsilon_k), \quad [P_{\epsilon_j}, P_{\epsilon_k}] = 0 = [Q_{\epsilon_j}, Q_{\epsilon_k}]. \tag{2.4}$$

We also set

$$[P_{\epsilon_j}, \alpha] = [Q_{\epsilon_j}, \alpha] = 0, \quad [P_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = [Q_{\epsilon_j}, U(\widehat{\mathfrak{g}})] = 0 \tag{2.5}$$

where $P_\alpha = \sum_j c_j P_{\epsilon_j}$ for $\alpha = \sum_j c_j \epsilon_j$. We set $P_{\bar{\mathfrak{h}}} = \bigoplus_{j \in I} \mathbb{C} P_{\epsilon_j}$, $Q_{\bar{\mathfrak{h}}} = \bigoplus_{j \in I} \mathbb{C} Q_{\epsilon_j}$, $P_j = P_{\alpha_j^\vee}$, $P^j = P_{\bar{\Lambda}_j}$ and $Q_j = Q_{\alpha_j}$, $Q^j = Q_{\bar{\Lambda}_j^\vee}$. Here $\alpha_j^\vee = 2\alpha_j / (\alpha_j, \alpha_j)$.

For the abelian group $\mathcal{R}_Q = \sum_{j=1}^N \mathbb{Z} Q_{\alpha_j}$, we denote by $\mathbb{C}[\mathcal{R}_Q]$ the group algebra over \mathbb{C} of \mathcal{R}_Q . We denote by e^α the element of $\mathbb{C}[\mathcal{R}_Q]$ corresponding to $\alpha \in \mathcal{R}_Q$. These e^α satisfy $e^\alpha e^\beta = e^{\alpha+\beta}$ and $(e^\alpha)^{-1} = e^{-\alpha}$. In particular, $e^0 = 1$ is the identity element.

Now let us set $H = \bar{\mathfrak{h}} \oplus P_{\bar{\mathfrak{h}}} = \sum_j \mathbb{C}(P_{\epsilon_j} + h_{\epsilon_j}) + \sum_j \mathbb{C} P_{\epsilon_j} + \mathbb{C}c$ and denote its dual space by $H^* = \bar{\mathfrak{h}}^* \oplus Q_{\bar{\mathfrak{h}}}$. We define the pairing by Eq. (2.1), $\langle Q_\alpha, P_\beta \rangle = (\alpha, \beta)$ and $\langle Q_\alpha, h_\beta \rangle = \langle Q_\alpha, c \rangle = \langle Q_\alpha, d \rangle = 0 = \langle \alpha, P_\beta \rangle = \langle \delta, P_\beta \rangle = \langle \Lambda_0, P_\beta \rangle$. We define $\mathbb{F} = \mathcal{M}_{H^*}$ to be the field of meromorphic functions on H^* . We regard a function of $P + h = \sum_j a_j (P_{\epsilon_j} + h_{\epsilon_j})$, $P = \sum_j b_j P_{\epsilon_j}$ and c , $\widehat{f} = f(P + h, P, c)$, as an element in \mathbb{F} by $\widehat{f}(\mu) = f(\langle \mu, P + h \rangle, \langle \mu, P \rangle, \langle \mu, c \rangle)$ for $\mu \in H^*$.

We use the following notations.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_j = \frac{q^n - q^{-n}}{q_j - q_j^{-1}},$$

$$[n]_i! = [n]_i [n-1]_i \cdots [1]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[n]_i! [m-n]_i!},$$

$$(x; q)_\infty = \prod_{n=0}^\infty (1 - xq^n), \quad (x; q, t)_\infty = \prod_{n,m=0}^\infty (1 - xq^n t^m),$$

$$\Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.$$

Definition 2.1 The elliptic algebra $U_{q,p}(\widehat{\mathfrak{g}})$ is a topological algebra over $\mathbb{F}[[p]]$ generated by \mathcal{M}_{H^*} , $e_{j,m}$, $f_{j,m}$, $\alpha_{j,n}^\vee$, K_j^\pm , ($j \in I, m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}$), d and the central element c . We assume K_j^\pm are invertible and set

$$e_j(z) = \sum_{m \in \mathbb{Z}} e_{j,m} z^{-m}, \quad f_j(z) = \sum_{m \in \mathbb{Z}} f_{j,m} z^{-m},$$

$$\psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \exp\left(- (q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,-n}^\vee}{1 - p^n} z^n\right) \exp\left((q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,n}^\vee}{1 - p^n} z^{-n}\right),$$

$$\psi_j^-(q^{\frac{c}{2}}z) = K_j^- \exp\left(- (q_j - q_j^{-1}) \sum_{n>0} \frac{p^n \alpha_{j,-n}^\vee}{1 - p^n} z^n\right) \exp\left((q_j - q_j^{-1}) \sum_{n>0} \frac{\alpha_{j,n}^\vee}{1 - p^n} z^{-n}\right).$$

Note that $\psi_j^\pm(z)$ are formal Laurent series in z , whose coefficients are well defined in the p -adic topology. We call $e_j(z)$, $f_j(z)$, $\psi_j^\pm(z)$ the elliptic currents. The defining relations are as follows. For $g(P), g(P + h) \in \mathcal{M}_{H^*}$,

$$g(P + h)e_j(z) = e_j(z)g(P + h), \quad g(P)e_j(z) = e_j(z)g(P - \langle Q_{\alpha_j}, P \rangle), \tag{2.6}$$

$$g(P + h)f_j(z) = f_j(z)g(P + h - \langle \alpha_j, P + h \rangle), \quad g(P)f_j(z) = f_j(z)g(P), \tag{2.7}$$

$$[g(P), \alpha_{i,m}^\vee] = [g(P + h), \alpha_{i,n}^\vee] = 0, \tag{2.8}$$

$$g(P)K_j^\pm = K_j^\pm g(P - \langle Q_{\alpha_j}, P \rangle), \tag{2.9}$$

$$g(P + h)K_j^\pm = K_j^\pm g(P + h - \langle Q_{\alpha_j}, P \rangle), \tag{2.10}$$

$$[d, g(P)] = [d, g(P + h)] = 0, \tag{2.11}$$

$$[d, \alpha_{j,n}^\vee] = n\alpha_{j,n}^\vee, \quad [d, e_j(z)] = -z \frac{\partial}{\partial z} e_j(z), \quad [d, f_j(z)] = -z \frac{\partial}{\partial z} f_j(z), \tag{2.12}$$

$$K_i^\pm e_j(z) = q_i^{\mp a_{ij}} e_j(z) K_i^\pm, \quad K_i^\pm f_j(z) = q_i^{\pm a_{ij}} f_j(z) K_i^\pm, \tag{2.13}$$

$$[\alpha_{i,m}^\vee, \alpha_{j,n}^\vee] = \delta_{m+n,0} \frac{[a_{ij}m]_i [cm]_j}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \tag{2.14}$$

$$[\alpha_{i,m}^\vee, e_j(z)] = \frac{[a_{ij}m]_i}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} z^m e_j(z), \tag{2.15}$$

$$[\alpha_{i,m}^\vee, f_j(z)] = -\frac{[a_{ij}m]_i}{m} z^m f_j(z), \tag{2.16}$$

$$z_1 \frac{(q^{b_{ij}} z_2 / z_1; p^*)_\infty}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_\infty} e_i(z_1) e_j(z_2) = -z_2 \frac{(q^{b_{ij}} z_1 / z_2; p^*)_\infty}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_\infty} e_j(z_2) e_i(z_1), \tag{2.17}$$

$$z_1 \frac{(q^{-b_{ij}} z_2 / z_1; p)_\infty}{(pq^{b_{ij}} z_2 / z_1; p)_\infty} f_i(z_1) f_j(z_2) = -z_2 \frac{(q^{-b_{ij}} z_1 / z_2; p)_\infty}{(pq^{b_{ij}} z_1 / z_2; p)_\infty} f_j(z_2) f_i(z_1), \tag{2.18}$$

$$[e_i(z_1), f_j(z_2)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi_j^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi_j^+(q^{-\frac{c}{2}} z_2) \right), \tag{2.19}$$

$$\sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(p^* q^2 z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p^*)_{\infty}} \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^* q^{b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}}{(p^* q^{-b_{ij}} w / z_{\sigma(m)}; p^*)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(p^* q^{b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_{\sigma(m)} / w; p^*)_{\infty}} \times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}) = 0, \tag{2.20}$$

$$\sum_{\sigma \in S_a} \prod_{1 \leq m < k \leq a} \frac{(pq^{-2} z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}}{(pq^2 z_{\sigma(k)} / z_{\sigma(m)}; p)_{\infty}} \times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(pq^{-b_{ij}} w / z_{\sigma(m)}; p)_{\infty}}{(pq^{b_{ij}} w / z_{\sigma(m)}; p)_{\infty}} \prod_{s+1 \leq m \leq a} \frac{(pq^{-b_{ij}} z_{\sigma(m)} / w; p)_{\infty}}{(pq^{b_{ij}} z_{\sigma(m)} / w; p)_{\infty}} \times f_i(z_{\sigma(1)}) \cdots f_i(z_{\sigma(s)}) f_j(w) f_i(z_{\sigma(s+1)}) \cdots f_i(z_{\sigma(a)}) = 0 \quad (i \neq j, a = 1 - a_{ij}), \tag{2.21}$$

where $p^* = pq^{-2c}$ and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We also denote by $U'_{q,p}(\widehat{\mathfrak{g}})$ the subalgebra obtained by removing d .

We treat the relations (2.12), (2.15)–(2.21) as formal Laurent series in z, w and z_j 's. In each term of Eqs. (2.17)–(2.21), the expansion direction of the structure function given by a ratio of infinite products is chosen according to the order of the accompanied product of the elliptic currents. For example, in the l.h.s of Eq. (2.17), $\frac{(q^{b_{ij}} z_2 / z_1; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_2 / z_1; p^*)_{\infty}}$ should be expanded in z_2 / z_1 , whereas in the r.h.s $\frac{(q^{b_{ij}} z_1 / z_2; p^*)_{\infty}}{(p^* q^{-b_{ij}} z_1 / z_2; p^*)_{\infty}}$ should be expanded in z_1 / z_2 . In each term in Eq. (2.20), the coefficient function is expanded in $z_{\sigma(k)} / z_{\sigma(m)}$ ($m < k$), $w / z_{\sigma(m)}$ ($m \leq s$) and $z_{\sigma(m)} / w$ ($m \geq s + 1$). All the coefficients in z_j 's are well defined in the p -adic topology.

Remark. In [1, 2, 35, 36], assuming that q is a transcendental complex number satisfying $|q| < 1$, we wrote (2.17), (2.18) as

$$z_1 \Theta_{p^*}(q^{b_{ij}} z_2 / z_1) e_i(z_1) e_j(z_2) = -z_2 \Theta_{p^*}(q^{b_{ij}} z_1 / z_2) e_j(z_2) e_i(z_1),$$

$$z_1 \Theta_p(q^{-b_{ij}} z_2 / z_1) f_i(z_1) f_j(z_2) = -z_2 \Theta_p(q^{-b_{ij}} z_1 / z_2) f_j(z_2) f_i(z_1),$$

in the sense of analytic continuation.

Let $U_q(\widehat{\mathfrak{g}})$ be the quantum affine algebra associated with $\widehat{\mathfrak{g}}$ in the Drinfeld realization [3]. See Appendix A. $U_{q,p}(\widehat{\mathfrak{g}})$ is a natural face type (i.e. dynamical) elliptic deformation of $U_q(\widehat{\mathfrak{g}})$ in the following sense.

Theorem 2.2

$$U_{q,p}(\widehat{\mathfrak{g}}) / pU_{q,p}(\widehat{\mathfrak{g}}) \cong (\mathbb{F} \otimes_{\mathbb{C}} U_q(\widehat{\mathfrak{g}})) \sharp \mathbb{C}[\mathcal{R}_Q].$$

Here the smash product \sharp is defined as follows.

$$g(P, P + h)x \otimes e^{\alpha} \cdot f(P, P + h)y \otimes e^{\beta}$$

$$= g(P, P + h)f(P - \langle \alpha, P \rangle, P + h - \langle \alpha + wt(x), P + h \rangle)xy \otimes e^{\alpha + \beta}$$

where $wt(x) \in \bar{\mathfrak{h}}^*$ s.t. $q^h x q^{-h} = q^{\langle wt(x), h \rangle} x$ for $x, y \in U_q(\widehat{\mathfrak{g}})$, $f(P), g(P) \in \mathbb{F}$, $e^{\alpha}, e^{\beta} \in \mathbb{C}[\mathcal{R}_Q]$.

Proof At $p = 0$, the relations for $\alpha_{j,m}^\vee, e_j(z), f_j(z)$ (2.12)–(2.21) coincide with those for $a_{i,m}^\vee, x_j^+(z), x_j^-(z)$ (6)–(11) of $U_q(\widehat{\mathfrak{g}})$. Therefore from Eqs. (2.6)–(2.10), one has the isomorphism

$$e_j(z) \mapsto x_j^+(z)e^{-Q_{\alpha_j}}, \quad f_j(z) \mapsto x_j^-(z), \quad K_j^\pm \mapsto q_j^{\mp h_j}e^{-Q_{\alpha_j}}, \quad \alpha_{j,m}^\vee \mapsto a_{j,m}^\vee \pmod{pU_{q,p}(\widehat{\mathfrak{g}})}.$$

□

2.2 H -algebra $U_{q,p}(\widehat{\mathfrak{g}})$

Let \mathcal{A} be a complex associative algebra, \mathcal{H} be a finite dimensional commutative subalgebra of \mathcal{A} , and $\mathcal{M}_{\mathcal{H}^*}$ be the field of meromorphic functions on \mathcal{H}^* the dual space of \mathcal{H} .

Definition 2.3 (\mathcal{H} -algebra) An \mathcal{H} -algebra is an associative algebra \mathcal{A} with 1, which is bigraded over \mathcal{H}^* , $\mathcal{A} = \bigoplus_{\alpha, \beta \in \mathcal{H}^*} \mathcal{A}_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathcal{M}_{\mathcal{H}^*} \rightarrow \mathcal{A}_{00}$ (the left and right moment maps), such that

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathcal{A}_{\alpha\beta}, \quad \widehat{f} \in \mathcal{M}_{\mathcal{H}^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$ of $\mathcal{M}_{\mathcal{H}^*}$.

Proposition 2.4 $U = U_{q,p}(\widehat{\mathfrak{g}})$ is an H -algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha, \beta}$$

$$U_{\alpha\beta} = \left\{ x \in U \mid \begin{aligned} q^{P+h} x q^{-(P+h)} &= q^{<\alpha, P+h>} x, \\ q^P x q^{-P} &= q^{<\beta, P>} x \quad \forall P + h, P \in H \end{aligned} \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{0,0}$ defined by

$$\mu_l(\widehat{f}) = f(P + h, p) \in \mathbb{F}[[p]], \quad \mu_r(\widehat{f}) = f(P, p^*) \in \mathbb{F}[[p]].$$

2.3 Dynamical Representations

Let us consider a vector space \mathcal{V} over \mathbb{F} , which is H -diagonalizable, i.e.

$$\mathcal{V} = \bigoplus_{\lambda, \mu \in H^*} \mathcal{V}_{\lambda, \mu}, \quad \mathcal{V}_{\lambda, \mu} = \{v \in \mathcal{V} \mid q^{P+h} \cdot v = q^{<\lambda, P+h>} v, \\ q^P \cdot v = q^{<\mu, P>} v \quad \forall P + h, P \in H\}.$$

Let us define the H -algebra $\mathcal{D}_{H, \mathcal{V}}$ of the \mathbb{C} -linear operators on \mathcal{V} by

$$\mathcal{D}_{H, \mathcal{V}} = \bigoplus_{\alpha, \beta \in H^*} (\mathcal{D}_{H, \mathcal{V}})_{\alpha\beta},$$

$$(\mathcal{D}_{H, \mathcal{V}})_{\alpha\beta} = \left\{ X \in \text{End}_{\mathbb{C}} \mathcal{V} \mid \begin{aligned} f(P+h)X &= Xf(P+h + <\alpha, P+h>), \\ f(P)X &= Xf(P + <\beta, P>), \\ f(P), f(P+h) &\in \mathbb{F}, X \cdot \mathcal{V}_{\lambda, \mu} \subseteq \mathcal{V}_{\lambda+\alpha, \mu+\beta} \end{aligned} \right\},$$

$$\mu_l^{\mathcal{D}_{H, \mathcal{V}}}(\widehat{f})v = f(<\lambda, P+h>, p)v, \quad \mu_r^{\mathcal{D}_{H, \mathcal{V}}}(\widehat{f})v = f(<\mu, P>, p^*)v, \\ \widehat{f} \in \mathcal{M}_{H^*}, v \in \mathcal{V}_{\lambda, \mu}.$$

Definition 2.5 We define a dynamical representation of $U_{q,p}(\widehat{\mathfrak{g}})$ on \mathcal{V} to be an H -algebra homomorphism $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H,\mathcal{V}}$. By the action π of $U_{q,p}(\widehat{\mathfrak{g}})$ we regard \mathcal{V} as a $U_{q,p}(\widehat{\mathfrak{g}})$ -module.

Definition 2.6 For $k \in \mathbb{C}$, we say that a $U_{q,p}(\widehat{\mathfrak{g}})$ -module has level k if c act as the scalar k on it.

Remark For the level-0 representations, Definition 2.5 is essentially the same as in [8], by identifying P and $P + h$ with λ and $\lambda - \gamma h$, respectively. This definition is valid also for the non-zero level cases [10].

Definition 2.7 For $\omega \in \mathbb{C}$, we set

$$\mathcal{V}_\omega = \{v \in \mathcal{V} \mid -d \cdot v = \omega v\}$$

and we call \mathcal{V}_ω the space of elements homogeneous of degree ω . We also say that $X \in \mathcal{D}_{H,\mathcal{V}}$ is homogeneous of degree $\omega \in \mathbb{C}$ if

$$[-d, X] = \omega X$$

and denote by $(\mathcal{D}_{H,\mathcal{V}})_\omega$ the space of all endomorphisms homogeneous of degree ω .

Definition 2.8 Let $\mathcal{H}, \mathcal{N}_+, \mathcal{N}_-$ be the subalgebras of $U_{q,p}(\widehat{\mathfrak{g}})$ generated by c, d, K_i^\pm ($i \in I$), by $\alpha_{i,n}^\vee$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$) $f_{i,n}$ ($i \in I, n \in \mathbb{Z}_{>0}$) and by $\alpha_{i,-n}^\vee$ ($i \in I, n \in \mathbb{Z}_{>0}$), $e_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{>0}$), $f_{i,-n}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$), respectively.

Definition 2.9 For $k \in \mathbb{C}, \lambda \in \mathfrak{h}^*$ and $\mu \in H^*$, a (dynamical) $U_{q,p}(\widehat{\mathfrak{g}})$ -module $\mathcal{V}(\lambda, \mu)$ is called the level- k highest weight module with the highest weight (λ, μ) , if there exists a vector $v \in \mathcal{V}(\lambda, \mu)$ such that

$$\begin{aligned} \mathcal{V}(\lambda, \mu) &= U_{q,p}(\widehat{\mathfrak{g}}) \cdot v, & \mathcal{N}_+ \cdot v &= 0, \\ c \cdot v &= kv, & f(P) \cdot v &= f(\langle \mu, P \rangle)v, & f(P + h) \cdot v &= f(\langle \lambda, P + h \rangle)v. \end{aligned}$$

We define the category \mathfrak{C}_k in the analogous way to the classical affine Lie algebra case [18].

Definition 2.10 For $k \in \mathbb{C}$, \mathfrak{C}_k is the full subcategory of the category of $U_{q,p}(\widehat{\mathfrak{g}})$ -modules consisting of those modules \mathcal{V} such that

- (i) \mathcal{V} has level k
- (ii) $\mathcal{V} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{V}_\omega$
- (iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0, \mathcal{V}_{\omega+n} = 0$.

Since $\pi \mathcal{N}_+ \subset \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{D}_{H,\mathcal{V}})_n$, any level- k highest weight $U_{q,p}(\widehat{\mathfrak{g}})$ -modules belong to \mathfrak{C}_k .

3 The Dynamical Quantum Z-Algebras

In this section we introduce a quantum and dynamical analogue \mathcal{Z}_k of Lepowsky-Wilson’s Z -algebra associated with the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -modules and define a category \mathfrak{D}_k of the

\mathcal{Z}_k -modules. Each representation of \mathcal{Z}_k in \mathfrak{D}_k turns out to be a dynamical analogue of the quantum Z -algebra derived by Jing [23] from the level- k representation in the $U_q(\widehat{\mathfrak{g}})$ counterpart of \mathfrak{D}_k . See sec.4.1. We also provide the Serre relations (3.23) which are not written in [23] explicitly.

3.1 The Heisenberg algebra $U_{q,p}(\mathcal{H})$

Let $U_{q,p}(\mathcal{H})$ be the subalgebra of $U_{q,p}(\widehat{\mathfrak{g}})$ generated by $\alpha_{i,n}^\vee$ ($i \in I, n \in \mathbb{Z}_{\neq 0}$) and c . It is convenient to introduce the simple root type generators $\alpha_{j,m}$ and $\alpha'_{j,m}$ defined by $\alpha_{j,m} = [d_j]\alpha_{j,m}^\vee$ and $\alpha'_{j,m} = \frac{1 - p^{*m}}{1 - p^m} q^{cm} \alpha_{j,m}$, ($j \in I, m \neq 0$). From Eqs. (2.14), (2.15), (2.16), we have

$$[\alpha_{i,m}, \alpha_{j,n}] = \frac{[b_{ijm}][cm]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} \delta_{m+n,0}, \tag{3.1}$$

$$[\alpha'_{i,m}, \alpha'_{j,n}] = \frac{[b_{ijm}][cm]}{m} \frac{1 - p^{*m}}{1 - p^m} q^{cm} \delta_{m+n,0}, \tag{3.2}$$

$$[\alpha_{i,m}, \alpha'_{j,n}] = \frac{[b_{ijm}][cm]}{m} \delta_{m+n,0}, \tag{3.3}$$

$$[\alpha_{i,m}, e_j(z)] = \frac{[b_{ijm}]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} z^m e_j(z), \tag{3.4}$$

$$[\alpha'_{i,m}, f_j(z)] = -\frac{[b_{ijm}]}{m} \frac{1 - p^{*m}}{1 - p^m} q^{cm} z^m f_j(z). \tag{3.5}$$

Let $U_{q,p}(\mathcal{H}_+)$ (resp. $U_{q,p}(\mathcal{H}_-)$) be the commutative subalgebras of $U_{q,p}(\mathcal{H})$ generated by $\{c, \alpha_{i,n} (i \in I, n \in \mathbb{Z}_{>0})\}$ (resp. $\{\alpha_{i,-n} (i \in I, n \in \mathbb{Z}_{>0})\}$). We have

$$U_{q,p}(\mathcal{H}) = U_{q,p}(\mathcal{H}_-)U_{q,p}(\mathcal{H}_+).$$

Let $\mathbb{C}1_k$ be the one-dimensional $U_{q,p}(\mathcal{H}_+)$ -module generated by the vacuum vector 1_k defined by

$$c \cdot 1_k = k1_k \quad \alpha_{i,n} \cdot 1_k = 0 \quad (n > 0).$$

Then we have the induced $U_{q,p}(\mathcal{H})$ -module

$$\mathcal{F}_{\alpha,k} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathbb{C}1_k.$$

We identify $\mathcal{F}_{\alpha,k}$ with a polynomial ring $\mathbb{C}[\alpha_{i,-m} (i \in I, m > 0)]$ by

$$\begin{aligned} c \cdot u &= ku, & \alpha_{i,-n} \cdot u &= \alpha_{i,-n}u, \\ \alpha_{i,n} \cdot u &= \sum_j \frac{[b_{ijn}][kn]}{n} \frac{1 - p^n}{1 - p^{*n}} q^{-kn} \frac{\partial}{\partial \alpha_{j,-n}} u \end{aligned} \quad (n > 0)$$

for $u \in \mathbb{C}[\alpha_{i,-m} (i \in I, m > 0)]$.

3.2 The dynamical quantum Z -algebra $\mathcal{Z}_{\mathcal{V}}$

Let $k \in \mathbb{C}^\times$ and $(\mathcal{V}, \pi) \in \mathfrak{C}_k$. We call $\pi U_{q,p}(\mathcal{H}) \subset (D_{H,\mathcal{V}})_{00}$ the level- k Heisenberg algebra. We define the following vertex operators in $(D_{H,\mathcal{V}})_{00}[[z, z^{-1}]]$.

$$E^\pm(\alpha_j, z) = \exp\left(\pm \sum_{n>0} \frac{\pi(\alpha_{j,\pm n})}{[kn]} z^{\mp n}\right), \quad E^\pm(\alpha'_j, z) = \exp\left(\mp \sum_{n>0} \frac{\pi(\alpha'_{j,\pm n})}{[kn]} z^{\mp n}\right).$$

These satisfy the following relations.

Proposition 3.1

$$E^+(\alpha_i, z)E^-(\alpha_j, w) = \frac{(q^{-b_{ij}+2k}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p^*)_\infty}{(q^{b_{ij}+2k}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p^*)_\infty} E^-(\alpha_j, w)E^+(\alpha_i, z), \quad (3.6)$$

$$E^+(\alpha'_i, z)E^-(\alpha'_j, w) = \frac{(q^{-b_{ij}}w/z; q^{2k})_\infty (q^{b_{ij}}w/z; p)_\infty}{(q^{b_{ij}}w/z; q^{2k})_\infty (q^{-b_{ij}}w/z; p)_\infty} E^-(\alpha'_j, w)E^+(\alpha'_i, z), \quad (3.7)$$

$$E^+(\alpha_i, z)E^-(\alpha'_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha'_j, w)E^+(\alpha_i, z), \quad (3.8)$$

$$E^+(\alpha'_i, z)E^-(\alpha_j, w) = \frac{(q^{b_{ij}+k}w/z; q^{2k})_\infty}{(q^{-b_{ij}+k}w/z; q^{2k})_\infty} E^-(\alpha_j, w)E^+(\alpha'_i, z), \quad (3.9)$$

$$E^\pm(\alpha_i, z)e_j(w) = \frac{(q^{\pm b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p^*)_\infty}{(q^{\mp b_{ij}+2k}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p^*)_\infty} e_j(w)E^\pm(\alpha_i, z), \quad (3.10)$$

$$E^\pm(\alpha'_i, z)f_j(w) = \frac{(q^{\pm b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\pm b_{ij}}(w/z)^{\pm 1}; p)_\infty}{(q^{\mp b_{ij}}(w/z)^{\pm 1}; q^{2k})_\infty (q^{\mp b_{ij}}(w/z)^{\pm 1}; p)_\infty} f_j(w)E^\pm(\alpha'_i, z), \quad (3.11)$$

$$E^\pm(\alpha'_i, z)e_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} e_j(w)E^\pm(\alpha'_i, z), \quad (3.12)$$

$$E^\pm(\alpha_i, z)f_j(w) = \frac{(q^{\mp b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty}{(q^{\pm b_{ij}+k}(w/z)^{\pm 1}; q^{2k})_\infty} f_j(w)E^\pm(\alpha_i, z). \quad (3.13)$$

Definition 3.2 We define $\mathcal{Z}_j^\pm(z; \mathcal{V}) \in \mathcal{D}_{H,\mathcal{V}}[[z, z^{-1}]]$ by

$$\mathcal{Z}_j^+(z; \mathcal{V}) := E^-(\alpha_j, z)\pi(e_j(z))E^+(\alpha_j, z), \quad (3.14)$$

$$\mathcal{Z}_j^-(z; \mathcal{V}) := E^-(\alpha'_j, z)\pi(f_j(z))E^+(\alpha'_j, z). \quad (3.15)$$

for $j \in I$ and call them the dynamical quantum Z operators associated with $(\mathcal{V}, \pi) \in \mathfrak{C}_k$.

Note that due to the truncation property of the grading of $\mathcal{V} \in \mathfrak{C}_k$ w.r.t $-d$, $\mathcal{Z}_j^\pm(z; \mathcal{V})$ are well defined i.e. the coefficients $\mathcal{Z}_{j,n}^\pm(\mathcal{V})$ of $\mathcal{Z}_j^\pm(z; \mathcal{V}) = \sum_{n \in \mathbb{Z}} \mathcal{Z}_{j,n}^\pm(\mathcal{V})z^{-n}$ in z are well defined elements in $(\mathcal{D}_{H,\mathcal{V}})_n$ for all $n \in \mathbb{Z}$. For the sake of simplicity of the presentation, we often drop π to denote the elements in $\mathcal{D}_{H,\mathcal{V}}$.

From the defining relations of $U_{q,p}(\widehat{\mathfrak{g}})$, we obtain the following relations of the dynamical quantum Z operators.

Theorem 3.3

$$g(P + h)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P + h),$$

$$g(P)\mathcal{Z}_i^+(z; \mathcal{V}) = \mathcal{Z}_i^+(z; \mathcal{V})g(P - \langle Q_{\alpha_i}, P \rangle), \tag{3.16}$$

$$g(P + h)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P + h - \langle \alpha_i, P + h \rangle),$$

$$g(P)\mathcal{Z}_i^-(z; \mathcal{V}) = \mathcal{Z}_i^-(z; \mathcal{V})g(P), \tag{3.17}$$

$$[d, \mathcal{Z}_j^\pm(z; \mathcal{V})] = -z \frac{\partial}{\partial z} \mathcal{Z}_j^\pm(z; \mathcal{V}), \tag{3.18}$$

$$[\alpha_{i,m}, \mathcal{Z}_j^\pm(w; \mathcal{V})] = 0, \tag{3.19}$$

$$K_i^\pm \mathcal{Z}_j^\pm(z; \mathcal{V}) = q^{\mp b_{ij}} \mathcal{Z}_j^\pm(z; \mathcal{V}) K_i^\pm, \quad K_i^\pm \mathcal{Z}_j^\mp(z; \mathcal{V}) = q^{\pm b_{ij}} \mathcal{Z}_j^\mp(z; \mathcal{V}) K_i^\pm, \tag{3.20}$$

$$z \frac{(q^{-b_{ij}} w/z; q^{2k})_\infty}{(q^{b_{ij}+2k} w/z; q^{2k})_\infty} \mathcal{Z}_i^\pm(z; \mathcal{V}) \mathcal{Z}_j^\pm(w; \mathcal{V}) = -w \frac{(q^{-b_{ij}} z/w; q^{2k})_\infty}{(q^{b_{ij}+2k} z/w; q^{2k})_\infty} \mathcal{Z}_j^\pm(w; \mathcal{V}) \mathcal{Z}_i^\pm(z; \mathcal{V}), \tag{3.21}$$

$$\frac{(q^{b_{ij}+k} w/z; q^{2k})_\infty}{(q^{-b_{ij}+k} w/z; q^{2k})_\infty} \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^-(w; \mathcal{V}) - \frac{(q^{b_{ij}+k} z/w; q^{2k})_\infty}{(q^{-b_{ij}+k} z/w; q^{2k})_\infty} \mathcal{Z}_j^-(w; \mathcal{V}) \mathcal{Z}_i^+(z; \mathcal{V})$$

$$= \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(K_i^- \delta(q^{-k} z/w) - K_i^+ \delta(q^k z/w) \right), \tag{3.22}$$

$$\sum_{\sigma \in S_a} \prod_{1 \leq m < l \leq a} \frac{(q^{2+k\mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty}{(q^{-2+k\mp k} z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty}$$

$$\times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(q^{-b_{ij}+k\mp k} w/z_{\sigma(m)}; q^{2k})_\infty}{(q^{b_{ij}+k\mp k} w/z_{\sigma(m)}; q^{2k})_\infty} \prod_{s+1 \leq m \leq a} \frac{(q^{-b_{ij}+k\mp k} z_{\sigma(m)}/w; q^{2k})_\infty}{(q^{b_{ij}+k\mp k} z_{\sigma(m)}/w; q^{2k})_\infty}$$

$$\times \mathcal{Z}_i^\pm(z_{\sigma(1)}; \mathcal{V}) \cdots \mathcal{Z}_i^\pm(z_{\sigma(s)}; \mathcal{V}) \mathcal{Z}_j^\pm(w; \mathcal{V}) \mathcal{Z}_i^\pm(z_{\sigma(s+1)}; \mathcal{V}) \cdots \mathcal{Z}_i^\pm(z_{\sigma(a)}; \mathcal{V}) = 0$$

$$(i \neq j, a = 1 - a_{ij}). \tag{3.23}$$

Proof The relations (3.17) and (3.18) follow from Eqs. (2.6)–(2.10) and (2.12), respectively. Let us show the relation (3.19). For $m > 0$, we have

$$[\alpha_{i,m}, \mathcal{Z}_j^+(z; \mathcal{V})] = [\alpha_{i,m}, E^-(\alpha_j, z)] e_j(z) E^+(\alpha_j, z) + E^-(\alpha_j, z) [\alpha_{i,m}, e_j(z)] E^+(\alpha_j, z).$$

This vanishes due to Eq. (3.4) and

$$[\alpha_{i,m}, E^-(\alpha_j, z)] = -\frac{[b_{ij}m]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-km} z^m,$$

where $p^* = pq^{-2k}$. Similarly, $[\alpha_{i,m}, \mathcal{Z}_j^-(z; \mathcal{V})] = 0$ follows from Eq. (3.5) and

$$[\alpha'_{i,m}, E^-(\alpha'_j, z)] = \frac{[b_{ij}m]}{m} \frac{1 - p^{*m}}{1 - p^m} q^{km} z^m.$$

The case $m < 0$ can be proved in a similar way.

The relation (3.21) follows from

$$\begin{aligned}
 & \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^+(w; \mathcal{V}) \\
 &= E^-(\alpha_i, z) e_i(z) E^+(\alpha_i, z) E^-(\alpha_j, w) e_j(w) E^+(\alpha_j, w) \\
 &= \frac{(q^{-b_{ij}+2k} w/z; q^{2k})_\infty (q^{-b_{ij}} w/z; p^*)_\infty}{(q^{b_{ij}+2k} w/z; q^{2k})_\infty (q^{b_{ij}} w/z; p^*)_\infty} E^-(\alpha_i, z) e_i(z) E^-(\alpha_j, w) E^+(\alpha_i, z) e_j(w) E^+(\alpha_j, w) \\
 &= \frac{(q^{b_{ij}+2k} w/z; q^{2k})_\infty (q^{b_{ij}} w/z; p^*)_\infty}{(q^{-b_{ij}+2k} w/z; q^{2k})_\infty (q^{-b_{ij}} w/z; p^*)_\infty} E^-(\alpha_i, z) E^-(\alpha_j, w) e_i(z) e_j(w) E^+(\alpha_i, z) E^+(\alpha_j, w) \\
 &= -\frac{w}{z(1 - q^{-b_{ij}} w/z)} \frac{(q^{b_{ij}+2k} w/z; q^{2k})_\infty (q^{b_{ij}} z/w; p^*)_\infty}{(q^{-b_{ij}+2k} w/z; q^{2k})_\infty (p^* q^{-b_{ij}} z/w; p^*)_\infty} \\
 &\quad \times E^-(\alpha_i, z) E^-(\alpha_j, w) e_j(w) e_i(z) E^+(\alpha_i, z) E^+(\alpha_j, w) \\
 &= -\frac{w(1 - q^{-b_{ij}} z/w)}{z(1 - q^{-b_{ij}} w/z)} \frac{(q^{-b_{ij}+2k} z/w; q^{2k})_\infty^2 (q^{b_{ij}+2k} w/z; q^{2k})_\infty (q^{-b_{ij}} z/w; p^*)_\infty}{(q^{b_{ij}+2k} z/w; q^{2k})_\infty^2 (q^{-b_{ij}+2k} w/z; q^{2k})_\infty (q^{b_{ij}} z/w; p^*)_\infty} \\
 &\quad \times E^-(\alpha_j, w) e_j(w) E^-(\alpha_i, z) E^+(\alpha_j, w) e_i(z) E^+(\alpha_i, z) \\
 &= -\frac{w}{z} \frac{(q^{-b_{ij}} z/w; q^{2k})_\infty (q^{b_{ij}+2k} w/z; q^{2k})_\infty}{(q^{b_{ij}+2k} z/w; q^{2k})_\infty (q^{-b_{ij}} w/z; q^{2k})_\infty} \mathcal{Z}_j^+(w; \mathcal{V}) \mathcal{Z}_i^+(z; \mathcal{V}).
 \end{aligned}$$

We also derive (3.22) as follows.

$$\begin{aligned}
 & \frac{(q^{b_{ij}+k} w/z; q^{2k})_\infty}{(q^{-b_{ij}+k} w/z; q^{2k})_\infty} \mathcal{Z}_i^+(z; \mathcal{V}) \mathcal{Z}_j^-(w; \mathcal{V}) \\
 &= \frac{(q^{b_{ij}+k} w/z; q^{2k})_\infty}{(q^{-b_{ij}+k} w/z; q^{2k})_\infty} E^-(\alpha_i, z) e_i(z) E^+(\alpha_i, z) E^-(\alpha'_j, w) f_j(w) E^+(\alpha'_j, w) \\
 &= E^-(\alpha_i, z) E^-(\alpha'_j, w) e_i(z) f_j(w) E^+(\alpha_i, z) E^+(\alpha'_j, w) \\
 &= E^-(\alpha_i, z) E^-(\alpha'_j, w) \left[f_j(w) e_i(z) + \frac{\delta_{ij}}{q_i - q_i^{-1}} \left(\delta \left(q^{-k} \frac{z}{w} \right) \psi_i^-(q^{k/2} w) \right. \right. \\
 &\quad \left. \left. - \delta \left(q^k \frac{z}{w} \right) \psi_i^+(q^{-k/2} w) \right) \right] E^+(\alpha_i, z) E^+(\alpha'_j, w).
 \end{aligned}$$

Then use

$$\psi_i^\pm(q^{\mp k/2} w) = K_i^\pm E^-(\alpha_i, q^{\mp k} w)^{-1} E^-(\alpha'_j, w)^{-1} E^+(\alpha_i, q^{\mp k} w)^{-1} E^+(\alpha'_j, w)^{-1}$$

and the property of the delta function.

To prove the Serre relation (3.23) for $\mathcal{Z}_j^+(z)$ we use Eqs. (3.14) and (3.19) and obtain

$$e_i(z) = E(\alpha_i, z) \mathcal{Z}_i^+(z; \mathcal{V}) \tag{3.24}$$

where we set

$$E(\alpha_i, z) = E^-(\alpha_i, z)^{-1} E^+(\alpha_i, z)^{-1}.$$

From Eq. (3.6), we have

$$\begin{aligned}
 & E(\alpha_i, z) E(\alpha_j, w) \\
 &= \frac{(q^{-2} w/z; q^{2k})_\infty (q^2 z/w; q^{2k})_\infty (p^* q^{-2} w/z; p^*)_\infty (p^* q^2 z/w; p^*)_\infty}{(q^2 w/z; q^{2k})_\infty (q^{-2} z/w; q^{2k})_\infty (p^* q^2 w/z; p^*)_\infty (p^* q^{-2} z/w; p^*)_\infty} E(\alpha_j, w) E(\alpha_i, z).
 \end{aligned} \tag{3.25}$$

Next note that (2.20) is equivalent to

$$\begin{aligned} 0 &= \prod_{1 \leq m < l \leq a} \frac{(p^*q^2z_l/z_m; p^*)_\infty}{(p^*q^{-2}z_l/z_m; p^*)_\infty} \prod_{1 \leq i \leq a} \frac{(p^*q^{b_{ij}}z_i/w; p^*)_\infty}{(p^*q^{-b_{ij}}z_i/w; p^*)_\infty} \\ &\times \sum_{\sigma \in \mathcal{S}_a} \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(p^*q^{-2}z_l/z_m; p^*)_\infty}{(p^*q^2z_l/z_m; p^*)_\infty} \frac{(p^*q^2z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-2}z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty} \\ &\times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^*q^{b_{ij}}w/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-b_{ij}}w/z_{\sigma(m)}; p^*)_\infty} \frac{(p^*q^{-b_{ij}}z_{\sigma(m)}/w; p^*)_\infty}{(p^*q^{b_{ij}}z_{\sigma(m)}/w; p^*)_\infty} \\ &\times e_i(z_{\sigma(1)}) \cdots e_i(z_{\sigma(s)}) e_j(w) e_i(z_{\sigma(s+1)}) \cdots e_i(z_{\sigma(a)}). \end{aligned}$$

Substitute (3.24) into this, and move all $E(\alpha_i, z_j)$ and $E(\alpha_j, w)$ to the left. Then we get

$$\begin{aligned} 0 &= \prod_{1 \leq m < l \leq a} \frac{(p^*q^2z_l/z_m; p^*)_\infty}{(p^*q^{-2}z_l/z_m; p^*)_\infty} \prod_{1 \leq i \leq a} \frac{(p^*q^{b_{ij}}z_i/w; p^*)_\infty}{(p^*q^{-b_{ij}}z_i/w; p^*)_\infty} \\ &\times \sum_{\sigma \in \mathcal{S}_a} \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(p^*q^{-2}z_l/z_m; p^*)_\infty}{(p^*q^2z_l/z_m; p^*)_\infty} \frac{(p^*q^2z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-2}z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty} \\ &\times \sum_{s=0}^a (-1)^s \begin{bmatrix} a \\ s \end{bmatrix}_i \prod_{1 \leq m \leq s} \frac{(p^*q^{b_{ij}}w/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-b_{ij}}w/z_{\sigma(m)}; p^*)_\infty} \frac{(p^*q^{-b_{ij}}z_{\sigma(m)}/w; p^*)_\infty}{(p^*q^{b_{ij}}z_{\sigma(m)}/w; p^*)_\infty} \\ &\times \varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(a)}) \\ &\times \mathcal{Z}_i^+(z_{\sigma(1)}; \mathcal{V}) \cdots \mathcal{Z}_i^+(z_{\sigma(s)}; \mathcal{V}) \mathcal{Z}_j^+(w; \mathcal{V}) \mathcal{Z}_i^+(z_{\sigma(s+1)}; \mathcal{V}) \cdots \mathcal{Z}_i^+(z_{\sigma(a)}; \mathcal{V}), \end{aligned} \tag{3.26}$$

where we set

$$\begin{aligned} &\varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(a)}) \\ &= E(\alpha_i, z_{\sigma(1)}) \cdots E(\alpha_i, z_{\sigma(s)}) E(\alpha_j, w) E(\alpha_i, z_{\sigma(s+1)}) \cdots E(\alpha_i, z_{\sigma(a)}). \end{aligned}$$

Then moving $E(\alpha_j, w)$ to the left end by Eq. (3.25), we have

$$\begin{aligned} &\varepsilon(z_{\sigma(1)}, \dots, z_{\sigma(s)}, w, z_{\sigma(s+1)}, \dots, z_{\sigma(N)}) \\ &= \prod_{1 \leq i \leq s} \frac{(q^{-b_{ij}}w/z_{\sigma(i)}; q^{2k})_\infty (q^{b_{ij}}z_{\sigma(i)}/w; q^{2k})_\infty}{(q^{b_{ij}}w/z_{\sigma(i)}; q^{2k})_\infty (q^{-b_{ij}}z_{\sigma(i)}/w; q^{2k})_\infty} \frac{(p^*q^{-b_{ij}}w/z_{\sigma(i)}; p^*)_\infty (p^*q^{b_{ij}}z_{\sigma(i)}/w; p^*)_\infty}{(p^*q^{b_{ij}}w/z_{\sigma(i)}; p^*)_\infty (p^*q^{-b_{ij}}z_{\sigma(i)}/w; p^*)_\infty} \\ &\times \varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)}). \end{aligned}$$

Substituting this into Eq. (3.26), we can factor out $\varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)})$ from $\sum_{s=0}^a$. Then exchanging the order of $E(\alpha_i, z_l)$'s by Eq. (3.25), we have

$$\begin{aligned} &\varepsilon(w, z_{\sigma(1)}, \dots, z_{\sigma(a)}) \\ &= \prod_{\substack{1 \leq m < l \leq a \\ \sigma^{-1}(m) > \sigma^{-1}(l)}} \frac{(q^{-2}z_l/z_m; q^{2k})_\infty (q^2z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty}{(q^2z_l/z_m; q^{2k})_\infty (q^{-2}z_{\sigma(l)}/z_{\sigma(m)}; q^{2k})_\infty} \frac{(p^*q^2z_l/z_m; p^*)_\infty (q^{-2}p^*z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty}{(p^*q^{-2}z_l/z_m; p^*)_\infty (p^*q^2z_{\sigma(l)}/z_{\sigma(m)}; p^*)_\infty} \\ &\times \varepsilon(w, z_1, \dots, z_a). \end{aligned}$$

Substituting this into Eq. (3.26), we can factor out $\varepsilon(w, z_1, \dots, z_a)$ completely from $\sum_{\sigma \in S_a}$. Multiply

$$\prod_{1 \leq m < l \leq a} \frac{(q^2 z_l / z_m; q^{2k})_\infty}{(q^{-2} z_l / z_m; q^{2k})_\infty} \prod_{1 \leq m \leq a} \frac{(q^{-b_{ij}} z_m / w; q^{2k})_\infty}{(q^{b_{ij}} z_m / w; q^{2k})_\infty},$$

and drop the overall factor depending on p^* , one gets the desired relation. One can prove the $\mathcal{Z}_j^-(z; \mathcal{V})$ case in the same way. \square

Definition 3.4 For $k \in \mathbb{C}^\times$ and $(\mathcal{V}, \pi) \in \mathfrak{C}_k$, we call the H -subalgebra of $\mathcal{D}_{H, \mathcal{V}}$ generated by $\mathcal{Z}_{i,m}^\pm(\mathcal{V})$, K_i^\pm ($i \in I, m \in \mathbb{Z}$), \mathcal{M}_{H^*} and d the dynamical quantum Z -algebra $\mathcal{Z}_\mathcal{V}$ associated with (\mathcal{V}, π) .

3.3 The universal algebra \mathcal{Z}_k

Using the relations in Theorem 3.3, we define the universal dynamical quantum Z -algebra as follows.

Definition 3.5 Let $\mathcal{Z}_{i,m}^\pm$ ($i \in I, m \in \mathbb{Z}$) be abstract symbols. We set $\mathcal{Z}_i^\pm(z) = \sum_{m \in \mathbb{Z}} \mathcal{Z}_{i,m}^\pm z^{-m}$. We define the universal dynamical quantum Z -algebra \mathcal{Z}_k to be a topological algebra over $\mathbb{F}[[q^{2k}]]$ generated by $\mathcal{Z}_{i,m}^\pm$, K_i^\pm ($i \in I, m \in \mathbb{Z}$), d , \mathcal{M}_{H^*} subject to the relations obtained by replacing $\mathcal{Z}_i^\pm(z; \mathcal{V})$ by $\mathcal{Z}_i^\pm(z)$ in Theorem 3.3.

We treat the relations as formal Laurent series in z, w and z_j 's in a similar way to those of $U_{q,p}(\widehat{\mathfrak{g}})$ in section 2.1. The defining relations are well-defined in the q^{2k} -adic topology.

Proposition 3.6 \mathcal{Z}_k is an H -algebra with the same μ_l, μ_r as in $U_{q,p}(\widehat{\mathfrak{g}})$.

Note that for $(\mathcal{V}, \pi) \in \mathfrak{C}_k$ we extend π to the map $\pi : \mathcal{Z}_k \rightarrow \mathcal{D}_{H, \mathcal{V}}$ by $\pi(\mathcal{Z}_{i,m}^\pm) = \mathcal{Z}_{i,m}^\pm(\mathcal{V})$. Then \mathcal{V} is a \mathcal{Z}_k -module by π .

Definition 3.7 For $k \in \mathbb{C}^\times$, we denote by \mathfrak{D}_k the full subcategory of the category of \mathcal{Z}_k -modules consisting of those modules (\mathcal{W}, σ) such that

- (i) \mathcal{W} has level k .
- (ii) $\mathcal{W} = \bigsqcup_{\omega \in \mathbb{C}} \mathcal{W}_\omega$, where $\mathcal{W}_\omega = \{w \in \mathcal{W} \mid -\sigma(d)w = \omega w\}$
- (iii) For every $\omega \in \mathbb{C}$, there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{W}_{\omega+n} = 0$.

Let us consider $(\mathcal{V}, \pi) \in \mathfrak{C}_k$. Following Lepowsky and Wilson [18], we define the vacuum space $\Omega_\mathcal{V}$ by

$$\Omega_\mathcal{V} = \{v \in \mathcal{V} \mid \pi(\alpha_{i,n})v = 0 \quad \forall i \in I, n \in \mathbb{Z}_{>0}\}.$$

From Theorem 3.3, $\Omega_\mathcal{V}$ is stable under the action of $\mathcal{Z}_\mathcal{V}$. For a morphism $f : \mathcal{V} \rightarrow \mathcal{V}'$ in \mathfrak{C}_k , we have

$$f(\Omega_\mathcal{V}) \subset \Omega_{\mathcal{V}'}$$

Proposition 3.8 For $(\mathcal{V}, \pi) \in \mathfrak{C}_k$, there is a unique representation σ of \mathcal{Z}_k on $\Omega_{\mathcal{V}}$ such that $(\Omega_{\mathcal{V}}, \sigma) \in \mathfrak{D}_k$,

$$\sigma(K_i^{\pm}) = \pi(K_i^{\pm}), \quad \sigma(\mathcal{Z}_{i,m}^{\pm}) = \mathcal{Z}_{i,m}^{\pm}(\mathcal{V}) \quad \forall i \in I, m \in \mathbb{Z}.$$

We hence define a functor $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$ by

$$\Omega(\mathcal{V}, \pi) = (\Omega_{\mathcal{V}}, \sigma), \quad \Omega(f) = f|_{\Omega_{\mathcal{V}}} : \Omega_{\mathcal{V}} \rightarrow \Omega_{\mathcal{V}'}$$

3.4 The functor Λ

We define a reverse functor $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$ as follows. Let $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$ be a \mathcal{Z}_k -module. We define $U_{q,p}(\mathcal{H})$ -module $\text{Ind } \mathcal{W}$ by requiring $\alpha_{i,m} \cdot \mathcal{W} = 0$ and

$$\text{Ind } \mathcal{W} = U_{q,p}(\mathcal{H}) \otimes_{U_{q,p}(\mathcal{H}_+)} \mathcal{W}.$$

Let $\mathcal{F}_{\alpha,k}$ be the level- k Fock module defined in section 3.1. We have a natural isomorphism $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \cong \text{Ind } \mathcal{W}$ by $(u \otimes 1_k) \otimes w \mapsto u \otimes w$ [18]. We thus identify the $U_{q,p}(\mathcal{H})$ -module $\text{Ind } \mathcal{W}$ with $\mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$, with the action π of $U_{q,p}(\mathcal{H})$

$$\pi(c) = 1 \otimes c, \quad \pi(K_i^{\pm}) = 1 \otimes \sigma(K_i^{\pm}), \quad \pi(\alpha_{i,m}) = \alpha_{i,m} \otimes 1.$$

For $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$ and $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}$, we define $e'_j(z), f'_j(z) \in \mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$ by

$$\begin{aligned} e'_j(z) &= E^-(\alpha_j, z)^{-1} E^+(\alpha_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^+(z)), \\ f'_j(z) &= E^-(\alpha'_j, z)^{-1} E^+(\alpha'_j, z)^{-1} \otimes \sigma(\mathcal{Z}_j^-(z)). \end{aligned}$$

These are well-defined elements of $\mathcal{D}_{H, \text{Ind } \mathcal{W}}[[z, z^{-1}]]$. By a similar argument to the proof of Theorem 3.3 one can show that $e'_j(z)$ and $f'_j(z)$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{g}})$ with $c = k$. We hence extend $\pi : U_{q,p}(\mathcal{H}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$ to $\pi : U_{q,p}(\widehat{\mathfrak{g}}) \rightarrow \mathcal{D}_{H, \text{Ind } \mathcal{W}}$ as an H -algebra homomorphism by

$$\begin{aligned} \pi(e_j(z)) &= e'_j(z), \quad \pi(f_j(z)) = f'_j(z), \\ \pi(d) &= d \otimes 1 + 1 \otimes \sigma(d). \end{aligned}$$

By construction, the latter map is uniquely determined.

Proposition 3.9 For $(\mathcal{W}, \sigma) \in \mathfrak{D}_k$, there is a unique level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module $(\text{Ind } \mathcal{W}, \pi) \in \mathfrak{C}_k$.

We thus reach the following definition.

Definition 3.10 We define a functor $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$ by

- (i) $\Lambda(\mathcal{W}, \sigma) = (\text{Ind } \mathcal{W}, \pi)$
- (ii) For a morphism $f : \mathcal{W} \rightarrow \mathcal{W}'$ in \mathfrak{D}_k , define $\Lambda(f) : \text{Ind } \mathcal{W} \rightarrow \text{Ind } \mathcal{W}'$ to be the induced $U_{q,p}(\mathcal{H})$ -module map. Then $\Lambda(f)$ is a $U_{q,p}(\widehat{\mathfrak{g}})$ -module map.

We obtain the following theorem analogously to the case of the affine Lie algebras [18].

Theorem 3.11 For $k \in \mathbb{C}^{\times}$, the two categories \mathfrak{C}_k and \mathfrak{D}_k are equivalent by the functors $\Omega : \mathfrak{C}_k \rightarrow \mathfrak{D}_k$ and $\Lambda : \mathfrak{D}_k \rightarrow \mathfrak{C}_k$. In particular, the level- k $U_{q,p}(\widehat{\mathfrak{g}})$ -module $\text{Ind } \mathcal{W} = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W} \in \mathfrak{C}_k$ is irreducible if and only if $\mathcal{W} \in \mathfrak{D}_k$ is an irreducible \mathcal{Z}_k -module.

4 The Induced $U_{q,p}(\widehat{\mathfrak{g}})$ -Modules

In this section we give a simple realization of the dynamical quantum Z -algebra \mathcal{Z}_k in terms of the quantum Z -algebra Z_k associated with $U_q(\widehat{\mathfrak{g}})$ and construct the level- k induced $U_{q,p}(\widehat{\mathfrak{g}})$ -modules. We also give some examples of the level-1 irreducible representations.

4.1 The quantum Z -algebra Z_k associated with $U_q(\widehat{\mathfrak{g}})$

One can apply the arguments similar to those in sections 3.1–3.3 to the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization and define the corresponding quantum Z -algebras Z_V associated with the level- k $U_q(\widehat{\mathfrak{g}})$ -module V [23] and the universal one Z_k . See Appendix A. We also denote by C_k and D_k the $U_q(\widehat{\mathfrak{g}})$ counterparts of the categories \mathfrak{C}_k and \mathfrak{D}_k .

Comparing the defining relations of \mathcal{Z}_k with those of Z_k , we obtain the following isomorphism.

Proposition 4.1 *We have the isomorphism*

$$\mathcal{Z}_k \cong (\mathbb{F} \otimes_{\mathbb{C}} Z_k) \sharp \mathbb{C}[\mathcal{R}_Q]$$

as an H -algebra by

$$\begin{aligned} \mathcal{Z}_{j,m}^+ &\mapsto Z_{j,m}^+ e^{-Q_{\alpha_j}}, & \mathcal{Z}_{j,m}^- &\mapsto Z_{j,m}^-, \\ K_i^{\pm} &\mapsto q_i^{\mp h_i} e^{-Q_{\alpha_j}} \quad (i \in I, m \in \mathbb{Z}), & d &\mapsto \bar{d}, \end{aligned}$$

where $Z_{j,m}^{\pm}$ denotes the generators in Z_k (Definition A.3).

Theorem 4.2 *For $(W, \bar{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, there is a dynamical representation σ of \mathcal{Z}_k on $\mathcal{W}_{H,Q}(\mu) := (\mathbb{F} \otimes_{\mathbb{C}} W) \otimes_{\mathbb{C}} e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ such that $(\mathcal{W}_{H,Q}(\mu), \sigma) \in \mathfrak{C}_k$ and*

$$\begin{aligned} \sigma(\mathcal{Z}_{j,m}^+) &= \bar{\sigma}(Z_{j,m}^+) \otimes e^{-Q_{\alpha_j}}, & \sigma(\mathcal{Z}_{j,m}^-) &= \bar{\sigma}(Z_{j,m}^-) \otimes 1, \\ \sigma(K_j^{\pm}) &= \bar{\sigma}(q_j^{\mp h_j}) \otimes e^{-Q_{\alpha_j}}, & \sigma(d) &= \bar{\sigma}(\bar{d}) \otimes 1 + 1 \otimes P_d, \end{aligned}$$

where P_d denotes a \mathbb{C} -linear operator on $1 \otimes e^{Q_{\bar{\mu}}} \mathbb{C}[\mathcal{R}_Q]$ such that

$$[1 \otimes P_d, \sigma(\mathcal{Z}_{j,m}^{\pm})] = 0.$$

Proposition 4.3 *The representation $(\mathcal{W}_{H,Q}(\mu), \sigma)$ of \mathcal{Z}_k is irreducible if and only if W is an irreducible Z_k -module.*

From this and Theorem 3.11, we obtain:

Proposition 4.4 *For a Z_k -module $(W, \bar{\sigma}) \in D_k$ and generic $\mu \in \mathfrak{h}^*$, let $(\mathcal{W}_{H,Q}(\mu), \sigma)$ be the \mathcal{Z}_k -module constructed in Theorem 4.2 and $\text{Ind} \mathcal{W}_{H,Q}(\mu) = \mathcal{F}_{\alpha,k} \otimes_{\mathbb{C}} \mathcal{W}_{H,Q}(\mu)$ be the level- k induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module given in Proposition 3.9. Then $(\text{Ind} \mathcal{W}_{H,Q}(\mu), \pi)$ is irreducible if and only if $(W, \bar{\sigma})$ is irreducible.*

4.2 Examples of the irreducible representations

We here give some examples of the level-1 irreducible induced representations of $U_{q,p}(\widehat{\mathfrak{g}})$ of types $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$.

4.2.1 The simply laced case :

Let $\mathbb{C}[\mathcal{Q}]$ be the group algebra of the root lattice $\mathcal{Q} = \oplus_i \mathbb{Z}\alpha_i$ with the central extension:

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j)} e^{\alpha_j} e^{\alpha_i} \quad (i, j \in I).$$

Let us consider the fundamental weight Λ_a of $\widehat{\mathfrak{g}}$ with $0 \leq a \leq l$ for $A_l^{(1)}$, $a = 0, 1, l - 1, l$ for $D_l^{(1)}$, $a = 0, 1, 2$ for $E_6^{(1)}$, $a = 0, 1$ for $E_7^{(1)}$, $a = 0$ for $E_8^{(1)}$.

Theorem 4.5 [23, 39] *An inequivalent set of the level-1 irreducible $Z_1(\widehat{\mathfrak{g}})$ -modules is given by $W(\Lambda_a) = e^{\Lambda_a} \mathbb{C}[\mathcal{Q}]$, on which the actions of $Z_j^\pm(z)$ are given by*

$$Z_j^\pm(z) = e^{\pm\alpha_j} z^{\pm h_j + 1} \tag{4.1}$$

with

$$z^{\pm h_i} e^{\pm\alpha_j} e^{\bar{\Lambda}_a} = z^{\pm(\alpha_i^\vee, \alpha_j + \bar{\Lambda}_a)} e^{\pm\alpha_j} e^{\bar{\Lambda}_a} \quad (i, j \in I).$$

Then for generic $\mu \in \mathfrak{h}^*$, we have from Theorem 4.2 a level-1 irreducible $Z_1(\widehat{\mathfrak{g}})$ module $\mathcal{W}_{H, \mathcal{Q}}(\Lambda_a, \mu) := (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{\mathcal{Q}\bar{\mu}} \mathbb{C}[\mathcal{R}_{\mathcal{Q}}]$ with the action given by

$$\mathcal{Z}_j^+(z) = Z_j^+(z) \otimes e^{-\mathcal{Q}\alpha_j}, \quad \mathcal{Z}_j^-(z) = Z_j^-(z) \otimes 1. \tag{4.2}$$

Then from Proposition 4.4 we obtain:

Theorem 4.6 *A level-1 irreducible highest weight representations of $U_{q,p}(\widehat{\mathfrak{g}})$ is given by $\mathcal{V}(\Lambda_a + \mu, \mu) := \text{Ind } \mathcal{W}_{H, \mathcal{Q}}(\Lambda_a, \mu)$ with the highest weight $(\Lambda_a + \mu, \mu)$:*

$$\mathcal{V}(\Lambda_a + \mu, \mu) = \mathcal{F}_{\alpha, 1} \otimes \mathcal{W}_{H, \mathcal{Q}}(\Lambda_a, \mu) = \bigoplus_{\gamma, \kappa \in \mathcal{Q}} \mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu),$$

where

$$\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu) = \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha, 1} \otimes e^{\bar{\Lambda}_a + \gamma}) \otimes e^{\mathcal{Q}\bar{\mu} + \kappa},$$

The highest weight vector is $1_1 \otimes e^{\bar{\Lambda}_a} \otimes e^{\mathcal{Q}\bar{\mu}}$. The derivation operator d is realized as

$$d = -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2)(P + h)^j,$$

$$N^\alpha = \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j,$$

where $r, r^* \in \mathbb{C}^\times$, and A_m^j are the fundamental weight type elliptic bosons given in Sec.5.1.

One can easily calculate the character of $\mathcal{V}(\Lambda_a, \mu)$:

$$ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} = \text{tr}_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c(W(\mathfrak{g}))}{24}} = \sum_{\gamma, \kappa \in \mathcal{Q}} ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)},$$

$$ch_{\mathcal{F}_{\gamma, \kappa}(\Lambda_a, \mu)} = \frac{1}{\eta(q)^l} q^{\frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \bar{\rho})|^2}.$$

Here $c(W(\mathfrak{g})) = l \left(1 - \frac{g(g+1)}{r r^*} \right)$, and $\eta(q)$ denotes Dedekind's η -function given by

$$\eta(q) = q^{\frac{1}{24}} (q; q)_\infty.$$

One should note that the character $ch_{\mathcal{F}_{\gamma,\kappa}(\Lambda_a,\mu)}$ coincides with the one of the Verma module of the $W(\mathfrak{g})$ -algebras for $\mathfrak{g} = A_l, D_l, E_6, E_7, E_8$ with the highest weight $h = \frac{1}{2r^*} |r(\bar{\mu} + \kappa + \rho) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \rho)|^2$ and the central charge $c(W(\mathfrak{g}))$. In fact, for $\widehat{\mathfrak{g}} = A_l^{(1)}$ case, for example, one can construct an action of the deformed $W(A_l)$ algebra on $\mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu)$ explicitly.

Theorem 4.7 [26, 27] *For $p = q^{2r}$ and $p^* = pq^{-2} = q^{2r^*}$, i.e. $r^* = r - 1$, the deformed $W(A_l)$ -algebra acts on $\mathcal{F}_{\xi,\kappa}(\Lambda_i + \mu, \mu)$ by*

$$\Lambda_j(z) =: \exp \left\{ \sum_{m \neq 0} (q^m - q^{-m})(1 - p^{*m}) \mathcal{E}_m^{+j} (q^j z)^{-m} \right\} : \otimes p^{*h\bar{\epsilon}_j} \quad (1 \leq j \leq l),$$

$$T_n(z) = \sum_{1 \leq j_1 < \dots < j_n \leq l} : \Lambda_{j_1}(z) \Lambda_{j_2}(zq^{-2}) \dots \Lambda_{j_n}(zq^{-2(n-1)}) : \quad (1 \leq n \leq l).$$

Here $\mathcal{E}_m^{\pm j}$ denotes the orthonormal basis type elliptic boson given in Eq. (5.3), and $: \cdot :$ denotes the normal ordering of the enclosed expression such that the operators $\mathcal{E}_m^{\pm j}$ for $m < 0$ are to be placed to the left of the operators $\mathcal{E}_m^{\pm j}$ for $m > 0$. In addition, the level-1 elliptic currents $e_j(w)$ and $f_j(w)$ of $U_{q,p}(A_l^{(1)})$ obtained from Proposition 3.9, Eqs. (4.1) and (4.2) are the screening currents of the deformed $W(A_l)$ -algebra, i.e. they commute with $T_n(z)$ up to a total difference.

See also [2, 7, 29]. A similar statement is valid also for the deformed $W(D_l)$ [28] and $U_{q,p}(D_l^{(1)})$. We also expect that for $r \in \mathbb{Z}_{>0}$ satisfying $r > g + 1$ and for a level- $(r - g - 1)$ dominant integral weight μ , the space $\mathcal{F}_{\gamma,\kappa}(\Lambda_a, \mu)$ becomes completely degenerate with respect to the action of the corresponding deformed $W(\mathfrak{g})$ -algebra [26–28], although the $E_{6,7,8}$ -type deformed W algebras have not yet been constructed explicitly. In order to get the irreducible module one should make the BRST-resolution in terms of the BRST-charge constructed from the half currents of $U_{q,p}(\widehat{\mathfrak{g}})$. An explicit demonstration for the $A_1^{(1)}$ case has been discussed in [35].

Remark. In Theorem 4.7, we assumed $p = q^{2r}$ in order to make a connection to the deformed $W(A_l)$ -algebra. The same relation arises naturally when one considers the finite dimensional representations of the universal elliptic dynamical \mathcal{R} matrices [5, 40].

4.2.2 The $B_l^{(1)}$ case

We follow the work [41] and its quantum analogues [42, 43] with a slight modification in the Ramond sector according to [44]. Let e^{α_i} ($i \in I$) be the generators of the group algebra $\mathbb{C}[\mathcal{Q}]$ with the following central extension.

$$e^{\alpha_i} e^{\alpha_j} = (-1)^{(\alpha_i, \alpha_j) + (\alpha_i, \alpha_i)(\alpha_j, \alpha_j)} e^{\alpha_j} e^{\alpha_i}$$

As before we regard h_i ($i \in I$) as an operator such that

$$z^{\pm h_i} e^{\alpha_j} = z^{\pm(\alpha_i^\vee, \alpha_j)} e^{\alpha_j} z^{\pm h_i}$$

We also need the Neveu-Schwartz (NS) fermion $\{\Psi_n | n \in \mathbb{Z} + \frac{1}{2}\}$ and the Ramond (R) fermion $\{\Psi_n | n \in \mathbb{Z}\}$ satisfying the following anti-commutation relations.

$$\{\Psi_m, \Psi_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$$

with $\mathcal{N} = 1/(q^{\frac{1}{2}} + q^{-\frac{1}{2}})$. We define

$$\mathcal{F}^{NS} = \mathbb{C}[\Psi_{-\frac{1}{2}}, \Psi_{-\frac{3}{2}} \cdots], \quad \widetilde{\mathcal{F}}^R = \mathbb{C}[\Psi_{-1}, \Psi_{-2}, \dots]$$

and their submodules $\mathcal{F}_{even}^{NS,R}$ (reps. $\mathcal{F}_{odd}^{NS,R}$) generated by the even (reps. odd) number of Ψ_{-m} 's. One should note that for the R fermion $\Psi_0^2 = \mathcal{N}$ and $\{\Psi_m, \Psi_0\} = 0$ for $m \neq 0$. So we have two degenerate vacuum states 1 and $\Psi_0 1$. We hence consider the extended space

$$\widehat{\mathcal{F}}^R = \widetilde{\mathcal{F}}^R \otimes \mathbb{C}^2$$

and realize the R -fermions by

$$\widehat{\Psi}_m = \Psi_m \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (m \in \mathbb{Z}_{\neq 0}), \quad \widehat{\Psi}_0 = \mathcal{N}^{\frac{1}{2}} (1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}).$$

Note that $\{\widehat{\Psi}_m, \widehat{\Psi}_n\} = \delta_{m+n,0} \mathcal{N}(q^m + q^{-m})$. We set

$$\mathcal{F}^R = \mathcal{F}_{even}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \mathcal{F}_{odd}^R \otimes \mathbb{C} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The action of Ψ_m on \mathcal{F}^{NS} is given by

$$\Psi_{-m} \cdot u = \Psi_{-m} u, \quad \Psi_m \cdot u = \{\Psi_m, u\} \quad (m \in \mathbb{Z}_{>0}),$$

where $u \in \mathcal{F}^{NS}$, whereas $\widehat{\Psi}_m$ acts on \mathcal{F}^R as

$$\begin{aligned} \widehat{\Psi}_{-m} \cdot u \otimes v &= \Psi_{-m} u \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), & \widehat{\Psi}_0 \cdot u \otimes v &= u \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v, \\ \widehat{\Psi}_m \cdot u \otimes v &= \{\Psi_m, u\} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v \quad (m \in \mathbb{Z}_{>0}), \end{aligned}$$

where $u \in \widetilde{\mathcal{F}}^R, v \in \mathbb{C}^2$.

Let us define the fermion fields $\Psi^{NS}(z)$ and $\Psi^R(z)$ by

$$\Psi^{NS}(z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \Psi_n z^{-n}, \quad \Psi^R(z) = \sum_{n \in \mathbb{Z}} \widehat{\Psi}_n z^{-n}.$$

One can derive the following operator product expansions.

$$\Psi(z)\Psi(w) =: \Psi(z)\Psi(w) : + \langle \Psi(z)\Psi(w) \rangle,$$

where

$$\langle \Psi(z)\Psi(w) \rangle = \begin{cases} \frac{(zw)^{1/2}(z-w)}{(z-qw)(z-q^{-1}w)} & \text{for NS} \\ \mathcal{N} \frac{(z-w)(z+w)}{(z-qw)(z-q^{-1}w)} & \text{for R.} \end{cases}$$

Then the quantum Z -algebra $Z_1(B_l^{(1)})$ is realized as follows [23].

$$\begin{aligned} Z_i^\pm(z) &= e^{\pm\alpha_i} z^{\pm h_i + 1} \quad (1 \leq i \leq l-1), \\ Z_l^\pm(z) &= \frac{1}{\mathcal{N}^{1/2}} \Psi(z) e^{\pm\alpha_l} z^{\pm d_l h_l + d_l}. \end{aligned}$$

There are three irreducible $Z_1(B_l^{(1)})$ -modules given by

$$\begin{aligned} W(\Lambda_0) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0] \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0]e^{\bar{\Lambda}_1}, \\ W(\Lambda_1) &= \mathcal{F}_{even}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0]e^{\bar{\Lambda}_1} \oplus \mathcal{F}_{odd}^{NS} \otimes \mathbb{C}[\mathcal{Q}_0], \\ W(\Lambda_l) &= \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}]e^{\bar{\Lambda}_l} \cong \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0]e^{\bar{\Lambda}_l} \oplus \mathcal{F}^R \otimes \mathbb{C}[\mathcal{Q}_0]e^{\bar{\Lambda}_1 + \bar{\Lambda}_l}, \end{aligned}$$

where \mathcal{Q}_0 denotes the sublattice of \mathcal{Q} generated by the long roots. For generic $\mu \in \mathfrak{h}^*$ and $a = 0, 1, l$, we set $\mathcal{W}_{H, \mathcal{Q}}(\Lambda_a, \mu) = (\mathbb{F} \otimes_{\mathbb{C}} W(\Lambda_a)) \otimes e^{Q\bar{\mu}} \mathbb{C}[\mathcal{R}_{\mathcal{Q}}]$. From Proposition 3.9 we have the following three level-1 irreducible $U_{q,p}(\hat{B}_l^{(1)})$ -modules with the highest weight $(\Lambda_a + \mu, \mu)$:

$$\begin{aligned} \mathcal{V}(\Lambda_a + \mu, \mu) &= \mathcal{F}_{\alpha,1} \otimes_{\mathbb{C}} \mathcal{W}_{H, \mathcal{Q}}(\Lambda_a, \mu) \\ &= \bigoplus_{\gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}} \bigoplus_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta}} \mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu), \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\gamma}) \otimes e^{Q\bar{\mu} + \kappa}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_0, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q\bar{\mu} + \kappa}, \\ \mathcal{F}_{\Lambda_1, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{even}^{NS} \otimes e^{\bar{\Lambda}_1 + \gamma}) \otimes e^{Q\bar{\mu} + \kappa}, \\ \mathcal{F}_{\Lambda_0, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}_{odd}^{NS} \otimes e^{\gamma}) \otimes e^{Q\bar{\mu} + \kappa}, \\ \mathcal{F}_{\Lambda_l, \gamma, \kappa}(\Lambda_l, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \gamma}) \otimes e^{Q\bar{\mu} + \kappa}, \\ \mathcal{F}_{\Lambda_l - \alpha_l, \gamma, \kappa}(\Lambda_1, \mu) &= \mathbb{F} \otimes_{\mathbb{C}} (\mathcal{F}_{\alpha,1} \otimes \mathcal{F}^R \otimes e^{\bar{\Lambda}_l + \bar{\Lambda}_1 + \gamma}) \otimes e^{Q\bar{\mu} + \kappa}. \end{aligned}$$

The highest weight vectors are given by $1 \otimes 1 \otimes 1 \otimes e^{Q\bar{\mu}}$ for $\mathcal{V}(\Lambda_0 + \mu, \mu)$, $1 \otimes 1 \otimes e^{\bar{\Lambda}_1} \otimes e^{Q\bar{\mu}}$ for $\mathcal{V}(\Lambda_1 + \mu, \mu)$ and $1 \otimes 1 \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes e^{\bar{\Lambda}_l} \otimes e^{Q\bar{\mu}}$ for $\mathcal{V}(\Lambda_l + \mu, \mu)$, respectively.

It is also easy to calculate the characters of these modules:

$$ch_{\mathcal{V}(\Lambda_a + \mu, \mu)} = \text{tr}_{\mathcal{V}(\Lambda_a + \mu, \mu)} q^{-d - \frac{c_W}{24}} = \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0 + \mathbb{C}\delta \\ \gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}}} ch_{\mathcal{F}_{\lambda, \gamma, \kappa}(\Lambda_a, \mu)},$$

where $c_W = (l + \frac{1}{2}) \left(1 - \frac{2l(2l-1)}{rr^*}\right)$ is the central charge of the WB_l algebra by Fateev and Lukyanov [31], and the derivation operator d is realized as

$$\begin{aligned} d &= -\frac{1}{2} \sum_{j=1}^l h_j h^j - N^\alpha - N^\Psi + \frac{1}{2r^*} \sum_{j=1}^l (P_j + 2) P^j - \frac{1}{2r} \sum_{j=1}^l ((P + h)_j + 2) (P + h)^j, \\ N^\alpha &= \sum_{j=1}^l \sum_{m \in \mathbb{Z}_{>0}} \frac{m^2}{[m]} \frac{1 - p^{*m}}{1 - p^m} q^m \alpha_{j, -m} A_m^j, \quad N^\Psi = \sum_{m > 0} \frac{m(q^{\frac{1}{2}} + q^{-\frac{1}{2}})}{q^m + q^{-m}} \Psi_{-m} \Psi_m \end{aligned}$$

where $r, r^* \in \mathbb{C}^\times$, and A_m^j are the fundamental weight type elliptic bosons of the type B_l given in Sec.5.1, Ψ_m denotes Ψ_m on \mathcal{F}^{NS} and $\widehat{\Psi}_m$ on \mathcal{F}^R . We obtain:

$$\begin{aligned}
 ch_{\mathcal{V}(\Lambda_a+\mu,\mu)} &= \sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0+\mathbb{C}\delta \\ \gamma \in \mathcal{Q}_0, \kappa \in \mathcal{Q}}} ch_{\mathcal{F}_{\lambda,\gamma,\kappa}(\Lambda_a,\mu)}, \\
 ch_{\mathcal{F}_{\Lambda_0,\gamma,\kappa}(\Lambda_0,\mu)} &= c_{\Lambda_0}^{\Lambda_0} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2, \\
 ch_{\mathcal{F}_{\Lambda_1,\gamma,\kappa}(\Lambda_0,\mu)} &= c_{\Lambda_1}^{\Lambda_0} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\Lambda}_1+\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2, \\
 ch_{\mathcal{F}_{\Lambda_1,\gamma,\kappa}(\Lambda_1,\mu)} &= c_{\Lambda_1}^{\Lambda_1} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\Lambda}_1+\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2, \\
 ch_{\mathcal{F}_{\Lambda_0,\gamma,\kappa}(\Lambda_1,\mu)} &= c_{\Lambda_0}^{\Lambda_1} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2, \\
 ch_{\mathcal{F}_{\Lambda_l,\gamma,\kappa}(\Lambda_l,\mu)} &= c_{\Lambda_l}^{\Lambda_l} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\Lambda}_l+\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2, \\
 ch_{\mathcal{F}_{\Lambda_l-\alpha_l,\gamma,\kappa}(\Lambda_l,\mu)} &= c_{\Lambda_l-\alpha_l}^{\Lambda_l} q \frac{1}{2rr^*} \left| r(\bar{\mu}+\kappa+\bar{\rho})-r^*(\bar{\Lambda}_l+\bar{\mu}+\kappa+\gamma+\bar{\rho}) \right|^2,
 \end{aligned}$$

where

$$\begin{aligned}
 c_{\Lambda_0}^{\Lambda_0} &= c_{\Lambda_1}^{\Lambda_1} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left((-q^{\frac{1}{2}}; q)_\infty + (q^{\frac{1}{2}}; q)_\infty \right), \\
 c_{\Lambda_0}^{\Lambda_1} &= c_{\Lambda_1}^{\Lambda_0} = \frac{q^{-\frac{1}{48}}}{2\eta(q)^l} \left((-q^{\frac{1}{2}}; q)_\infty - (q^{\frac{1}{2}}; q)_\infty \right), \\
 c_{\Lambda_l}^{\Lambda_l} &= c_{\Lambda_l-\alpha_l}^{\Lambda_l} = \frac{q^{\frac{1}{24}}}{2\eta(q)^l} (-q; q)_\infty.
 \end{aligned}$$

$\sum_{\substack{\lambda \in \max(\Lambda_a) \\ \text{mod } \mathcal{Q}_0+\mathbb{C}\delta}} ch_{\mathcal{F}_{\lambda,\gamma,\kappa}(\Lambda_a,\mu)}$ coincides with the character of the Verma modules of the WB_l -algebra with the highest weight $h = \frac{1}{2rr^*} |r(\bar{\mu} + \kappa + \bar{\rho}) - r^*(\bar{\Lambda}_a + \bar{\mu} + \gamma + \kappa + \bar{\rho})|^2$ and the central charge c_W with $r, r^* = r - 1 \in \mathbb{C}$ being generic.

Conjecture 4.8 *There exists a deformation of the WB_l -algebra such that*

- i) *its generating functions commute with the level-1 elliptic currents $e_j(z)$ and $f_j(z)$ of $U_{q,p}(B_l^{(1)})$ modulo a total difference, i.e. $e_j(z)$ and $f_j(z)$ at $c = 1$ are the screening currents of the deformation of the WB_l -algebra,*
- ii) *for generic r and $\mu \in \mathfrak{h}^*$, $\mathcal{F}_{\lambda,\xi,\kappa}(\Lambda+\mu, \mu)$ is an irreducible module of the deformation of the WB_l -algebra.*

Remark All the algebras $W(\mathfrak{g})$ appearing in section 4.2.1 and WB_l in this subsection are the W -algebras associated with the coset $X_l^{(1)} \oplus X_l^{(1)} \supset (X_l^{(1)})_{\text{diag}}$ with level $(r - g - 1, 1)$. In particular, the WB_l is different from the one obtained from the quantum Hamiltonian reduction of the affine Lie algebra $B_l^{(1)}$. The W -algebras associated with such coset describe the critical behavior of the face type solvable lattice models introduced by Jimbo, Miwa and Okado [33].

5 Elliptic Bosons of Various Types

In this section we introduce elliptic bosons of the fundamental weight type A_m^j and the orthogonal basis type $\mathcal{E}_m^{\pm j}$ for $U_{q,p}(\widehat{\mathfrak{g}})$, $\widehat{\mathfrak{g}} = A_l^{(1)}, B_l^{(1)}, C_l^{(1)}, D_l^{(1)}$. The level-1 bosons A_m^j and $\mathcal{E}_m^{\pm j}$ are used to realize the derivation operator d and the generating function of the deformed $W(A_l)$ -algebra, respectively, in section 4.2.

5.1 Definition

Let us set $\eta = -tg/2$ ($t = (\text{long root})^2/2$).

	$A_n^{(1)}$	$B_n^{(1)}$	$C_n^{(1)}$	$D_n^{(1)}$
g	$n + 1$	$2n - 1$	$n + 1$	$2n - 2$
t	1	1	2	1
η	$-\frac{l+1}{2}$	$-\frac{2l-1}{2}$	$-(l + 1)$	$-(l - 1)$

Let $\alpha_{i,m}$ be the elliptic bosons of the simple root type as in section 2. We define the fundamental weight type elliptic bosons A_m^j ($1 \leq j \leq l, m \in \mathbb{Z}_{\neq 0}$) by

$$[\alpha_{i,m}, A_n^j] = -\delta_{i,j} \delta_{m+n,0} \frac{[cm]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} \quad (1 \leq i, j \leq l). \tag{5.1}$$

Note that using the matrix $B(m) = ([b_{i,jm}])_{1 \leq i, j \leq l}$, we have [28]

$$A_m^j = \sum_{k=1}^l (B(m)^{-1})_{kj} \alpha_{k,m}.$$

Solving Eq. (5.1) we obtain the following.

For $A_l^{(1)}$,

$$A_m^j = C_m \left([(2\eta + j)m] \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^l [(2\eta + k)m] \alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For $B_l^{(1)}$,

$$A_m^j = C_m \left((q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^l (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} \right) \quad (1 \leq j \leq l).$$

For $C_l^{(1)}$,

$$A_m^j = C_m \left((q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^{l-1} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} + [jm] \alpha_{l,m} \right), \quad (1 \leq j \leq l - 1),$$

$$A_m^l = C_m \left(\sum_{k=1}^{l-1} [km] \alpha_{k,m} + \frac{[m]}{[2m]} [lm] \alpha_{l,m} \right).$$

For $D_l^{(1)}$,

$$A_m^j = C_m \left((q^{(\eta+j)m} + q^{-(\eta+j)m}) \sum_{k=1}^j [km] \alpha_{k,m} + [jm] \sum_{k=j+1}^{l-2} (q^{(\eta+k)m} + q^{-(\eta+k)m}) \alpha_{k,m} + [jm] (\alpha_{l-1,m} + a_{l,m}) \right) \quad (1 \leq j \leq l-2),$$

$$A_m^{l-1} = C_m \left(\sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([lm] \alpha_{l-1,m} + [(l-2)m] a_{l,m}) \right),$$

$$A_m^l = C_m \left(\sum_{k=1}^{l-2} [km] \alpha_{k,m} + \frac{[m]}{[2m]} ([(l-2)m] \alpha_{l-1,m} + [lm] a_{l,m}) \right).$$

Here

$$C_m = \frac{1}{[m]^2 [2\eta m]} \quad \text{for } A_l^{(1)}$$

$$= \frac{[\eta m]}{[m]^2 [2\eta m]} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}.$$

We then divide A_m^j into two terms and define the elliptic bosons $\mathcal{E}_m^{\pm j}$ of the orthogonal basis type as follows.

For $A_l^{(1)}$,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \tag{5.2}$$

$$\mathcal{E}_m^{\pm j} = \pm q^{\pm jm} \frac{C_m}{q - q^{-1}} \left(q^{\pm 2\eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} + \sum_{k=j}^l [(2\eta + k)m] \alpha_{k,m} \right) \tag{5.3}$$

for $1 \leq j \leq l$. It is convenient to define $\mathcal{E}_m^{\pm(l+1)}$ by

$$\mathcal{E}_m^{\pm(l+1)} = \mp \frac{C_m}{q - q^{-1}} \sum_{k=1}^l [km] \alpha_{k,m}. \tag{5.4}$$

For $B_l^{(1)}$,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \tag{5.5}$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^l [(\eta + k)m]_+ \alpha_{k,m} \right) \tag{5.6}$$

for $1 \leq j \leq l$. Here we set

$$[m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

We also define

$$\mathcal{E}_m^0 = \frac{[m]}{[m]} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}). \tag{5.7}$$

For $C_l^{(1)}$,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j} \quad (5.8)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-1} [(\eta + k)m]_+ \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right) \quad (1 \leq j \leq l-1), \quad (5.9)$$

$$A_m^l = \frac{1}{q^m + q^{-m}} (\mathcal{E}_m^{+l} + \mathcal{E}_m^{-l}), \quad (5.10)$$

$$\mathcal{E}_m^{\pm l} = q^{\pm lm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{l-1} [km] \alpha_{k,m} \pm \frac{\alpha_{l,m}}{q - q^{-1}} \right). \quad (5.11)$$

For $D_l^{(1)}$,

$$A_m^j = \mathcal{E}_m^{+j} + \mathcal{E}_m^{-j}, \quad (5.12)$$

$$\mathcal{E}_m^{\pm j} = q^{\pm jm} C_m \left(q^{\pm \eta m} \sum_{k=1}^{j-1} [km] \alpha_{k,m} \pm \sum_{k=j}^{l-2} [(\eta + k)m]_+ \alpha_{k,m} \pm \frac{1}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right) \quad (1 \leq j \leq l-2), \quad (5.13)$$

$$\mathcal{E}_m^{\pm(l-1)} = C_m \left(\sum_{k=1}^{l-2} [km] \alpha_{k,m} \pm \frac{q^{\mp \eta m}}{q - q^{-1}} (\alpha_{l-1,m} + \alpha_{l,m}) \right), \quad (5.14)$$

$$\mathcal{E}_m^{\pm l} = q^{\pm lm} C_m \left(\sum_{k=1}^{l-2} [km] \alpha_{k,m} \mp \frac{1}{q - q^{-1}} (q^{\pm \eta m} \alpha_{l-1,m} - q^{\mp \eta m} \alpha_{l,m}) \right). \quad (5.15)$$

Proposition 5.1

$$\alpha_{j,m} = \pm [m]^2 (q - q^{-1}) (\mathcal{E}_m^{\pm j} - q^{\mp m} \mathcal{E}_m^{\pm(j+1)}), \quad (5.16)$$

where $1 \leq j \leq l$ for $A_l^{(1)}$, $1 \leq j \leq l-1$ for $B_l^{(1)}$, $C_l^{(1)}$, $D_l^{(1)}$, and

$$\alpha_{l,m} = [m] (q^{m/2} - q^{-m/2}) (q^{-m/2} \mathcal{E}_m^{+l} - q^{m/2} \mathcal{E}_m^{-l}) \quad \text{for } B_l^{(1)}, \quad (5.17)$$

$$= [m]^2 (q - q^{-1}) (q^m \mathcal{E}_m^{+l} - q^{-m} \mathcal{E}_m^{-l}) \quad \text{for } C_l^{(1)}, \quad (5.18)$$

$$= \pm [m]^2 (q - q^{-1}) (\mathcal{E}_m^{\pm(l-1)} - q^{\pm m} \mathcal{E}_m^{\mp l}) \quad \text{for } D_l^{(1)}. \quad (5.19)$$

Proposition 5.2 *The following relations hold.*

$$\mathcal{E}_m^{\pm 1} = \pm \frac{q^{\pm m}}{q^m - q^{-m}} A_m^1, \quad \mathcal{E}_m^{\pm j} = \pm \frac{1}{q^m - q^{-m}} (q^{\pm m} A_m^j - A_m^{j-1}), \quad (5.20)$$

where $2 \leq j \leq l$ for $A_l^{(1)}$, $2 \leq j \leq l$ for $B_l^{(1)}$, $2 \leq j \leq l-1$ for $C_l^{(1)}$ and $2 \leq j \leq l-2$ for $D_l^{(1)}$. In addition, we have

$$\mathcal{E}_m^{\pm(l+1)} = \mp \frac{1}{q^m - q^{-m}} A_m^l, \quad \sum_{j=1}^{l+1} q^{\pm(j-1)m} \mathcal{E}_m^{\pm j} = 0 \quad \text{for } A_l^{(1)}, \quad (5.21)$$

$$\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} ((q^m + q^{-m}) q^{\pm m} A_m^l - A_m^{l-1}) \quad \text{for } C_l^{(1)}, \quad (5.22)$$

and

$$\mathcal{E}_m^{\pm(l-1)} = \pm \frac{1}{q^m - q^{-m}} \left(q^{\pm m} A_m^{l-1} + q^{\pm m} A_m^l - A_m^{l-2} \right), \tag{5.23}$$

$$\mathcal{E}_m^{\pm l} = \pm \frac{1}{q^m - q^{-m}} \left(q^{\pm 2m} A_m^l - A_m^{l-1} \right) \quad \text{for } D_l^{(1)}. \tag{5.24}$$

Remark The level-1 case i.e. $c = 1$, the $A_l^{(1)}$ type relation was given in [26, 27] and the $D_l^{(1)}$ type was essentially given in [28], where parameters q and t should be identified with our $p^{*\frac{1}{2}} = p^{\frac{1}{2}}q^{-1}$ and $p^{\frac{1}{2}}$, respectively. However the $B_l^{(1)}$ and $C_l^{(1)}$ cases are different from those given in [28]. At least the formulas for $B_l^{(1)}$ and $C_l^{(1)}$ seem to be reversed. Our definitions and relations are valid for arbitrary level c .

Although the expressions of $\mathcal{E}_m^{\pm j}$ are complicated depending on the types of the affine Lie algebras, their commutation relations are rather universal:

Theorem 5.3 For $1 \leq j, k \leq l$, the following commutation relations hold. For $A_l^{(1)}$,

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = [\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \delta_{m+n,0} \frac{[cm][(2\eta + 1)m]}{m(q - q^{-1})^2[m]^3[2\eta m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \tag{5.25}$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \delta_{m+n,0} q^{\mp(\text{sgn}(k-j)2\eta+k-j)m} \frac{[cm]}{m(q - q^{-1})[m]^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \tag{5.26}$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = -\delta_{m+n,0} q^{\pm(2\eta+j+k)m} \frac{[cm]}{m(q - q^{-1})[m]^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}. \tag{5.27}$$

For $B_l^{(1)}, C_l^{(1)}, D_l^{(1)}$,

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm j}] = \delta_{m+n,0} \frac{[cm][\eta m][2(\eta + 1)m]}{m(q - q^{-1})^2[m]^3[2\eta m][(\eta + 1)m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \tag{5.28}$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp j}] = \mp \delta_{m+n,0} \frac{q^{\pm jm}[cm][\eta m]}{m[m]^3(q - q^{-1})^2[2\eta m]} \frac{[pm]}{[p^*m]} \left(q^{\pm(\eta+j)m}[m] \pm q^{\mp(j-1)m}[\eta m]_+ \right), \tag{5.29}$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\pm k}] = \mp \text{sgn}(k - j) \delta_{m+n,0} q^{\mp(\text{sgn}(k-j)\eta+k-j)m} \frac{[cm][\eta m]}{m(q - q^{-1})[m]^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \tag{5.30}$$

$$[\mathcal{E}_m^{\pm j}, \mathcal{E}_n^{\mp k}] = \mp \delta_{m+n,0} q^{\pm(\eta+j+k)m} \frac{[cm][\eta m]}{m(q - q^{-1})[m]^2[2\eta m]} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}. \tag{5.31}$$

Here

$$\text{sgn}(l - j) = \begin{cases} + & (l > j), \\ - & (l < j). \end{cases}$$

Proof Straightforward calculation using Proposition 5.2 and Eq. (5.1). □

Proposition 5.4 For $1 \leq i \leq l$, the following commutation relations hold.

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm j}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,j} - \delta_{i,j-1}) \tag{5.32}$$

where $1 \leq j \leq l$ for $A_l^{(1)}, B_l^{(1)}$, $1 \leq j \leq l-1$ for $C_l^{(1)}$, $1 \leq j \leq l-2$ for $D_l^{(1)}$. In addition,

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm l}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} (q^{\mp m} (q^m + q^{-m}) \delta_{i,l} - \delta_{i,l-1}) \text{ for } C_l^{(1)}, \quad (5.33)$$

and

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm(l-1)}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} (q^{\mp m} \delta_{i,l-1} + q^{\mp m} \delta_{i,l} - \delta_{i,l-2}), \quad (5.34)$$

$$[\alpha_{i,m}, \mathcal{E}_n^{\pm l}] = \pm \frac{[cm]}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} (q^{\mp 2m} \delta_{i,l} - \delta_{i,l-1}) \quad \text{for } D_l^{(1)}. \quad (5.35)$$

From Eqs. (2.15) and (2.16) we also obtain the following relations.

Proposition 5.5 For $1 \leq j \leq l$,

$$[\mathcal{E}_m^{\pm i}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}), \quad (5.36)$$

$$[\mathcal{E}_m^{\pm i}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{i,j} - \delta_{i-1,j}) \quad (5.37)$$

where $1 \leq i \leq l$ for $A_l^{(1)}, B_l^{(1)}$, $1 \leq i \leq l-1$ for $C_l^{(1)}$, $1 \leq i \leq l-2$ for $D_l^{(1)}$. In addition,

$$[\mathcal{E}_m^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}), \quad (5.38)$$

$$[\mathcal{E}_m^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} (q^m + q^{-m}) \delta_{l,j} - \delta_{l-1,j}) \quad \text{for } C_l^{(1)}, \quad (5.39)$$

and

$$[\mathcal{E}_m^{\pm(l-1)}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm m} \delta_{l-1,j} + q^{\pm m} \delta_{l,j} - \delta_{l-2,j}), \quad (5.40)$$

$$[\mathcal{E}_m^{\pm(l-1)}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm m} \delta_{l-1,j} + q^{\pm m} \delta_{l,j} - \delta_{l-2,j}), \quad (5.41)$$

$$[\mathcal{E}_m^{\pm l}, e_j(z)] = \pm \frac{q^{-cm} z^m}{m(q^m - q^{-m})} \frac{1 - p^m}{1 - p^{*m}} e_j(z) (q^{\pm 2m} \delta_{l,j} - \delta_{l-1,j}), \quad (5.42)$$

$$[\mathcal{E}_m^{\pm l}, f_j(z)] = \mp \frac{z^m}{m(q^m - q^{-m})} f_j(z) (q^{\pm 2m} \delta_{l,j} - \delta_{l-1,j}) \quad \text{for } D_l^{(1)}. \quad (5.43)$$

5.2 The Elliptic Currents $k_{\pm j}(z)$

Let us set

$$\psi_j(z) = : \exp \left\{ (q - q^{-1}) \sum_{m \neq 0} \frac{\alpha_{j,m}}{1 - p^m} p^m z^{-m} \right\} : . \tag{5.44}$$

Then the elliptic currents $\psi_j^{\pm}(z)$ in Definition 2.1 can be written as

$$\psi_j^+(q^{-\frac{c}{2}}z) = K_j^+ \psi_j(z), \quad \psi_j^-(q^{-\frac{c}{2}}z) = K_j^- \psi_j(pq^{-c}z). \tag{5.45}$$

Let us introduce the new currents $k_{\pm j}(z)$ ($1 \leq j \leq l$) associated with $\mathcal{E}_m^{\pm j}$ by

$$k_{\pm j}(z) = : \exp \left\{ \sum_{m \neq 0} \frac{[m]^2 (q - q^{-1})^2}{1 - p^m} p^m \mathcal{E}_m^{\pm j} z^{-m} \right\} : \tag{5.46}$$

and in addition we define $k_0(z)$ for $B_l^{(1)}$ by

$$k_0(z) = : k_{-l}(q^{-1/2}z) \psi_l(q^{-1/2}z) :=: k_{+l}(q^{1/2}z) \psi_l(q^{1/2}z)^{-1} : . \tag{5.47}$$

Then from Proposition 5.1 we have the following decompositions.

Proposition 5.6

$$\psi_j(z) = : k_{+j}(z) k_{+(j+1)}(qz)^{-1} :=: k_{-j}(z)^{-1} k_{-(j+1)}(q^{-1}z) : \tag{5.48}$$

where $1 \leq j \leq l - 1$ for $A_l^{(1)}$, $1 \leq j \leq l - 1$ for $B_l^{(1)}$, $C_l^{(1)}$ and $D_l^{(1)}$. In addition,

$$\psi_l(z) = : k_{+l}(z) k_0(q^{-1/2}z)^{-1} :=: k_{-l}(z)^{-1} k_0(q^{1/2}z) : \quad \text{for } B_l^{(1)}, \tag{5.49}$$

$$= : k_{+l}(q^{-1}z) k_{-l}(qz)^{-1} : \quad \text{for } C_l^{(1)}, \tag{5.50}$$

$$= : k_{+(l-1)}(z) k_{-l}(q^{-1}z)^{-1} :=: k_{-(l-1)}(z)^{-1} k_{+l}(qz) : \quad \text{for } D_l^{(1)}. \tag{5.51}$$

Now let us introduce the functions $\tilde{\rho}^+(z)$, which appear associated with the elliptic dynamical R -matrices [40]:

$$\tilde{\rho}^+(z) = \frac{\{q^2z\} \{\xi^2q^{-2}z\}}{\{\xi^2z\} \{z\}} \frac{\{p\xi^2/z\} \{p/z\}}{\{p\xi^2q^{-2}/z\} \{pq^2/z\}} \quad \text{for } A_l^{(1)}, \tag{5.52}$$

$$= \frac{\{\xi z\}^2 \{\xi^2q^{-2}z\} \{q^2z\}}{\{\xi^2z\} \{z\} \{\xi q^2z\} \{\xi q^{-2}z\}} \frac{\{p\xi^2/z\} \{p/z\} \{p\xi q^2/z\} \{p\xi q^{-2}/z\}}{\{p\xi/z\}^2 \{p\xi^2q^{-2}/z\} \{pq^2/z\}} \quad \text{for } B_l^{(1)}, C_l^{(1)}, D_l^{(1)}, \tag{5.53}$$

where $\xi = q^{-2\eta}$, $\{z\} = (z; p, \xi^2)_\infty$. The following Theorem indicates a deep relationship between $k_{\pm j}(z)$'s and elliptic dynamical R -matrices.

Theorem 5.7

$$k_{\pm j}(z_1)k_{\pm j}(z_2) = \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)}k_{\pm j}(z_2)k_{\pm j}(z_1), \quad (1 \leq j \leq l),$$

$$k_{+j}(q^j z_1)k_{+k}(q^k z_2) = \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)}k_{+k}(q^k z_2)k_{+j}(q^j z_1) \quad (1 \leq j < k \leq l),$$

$$k_{-j}(q^{-j} z_1)k_{-k}(q^{-k} z_2) = \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)}k_{-k}(q^{-k} z_2)k_{-j}(q^{-j} z_1) \quad (1 \leq k < j \leq l),$$

$$k_{+j}(q^j z_1)k_{-k}(q^{-k} \xi z_2) = \frac{\tilde{\rho}^{+*}(z)}{\tilde{\rho}^+(z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)}k_{-k}(q^{-k} \xi z_2)k_{+j}(q^j z_1) \quad (j \neq k),$$

$$k_{+j}(q^j z_1)k_{-j}(q^{-j} \xi z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{2j-2}\xi^{-1}z)\Theta_p(q^{2j}\xi^{-1}z)}{\Theta_{p^*}(q^{2j}\xi^{-1}z)\Theta_p(q^{2j-2}\xi^{-1}z)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)}k_{-j}(q^{-j} \xi z_2)k_{+j}(q^j z_1),$$

where $z = z_1/z_2$ and $\tilde{\rho}^{+*}(z) = \tilde{\rho}^+(z)|_{p \mapsto p^*}$. In addition, for $B_l^{(1)}$ we have

$$k_0(z_1)k_0(z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(q^2z)\Theta_{p^*}(qz)\Theta_p(q^{-1}z)}{\Theta_{p^*}(q^2z)\Theta_p(q^{-2}z)\Theta_{p^*}(q^{-1}z)\Theta_p(qz)}k_0(z_2)k_0(z_1),$$

$$k_{+j}(q^j z_1)k_0(q^{l-1/2} z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(q^{-2}z)\Theta_p(z)}{\Theta_{p^*}(z)\Theta_p(q^{-2}z)}k_0(q^{l-1/2} z_2)k_{+j}(q^j z_1) \quad (1 \leq j \leq l),$$

$$k_{-j}(\xi q^{-j} z_1)k_0(q^{l-1/2} z_2) = \frac{\tilde{\rho}^{+*}(u)}{\tilde{\rho}^+(u)} \frac{\Theta_{p^*}(z)\Theta_p(q^2z)}{\Theta_{p^*}(q^2z)\Theta_p(z)}k_0(q^{l-1/2} z_2)k_{-j}(\xi q^{-j} z_1) \quad (1 \leq j \leq l).$$

Proof Straightforward calculation using Theorem 5.3. \square

In addition from Proposition 5.5, we obtain:

Proposition 5.8

$$k_{\pm j}(z_1)e_j(z_2) = \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c \mp 2}z)}e_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l),$$

$$k_{\pm j}(z_1)e_{j-1}(z_2) = \frac{\Theta_{p^*}(q^{-c \mp 1}z)}{\Theta_{p^*}(q^{-c \pm 1}z)}e_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l),$$

$$k_{\pm j}(z_1)e_k(z_2) = e_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1),$$

$$k_{\pm j}(z_1)f_j(z_2) = \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)}f_j(z_2)k_{\pm j}(z_1) \quad (1 \leq j \leq l),$$

$$k_{\pm j}(z_1)f_{j-1}(z_2) = \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{j-1}(z_2)k_{\pm j}(z_1) \quad (2 \leq j \leq l),$$

$$k_{\pm j}(z_1)f_k(z_2) = f_k(z_2)k_{\pm j}(z_1) \quad (k \neq j, j-1)$$

for $A_l^{(1)}$, $B_l^{(1)}$ with $1 \leq i \leq l$, $C_l^{(1)}$ with $1 \leq i \leq l - 1$, $D_l^{(1)}$ with $1 \leq i \leq l - 2$. In addition, we have

$$\begin{aligned}
 k_0(q^{l-1/2}z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c+l}z)\Theta_{p^*}(q^{-c+l-1}z)}{\Theta_{p^*}(q^{-c+l-2}z)\Theta_{p^*}(q^{-c+l+1}z)}e_l(z_2)k_0(q^{l-1/2}z_1), \\
 k_0(q^{l-1/2}z_1)e_j(z_2) &= e_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l - 1), \\
 k_0(q^{l-1/2}z_1)f_l(z_2) &= \frac{\Theta_p(q^{l-2}z)\Theta_p(q^{l+1}z)}{\Theta_p(q^l z)\Theta_p(q^{l-1}z)}f_l(z_2)k_0(q^{l-1/2}z_1), \\
 k_0(q^{l-1/2}z_1)f_j(z_2) &= f_j(z_2)k_0(q^{l-1/2}z_1) \quad (1 \leq j \leq l - 1) \quad \text{for } B_l^{(1)}, \\
 k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\pm 1}z)}{\Theta_{p^*}(q^{-c\mp 3}z)}e_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1), \\
 k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_p(q^{\mp 3}z)}{\Theta_p(q^{\pm 1}z)}f_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1) \quad \text{for } C_l^{(1)}, \\
 k_{\pm(l-1)}(z_1)e_j(z_2) &= \frac{\Theta_{p^*}(q^{-c}z)}{\Theta_{p^*}(q^{-c\mp 2}z)}e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l - 1), \\
 k_{\pm(l-1)}(z_1)e_{l-2}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
 k_{\pm(l-1)}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l - 1, l - 2), \\
 k_{\pm l}(z_1)e_l(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\mp 3}z)}e_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_{l-1}(z_2) &= \frac{\Theta_{p^*}(q^{-c\mp 1}z)}{\Theta_{p^*}(q^{-c\pm 1}z)}e_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)e_j(z_2) &= e_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1), \\
 k_{\pm(l-1)}(z_1)f_j(z_2) &= \frac{\Theta_p(q^{\mp 2}z)}{\Theta_p(z)}f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j = l, l - 1), \\
 k_{\pm(l-1)}(z_1)f_{l-2}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-2}(z_2)k_{\pm(l-1)}(z_1), \\
 k_{\pm(l-1)}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm(l-1)}(z_1) \quad (j \neq l, l - 1, l - 2), \\
 k_{\pm l}(z_1)f_l(z_2) &= \frac{\Theta_p(q^{\mp 3}z)}{\Theta_p(q^{\mp 1}z)}f_l(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_{l-1}(z_2) &= \frac{\Theta_p(q^{\pm 1}z)}{\Theta_p(q^{\mp 1}z)}f_{l-1}(z_2)k_{\pm l}(z_1), \\
 k_{\pm l}(z_1)f_j(z_2) &= f_j(z_2)k_{\pm l}(z_1) \quad (j \neq l, l - 1) \quad \text{for } D_l^{(1)}.
 \end{aligned}$$

The elliptic bosons $\mathcal{E}_m^{\pm j}$ and their elliptic currents $k_{\pm j}(z)$ are useful to realize the L -operators and the vertex operators for $U_{q,p}(\widehat{\mathfrak{g}})$ as well as deformation of the W -algebras. We will discuss this subject in separate papers.

Acknowledgments H.K is supported by the Grant-in -Aid for Scientific Research (C) 22540022 JSPS, Japan. R.M.F is grateful to the Egyptian government for a scholarship.

Appendix A: The Drinfeld Realization of $U_q(\widehat{\mathfrak{g}})$

Let $\widehat{\mathfrak{g}}$ be an untwisted affine Lie algebra.

Definition A.1 The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ in the Drinfeld realization is a unital \mathbb{C} -algebra generated by q^h ($h \in \mathfrak{h}$), $a_{i,n}^\vee$, $x_{i,m}^\pm$ ($i \in I$, $n \in \mathbb{Z}_{\neq 0}$, $m \in \mathbb{Z}$) \bar{d} and the central element c . We set

$$x_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m}^\pm z^{-m}, \quad (6)$$

$$\psi_i(z) = q_i^{h_i} \exp \left((q_i - q_i^{-1}) \sum_{n>0} a_{i,n}^\vee z^{-n} \right), \quad (7)$$

$$\varphi_i(z) = q_i^{-h_i} \exp \left(-(q_i - q_i^{-1}) \sum_{n>0} a_{i,-n}^\vee z^n \right). \quad (8)$$

The defining relations are as follows.

$$[q_i^{\pm h_i}, \bar{d}] = 0, \quad [\bar{d}, a_{i,n}] = na_{i,n}, \quad [\bar{d}, x_{i,n}^\pm] = nx_{i,n}^\pm, \quad (9)$$

$$[q_i^{\pm h_i}, a_{j,n}] = 0, \quad q_i^{h_i} x_j^\pm(z) = q_i^{\pm a_{ij}} x_j^\pm(z) q_i^{h_i}, \quad (10)$$

$$[a_{i,n}^\vee, a_{j,m}^\vee] = \frac{[a_{ij}n]_i [cn]_j}{n} q^{-c|n|} \delta_{n+m,0}, \quad (11)$$

$$[a_{i,n}^\vee, x_j^+(z)] = \frac{[a_{ij}n]_i}{n} q^{-c|n|} z^n x_j^+(z), \quad (12)$$

$$[a_{i,n}^\vee, x_j^-(z)] = -\frac{[a_{ij}n]_i}{n} z^n x_j^-(z), \quad (13)$$

$$(z - q^{\pm b_{ij}} w) x_i^\pm(z) x_j^\pm(w) = (q^{\pm b_{ij}} z - w) x_j^\pm(w) x_i^\pm(z), \quad (14)$$

$$[x_i^+(z), x_j^-(w)] = \frac{\delta_{i,j}}{q_i - q_i^{-1}} \left(\delta \left(q^{-k} \frac{z}{w} \right) \psi_i(q^{k/2} w) - \delta \left(q^k \frac{z}{w} \right) \varphi_i(q^{-k/2} w) \right), \quad (15)$$

$$\sum_{\sigma \in S_a} \sum_{s=0}^a (-)^s \begin{bmatrix} a \\ s \end{bmatrix}_i x_i^\pm(z_{\sigma(1)}) \cdots x_i^\pm(z_{\sigma(s)}) x_j^\pm(w) x_i^\pm(z_{\sigma(s+1)}) \cdots x_i^\pm(z_{\sigma(a)}) = 0, \quad (16)$$

$(i \neq j, a = 1 - a_{ij}).$

For $k \in \mathbb{C}$, we define the category C_k of the level- k $U_q(\widehat{\mathfrak{g}})$ -modules in the same way as \mathfrak{C}_k of $U_{q,p}(\widehat{\mathfrak{g}})$ in section 2. Let $a_{i,n} = [d_i] a_{i,n}^\vee$ ($i \in I$, $n \in \mathbb{Z}_{\neq 0}$) be the simple root type level- k Drinfeld bosons. They satisfy

$$[a_{i,n}, a_{j,m}] = \frac{[b_{ij}n][kn]}{n} q^{-k|n|} \delta_{n+m,0}.$$

For $(V, \bar{\pi}) \in C_k$, we define the Z -operators associated with the level- k $U_q(\widehat{\mathfrak{g}})$ -module V by

$$Z_i^\pm(z; V) = \exp\left(\mp \sum_{n \geq 1} \frac{\bar{\pi}(a_{i,-n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^n\right) \bar{\pi}(x_i^\pm(z)) \exp\left(\pm \sum_{n \geq 1} \frac{\bar{\pi}(a_{i,n})}{[kn]} q^{\frac{1 \mp 1}{2} kn} z^{-n}\right).$$

The coefficients $Z_{i,n}^\pm(V)$ of $Z_i^\pm(z; V) = \sum_{n \in \mathbb{Z}} Z_{i,n}^\pm(V) z^{-n}$ in z are well defined elements in $\text{End}_{\mathbb{C}} V$.

Theorem A.2 *The Z -operators $Z_i^\pm(z; V)$ satisfy the same relations in Theorem 3.3 except for Eqs. (3.16), (3.17) with replacement $\mathcal{Z}_j^\pm(z; \mathcal{V})$, $\alpha_{j,m}$, d and K_j^\pm by $Z_i^\pm(z; V)$, $a_{j,m}$, \bar{d} and $q_j^{\mp h_j}$, respectively.*

Remark This theorem is essentially due to Jing [23]. However, in [23] no Serre relations are written explicitly. There are also some misprints in Theorem 2.2 in [23]:

- $(1 - q^\mp w/z)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$ should be read as $(1 - q^\mp w/z)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$
- $(1 - q^\mp z/w)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$ should be read as $(1 - q^\mp z/w)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$
- $(1 - w/z)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$ should be read as $(1 - w/z)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$
- $(1 - z/w)_{q^{2k}}^{(\alpha_i|\alpha_j)/k}$ should be read as $(1 - z/w)_{q^{2k}}^{-(\alpha_i|\alpha_j)/k}$

Definition A.3 For $k \in \mathbb{C}^\times$ and $(V, \bar{\pi}) \in C_k$, we call the subalgebra of $\text{End}_{\mathbb{C}} V$ generated by $Z_{i,m}^\pm(V)$, $q_i^{\pm h_i}$ ($i \in I, m \in \mathbb{Z}$) and \bar{d} the quantum Z -algebra Z_V associated with $(V, \bar{\pi})$. We also define the universal quantum Z algebra Z_k as a topological algebra over $\mathbb{C}[[q^{2k}]]$ in the same way as \mathcal{Z}_k in Definition 3.5. We denote the generators in Z_k by $Z_{j,m}^\pm$ ($j \in I$).

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Higher level representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and its integrability

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(Received Xxx 00, 0000)

ABSTRACT. By using an elliptic analogue of the Drinfeld coproduct, we construct the level- $k + 1$ representation of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ from the level-1 highest weight representation. The quantum Z -algebra of level- $k + 1$ is realized. We also find the elliptic analogue of the condition of integrability for higher level modules constructed by the Drinfeld coproduct. This also enables us to express $\Delta^k(e(z))\Delta^k(e(zq^2))\cdots\Delta^k(e(zq^{2(N-1)}))$ and $\Delta^k(f(z))\Delta^k(f(zq^{-2}))\cdots\Delta^k(f(zq^{-2(N-1)}))$ as vertex operators of the level- $k + 1$ bosons.

1. Introduction

Lepowsky and Primic [15] studied the condition of integrability of higher level representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Ding and Feigin [1], Ding and Miwa [3] studied the quantum integrable condition and the q -parafermion of $U_q(\widehat{\mathfrak{sl}}_2)$ by using the Drinfeld coproduct [1] for the Drinfeld realization of $U_q(\widehat{\mathfrak{sl}}_2)$ [4]. The universal R matrix R_∞ associated with the Drinfeld coproduct is given in [2] for $U_q(\widehat{\mathfrak{g}})$ for general untwisted affine Lie algebra $\widehat{\mathfrak{g}}$. In [11, 8], Jimbo, Konno, Odake, Shiraishi gave an elliptic analogue $U_{q,p}(\widehat{\mathfrak{g}})$ of the Drinfeld realization of $U_q(\widehat{\mathfrak{g}})$. In particular in [8], the authors introduced the elliptic analogue of the Drinfeld coproduct for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Konno [13] defined the H -Hopf algebroid structure [5, 6, 10] of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in term of the coproduct of the L -operator of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and defined the associated elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Farghly, Konno and Oshima [16] gave a new definition of $U_{q,p}(\widehat{\mathfrak{g}})$ as a certain topological algebra over the ring of formal power series in p and studied the dynamical quantum Z -algebra structure associated with the level- k highest weight representation of $U_{q,p}(\widehat{\mathfrak{g}})$. Also the authors constructed the induced $U_{q,p}(\widehat{\mathfrak{g}})$ -module from the dynamical quantum Z -module. The level-1 standard representations of $U_{q,p}(\widehat{\mathfrak{g}})$ for $\widehat{\mathfrak{g}} = A_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ and $B_l^{(1)}$ were also given. The purpose of this paper is to construct the higher level realization of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ from its standard level-1 realization [16] by using the elliptic Drinfeld coproduct [8, 14]. The higher level elliptic currents are expressed in term of the level-1 currents. In particular, we obtain the level- $k + 1$ Heisenberg algebra, then we introduce the vertex operators $E_{(k)}^\pm(\alpha, z), E_{(k)}^\pm(\alpha', z)$ and we define the

2010 *Mathematics Subject Classification*. Primary 12A34, 98B76; Secondary 23C57.

Key words and phrases. Elliptic quantum group, Quantum affine algebra, Integrable module.

level- $k + 1$ quantum Z -operators from the level- $k + 1$ elliptic currents. Also, we give the elliptic analogue of the quantum integrable condition for level- $k + 1$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

This paper is organized as follows. In section 2, we define the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ in term of the elliptic Drinfeld generators. We use the Drinfeld coproduct to define the H -Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and formulate it as an elliptic quantum group. Also we recall the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ following [16]. In section 3, we show a construction of the level- $k + 1$ realization ($k \in \mathbb{Z}_{>0}$) of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Also, we give a realization of the level- $k + 1$ Z -algebra. In section 4, we present the elliptic analogue of quantum integrable condition for any level- $k + 1$ integrable module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

2. Elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we expose the definition of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and the level-1 realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ which we are going to use in the following sections.

2.1. Definition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [16]. Let $\mathfrak{h} = \widetilde{\mathfrak{h}} \oplus \mathbb{C}d$, $\widetilde{\mathfrak{h}} = \mathbb{C}h \oplus \mathbb{C}c$ be the Cartan subalgebra of $\widehat{\mathfrak{sl}}_2$. Define $\delta, \Lambda_0, \alpha \in \mathfrak{h}^*$ by

$$\langle \alpha, h \rangle = 2, \quad \langle \delta, d \rangle = 1 = \langle \Lambda_0, c \rangle, \quad (2.1)$$

the other pairings are 0. We also define $\bar{\Lambda}_1 \in \mathfrak{h}^*$ by

$$\langle \bar{\Lambda}_1, h \rangle = 1$$

We set $\widetilde{\mathfrak{h}}^* = \mathbb{C}\Lambda_0 \oplus \mathbb{C}\bar{\Lambda}_1$, $\mathfrak{Q} = \mathbb{Z}\alpha$ and $\mathfrak{P} = \mathbb{Z}\bar{\Lambda}_1$.

We introduce another Heisenberg algebra generated by P and Q with the pairing $\langle P, Q \rangle = 1$. Now let us set $H = \widetilde{\mathfrak{h}} \oplus \mathbb{C}P$ and denote its dual space by $H^* = \widetilde{\mathfrak{h}}^* \oplus \mathbb{C}Q$. We define the pairing by equation (2.1), and $\langle Q, h \rangle = \langle Q, c \rangle = \langle Q, d \rangle = 0 = \langle \alpha, P \rangle = \langle \delta, P \rangle = \langle \Lambda_0, P \rangle$. We define $\mathbb{F} = \mathfrak{M}_{H^*}$ to be the field of meromorphic functions on H^* . We regard a function of $P + h$, P and c , $\widehat{f} = f(P + h, P, c)$, as an element in \mathbb{F} by $\widehat{f}(\mu) = f(\langle \mu, P + h \rangle, \langle \mu, P \rangle, \langle \mu, c \rangle)$ for $\mu \in H^*$.

We use the following notations.

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n),$$

$$(x; q, t)_\infty = \prod_{n,m=0}^{\infty} (1 - xq^n t^m), \quad \Theta_p(z) = (z; p)_\infty (p/z; p)_\infty (p; p)_\infty.$$

DEFINITION 2.1. [16] *The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is a topological algebra over $\mathbb{F}[[p]]$ generated by \mathfrak{M}_{H^*} , $e_m, f_m, \alpha_n, (m \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}), K^\pm, d$ and the*

central element c . Let

$$\begin{aligned} e(z) &= \sum_{m \in \mathbb{Z}} e_m z^{-m}, & f(z) &= \sum_{m \in \mathbb{Z}} f_m z^{-m} \\ \psi^+(z) &= K^+ \exp \left(-(q - q^{-1}) \sum_{n>0} \frac{\alpha_{-n}}{1 - p^n} (zq^{\frac{c}{2}})^n \right) \\ &\quad \times \exp \left((q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_n}{1 - p^n} (zq^{\frac{c}{2}})^{-n} \right), \\ \psi^-(z) &= K^- \exp \left(-(q - q^{-1}) \sum_{n>0} \frac{p^n \alpha_{-n}}{1 - p^n} (zq^{-\frac{c}{2}})^n \right) \\ &\quad \times \exp \left((q - q^{-1}) \sum_{n>0} \frac{\alpha_n}{1 - p^n} (zq^{-\frac{c}{2}})^{-n} \right). \end{aligned}$$

We call $e(z), f(z), \psi^\pm(z)$ the elliptic currents. They are formal Laurent series in z . The defining relations are

$$g(P + h)e(z) = e(z)g(P + h), \quad g(P)e(z) = e(z)g(P - \langle Q, P \rangle), \quad (2.2)$$

$$g(P + h)f(z) = f(z)g(P + h - \langle \alpha, P + h \rangle), \quad g(P)f(z) = f(z)g(P), \quad (2.3)$$

$$[g(P), \alpha_m] = [g(P + h), \alpha_m] = 0, \quad (2.4)$$

$$g(P)K^\pm = K^\pm g(P - \langle Q, P \rangle), \quad (2.5)$$

$$g(P + h)K^\pm = K^\pm g(P + h - \langle Q, P \rangle), \quad (2.6)$$

$$[d, g(P + h, P)] = 0, \quad (2.7)$$

$$[d, \alpha_n] = n\alpha_n, \quad [d, e(z)] = -z \frac{\partial}{\partial z} e(z), \quad [d, f(z)] = -z \frac{\partial}{\partial z} f(z), \quad (2.8)$$

$$K^\pm e(z) = q^{\mp 2} e(z) K^\pm, \quad K^\pm f(z) = q^{\pm 2} f(z) K^\pm, \quad (2.9)$$

$$[\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{[2m][cm]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm}, \quad (2.10)$$

$$[\alpha_m, e(z)] = \frac{[2m]}{m} \frac{1 - p^m}{1 - p^{*m}} q^{-cm} z^m e(z), \quad (2.11)$$

$$[\alpha_m, f(z)] = -\frac{[2m]}{m} z^m f(z), \quad (2.12)$$

$$z_1 \frac{(q^2 z_2 / z_1; p^*)_\infty}{(p^* q^{-2} z_2 / z_1; p^*)_\infty} e(z_1) e(z_2) = -z_2 \frac{(q^2 z_1 / z_2; p^*)_\infty}{(p^* q^{-2} z_1 / z_2; p^*)_\infty} e(z_2) e(z_1), \quad (2.13)$$

$$z_1 \frac{(q^{-2} z_2 / z_1; p)_\infty}{(pq^2 z_2 / z_1; p)_\infty} f(z_1) f(z_2) = -z_2 \frac{(q^{-2} z_1 / z_2; p)_\infty}{(pq^2 z_1 / z_2; p)_\infty} f(z_2) f(z_1), \quad (2.14)$$

$$[e(z_1), f(z_2)] = \frac{1}{q - q^{-1}} \left(\delta(q^{-c} z_1 / z_2) \psi^-(q^{\frac{c}{2}} z_2) - \delta(q^c z_1 / z_2) \psi^+(q^{-\frac{c}{2}} z_2) \right), \quad (2.15)$$

where $p^* = pq^{-2c}$ and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.

2.2. Hopf algebroid structure of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Here we follow [13, 12, 14] to present the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ using the Drinfeld coproduct of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ [8].

2.2.1. H -Hopf algebroid. Let \mathfrak{A} be a complex associative algebra, \mathfrak{H} be a finite dimensional commutative subalgebra of \mathfrak{A} , and $\mathfrak{M}_{\mathfrak{H}^*}$ be the field of meromorphic functions on \mathfrak{H}^* the dual space of \mathfrak{H} .

DEFINITION 2.2 (\mathfrak{H} -algebra). An \mathfrak{H} -algebra is an associative algebra \mathfrak{A} with 1, which is bigraded over \mathfrak{H}^* , $\mathfrak{A} = \bigoplus_{\alpha, \beta \in \mathfrak{H}^*} \mathfrak{A}_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l, \mu_r : \mathfrak{M}_{\mathfrak{H}^*} \rightarrow \mathfrak{A}_{00}$ (the left and right moment maps), such that

$$\mu_l(\widehat{f})a = a\mu_l(T_\alpha \widehat{f}), \quad \mu_r(\widehat{f})a = a\mu_r(T_\beta \widehat{f}), \quad a \in \mathfrak{A}_{\alpha\beta}, \quad \widehat{f} \in \mathfrak{M}_{\mathfrak{H}^*},$$

where T_α denotes the automorphism $(T_\alpha \widehat{f})(\lambda) = \widehat{f}(\lambda + \alpha)$ of $\mathfrak{M}_{\mathfrak{H}^*}$.

DEFINITION 2.3. An \mathfrak{H} -algebra homomorphism is an algebra homomorphism $\pi : A \rightarrow B$ between two \mathfrak{H} -algebras A and B such that for $\alpha, \beta \in \mathfrak{H}^*$

$$\pi(A_{\alpha\beta}) \subseteq B_{\alpha\beta}, \quad \pi(\mu_l^A(\widehat{f})) = \mu_l^B(\widehat{f}), \quad \pi(\mu_r^A(\widehat{f})) = \mu_r^B(\widehat{f}).$$

The tensor product $A \widetilde{\otimes} B = \bigoplus_{\alpha, \beta \in \mathfrak{H}^*} (A \widetilde{\otimes} B)_{\alpha\beta} = \bigoplus_{\alpha, \beta \in \mathfrak{H}^*} \left(\bigoplus_{\gamma \in \mathfrak{H}^*} (A_{\alpha\gamma} \otimes_{\mathfrak{M}_{\mathfrak{H}^*}} B_{\gamma\beta}) \right)$ is again an \mathfrak{H} -algebra with the multiplication $(a \otimes b)(c \otimes d) = ac \otimes bd$. The tensor product $\otimes_{\mathfrak{M}_{\mathfrak{H}^*}}$ refers to the usual tensor product modulo the following rule:

$$\mu_r^A(\widehat{f})a \otimes b = a \otimes \mu_l^B(\widehat{f})b, \quad a \in A, b \in B, \widehat{f} \in \mathfrak{M}_{\mathfrak{H}^*}. \quad (2.16)$$

The unit object \mathfrak{D} in the category of \mathfrak{H} -algebras is an algebra of automorphisms $\mathfrak{M}_{\mathfrak{H}^*} \rightarrow \mathfrak{M}_{\mathfrak{H}^*}$

$$\mathfrak{D} = \left\{ \sum_i \widehat{f}_i T_{\beta_i} \mid \widehat{f}_i \in \mathfrak{M}_{\mathfrak{H}^*}, \beta_i \in \mathfrak{H}^* \right\} = \bigoplus_{\alpha \in \mathfrak{H}^*} \mathfrak{D}_{\alpha\alpha} \quad (2.17)$$

where $\mathfrak{D}_{\alpha\alpha} = \{ \widehat{f} T_{-\alpha} \mid \widehat{f} \in \mathfrak{M}_{\mathfrak{H}^*}, \alpha \in \mathfrak{H}^* \}$ and the moment maps $\mu_l^{\mathfrak{D}}, \mu_r^{\mathfrak{D}} : \mathfrak{M}_{\mathfrak{H}^*} \rightarrow \mathfrak{D}_{00}$ are defined by $\mu_l^{\mathfrak{D}}(\widehat{f}) = \mu_r^{\mathfrak{D}}(\widehat{f}) = \widehat{f} T_0$.

DEFINITION 2.4. An \mathfrak{H} -Hopf algebroid is an \mathfrak{H} -algebra A equipped with two \mathfrak{H} -algebra homomorphisms: coproduct $\Delta : A \rightarrow A \widetilde{\otimes} A$, counit $\varepsilon : A \rightarrow \mathfrak{D}$ and a

\mathbb{C} -linear map: antipode $a : A \rightarrow A$. Δ, ε, a satisfy the following

$$(\Delta \widetilde{\otimes} id) \circ \Delta = (id \widetilde{\otimes} \Delta) \circ \Delta \quad (2.18)$$

$$(\varepsilon \widetilde{\otimes} id) \circ \Delta = (id \widetilde{\otimes} \varepsilon) \circ \Delta \quad (2.19)$$

$$m \circ (id \widetilde{\otimes} a) \circ \Delta(x) = \mu_l(\varepsilon(x)1), \quad \forall x \in A \quad (2.20)$$

$$m \circ (a \widetilde{\otimes} id) \circ \Delta(x) = \mu_r(T_\alpha(\varepsilon(x)1)), \quad \forall x \in A_{\alpha\beta}. \quad (2.21)$$

$m : A \widetilde{\otimes} A \rightarrow A$ refers the multiplication and $\varepsilon(x)1 (x \in A)$ refers the action of the operator $\varepsilon(x)$ on the constant function $1 \in \mathfrak{M}_{\mathfrak{H}^*}$.

2.2.2. H -Hopf algebroid structure of $U = U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

PROPOSITION 2.5. $U = U_{q,p}(\widehat{\mathfrak{sl}}_2)$ is an H -algebra by

$$U = \bigoplus_{\alpha, \beta \in H^*} U_{\alpha\beta},$$

$$U_{\alpha\beta} = \left\{ x \in U \mid q^{P+h} x q^{-(P+h)} = q^{\langle \alpha, P+h \rangle} x, \quad q^P x q^{-P} = q^{\langle \beta, P \rangle} x \right. \\ \left. \forall P+h, P \in H \right\}$$

and $\mu_l, \mu_r : \mathbb{F} \rightarrow U_{00}$ defined by

$$\mu_l(\widehat{f}) = f(P+h, p) \in \mathbb{F}[[p]], \quad \mu_r(\widehat{f}) = f(P, p^*) \in \mathbb{F}[[p]].$$

The tensor product $U \widetilde{\otimes} U = \bigoplus_{\alpha, \beta \in H^*} (U \widetilde{\otimes} U)_{\alpha\beta}$ is an H^* bigraded algebra.

The H -algebra \mathfrak{D} of the shift operators is

$$\mathfrak{D} = \left\{ \sum_i \widehat{f}_i T_{\alpha_i} \mid \widehat{f}_i \in \mathfrak{M}_{H^*}, \alpha_i \in H^* \right\}.$$

with the bigraded structure and the moments map as in Definition 2.2.

In [13], Konno defined the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the co-product of L -operator. Here we define the Hopf algebroid structure on $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by the Drinfeld coproduct [8, 14].

THEOREM 2.6. [14] *The elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ has an elliptic analogue of the Drinfeld coproduct $\Delta : U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow U_{q,p}(\widehat{\mathfrak{sl}}_2) \widetilde{\otimes} U_{q,p}(\widehat{\mathfrak{sl}}_2)$, the counit $\varepsilon :$*

$U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow \mathfrak{D}$ and the antipode $a : U_{q,p}(\widehat{\mathfrak{sl}}_2) \rightarrow U_{q,p}(\widehat{\mathfrak{sl}}_2)$

$$\Delta(q^c) = q^c \widetilde{\otimes} q^c, \Delta(q^h) = q^h \widetilde{\otimes} q^h \quad (2.22)$$

$$\Delta(\psi^\pm(z)) = \psi^\pm(q^{\pm \frac{c(2)}{2}} z) \widetilde{\otimes} \psi^\pm(q^{\mp \frac{c(1)}{2}} z) \quad (2.23)$$

$$\Delta(\mu_r(\widehat{f})) = 1 \widetilde{\otimes} \mu_r(\widehat{f}), \Delta(\mu_l(\widehat{f})) = \mu_l(\widehat{f}) \widetilde{\otimes} 1 \quad (2.24)$$

$$\Delta(e(z)) = e(q^{-c(2)} z) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z) \quad (2.25)$$

$$\Delta(f(z)) = f(z) \widetilde{\otimes} 1 + \psi^+(q^{-\frac{c(1)}{2}} z) \widetilde{\otimes} f(zq^{-c(1)}) \quad (2.26)$$

$$\varepsilon(q^c) = 1, \varepsilon(\psi^+(z)) = \varepsilon(\psi^-(z)) = 1 \quad (2.27)$$

$$\varepsilon(\mu_r(\widehat{f})) = \varepsilon(\mu_l(\widehat{f})) = \widehat{f}T_0 \quad (2.28)$$

$$\varepsilon(e(z)) = \varepsilon(f(z)) = 0, \varepsilon(\alpha_n) = 0 \quad (2.29)$$

$$a(q^c) = q^{-c}, a(\psi^\pm(z)) = \psi^\pm(z)^{-1} \quad (2.30)$$

$$a(\mu_r(\widehat{f})) = \mu_l(\widehat{f}), a(\mu_l(\widehat{f})) = \mu_r(\widehat{f}) \quad (2.31)$$

$$a(e(z)) = -\psi^-(zq^{\frac{c}{2}})^{-1} e(q^c z) \quad (2.32)$$

$$a(f(z)) = -f(q^c z) \psi^+(zq^{\frac{c}{2}})^{-1}. \quad (2.33)$$

Namely, the maps Δ, ε are algebra homomorphism and a is an anti-algebra homomorphism satisfying the relations (2.18)-(2.21) in Definition 2.4. Therefore the H - algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with Δ, ε, a is an H -Hopf algebroid.

PROOF. Let's check (2.19)

$$\begin{aligned} m \circ (\varepsilon \widetilde{\otimes} id) \circ \Delta(e(z)) &= m(\varepsilon \widetilde{\otimes} id) \circ (e(q^{-c(2)} z) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z)) \\ &= m \circ (\varepsilon(e(q^{-c(2)} z)) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z)) \\ &= m \circ (1 \widetilde{\otimes} e(z)) = e(z), \end{aligned}$$

$$\begin{aligned} m \circ (id \widetilde{\otimes} \varepsilon) \circ \Delta(e(z)) &= m(id \widetilde{\otimes} \varepsilon) \circ (e(q^{-c(2)} z) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z)) \\ &= m \circ (e(\varepsilon(q^{-c(2)} z)) \widetilde{\otimes} \varepsilon(\psi^-(q^{-\frac{c(2)}{2}} z)) + 1 \widetilde{\otimes} \varepsilon(e(z))) \\ &= m \circ (e(z) \widetilde{\otimes} 1) = e(z). \end{aligned}$$

For (2.20)

$$\begin{aligned} m \circ (id \otimes a) \circ \Delta(e(z)) &= m \circ (id \otimes a) \circ (e(q^{-c(2)} z) \widetilde{\otimes} \psi^-(q^{-\frac{c(2)}{2}} z) + 1 \widetilde{\otimes} e(z)) \\ &= m \circ (e(a(q^{-c(2)} z)) \widetilde{\otimes} a(\psi^-(q^{-\frac{c(2)}{2}} z)) + 1 \widetilde{\otimes} a(e(z))) \\ &= e(q^{c(2)} z) \psi^-(q^{\frac{c(2)}{2}} z)^{-1} - \psi^+(q^{-\frac{c(1)}{2}} z)^{-1} e(q^{c(2)} z) = 0 \\ &= \mu_l(\varepsilon(e(z))1). \end{aligned}$$

We call the H -Hopf algebroid $(U_{q,p}(\widehat{\mathfrak{sl}}_2), H, \mathfrak{M}_{H^*}, \mu_l, \mu_r, \Delta, \varepsilon, a)$ the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

From (2.24), a straight forward calculation shows the following relation

$$\Delta \left(\frac{\mu_l(\widehat{f})}{\mu_r(\widehat{f})} \right) = \frac{\mu_l(\widehat{f})}{\mu_r(\widehat{f})} \widetilde{\otimes} \frac{\mu_l(\widehat{f})}{\mu_r(\widehat{f})}. \quad (2.34)$$

2.3. Level-1 highest weight representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

DEFINITION 2.7. [16] Let \mathfrak{H} , \mathfrak{N}_+ , \mathfrak{N}_- be the subalgebras of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ generated by c, d, K^\pm , by α_n ($n \in \mathbb{Z}_{>0}$), e_n ($n \in \mathbb{Z}_{\geq 0}$), f_n ($n \in \mathbb{Z}_{>0}$) and by α_{-n} ($n \in \mathbb{Z}_{>0}$), e_{-n} ($n \in \mathbb{Z}_{>0}$), f_{-n} ($n \in \mathbb{Z}_{\geq 0}$), respectively.

The Heisenberg algebra $U_{q,p}(\mathfrak{H})$ is a subalgebra of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ generated by α_m , ($m \neq 0$) and c . From defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we have

$$[\alpha_m, \alpha_n] = \frac{[2m][cm]}{m} \frac{1-p^m}{1-p^{*m}} q^{-cm} \delta_{m+n,0}, \quad (2.35)$$

$$[\alpha'_m, \alpha'_n] = \frac{[2m][cm]}{m} \frac{1-p^{*m}}{1-p^m} q^{cm} \delta_{m+n,0}, \quad (2.36)$$

$$[\alpha_m, \alpha'_n] = \frac{[2m][cm]}{m} \delta_{m+n,0}, \quad (2.37)$$

where $\alpha'_m = \frac{1-p^{*m}}{1-p^m} q^{cm} \alpha_m$, ($m \neq 0$).

DEFINITION 2.8. For $k \in \mathbb{C}$, a $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ -module $V(\lambda, \mu)$ is called the level- k highest weight module with the highest weight (λ, μ) , if there exists a highest weight vector $v \in V(\lambda, \mu)$ such that

$$\begin{aligned} V(\lambda, \mu) &= U_{q,p}(\widehat{\mathfrak{sl}}_2) \cdot v, & \mathfrak{N}_+ \cdot v &= 0, \\ c \cdot v &= kv, & f(P) \cdot v &= f(\langle \mu, P \rangle)v, & f(P+h) \cdot v &= f(\langle \lambda, P+h \rangle)v. \end{aligned}$$

DEFINITION 2.9. Define Λ_a ($a = 0, 1$) $\in \mathfrak{h}^*$ by

$$\langle \Lambda_a, h \rangle = \delta_{a,1}, \quad \langle \Lambda_a, c \rangle = \delta_{a,0},$$

and the other pairings are 0.

THEOREM 2.10. [16] For $a = 0, 1$. Define $V(\Lambda_a + \mu, \mu) = \bigoplus_{\gamma, \kappa \in \Omega} (\mathbb{F} \otimes_{\mathbb{C}} (F_{\alpha,1} \otimes e^{\bar{\Lambda}_a + \gamma}) \otimes e^{Q_{\mu+\kappa}})$. Let $\rho : U_{q,p}(\widehat{\mathfrak{sl}}_2) \longrightarrow \text{End}(V(\Lambda_a + \mu, \mu))$ by

$$\begin{aligned} \rho(\psi^+(z)) &= q^{-h} e^{-2Q} \exp \left(-(q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_{-n})}{1 - p^n} (zq^{\frac{1}{2}})^n \right) \\ &\quad \times \exp \left((q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_n)}{1 - p^n} (zq^{\frac{1}{2}})^{-n} \right) \end{aligned} \quad (2.38)$$

$$\begin{aligned} \rho(\psi^-(z)) &= q^h e^{-2Q} \exp \left(-(q - q^{-1}) \sum_{n>0} \frac{p^n \rho(\alpha_{-n})}{1 - p^n} (zq^{-\frac{1}{2}})^n \right) \\ &\quad \times \exp \left((q - q^{-1}) \sum_{n>0} \frac{\rho(\alpha_n)}{1 - p^n} (zq^{-\frac{1}{2}})^{-n} \right) \end{aligned} \quad (2.39)$$

$$\begin{aligned} \rho(e(z)) &=: \exp \left(- \sum_{n \neq 0} \frac{\rho(\alpha_n)}{[n]} z^{-n} \right) : e^\alpha z^{h+1} \\ \rho(f(z)) &=: \exp \left(\sum_{n \neq 0} \frac{\rho(\alpha'_n)}{[n]} z^{-n} \right) : e^{-\alpha} z^{-h+1}, \end{aligned} \quad (2.40)$$

where $F_{\alpha,1}$ is the polynomial ring $\mathbb{C}[\alpha_{-m} \ (m > 0)]$. For $u \in \mathbb{C}[\alpha_{-m} \ (m > 0)]$

$$\begin{aligned} \rho(c) \cdot u &= u, \quad \rho(\alpha_{-n}) \cdot u = \alpha_{-n} u, \\ \rho(\alpha_n) \cdot u &= \frac{[2n][n]}{n} \frac{1 - p^n}{1 - p^{*n}} q^{-n} \frac{\partial}{\partial \alpha_{-n}} u \quad (n > 0). \end{aligned}$$

Then $V(\Lambda_a + \mu, \mu)$ is the level-1 irreducible highest weight module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the highest weight $(\Lambda_a + \mu, \mu)$ and the highest weight vector $v_0 = 1 \otimes 1 \otimes e^{\Lambda_a} \otimes e^Q$.

For convention, we will drop ρ to refer the elements in $\text{End}(V(\Lambda_a + \mu\mu))$.

PROPOSITION 2.11. *The level-1 elliptic operators satisfy the following relations*

$$e(z)e(w) = \frac{(q^{-2}p^*\frac{w}{z}; p^*)_\infty (q^{-2}\frac{w}{z}; q^{2c})_\infty}{(q^2p^*\frac{w}{z}; p^*)_\infty (q^2\frac{w}{z}; q^{2c})_\infty} : e(z)e(w) : \quad (2.41)$$

$$\psi^-(z)e(w) = \frac{(q^{-2-\frac{c}{2}}\frac{w}{z}; pq^{-2c})_\infty}{(q^{2-\frac{c}{2}}\frac{w}{z}; pq^{-2c})_\infty} : \psi^-(z)e(w) : \quad (2.42)$$

$$f(z)f(w) = \frac{(q^{-2}\frac{w}{z}; q^{2c})_\infty (q^2\frac{w}{z}; p)_\infty}{(q^2\frac{w}{z}; q^{2c})_\infty (q^{-2}\frac{w}{z}; p)_\infty} : f(z)f(w) : \quad (2.43)$$

$$f(z)\psi^+(w) = \frac{(q^{-2+\frac{c}{2}}\frac{w}{z}, p)_\infty}{(q^{2+\frac{c}{2}}\frac{w}{z}, p)_\infty} : f(z)\psi^+(w) : \quad (2.44)$$

$$\psi^\pm(z)\psi^\pm(w) = \frac{(q^{-2}\frac{w}{z}; pq^{-2c})_\infty (q^2\frac{w}{z}; p)_\infty}{(q^2\frac{w}{z}; pq^{-2c})_\infty (q^{-2}\frac{w}{z}; p)_\infty} : \psi^\pm(z)\psi^\pm(w) :, \quad (2.45)$$

where c acts on $V(\Lambda_a + \mu, \mu)$ by 1.

3. Higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$

In this section we show a construction of the higher level realization of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ by using the Drinfeld coproduct of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. Also, we will present the associated Z -operators.

3.1. Higher level representation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$. For $k > 0, \lambda_i \in \mathfrak{h}^*, \mu^{(i)} \in H^*(i \in \{0, 1, \dots, k+1\})$. Let's consider a tensor product of $k+1$ copies of the level-1 highest weight modules $V(\Lambda_a + \mu, \mu)(a = 0, 1)$

$$\begin{aligned} V_{k+1}(\lambda_i, \mu) &= V(\Lambda_{a^{(1)}} + \mu^{(1), \mu^{(1)}}) \otimes \dots \otimes V(\Lambda_{a^{(i)}} + \mu^{(i), \mu^{(i)}}) \\ &\quad \otimes V(\Lambda_{a^{(i+1)}} + \mu^{(i+1), \mu^{(i+1)}}) \otimes \dots \otimes V(\Lambda_{a^{(k+1)}} + \mu^{(k+1), \mu^{(k+1)}}), \end{aligned} \quad (3.1)$$

such that $a^{(1)}, \dots, a^{(k+1)} \in \{0, 1\}$ and take i of a 's as 0 and $k+1-i$ of a 's as 1.

THEOREM 3.1. *The space $V_{k+1}(\lambda_i, \mu)$ is the level- $k+1$ module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ with the highest weight*

$$(\lambda_i, \mu) = (i\Lambda_0 + (k+1-i)\Lambda_1 + \sum_{j=1}^{k+1} \mu^{(j)}, \sum_{j=1}^{k+1} \mu^{(j)})$$

by the action

$$\begin{aligned}\Delta^k(e(z)) &= \sum_{i=1}^{k+1} e^i(z), \\ e^i(z) &= 1 \otimes \cdots \otimes 1 \otimes e(zq^{-(c^{(i+1)} + \cdots + c^{(k+1)})}) \\ &\quad \otimes \psi^-(zq^{-(\frac{c^{(i+1)}}{2} + c^{(i+2)} + \cdots + c^{(k+1)})}) \\ &\quad \otimes \psi^-(zq^{-(\frac{c^{(i+2)}}{2} + c^{(i+3)} + \cdots + c^{(k+1)})}) \otimes \cdots \otimes \psi^-(zq^{-\frac{c^{(k+1)}}{2}}),\end{aligned}\tag{3.2}$$

$$\begin{aligned}\Delta^k(f(z)) &= \sum_{i=1}^{k+1} f^i(z), \\ f^i(z) &= \psi^+(zq^{-\frac{c^{(1)}}{2}}) \otimes \psi^+(zq^{-(c^{(1)} + \frac{c^{(2)}}{2})}) \otimes \cdots \\ &\quad \otimes \psi^+(zq^{-(c^{(1)} + \cdots + c^{(i-2)} + \frac{c^{(i-1)}}{2})}) \otimes f(zq^{-(c^{(1)} + \cdots + c^{(i-2)} + c^{(i-1)})}) \\ &\quad \otimes 1 \cdots \otimes 1,\end{aligned}\tag{3.3}$$

$$\begin{aligned}\Delta^k(\psi^\pm(z)) &= \psi^\pm(zq^{\pm \frac{c^{(2)} + c^{(3)} + \cdots + c^{(k+1)}}{2}}) \otimes \psi^\pm(zq^{\mp \frac{c^{(1)}}{2} \pm \frac{c^{(3)} + c^{(4)} + \cdots + c^{(k+1)}}{2}}) \\ &\quad \otimes \psi^\pm(zq^{\mp \frac{c^{(1)} + c^{(2)}}{2} \pm \frac{c^{(4)} + c^{(5)} + \cdots + c^{(k+1)}}{2}}) \otimes \cdots \otimes \psi^\pm(zq^{-\frac{c^{(1)} + \cdots + c^{(k)}}{2}}),\end{aligned}\tag{3.4}$$

where $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$ and $c^{(i)}$ acts on $V(\Lambda_{a^{(i)}} + \mu^{(i)}, \mu^{(i)})$ as 1.

In order to show the proof of Theorem 3.1, we need the following OPE relations for $e^i(z)$ and $f^i(z)$ in the expansion of $\Delta^k(e(z))$ and $\Delta^k(f(z))$ respectively.

LEMMA 3.2. Set $i < j$. For $e^i(z)$

$$e^i(z)e^j(w) = \frac{(q^{-2}\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty}{(q^2\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^i(z)e^j(w) :, \tag{3.5}$$

$$e^j(z)e^i(w) = \frac{(q^{-2+2c^{(j)}}\frac{w}{z}; q^{2c^{(j)}})_\infty}{(q^{2+2c^{(j)}}\frac{w}{z}; q^{2c^{(j)}})_\infty} \frac{(q^{-2}\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty}{(q^2\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^j(z)e^i(w) :, \tag{3.6}$$

$$e^j(z)e^i(w) = \frac{(q^{-2-2\Delta^k(c)}p\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty}{(q^{2-2\Delta^k(c)}p\frac{w}{z}; pq^{-2\Delta^k(c)})_\infty} : e^j(z)e^i(w) :. \tag{3.7}$$

For $f^i(z)$

$$f^i(z)f^j(w) = \frac{(q^2\frac{w}{z}; p)_\infty}{(q^{-2}\frac{w}{z}; p)_\infty} : f^i(z)f^j(w) :, \tag{3.8}$$

$$f^j(z)f^i(w) = \frac{(q^2\frac{w}{z}; p)_\infty}{(q^{-2}\frac{w}{z}; p)_\infty} \frac{(q^{-2}\frac{w}{z}; q^{2c^{(j)}})_\infty}{(q^2\frac{w}{z}; q^{2c^{(j)}})_\infty} : f^j(z)f^i(w) :, \tag{3.9}$$

$$f^j(z)f^i(w) = \frac{(q^{-2}p\frac{w}{z}; p)_\infty}{(q^2p\frac{w}{z}; p)_\infty} : f^j(z)f^i(w) : . \quad (3.10)$$

PROOF. This follows from Proposition 2.11.

PROOF. Proof of Theorem 3.1. We can check directly that $\Delta^k(e(z))$, $\Delta^k(f(z))$ and $\Delta^k(\psi^\pm(z))$ satisfy the defining relations of the elliptic algebra $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

Let's show that $\Delta^k(e(z))$ satisfies (2.13). By using the tensor product rules, relations (3.5)-(3.7) and (2.16), we have

$$\begin{aligned} & z_1 \frac{(q^2 \frac{z_2}{z_1}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_2}{z_1}; pq^{-2\Delta^k(c)})_\infty} \Delta^k(e(z_1)) \Delta^k(e(z_2)) \\ &= z_1 \frac{(q^2 \frac{z_2}{z_1}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_2}{z_1}; pq^{-2\Delta^k(c)})_\infty} \sum_{i=1}^{k+1} e^i(z_1) \sum_{j=1}^{k+1} e^j(z_2) \\ &= z_1 q^2 \frac{(1 - q^{-2} \frac{z_2}{z_1})}{(1 - q^2 \frac{z_1}{z_2})} \frac{(q^2 \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty} \sum_{i < j}^{k+1} e^j(z_2) e^i(z_1) \\ &+ z_1 \frac{(q^2 \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty} \\ &\quad \times \sum_{i=j=1}^{k+1} \frac{(1 - q^{-2} \frac{z_2}{z_1})}{(1 - q^{-2} \frac{z_1}{z_2})} \frac{(q^{-2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty}{(q^{2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty} \frac{(q^{2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty}{(q^{-2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty} e^i(z_2) e^i(z_1) \\ &+ z_1 q^{-2} \frac{(1 - q^2 \frac{z_2}{z_1})}{(1 - q^{-2} \frac{z_1}{z_2})} \frac{(q^2 \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty} \sum_{i > j}^{k+1} e^j(z_2) e^i(z_1) \\ &= -z_2 \frac{(q^2 \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty}{(q^{-2}pq^{-2\Delta^k(c)} \frac{z_1}{z_2}; pq^{-2\Delta^k(c)})_\infty} \\ &\quad \times \left\{ -q^2 \frac{z_1}{z_2} \frac{(1 - q^{-2} \frac{z_2}{z_1})}{(1 - q^2 \frac{z_1}{z_2})} \sum_{i < j}^{k+1} e^j(z_2) e^i(z_1) \right. \\ &\quad \left. - \left(\frac{z_1}{z_2} \right) \frac{(1 - q^{-2} \frac{z_2}{z_1})}{(1 - q^{-2} \frac{z_1}{z_2})} \sum_{i=j=1}^{k+1} \frac{(q^{-2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty}{(q^{2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty} \frac{(q^{2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty}{(q^{-2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty} e^i(z_2) e^i(z_1) \right. \\ &\quad \left. - q^{-2} \frac{z_1}{z_2} \frac{(1 - q^2 \frac{z_2}{z_1})}{(1 - q^{-2} \frac{z_1}{z_2})} \sum_{i > j}^{k+1} e^j(z_2) e^i(z_1) \right\}. \end{aligned}$$

The factor

$$\begin{aligned} & -\left(\frac{z_1}{z_2}\right) \frac{(1 - q^{-2} \frac{z_2}{z_1})}{(1 - q^{-2} \frac{z_1}{z_2})} \sum_{i=j=1}^{k+1} \frac{(q^{-2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty}{(q^{2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty} \frac{(q^{2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty}{(q^{-2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty} \\ & = -\left(\frac{z_1}{z_2}\right) \sum_{i=j=1}^{k+1} \frac{(q^{-2} \frac{z_2}{z_1}; q^{2c^i})_\infty}{(q^{2+2c^i} \frac{z_2}{z_1}; q^{2c^i})_\infty} \frac{(q^{2+2c^i} \frac{z_1}{z_2}; q^{2c^i})_\infty}{(q^{-2} \frac{z_1}{z_2}; q^{2c^i})_\infty} \end{aligned}$$

becomes 1 on account of the notation $\Theta_{q^{2c^i}}(z_1/z_2) = -(z_1/z_2)\Theta_{q^{2c^i}}(z_2/z_1)$.

Similarly, we can show that $\Delta^k(f(z))$ realizes (2.14).

Also, we can prove that $\Delta^k(e(z))$ and $\Delta^k(f(z))$ satisfy (2.15)

$$\begin{aligned} & [\Delta^k(e(z_1)), \Delta^k(f(z_2))] \\ & = \frac{1}{q - q^{-1}} \left(\delta(q^{-\Delta^k(c)} \frac{z_1}{z_2}) \psi^-(q^{\frac{c(1)}{2}} z_2) - \delta(q^{2c^{(1)} - \Delta^k(c)} \frac{z_1}{z_2}) \psi^+(q^{-\frac{c(1)}{2}} z_2) \right) \\ & \quad \otimes \psi^-(z_1 q^{-(\frac{c(2)}{2} + c^{(3)} + \dots + c^{(k+1)})}) \otimes \dots \otimes \psi^-(z_1 q^{-\frac{c^{(k+1)}}{2}}) \\ & \quad + \psi^+(z_2 q^{-\frac{c(1)}{2}}) \otimes \psi^+(z_2 q^{-(c^{(1)} + \frac{c(2)}{2})}) \otimes \dots \otimes \psi^+(z_2 q^{-(c^{(1)} + \dots + c^{(i-2)} + \frac{c^{(i-1)}}{2})}) \\ & \quad \otimes \frac{1}{q - q^{-1}} \left(\delta(q^{-2c^{(k+1)} + \Delta^k(c)} \frac{z_1}{z_2}) \psi^-(q^{-\frac{c^{(k+1)}}{2} + \Delta^k(c)} z_2) - \delta(q^{\Delta^k(c)} \frac{z_1}{z_2}) \right. \\ & \quad \left. \times \psi^+(q^{-\frac{c^{(k+1)}}{2} - (c^{(1)} + \dots + c^{(k)})} z_2) \right). \end{aligned}$$

Then use the property of the delta function and (3.4).

Denote by $v^{(k+1)} \in V_{k+1}(\lambda, \mu)$ the tensor product of the highest weight vectors in the tensor factors in relation (3.1). We calculate the highest weight by using the action of \mathfrak{M}_{H^*} (2.34) on $v^{(k+1)}$ as follows

$$\begin{aligned} & \Delta^k\left(\frac{f(P)}{f(P+h)}\right) \cdot v^{(k+1)} \\ & = \frac{f(\langle \mu^{(1)} + \mu^{(2)} + \dots + \mu^{(k+1)}, P \rangle)}{f(\langle i\Lambda_0 + (k+1-i)\Lambda_1 + \mu^{(1)} + \mu^{(2)} + \dots + \mu^{(k+1)}, P+h \rangle)} v^{(k+1)}. \end{aligned}$$

We also obtain the comultiplication formula Δ^k of boson operator $\alpha_n (n \neq 0)$ from $\Delta^k(\psi^\pm(z))$.

COROLLARY 3.3. For $k \geq 1, n \neq 0$. The boson operator is

$$\begin{aligned} \Delta^k(\alpha_n) = & \alpha_n \otimes 1 \cdots 1 \otimes 1 + \frac{(1-p^n)q^{-c^{(1)}n}}{1-p^nq^{-2c^{(1)}n}} \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)})n}} \otimes 1 \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \cdots + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(i-1)})n}} \otimes 1 \cdots \otimes \alpha_n \otimes 1 \cdots \otimes 1 \\ & + \cdots + \frac{(1-p^n)q^{-(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}}{1-p^nq^{-2(c^{(1)}+c^{(2)}+\cdots+c^{(k)})n}} \otimes 1 \cdots \otimes \alpha_n, \end{aligned} \quad (3.11)$$

where $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$.

PROOF. Based on the relations (2.16), (2.38)-(2.39) and (3.4) in Theorem 3.1, we can write

$$\begin{aligned} \Delta^k(\psi^+(z)) = & \Delta^k(q^{-h}e^{-2Q}) \exp\left(- (q-q^{-1}) \sum_{n>0} \frac{\Delta^k(\alpha_{-n})}{1-p^n} (zq^{\frac{\Delta^k(c)}{2}})^n\right) \\ & \times \exp\left((q-q^{-1}) \sum_{n>0} \frac{p^n \Delta^k(\alpha_n)}{1-p^n} (zq^{\frac{\Delta^k(c)}{2}})^{-n}\right), \end{aligned} \quad (3.12)$$

$$\begin{aligned} \Delta^k(\psi^-(z)) = & \Delta^k(q^he^{-2Q}) \exp\left(- (q-q^{-1}) \sum_{n>0} \frac{p^n \Delta^k(\alpha_{-n})}{1-p^n} (zq^{-\frac{\Delta^k(c)}{2}})^n\right) \\ & \times \exp\left((q-q^{-1}) \sum_{n>0} \frac{\Delta^k(\alpha_n)}{1-p^n} (zq^{-\frac{\Delta^k(c)}{2}})^{-n}\right), \end{aligned} \quad (3.13)$$

where $\Delta^k(q^{\pm h}e^{-2Q}) = (q^{\pm h}e^{-2Q}) \otimes \cdots \otimes (q^{\pm h}e^{-2Q})$. These imply Corollary 3.3.

The operators $\Delta^k(\alpha_n)(n \neq 0)$ give a level- $k+1$ realization of the Heisenberg algebra $U_{q,p}(\mathfrak{H})$.

PROPOSITION 3.4. The operators $\Delta^k(\alpha_n)(n \neq 0)$ and $\Delta^k(c)$ satisfy

$$[\Delta^k(\alpha_m), \Delta^k(\alpha_n)] = \frac{[2m][\Delta^k(c)m]}{m} \frac{1-p^m}{1-p^mq^{-2\Delta^k(c)m}} q^{-\Delta^k(c)m} \delta_{m+n,0}, \quad (3.14)$$

$$[\Delta^k(\alpha_m), \Delta^k(e(z))] = \frac{[2m]}{m} \frac{1-p^m}{1-p^mq^{-2\Delta^k(c)m}} q^{-\Delta^k(c)m} z^m \Delta^k(e(z)), \quad (3.15)$$

$$[\Delta^k(\alpha_m), \Delta^k(f(z))] = -\frac{[2m]}{m} \frac{1-p^mq^{-2\Delta^k(c)m}}{1-p^m} q^{\Delta^k(c)m} z^m \Delta^k(f(z)). \quad (3.16)$$

3.2. Z -algebra. Here we give a realization of the level- $k + 1$ Z -algebra. The form of the vertex operators in [16] section 3 led us to introduce $E_{(k)}^{\pm}(\alpha, z)$ and $E_{(k)}^{\pm}(\alpha', z)$ in the following definition.

DEFINITION 3.1. *By using the level- $k + 1$ elliptic bosons $\Delta^k(\alpha_n)(n \neq 0)$, we define the vertex operators*

$$E_{(k)}^{\pm}(\alpha, z) = \exp\left(\pm \sum_{n>0} \frac{\Delta^k(\alpha_{\pm n})}{[\Delta^k(c)n]} z^{\mp n}\right),$$

$$E_{(k)}^{\pm}(\alpha', z) = \exp\left(\mp \sum_{n>0} \frac{\Delta^k(\alpha'_{\pm n})}{[\Delta^k(c)n]} z^{\mp n}\right),$$

which are formal Laurent series in z with coefficient in $\text{End}V_{k+1}(\lambda_i, \mu)$.

The following proposition is a consequence of the commutation relations (3.14)-(3.16) in Proposition 3.4 with $\Delta^k(c)$ acts as the scalar $k + 1$.

PROPOSITION 3.5. *$E_{(k)}^{\pm}(\alpha, z)$ and $E_{(k)}^{\pm}(\alpha', z)$ satisfy the following relations:*

$$E_{(k)}^+(\alpha, z)E_{(k)}^-(\alpha, w) = \frac{(q^{-2+2(k+1)}w/z; q^{2(k+1)})_{\infty}(q^{-2}w/z; pq^{-2(k+1)})_{\infty}}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_{\infty}(q^2w/z; pq^{-2(k+1)})_{\infty}} \\ \times E_{(k)}^-(\alpha, w)E_{(k)}^+(\alpha, z), \quad (3.17)$$

$$E_{(k)}^+(\alpha', z)E_{(k)}^-(\alpha', w) = \frac{(q^{-2}w/z; q^{2(k+1)})_{\infty}(q^2w/z; p)_{\infty}}{(q^2w/z; q^{2(k+1)})_{\infty}(q^{-2}w/z; p)_{\infty}} \\ \times E_{(k)}^-(\alpha', w)E_{(k)}^+(\alpha', z), \quad (3.18)$$

$$E_{(k)}^+(\alpha, z)E_{(k)}^-(\alpha', w) = \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_{\infty}} E_{(k)}^-(\alpha', w)E_{(k)}^+(\alpha, z), \quad (3.19)$$

$$E_{(k)}^+(\alpha', z)E_{(k)}^-(\alpha, w) = \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_{\infty}} E_{(k)}^-(\alpha, w)E_{(k)}^+(\alpha', z), \quad (3.20)$$

$$E_{(k)}^{\pm}(\alpha, z) \Delta^k(e(w)) = \frac{(q^{\pm 2+2(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}}{(q^{\mp 2+2(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}} \\ \times \frac{(q^{\pm 2}(w/z)^{\pm 1}; pq^{-2(k+1)})_{\infty}}{(q^{\mp 2}(w/z)^{\pm 1}; pq^{-2(k+1)})_{\infty}} \Delta^k(e(w))E_{(k)}^{\pm}(\alpha, z), \quad (3.21)$$

$$E_{(k)}^{\pm}(\alpha', z) \Delta^k(f(w)) = \frac{(q^{\pm 2}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\pm 2}(w/z)^{\pm 1}; p)_{\infty}}{(q^{\mp 2}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}(q^{\mp 2}(w/z)^{\pm 1}; p)_{\infty}} \\ \times \Delta^k(f(w))E_{(k)}^{\pm}(\alpha', z), \quad (3.22)$$

$$E_{(k)}^{\pm}(\alpha', z) \Delta^k(e(w)) = \frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}} \Delta^k(e(w)) E_{(k)}^{\pm}(\alpha', z), \quad (3.23)$$

$$E_{(k)}^{\pm}(\alpha, z) \Delta^k(f(w)) = \frac{(q^{\mp 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}}{(q^{\pm 2+(k+1)}(w/z)^{\pm 1}; q^{2(k+1)})_{\infty}} \Delta^k(f(w)) E_{(k)}^{\pm}(\alpha, z). \quad (3.24)$$

DEFINITION 3.2. [16] For $k \in \mathbb{Z}_{>0}$. We define the level- $k+1$ quantum Z -operators by

$$\begin{aligned} \Delta^k(e(z)) &= E(k, \alpha, z) Z^+(z) \\ \Delta^k(f(z)) &= E(k, \alpha', z) Z^-(z) \end{aligned}$$

where

$$\begin{aligned} E(k, \alpha, z) &= E_{(k)}^{-}(-\alpha, z) E_{(k)}^{+}(\alpha, z) \\ &= \exp\left(\sum_{n>0} \frac{\Delta^k(\alpha_{-n})}{[\Delta^k(c)n]} z^n\right) \exp\left(-\sum_{n>0} \frac{\Delta^k(\alpha_n)}{[\Delta^k(c)n]} z^{-n}\right), \end{aligned} \quad (3.25)$$

$$\begin{aligned} E(k, \alpha', z) &= E_{(k)}^{-}(-\alpha', z) E_{(k)}^{+}(-\alpha', z) \\ &= \exp\left(-\sum_{n>0} \frac{\Delta^k(\alpha'_{-n})}{[\Delta^k(c)n]} z^n\right) \exp\left(\sum_{n>0} \frac{\Delta^k(\alpha'_n)}{[\Delta^k(c)n]} z^{-n}\right), \end{aligned} \quad (3.26)$$

$$Z^{\pm}(z) = \sum_{i=1}^{k+1} Z_i^{\pm}(z). \quad (3.27)$$

Since $\Delta^k(e(z))$ and $\Delta^k(f(z))$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we find that $Z^{\pm}(z)$ satisfy the following relations [16]:

THEOREM 3.6. [16]

$$g(P+h)Z^+(z) = Z^+(z)g(P+h), g(P)Z^+(z) = Z^+(z)g(P - \langle Q, P \rangle), \quad (3.28)$$

$$g(P+h)Z^-(z) = Z^-(z)g(P+h - \langle \alpha, P+h \rangle), g(P)Z^-(z) = Z^-(z)g(P), \quad (3.29)$$

$$[d, Z^{\pm}(z)] = -z \frac{\partial}{\partial z} Z^{\pm}(z), \quad (3.30)$$

$$[\Delta^k(\alpha_m), Z^{\pm}(w)] = 0, \quad (3.31)$$

$$\begin{aligned} \Delta^k(K^{\pm})Z^+(z) &= q^{\mp 2(k+1)} Z^+(z) \Delta^k(K^{\pm}), \\ \Delta^k(K^{\pm})Z^-(z) &= q^{\pm 2(k+1)} Z^-(z) \Delta^k(K^{\pm}), \end{aligned} \quad (3.32)$$

$$\begin{aligned}
z \frac{(q^{-2}w/z; q^{2(k+1)})_{\infty}}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_{\infty}} Z^{\pm}(z) Z^{\pm}(w) = \\
-w \frac{(q^{-2}z/w; q^{2(k+1)})_{\infty}}{(q^{2+2(k+1)}z/w; q^{2(k+1)})_{\infty}} Z^{\pm}(w) Z^{\pm}(z),
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_{\infty}} Z^{+}(z) Z^{-}(w) \\
- \frac{(q^{2+(k+1)}z/w; q^{2(k+1)})_{\infty}}{(q^{-2+(k+1)}z/w; q^{2(k+1)})_{\infty}} Z^{-}(w) Z^{+}(z) = \\
\frac{1}{q - q^{-1}} \left(\Delta^k(K^{-})\delta(q^{-(k+1)}z/w) - \Delta^k(K^{+})\delta(q^{(k+1)}z/w) \right).
\end{aligned} \tag{3.34}$$

PROOF. Let us show the relation (3.31). For $m > 0$, we have

$$\begin{aligned}
[\Delta^k(\alpha_m), Z^{+}(w)] &= [\Delta^k(\alpha_m), E_{(k)}^{-}(\alpha, w)] \Delta^k(e(w)) E_{(k)}^{+}(\alpha, w) \\
&\quad + E_{(k)}^{-}(\alpha, w) [\Delta^k(\alpha_m), \Delta^k(e(w))] E_{(k)}^{+}(\alpha, w).
\end{aligned}$$

This vanishes because of (3.15) and

$$[\Delta^k(\alpha_m), E_{(k)}^{-}(\alpha, w)] = -\frac{[2m]}{m} \frac{1 - p^m}{1 - p^m q^{-2(k+1)m}} q^{-(k+1)m} w^m E_{(k)}^{-}(\alpha, w).$$

By the same way, from relation (3.16) and

$$[\Delta^k(\alpha'_m), E_{(k)}^{-}(\alpha', w)] = \frac{[2m]}{m} \frac{1 - p^m q^{-2(k+1)m}}{1 - p^m} q^{(k+1)m} w^m E_{(k)}^{-}(\alpha', w),$$

we get $[\Delta^k(\alpha'_m), Z^{-}(w)] = 0$.

Similarly, the case $m < 0$ can be proved.

To prove the relation (3.33), we use equations (3.17) and (3.21) and obtain

$$\begin{aligned}
&Z^{+}(z) Z^{+}(w) \\
&= E_{(k)}^{-}(\alpha, z) \Delta^k(e(z)) E_{(k)}^{+}(\alpha, z) E_{(k)}^{-}(\alpha, w) \Delta^k(e(w)) E_{(k)}^{+}(\alpha, w) \\
&= \frac{(q^{-2+2(k+1)}w/z; q^{2(k+1)})_{\infty} (q^{-2}w/z; pq^{-2(k+1)})_{\infty}}{(q^{2+2(k+1)}w/z; q^{2(k+1)})_{\infty} (q^2w/z; pq^{-2(k+1)})_{\infty}} \\
&\times E_{(k)}^{-}(\alpha, z) \Delta^k(e(z)) E_{(k)}^{-}(\alpha, w) E_{(k)}^{+}(\alpha, z) \Delta^k(e(w)) E_{(k)}^{+}(\alpha, w) \\
&= \frac{(q^{2+2(k+1)}(w/z); q^{2(k+1)})_{\infty} (q^2(w/z) pq^{-2(k+1)})_{\infty}}{(q^{-2+2(k+1)}(w/z); q^{2(k+1)})_{\infty} (q^{-2}(w/z); pq^{-2(k+1)})_{\infty}} \\
&\times E_{(k)}^{-}(\alpha, z) E_{(k)}^{-}(\alpha, w) \Delta^k(e(z)) \Delta^k(e(w)) E_{(k)}^{+}(\alpha, z) E_{(k)}^{+}(\alpha, w).
\end{aligned}$$

Since $\Delta^k(e(z))$ satisfy the defining relations of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ and again use (3.21), we get the desired relation.

We also derive (3.34) as follows.

$$\begin{aligned}
& \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_\infty} Z^+(z)Z^-(w) \\
&= \frac{(q^{2+(k+1)}w/z; q^{2(k+1)})_\infty}{(q^{-2+(k+1)}w/z; q^{2(k+1)})_\infty} \\
&\times E_{(k)}^-(\alpha, z) \Delta^k(e(z)) E_{(k)}^+(\alpha, z) E_{(k)}^-(\alpha', w) \Delta^k(f(w)) E_{(k)}^+(\alpha', w) \\
&= E_{(k)}^-(\alpha, z) E_{(k)}^-(\alpha', w) \Delta^k(e(z)) \Delta^k(f(w)) E_{(k)}^+(\alpha, z) E_{(k)}^+(\alpha', w) \\
&= E^-(\alpha_i, z) E^-(\alpha'_j, w) [\Delta^k(f(w)) \Delta^k(e(z)) \\
&+ \frac{1}{q - q^{-1}} \delta\left(q^{-(k+1)} \frac{z}{w}\right) \Delta^k(\psi^-)(q^{(k+1)/2}w) \\
&- \frac{1}{q - q^{-1}} \delta\left(q^{(k+1)} \frac{z}{w}\right) \Delta^k(\psi^+)(q^{-(k+1)/2}w)] E_{(k)}^+(\alpha, z) E_{(k)}^+(\alpha', w).
\end{aligned}$$

In the second equality, we used the relation (3.19). In the third equality we used the defining relation of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ between $\Delta^k(e(z))$ and $\Delta^k(f(w))$.

By using

$$\begin{aligned}
\Delta^k(\psi^\pm)(q^{\mp(k+1)/2}w) &= \Delta^k(K^\pm) E_{(k)}^-(\alpha, q^{\mp(k+1)}w)^{-1} E_{(k)}^-(\alpha', q^{\mp 1/2}w)^{-1} \\
&\times E^+(\alpha, q^{\mp(k+1)}w)^{-1} E_k^+(\alpha', q^{\mp 1/2}w)^{-1}
\end{aligned}$$

and the property of the delta function, we obtain relation (3.34).

From Definition 3.2, Theorem 3.1 and Theorem 2.10 with $c^{(i)} = 1 \otimes \cdots \otimes c \otimes \cdots \otimes 1$, we express the level- $k+1$ Z -operators as follows

$$\begin{aligned}
Z^+(z) &= \sum_{i=1}^{k+1} E_{(k)}^-(\alpha, z) \mathfrak{e}_i^-(\alpha, z) \mathfrak{e}_i^+(\alpha, z) E_{(k)}^+(\alpha, z) \\
&\times (1 \otimes \cdots \otimes e^\alpha \otimes e^{-2Q} \otimes \cdots \otimes e^{-2Q}) \\
&\times (1 \otimes \cdots \otimes z^h q^{-(c^{(i+1)} + \cdots + c^{(k+1)})h} \otimes q^h \otimes \cdots \otimes q^h) \\
&\times z q^{-(c^{(i+1)} + \cdots + c^{(k+1)})} \\
Z^-(z) &= \sum_{i=1}^{k+1} E_{(k)}^-(\alpha', z) \mathfrak{f}_i^-(\alpha, z) \mathfrak{f}_i^+(\alpha, z) E_{(k)}^+(\alpha', z) \\
&\times (e^{-2Q} \otimes \cdots \otimes e^{-2Q} \otimes e^{-\alpha} \otimes 1 \cdots \otimes 1) \\
&\times (q^{-h} \otimes \cdots \otimes q^{-h} \otimes z^{-h} q^{(c^{(i+1)} + \cdots + c^{(k+1)})h} \otimes 1 \cdots \otimes 1) \\
&\times z q^{-(c^{(i+1)} + \cdots + c^{(k+1)})}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{e}_i^-(\alpha, z) &= \\
&\exp \left((q^{-1} - q) \sum_{n>0} \left\{ 1 \otimes \cdots \otimes \frac{\alpha_{-n} q^{c^{(i)}n}}{1 - q^{2c^{(i)}n}} q^{-(c^{(i+1)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{p^n \alpha_{-n}}{1 - p^n} q^{-(c^{(i+2)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + \cdots + 1 \otimes \cdots \otimes \frac{p^n \alpha_{-n}}{1 - p^n} \right\} z^n \right) \\
\mathbf{e}_i^+(\alpha, z) &= \\
&\exp \left((q - q^{-1}) \sum_{n>0} \left\{ 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - q^{2c^{(i)}n}} q^{(c^{(i)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - p^n} q^{-(c^{(i+2)} + \cdots + c^{(k+1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + \cdots + 1 \otimes \cdots \otimes \frac{\alpha_n}{1 - p^n} \right\} z^{-n} \right) \\
\mathbf{f}_i^-(\alpha, z) &= \exp \left(-(q - q^{-1}) \sum_{n>0} \left\{ \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(1)}n}} q^{-c^{(1)}n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(2)}n}} q^{-(c^{(1)} + c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - p^n q^{-2c^{(i-1)}n}} q^{-(c^{(1)} + c^{(2)} + \cdots + c^{(i-1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_{-n}}{1 - q^{2c^{(i)}n}} q^{-(c^{(1)} + \cdots + c^{(i-1)})n + c^{(i)}n} \otimes 1 \cdots \otimes 1 \right\} z^n \right) \\
\mathbf{f}_i^+(\alpha, z) &= \exp \left((q - q^{-1}) \sum_{n>0} \left\{ \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(1)}n}} q^{-c^{(1)}n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(2)}n}} q^{(c^{(1)} - c^{(2)})n} \otimes 1 \cdots \otimes 1 + \cdots \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{p^n \alpha'_n}{1 - p^n q^{-2c^{(i-1)}n}} q^{(c^{(1)} + \cdots + c^{(i-2)} - c^{(i-1)})n} \otimes 1 \cdots \otimes 1 \right. \right. \\
&\quad \left. \left. + 1 \otimes \cdots \otimes \frac{\alpha'_n}{1 - q^{2c^{(i)}n}} q^{(c^{(1)} + c^{(2)} + \cdots + c^{(i-1)} + c^{(i)})n} \otimes 1 \cdots \otimes 1 \right\} z^{-n} \right)
\end{aligned}$$

4. Integrable condition of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module

In this section we show that the products $\mathfrak{E}_N(z) = \Delta^k(e(z))\Delta^k(e(zq^2)) \cdots \Delta^k(e(zq^{2(N-1)}))$ and $\mathfrak{F}_N(z) = \Delta^k(f(z))\Delta^k(f(zq^{-2})) \cdots \Delta^k(f(zq^{-2(N-1)}))$ give the integrable condition for the level- $k + 1$ $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module at $N = k + 2$, namely

the nilpotent condition. Then at the same time we show that the products $\mathfrak{E}_N(z)$ and $\mathfrak{F}_N(z)$ at $N = k + 1$ give certain vertex operators associated with the level- $k + 1$ module of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

THEOREM 4.1. *For $k \geq 1$. On the level- $k + 1$ integrable module $V_{k+1}(\lambda_i, \mu)$ of $U_{q,p}(\widehat{\mathfrak{sl}}_2)$, we obtain a quantum analogue of the condition of integrability (an elliptic analogue of the Wheel condition) as*

$$\mathfrak{E}_{k+2}(z) = \Delta^k(e(z)) \Delta^k(e(zq^2)) \cdots \Delta^k(e(zq^{2(k+1)})) = 0 \quad (4.1)$$

$$\mathfrak{F}_{k+2}(z) = \Delta^k(f(z)) \Delta^k(f(zq^{-2m})) \cdots \Delta^k(f(zq^{-2(k+1)})) = 0. \quad (4.2)$$

On the other hand, $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ give the following vertex operators

$$\mathfrak{E}_{k+1}(z) =$$

$$\begin{aligned} & \mathfrak{S}(p, q)_e : \exp \left(\sum_{n \neq 0} -\frac{\Delta^k(\alpha_n)}{[n]} q^{-kn} z^{-n} \right) : (1 \otimes K^- \otimes K^- \otimes \cdots \otimes K^-) \\ & \times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{h+1} \otimes \cdots \otimes z^{h+1}) \\ & \times (q^{kh} \otimes q^{(k-1)h} \otimes \cdots \otimes 1) q^{\frac{k(k+1)}{2}}, \end{aligned} \quad (4.3)$$

$$\mathfrak{F}_{k+1}(z) =$$

$$\begin{aligned} & \mathfrak{S}(p, q)_f : \exp \left(\sum_{n \neq 0} \frac{\Delta^k(\alpha'_n)}{[n]} q^{kn} z^{-n} \right) : (K^+ \otimes K^+ \otimes \cdots \otimes K^+ \otimes 1) \\ & \times (e^\alpha \otimes \cdots \otimes e^\alpha)(z^{-h+1} \otimes \cdots \otimes z^{-h+1}) \\ & \times (q^{(k+1)h} \otimes q^{(k)h} \otimes \cdots \otimes q^h) q^{-\frac{(k+1)(k+2)}{2}}, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} \mathfrak{S}(p, q)_e = & \frac{(q^{-2}pq^{-2(\Delta(c))}; pq^{-2(\Delta(c))})_\infty}{(q^2pq^{-2(\Delta(c))}; pq^{-2(\Delta(c))})_\infty} \\ & \times \prod_{j=1}^k \prod_{i=1}^j \frac{(q^{-2}pq^{-2(\Delta^{(j)}(c)+(2i-1))}; pq^{-2(\Delta^{(j)}(c))})_\infty}{(q^2pq^{-2(\Delta^{(j)}(c)+(2i-1))}; pq^{-2(\Delta^{(j)}(c))})_\infty} \\ & \times \prod_{j \leq l=1}^{k-1} \prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty}{(q^{2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty} \\ & \times \frac{(q^{2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}, \end{aligned}$$

$$\begin{aligned}
\mathfrak{S}(p, q)_f = & \prod_{j=0}^{k-1} \prod_{i=1}^{k-j} \frac{(pq^{2-2i-2(c^{(1)}+\dots+c^{(j)})}); pq^{-2(c^{(1)}+\dots+c^{(j)})}_\infty}{(pq^{-2-2i-2(c^{(1)}+\dots+c^{(j)})}); pq^{-2(c^{(1)}+\dots+c^{(j)})}_\infty} \\
& \times \prod_{j \leq l=1}^{k-1} \prod_{i=1}^{k-l} \frac{(q^{-2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty}{(q^{2+2j}pq^{-2\Delta^{(i+l)}(c)}; pq^{-2\Delta^{(i+l)}(c)})_\infty} \\
& \times \frac{(q^{2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}{(q^{-2+2j}pq^{-2\Delta^{(i+l-1)}(c)}; pq^{-2\Delta^{(i+l-1)}(c)})_\infty}.
\end{aligned}$$

PROOF. Let us show the proof of (4.1). From the comultiplication (3.2) in Theorem 3.1, we have the following product on $V_{k+1}(\lambda_i, \mu)$ for some positive integer N over all possible decompositions

$$\sum_{i_1, \dots, i_N \in \{1, \dots, k+1\}} e^{i_1}(z_{i_1}) e^{i_2}(z_{i_2}) \cdots e^{i_N}(z_{i_N}), \quad (4.5)$$

where $c^{(i)} = 1$.

From the relations (3.5)-(3.7) in Lemma 3.2, one can show that for $z_{i_{j+1}}/z_{i_j} = q^2$ all terms in (4.5) are zero except for those with indices $i_1 > \dots > i_{k+1}$. Suppose $N = k + 2$ and $z_{i_{j+1}}/z_{i_j} = q^2$, then for $m \neq n$ there is $i_m = i_n$. Thus we get the first condition of integrability. Similarly one can prove the $\mathfrak{F}_{k+2}(z)$ case.

For the vertex operator $\mathfrak{E}_{k+1}(z)$, since the term with $i_1 > \dots > i_{k+1}$ in (4.5) is not zero, we have

$$\begin{aligned}
\mathfrak{E}_{k+1}(z) = & e(zq^k) \otimes e(zq^{k-1})\psi^-(zq^{k-\frac{1}{2}}) \otimes e(zq^{k-2})\psi^-(zq^{k-\frac{1}{2}})\psi^-(zq^{k+\frac{3}{2}}) \\
& \otimes \cdots \otimes e(z)\psi^-(zq^{\frac{3}{2}}) \cdots \psi^-(q^{2k-\frac{1}{2}}).
\end{aligned}$$

We used relations (2.42) and (2.45) in Proposition 2.11 to write each factor of the tensor product in a normal order form. Then we get

$$\begin{aligned}
\mathfrak{E}_{k+1}(z) = & \mathfrak{S}(p, q)_e \left(e(zq^k) \otimes : e(zq^{k-1})\psi^-(zq^{k-\frac{1}{2}}) : \right. \\
& \otimes : e(zq^{k-2})\psi^-(zq^{k-\frac{1}{2}})\psi^-(zq^{k+\frac{3}{2}}) : \\
& \left. \otimes \cdots \otimes : e(z)\psi^-(zq^{\frac{3}{2}}) \cdots \psi^-(q^{2k-\frac{1}{2}}) : \right). \quad (4.6)
\end{aligned}$$

Substitute (2.39) and (2.40) from Theorem 2.10 into the above relation and use (3.11), we get the desired relation (4.3). Relation (4.4) can be proved in a similar way.

PROPOSITION 4.2. *On $V_{k+1}(\lambda_i, \mu)$, the vertex operators $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ satisfy the following difference equations*

$$\begin{aligned} \mathfrak{E}_{k+1}(zq^2) &= \Delta^k(q^{h+1}) \exp\left((q - q^{-1}) \sum_{n>0} \Delta^k(\alpha_{-n})(q^{k+1}z)^n\right) \\ &\times \mathfrak{E}_{k+1}(z) \Delta^k(q^{h+1}) \exp\left(-(q - q^{-1}) \sum_{n>0} \Delta^k(\alpha_n)(q^{k+1}z)^{-n}\right), \end{aligned} \quad (4.7)$$

$$\begin{aligned} \mathfrak{F}_{k+1}(zq^2) &= \Delta^k(q^{-(h+1)}) \exp\left((q - q^{-1}) \sum_{n>0} \Delta^k(\alpha'_{-n})(q^{k+1}z)^n\right) \\ &\times \mathfrak{F}_{k+1}(z) \Delta^k(q^{-(h+1)}) \exp\left(-(q - q^{-1}) \sum_{n>0} \Delta^k(\alpha'_n)(q^{k+1}z)^{-n}\right). \end{aligned} \quad (4.8)$$

By means of an elliptic analogue of the Drinfeld coproduct, we have found the higher level module of the elliptic quantum group $U_{q,p}(\widehat{\mathfrak{sl}}_2)$.

A highest weight \mathfrak{sl}_2 -module is called integrable if the Chevalley generators are locally nilpotent on this module [9]. Proposition VI.5 in Ref. [15] shows that on the level- k standard $\widehat{\mathfrak{sl}}_2$ -module, the currents $x_{\pm\alpha}(z)$ are nilpotent operators at $k + 1$, $x_{\pm\alpha}(z)^{k+1} = 0$. The authors in [1, 3] found the nilpotent condition for $U_q(\widehat{\mathfrak{sl}}_2)$ integrable module. Here we obtained the elliptic analogue of the nilpotent condition for $U_{q,p}(\widehat{\mathfrak{sl}}_2)$ module. In quantum case, the vertex operators $x^{\pm k}(z)$ in [1] satisfy certain q -difference equations $x^{+k}(zq^2) = \Delta^k \phi^{-1}(zq^{\frac{m+1}{2}}) x^{+k}(z) \Delta^k \psi(zq^{\frac{3(m+1)}{2}})$, $x^{-k}(zq^2) = \Delta^k \phi(zq^{\frac{-3(m+1)}{2}}) x^{-k}(z) \Delta^k \psi^{-1}(zq^{\frac{-(m+1)}{2}})$, where $\phi(z)$ and $\psi(z)$ are the generating functions of the bosons $a_{-n}, a_n (n \in \mathbb{Z}_{>0})$ respectively. We found the elliptic analogue of these q -difference relations. It is clear that the operators $\Delta^k(\psi^{\pm}(z))$ do not appear on the both sides of $\mathfrak{E}_{k+1}(z)$ and $\mathfrak{F}_{k+1}(z)$ in (4.7) and (4.8) respectively unlikely in quantum case because the operators $\Delta^k(\psi^{\pm}(z))$ (3.12)-(3.13) are exponential functions of both annihilation operator $\Delta^k(\alpha_n)$ and creation operator $\Delta^k(\alpha_{-n})$ with p factors.

The authors in [7] compute the correlation function of $U_q(\widehat{\mathfrak{sl}}_2)$ perfect vertex operators using the wheel condition. We expect that we can make a similar application in the elliptic case.

Acknowledgement

The author is greatly indebted to Professor H. Konno, for his encouragement, invaluable advice, stimulating discussions, valuable suggestions and comprehensive, critical reading of the manuscript. The author is also grateful to the Egyptian government for a scholarship.

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