Reversible and Conservative Elementary Triangular Partitioned Cellular Automata

Kenichi Morita
Hiroshima University

May 2016

This work was supported by JSPS KAKENHI Grant Number 15K00019

Abstract

Eight-state isotropic *triangular partitioned cellular automata* (TPCAs) are called *elementary TPCAs* (ETPCAs). They are extremely simple, since each of their local transition functions is described by only four local rules. Among them, we study computational universality of *reversible* and *conservative* ETPCAs. There are nine kinds of such ETPCAs. We show six of them are universal, and three are non-universal. Universality is shown by giving a configuration that simulates a Fredkin gate, a universal reversible gate. Computer simulation results are also given as movies and in the attachment files.

Contents

1. Preliminaries

1.1. Reversible Elementary Triangular Partitioned Cellular Automata (RETPCAs)

1.2. Turing Universality

2. Universality of Conservative RETPCAs

2.1. Universality of RETPCA T_{RL}

2.2. Universality of RETPCA T_{RU}

2.3. Universality of RETPCA T_{UR}

2.4. Non-universality of RETPCA T_{RR}

2.5. Non-universality of RETPCA T_{UU}

References

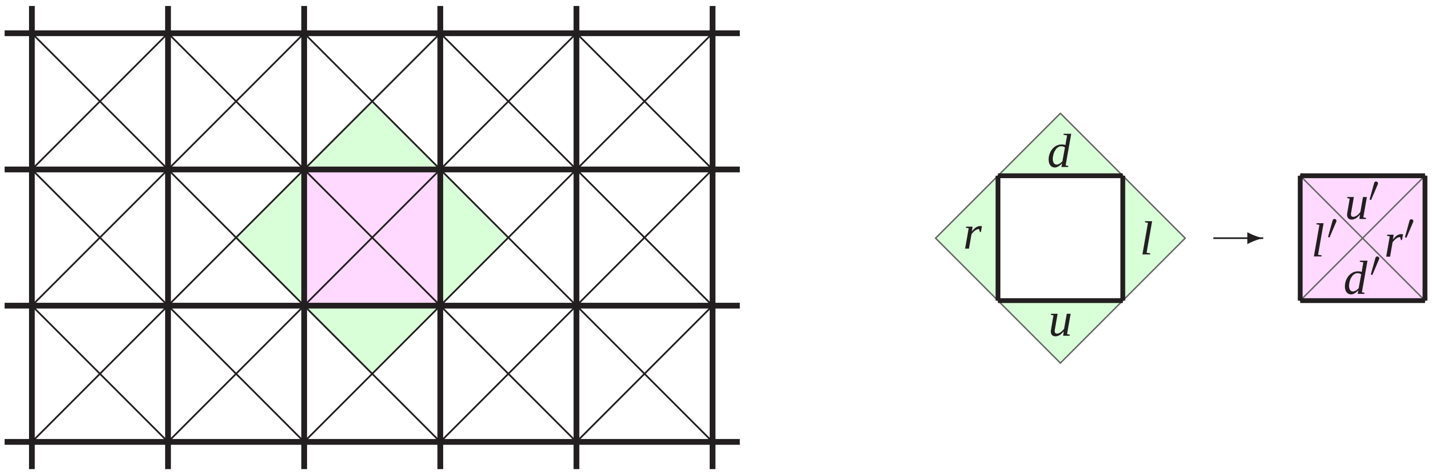
Appendices: List of Attachment files

1. Preliminaries

1.1. Reversible Elementary Triangular Partitioned Cellular Automata (RETPCAs)

Partitioned Cellular Automaton (PCA)

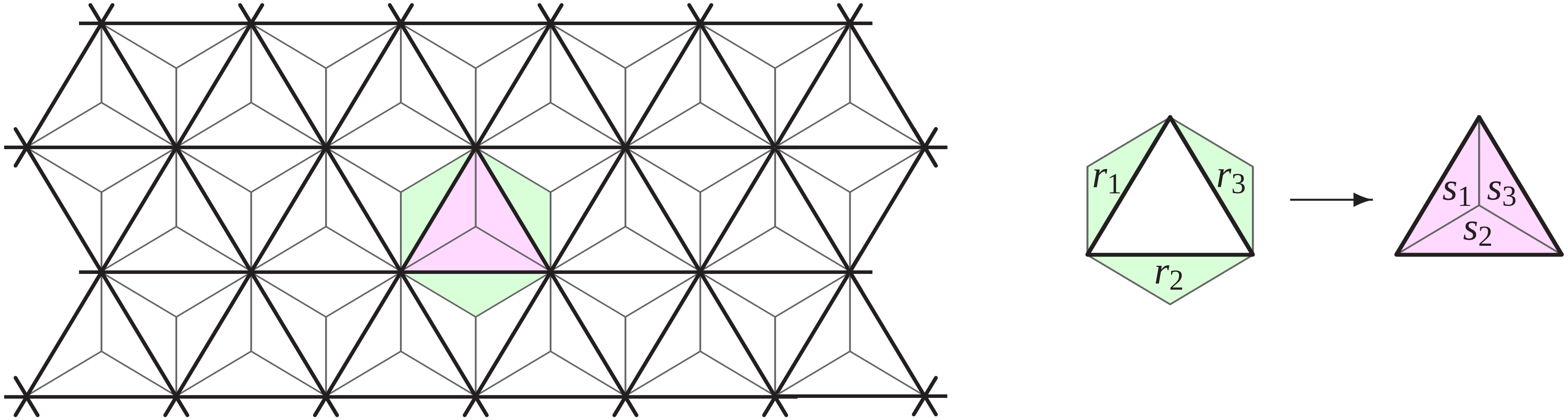
- We use the framework of *partitioned cellular automata* (PCAs), a subclass of standard CAs.
- In PCA, each cell is divided into several parts.
- The next state of each cell is determined by the states of the adjacent parts of the neighbor cells.



The above is the case of 2D 4-neighbor PCA.

Triangular partitioned cellular automaton (TPCA)

- TPCA T is a 2D PCA whose cell is triangular-shaped, and divided into three parts.

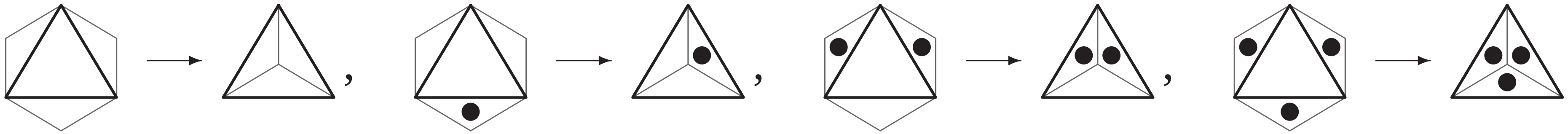


- The framework of PCAs makes it easy to design *reversible CAs*.

Local and global functions of PCA

- A *local function* f is described by *local rules*.

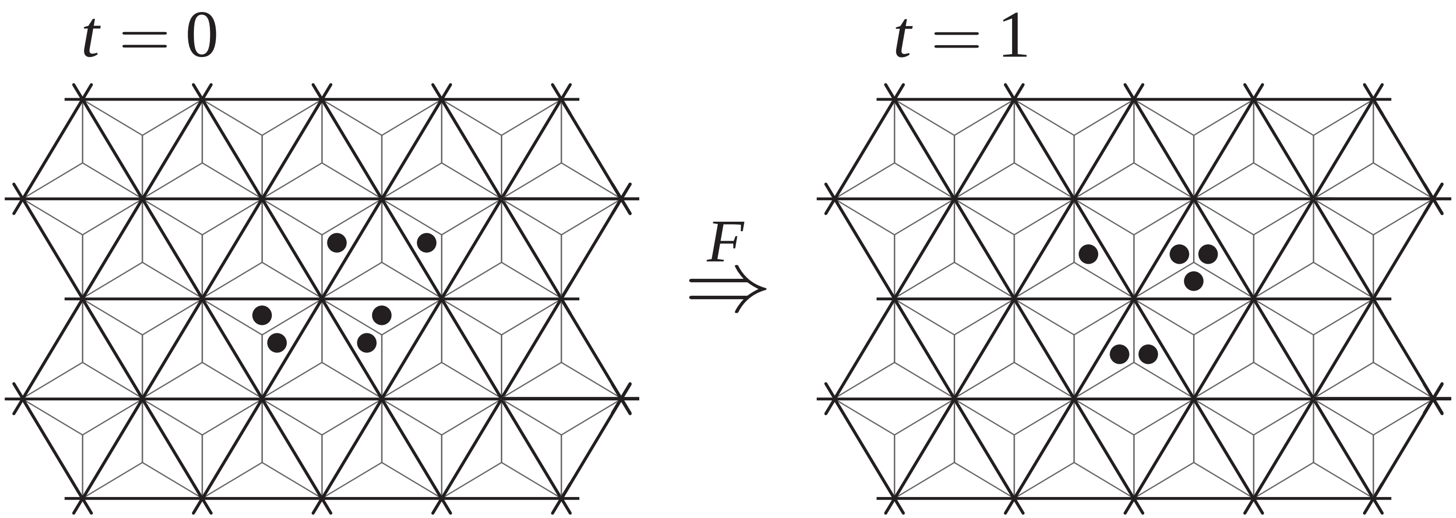
  Example:

  Note: Here, *isotropy* (or *rotation symmetry*) is assumed.

- Applying the local function f to all the cells in parallel, the *global function* F is obtained.

  Example:

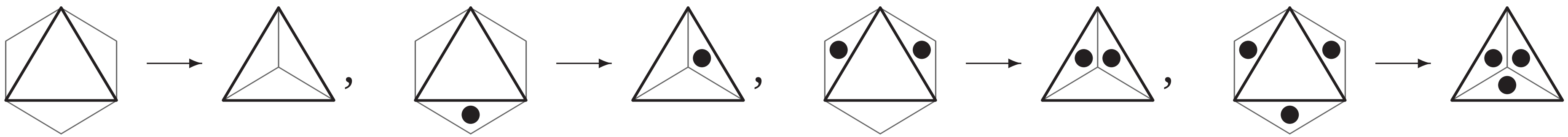
  

Reversible PCAs (RPCAs)

- A PCA is called *globally reversible*, if its global function is injective.
- A PCA is called *locally reversible*, if its local function is injective.

Lemma 1 [Morita, Harao, 1989] *A PCA is globally reversible iff it is locally reversible.*

- Such a PCA is simply called a *reversible PCA*.
- Example: A local function of a reversible TPCA.



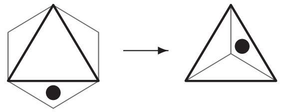
There is no pair of rules with the same right-hand sides.

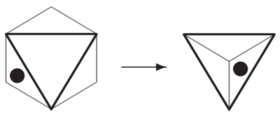
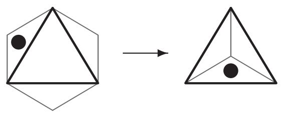
- They are closely related to physical reversibility.

Elementary TPCA (ETPCA)

An *eight-state* and *isotropic* TPCA is called an *elementary TPCA* (ETPCA)

- *Eight-state*:
  Each of three parts has the state set $\{0, 1\}$.
  Here, 0 and 1 are indicated by a *blank* and ●.
- *Isotropy* (or *rotation-symmetry*):
  For each rule, the rules obtained by rotating the both sides of it by a multiple of 60° exist.

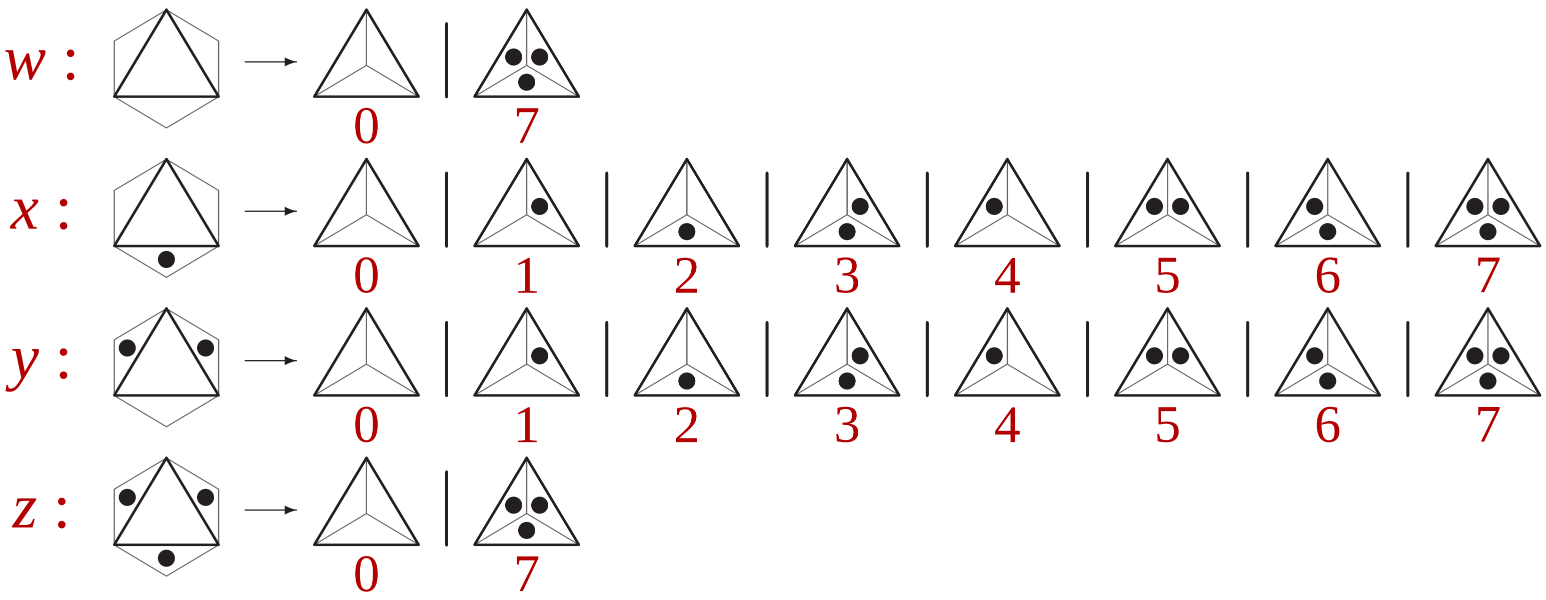
  Example: If there is a rule  ,

  then  ,  etc. also exist.

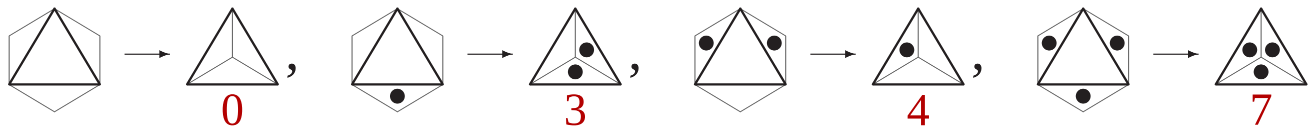
Note: ETPCAs are the simplest class of 2D PCAs that contains interesting ones, as the class of 1D ECAs [Wolfram, 2002].

Representing ETPCA by a number

- ETPCA T_{wxyz} is specified by a 4-digit number $wxyz$, where $w, z \in \{0, 7\}$ and $x, y \in \{0, 1, \ldots, 7\}$.



- Example: Local function of ETPCA T_{0347} :



Note: w and z must be 0 or 7 because of isotropy. Thus, there are 256 ETPCAs.

Reversible ETPCA (RETPCA)

- Let T_{wxyz} be an ETPCA.

$$T_{wxyz} \text{ is } \textit{reversible}$$
$$\text{iff}$$
$$(w,z) \in \{(0,7),(7,0)\} \wedge$$
$$(x,y) \in \{1,2,4\}\times\{3,5,6\} \cup \{3,5,6\}\times\{1,2,4\}$$

- There are 36 RETPCAs.

Conservative ETPCAs

- It is an ETPCA such that the total number of particles (i.e., ●'s) is conserved in each rule.
- Let T_{wxyz} be an ETPCA.

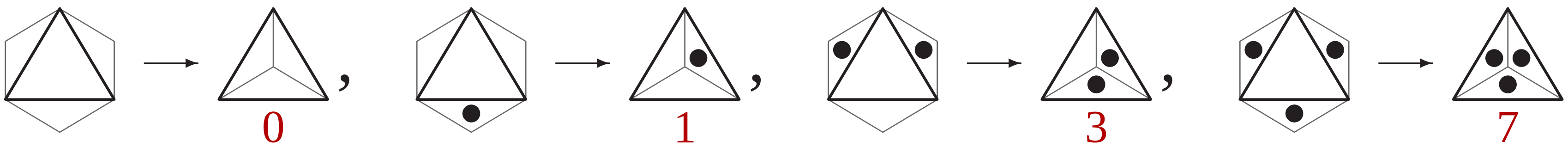
T_{wxyz} is *conservative* (or *bit-conserving*)

iff

$$w = 0 \ \wedge \ x \in \{1, 2, 4\} \ \wedge \ y \in \{3, 5, 6\} \ \wedge \ z = 7$$

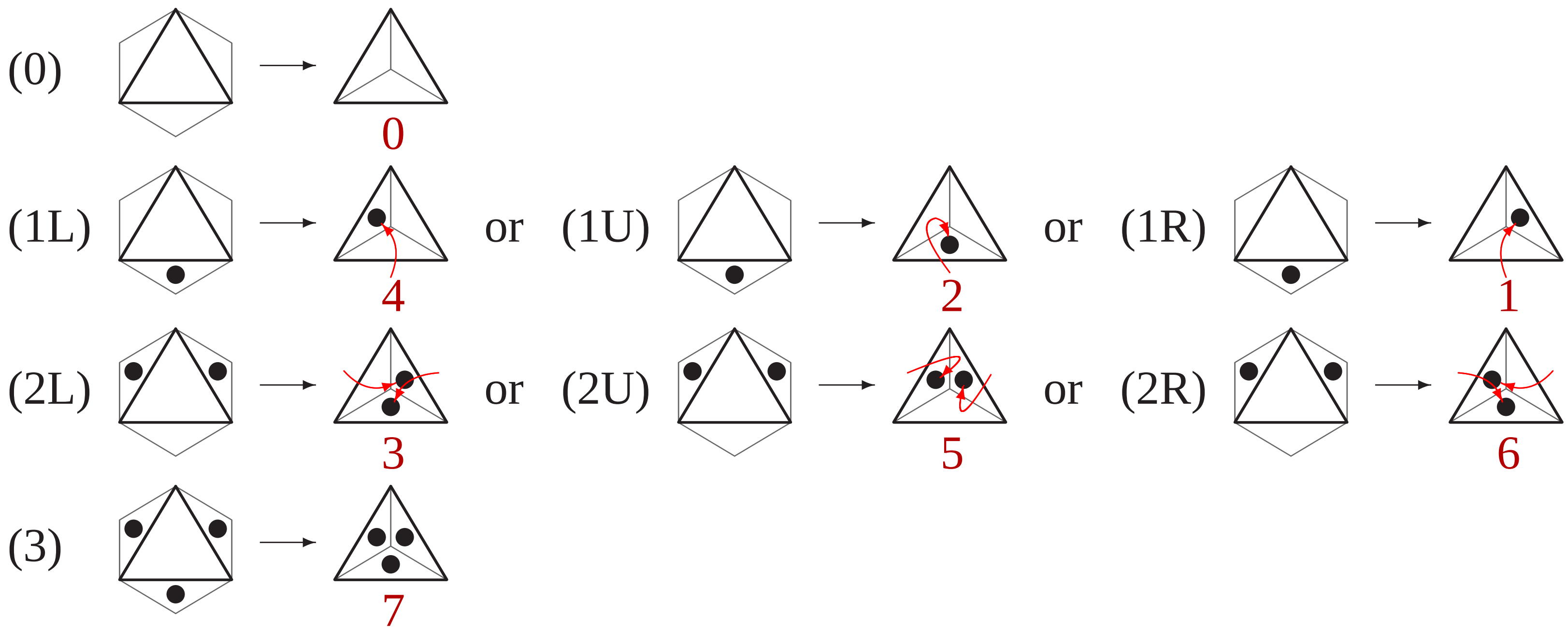
- We can see that, if an ETPCA is conservative, then it is reversible.

- Example: T_{0137} is a conservative RETPCA.



Aliases of nine conservative RETPCAs

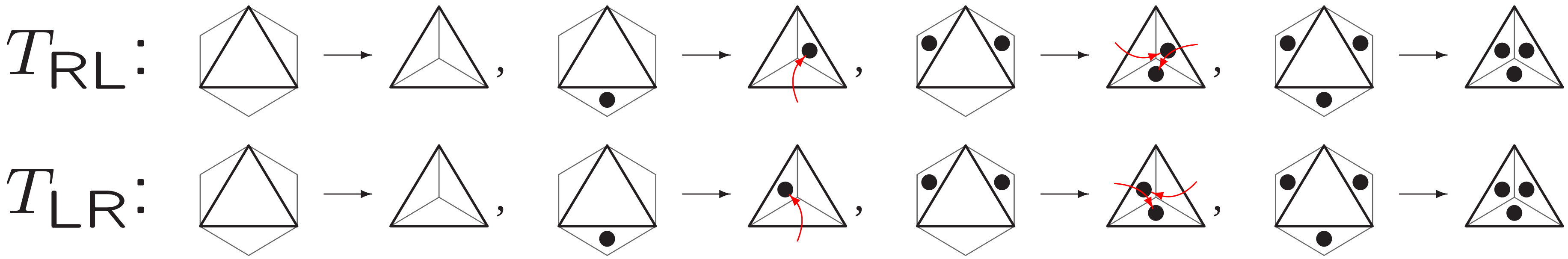
- Their local functions are:



- T_{XY} denotes RETPCA with the local function $f = \{(0), (1X), (2Y), (3)\}$, where $X, Y \in \{\text{L}, \text{U}, \text{R}\}$.
- L, U, and R stand for left-, U-, and right-turns of particles.
- Example: $T_{\text{RU}} = T_{0157}$

Duality of ETPCAs under reflection

- ETPCAs T and T' are called *dual under reflection* (denoted by $T \underset{\text{refl}}{\longleftrightarrow} T'$),if their local functions are mirror images of each other.
- Example: $T_{\text{RL}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{LR}}$

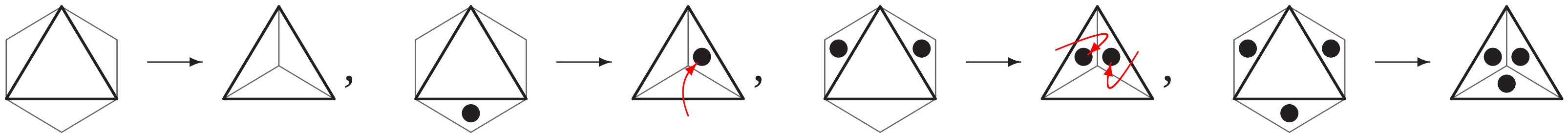


- $T_{\text{RL}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{LR}}$, $T_{\text{RU}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{LU}}$, $T_{\text{UR}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{UL}}$, $T_{\text{RR}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{LL}}$, $T_{\text{UU}} \underset{\text{refl}}{\longleftrightarrow} T_{\text{UU}}$.
- Dual ETPCAs are essentially the same. Hereafter, we study only T_{RL}, T_{RU}, T_{UR}, T_{RR} and T_{UU}.

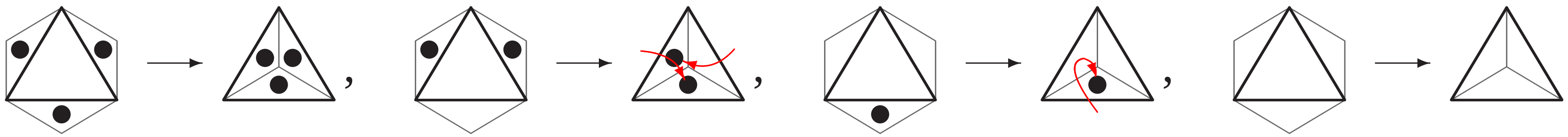
Duality of ETPCAs under conjugation

- ETPCAs T and T' are called *conjugate*, or *dual under conjugation* (denoted by $T \underset{\text{conj}}{\longleftrightarrow} T'$), if their local functions are *0-1 exchange* of the other.
- Example: $T_{\text{RU}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{UR}}$

T_{RU}:



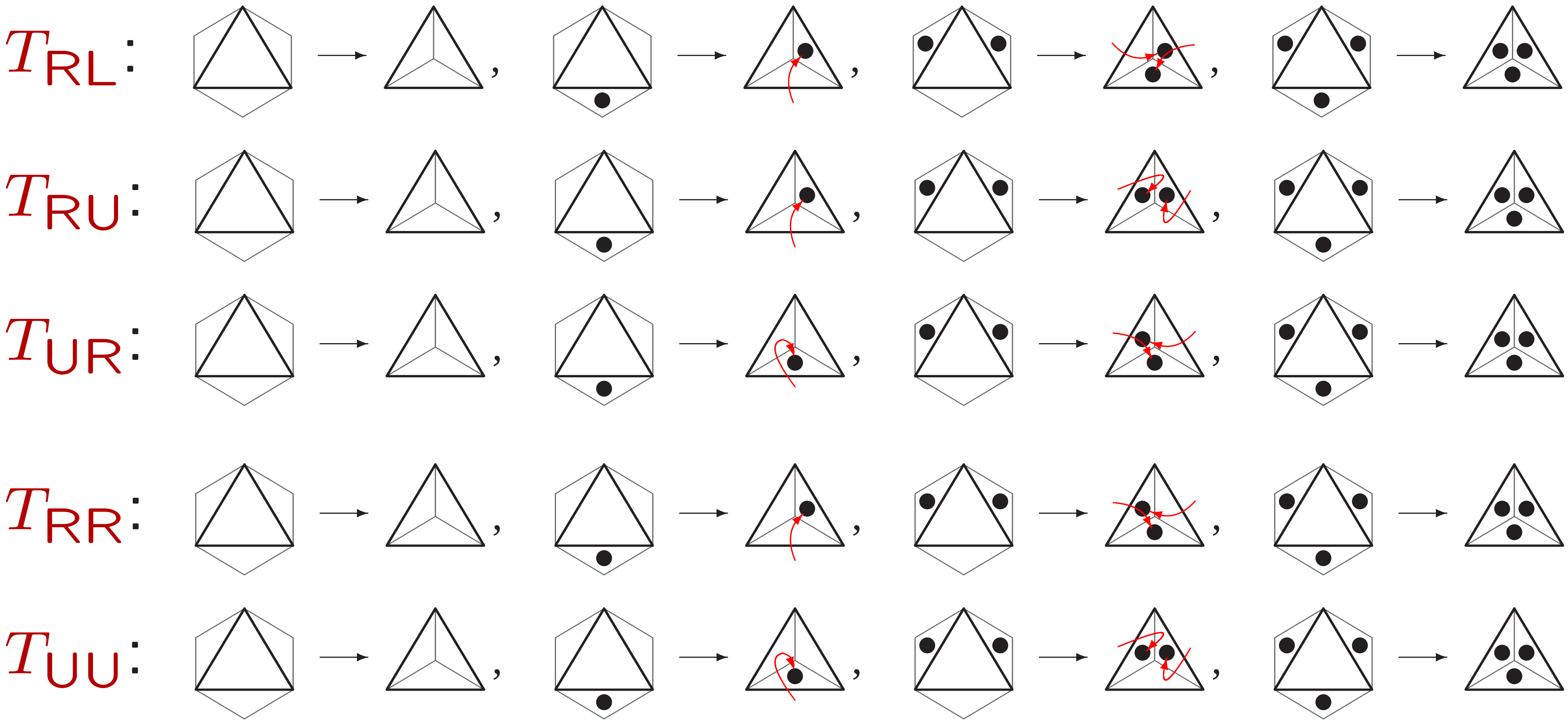
T_{UR}:



- $T_{\text{RU}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{UR}}$, $T_{\text{UL}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{LU}}$, $T_{\text{RL}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{LR}}$,
  $T_{\text{RR}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{RR}}$, $T_{\text{LL}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{LL}}$, $T_{\text{UU}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{UU}}$

Note: Though $T_{\text{UR}} \underset{\text{conj}}{\longleftrightarrow} T_{\text{RU}}$ holds, we also study T_{UR} as well as T_{RU}, since T_{UR} has a specific property explained later.

Conservative RETPCAs investigated here

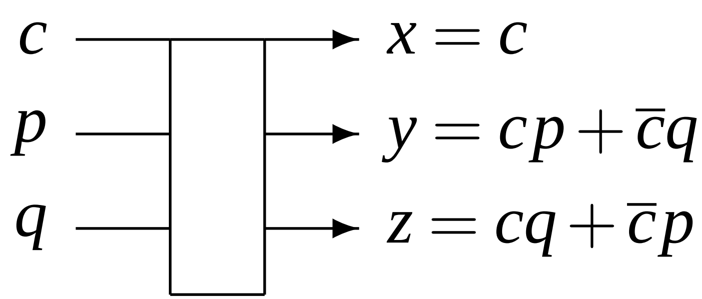


Note: Universality of T_{RU} has been shown in [Imai, Morita, 2000]

1.2. Turing universality

Turing universality of 2D RPCAs

- RPCA A is called *Turing universal*, if any Turing machine is simulated in it.
- To prove Turing universality of the RPCA A, it is sufficient to show that any circuit composed of *Fredkin gates* [Fredkin, Toffoli, 1982] and delay elements is simulated in it.



Fredkin gate

Showing Turing universality of an RPCA

If a specific RPCA A can simulate a Fredkin gate, then its universality is derived in the following way.

RPCA A

$\Downarrow$

Fredkin gate

$\Downarrow$ Lemma 2

Rotary element

$\Downarrow$ Lemma 3

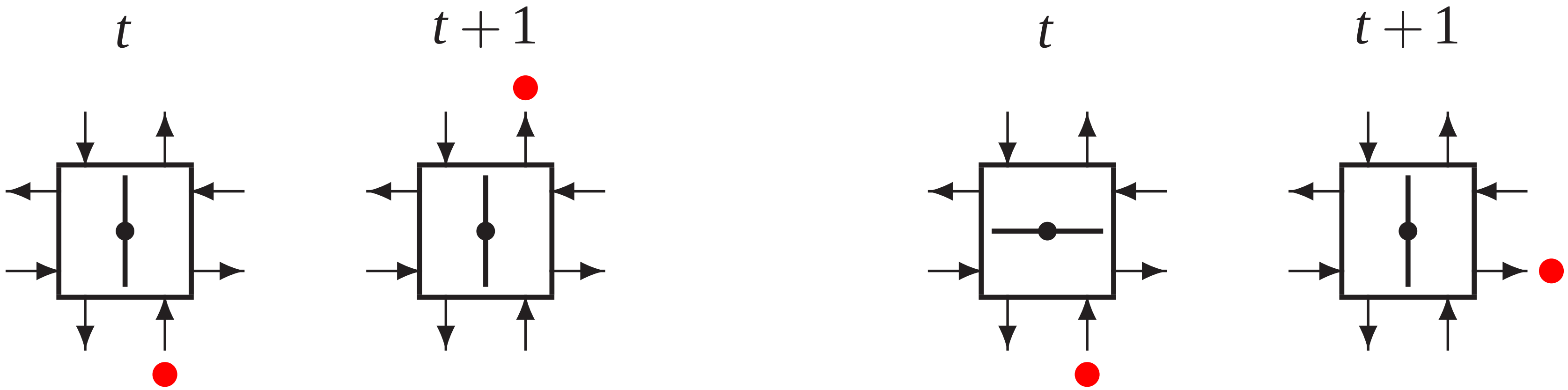
Any reversible Turing machine

$\Downarrow$ Lemma 4

Any Turing machine

Rotary element (RE)

- *Rotary element* is a 2-state 4-input 4-output *reversible sequential machine* (RSM).

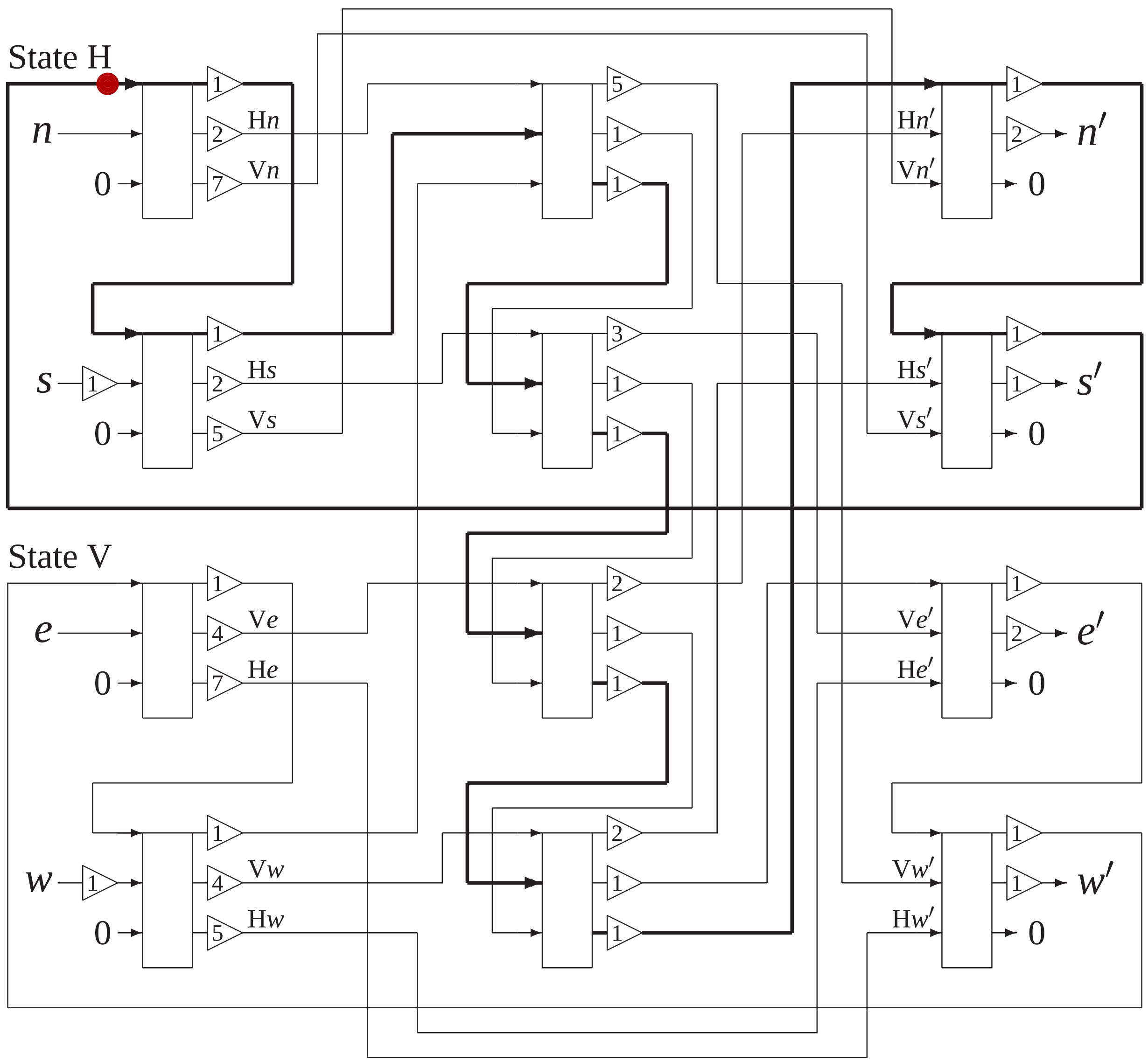


(a) Signal and bar are parallel (b) Signal and bar are orthogonal

Lemma 2 [Morita, 1990, 2012] *Any RSM (in particular RE) can be simulated by a garbage-less circuit composed of Fredkin gates and delay elements.*

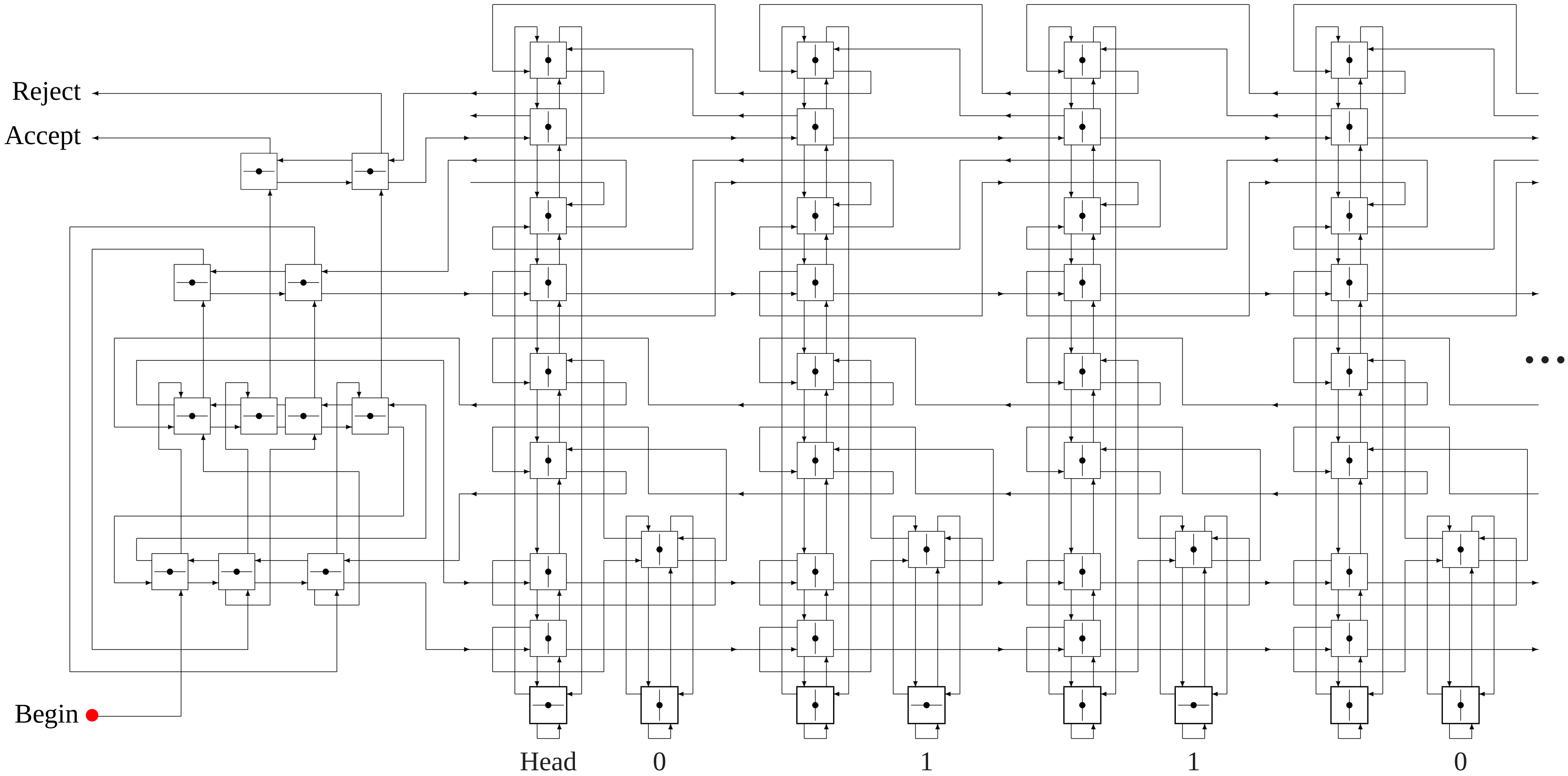
Rotary element composed of Fredkin gates

[Morita, 2012]



Making reversible Turing machines out of REs

Lemma 3 [Morita, 2001] *Any reversible Turing machine can be composed only of REs.*



Showing Turing universality of an RPCA

Lemma 4 [Bennett, 1973] *Any (irreversible) Turing machine can be simulated by a garbage-less reversible Turing machine.*

From Lemmas 2–4 we have the following.

Lemma 5 *An RPCA is Turing universal, if any circuit composed of Fredkin gates and delay elements is simulated in it.*

Note: Such an RPCA needs infinite (but ultimately periodic) configurations to simulate reversible Turing machines.

Known results

The following reversible CAs are Turing universal, since any circuit composed of Fredkin gates and delay elements are simulated in them.

- 2-state RCA with the Margolus neighborhood [Margolus, 1984]
- Two models of 16-state RPCAs on square grid [Morita, Ueno, 1992]
- RETPCA T_{UR} on triangular grid [Imai, Morita, 2000]
- RETPCA T_{0347} on triangular grid [Morita, 2016]

2. Universality of Conservative RETPCAs

Sufficient condition for universality

If the following are implemented in RETPCA, then any circuit composed of Fredkin gates and delay elements can be realized in it.

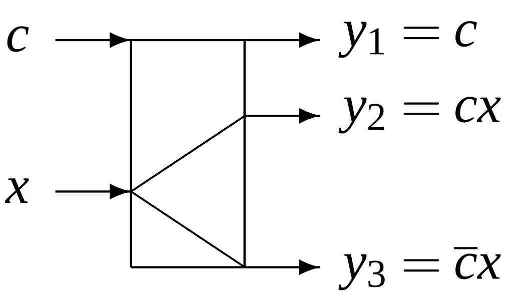
(1) Signal and transmission wire

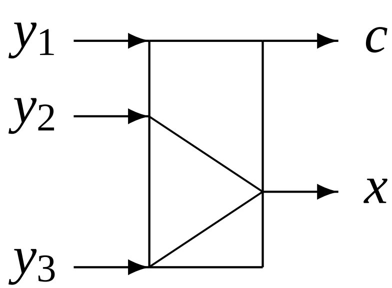
(2) Delay module (for fine adjustment of timing)

(3) Signal crossing module (in 2D space)

(4) Switch gate and inverse switch gate modules

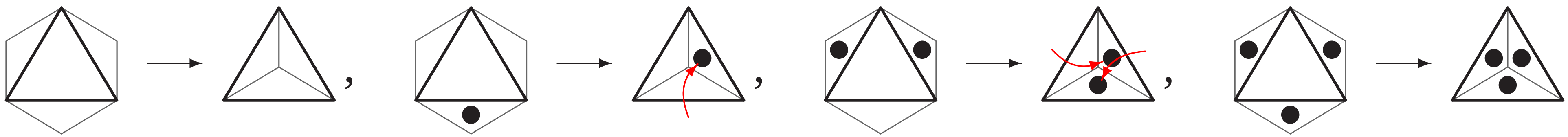
- It is known Fredkin gate is realized by *switch gate* and *inverse switch gate* [Fredkin, Toffoli, 1982].





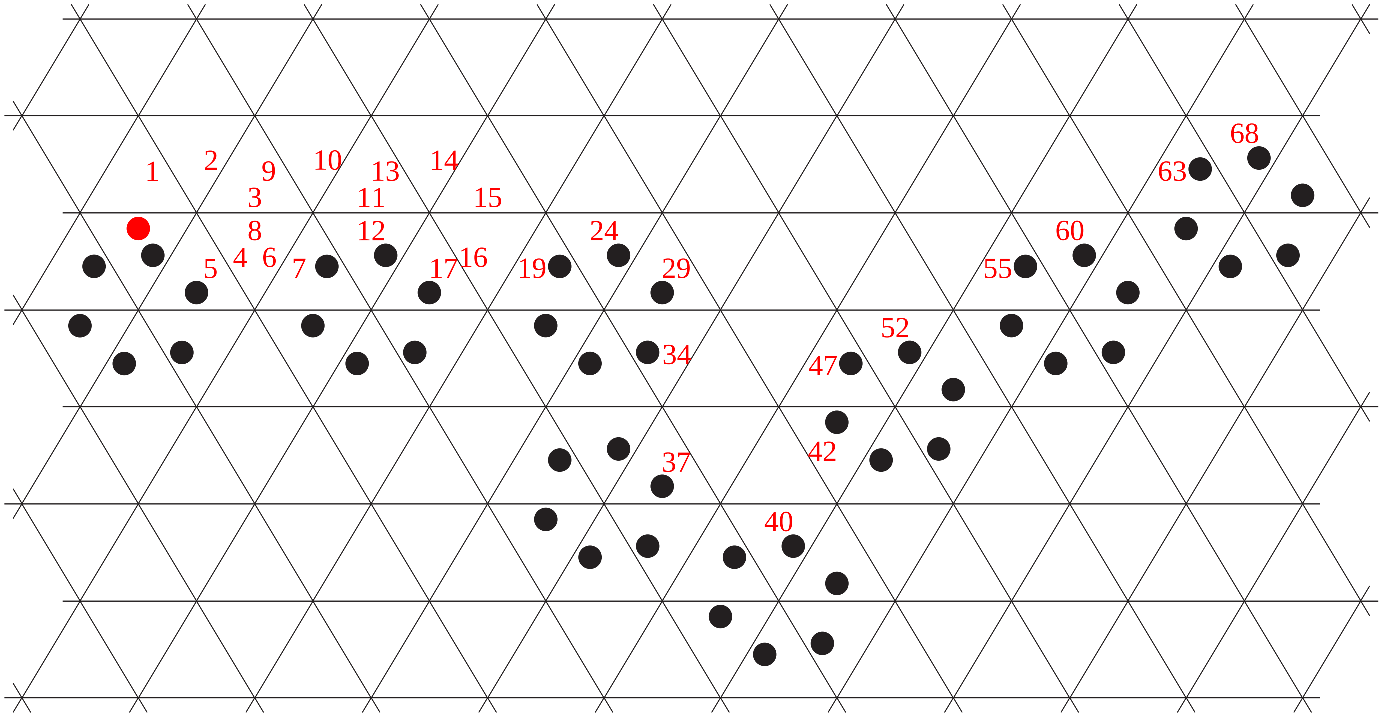
$c = y_1$ and $x = y_2 + y_3$
under the assumption
$(y_2 \rightarrow y_1) \wedge (y_3 \rightarrow \overline{y_1})$

2.1. Universality of RETPCA T_{RL}



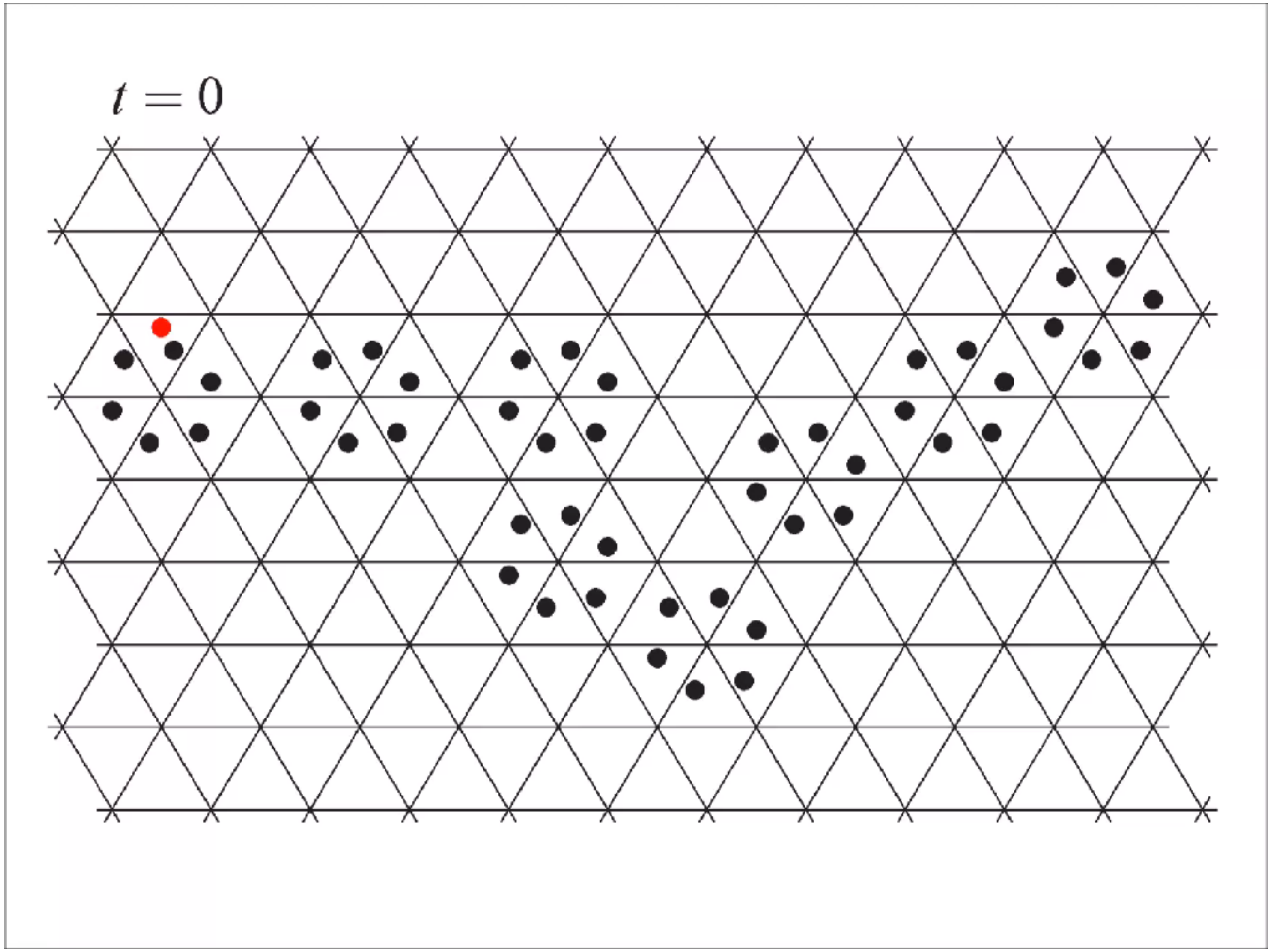
Signal and transmission wire in T_{RL}

- *Signal* is represented by a single particle.
- *Transmission wire* is an object made of stable *Blocks*, each of which consists of 6 particles.



A signal can travel along a wire. The number t $(1 \leq t \leq 68)$ shows the position of the signal at time t.

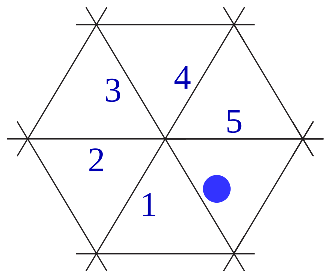
Signal and transmission wire in T_{RL} (Movie)

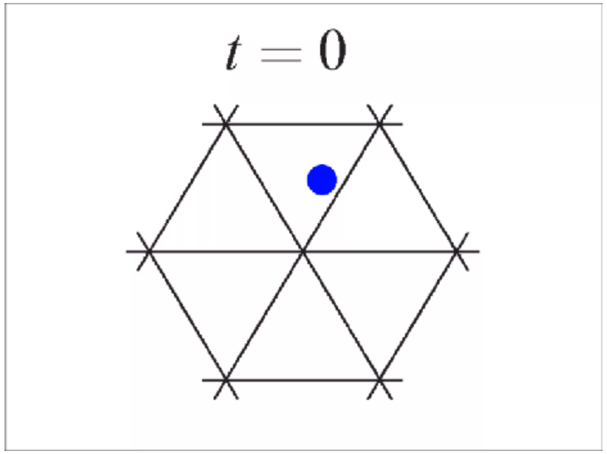


A signal is represented by a colored particle, but it is the same state as a black particle.

Signal control module in T_{RL}

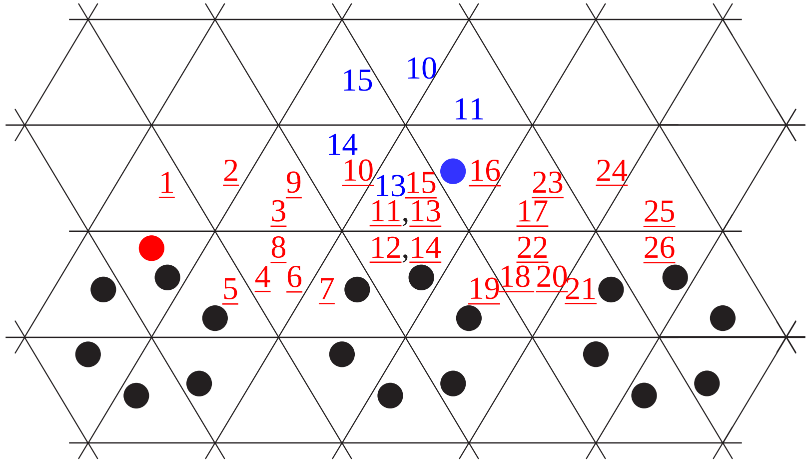
- *Signal control module* can alter the trajectory of a signal. It consists of a rotating particle.

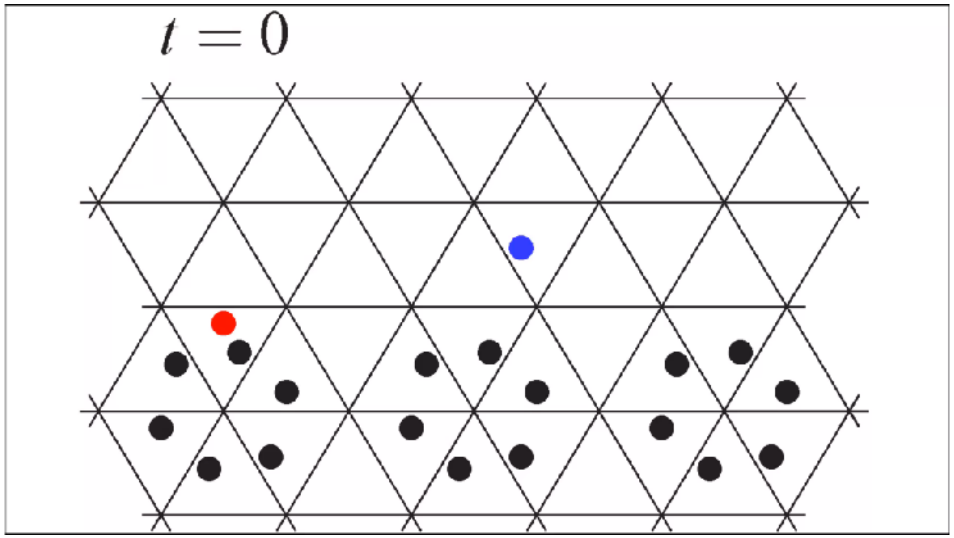




Delay module in T_{RL}

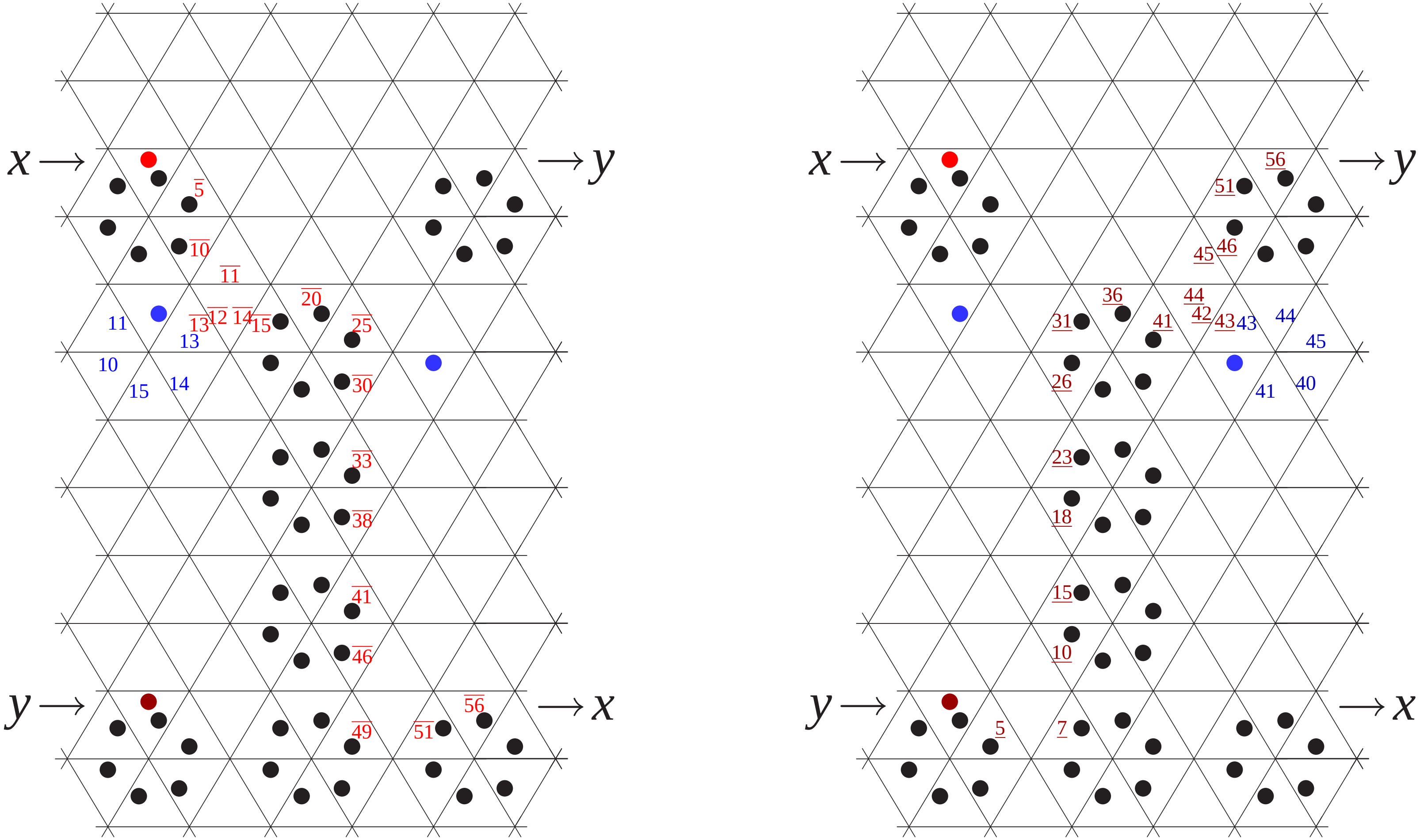
- *Delay module* of 2 steps is realized by attaching a signal control module to a transmission wire.





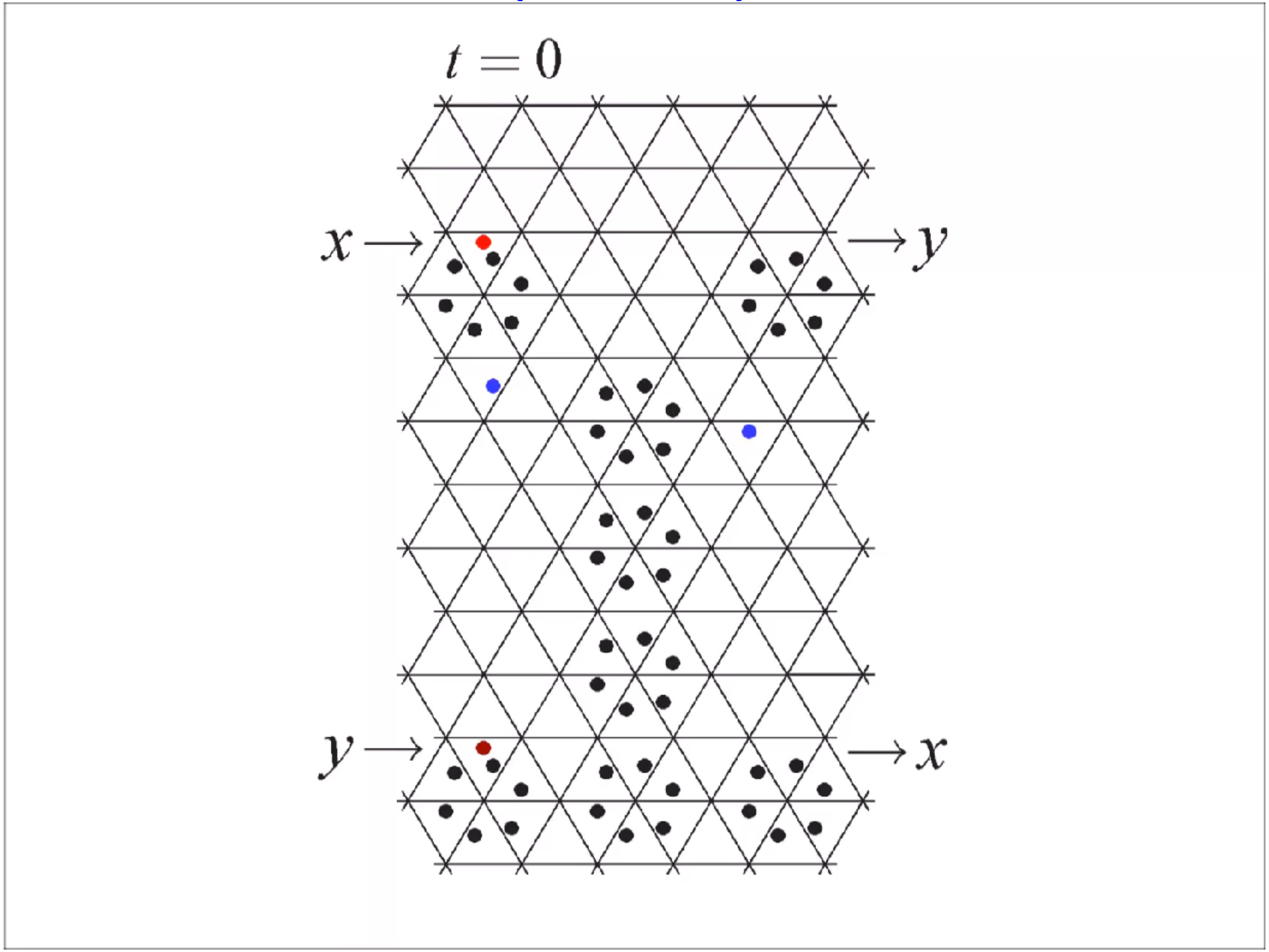
Signal crossing module in T_{RL}

- *Signal crossing module* is composed of 2 signal control modules, and transmission wires.



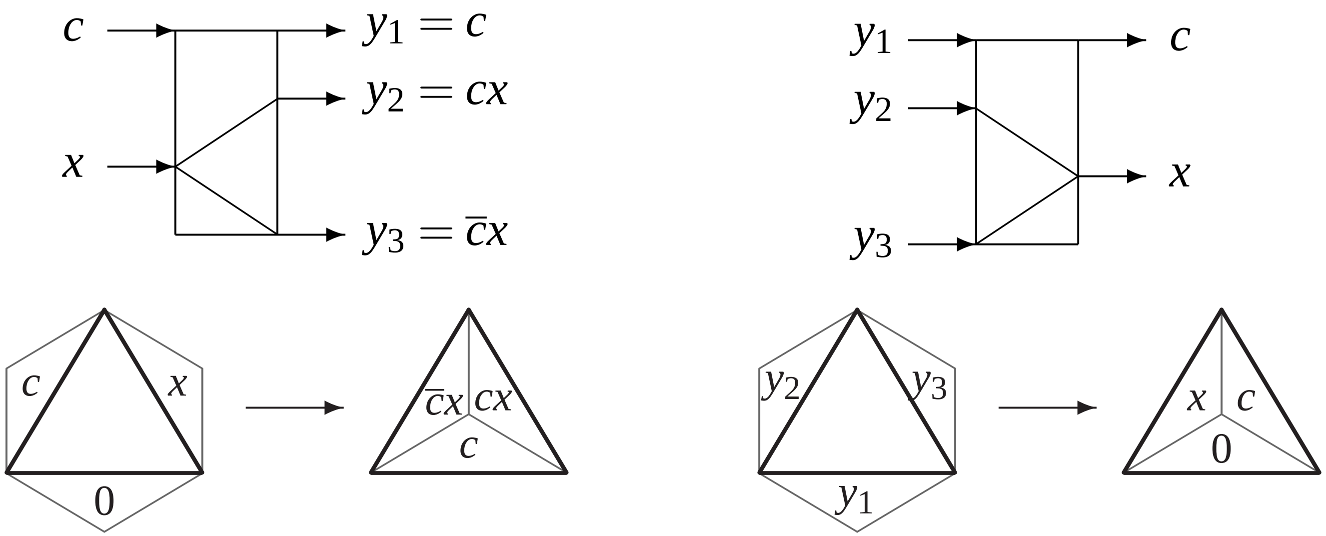
The trajectory of the input x (left), and that of y (right).

Signal crossing module in T_{RL} (Movie)



Implementing switch gate and inverse switch gate in T_{RL}

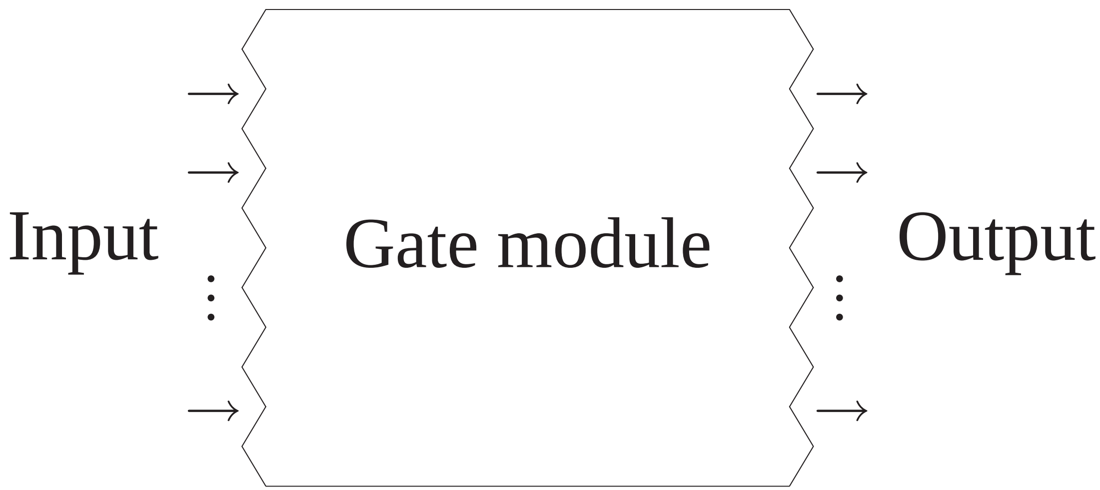
- A *single cell* of T_{RL} works as a switch gate and an inverse switch gate.



- But, it is not convenient to use it as a module.
- We implement these gates as *gate modules in the standard form*.

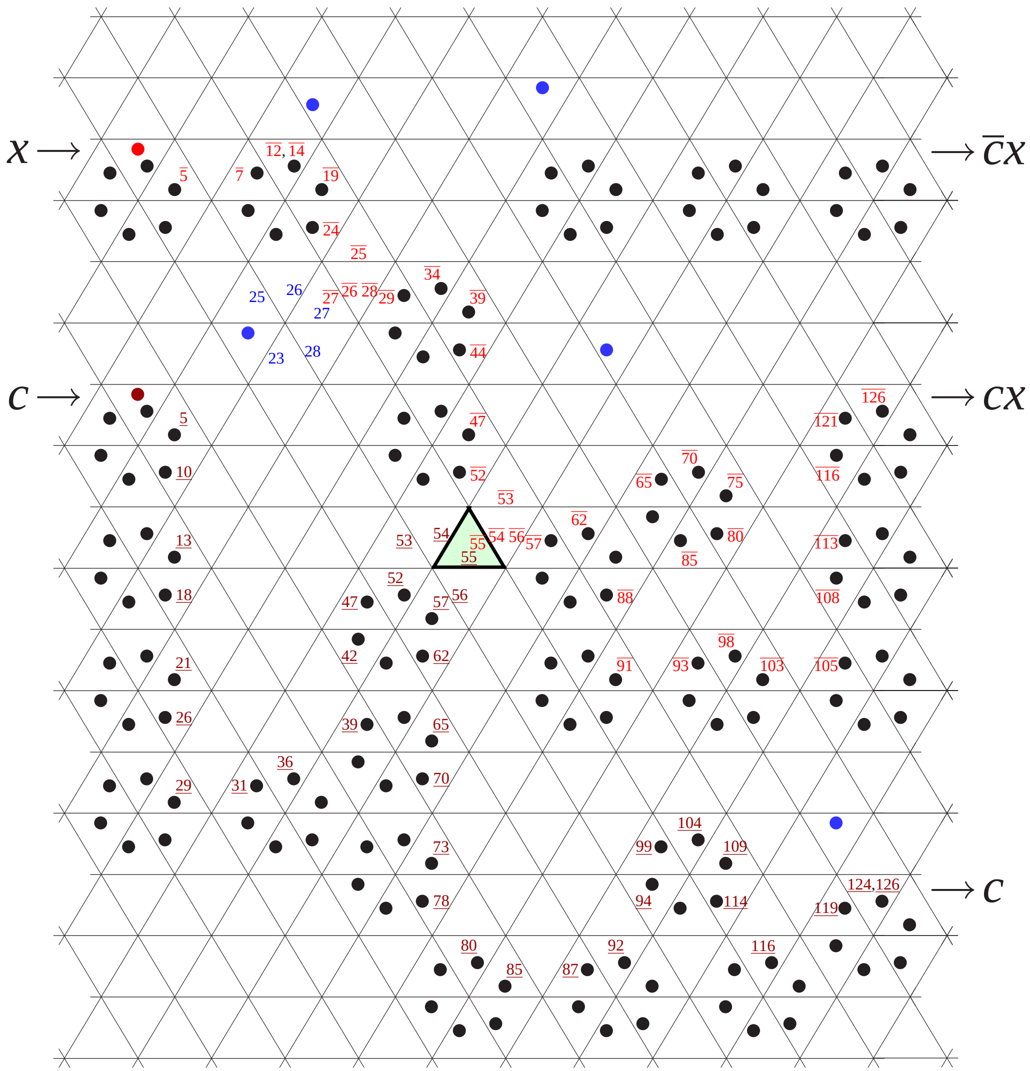
Gate module

- *Gate module* in the standard form is a pattern embedded in a rectangular-like region in the cellular space that satisfies the following.
  - It realizes a reversible logic gate.
  - Input ports are at the left end.
  - Output ports are at the right end.
  - Delay between input and output is constant.

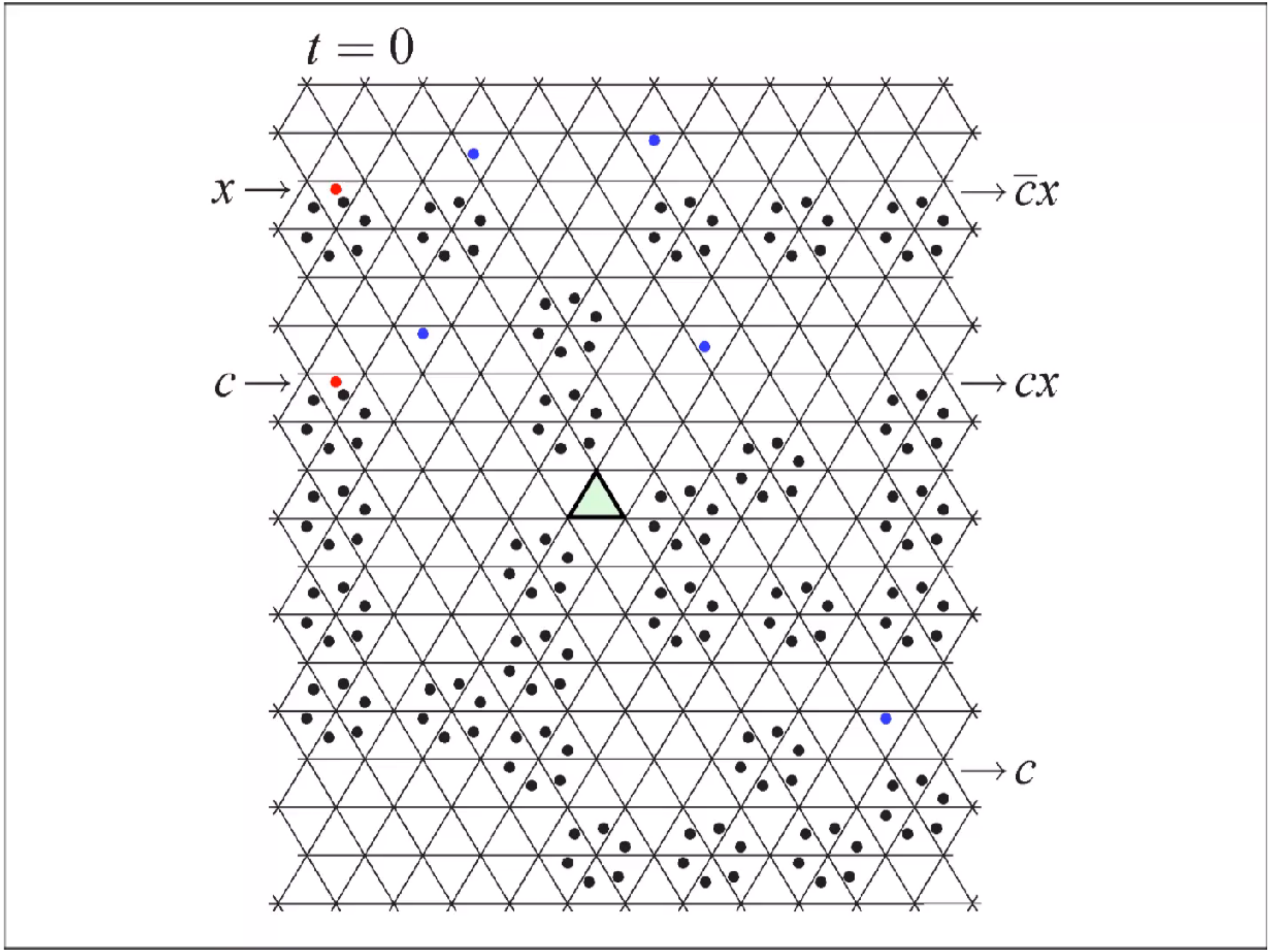


Switch gate module in T_{RL} $(c = 1,\ x = 1)$

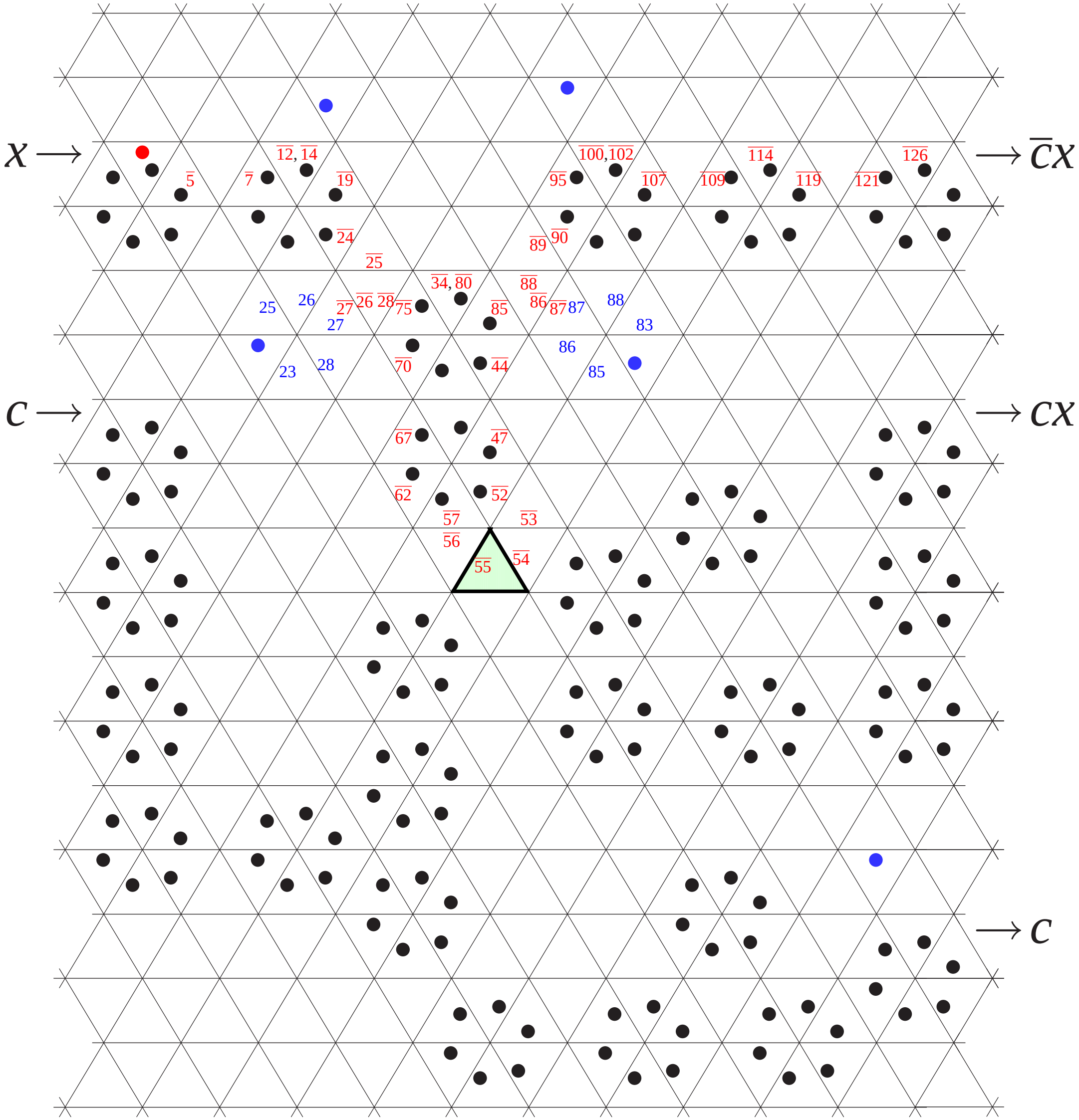
- It consists of a signal crossing module, 3 delay modules, and transmission wires.



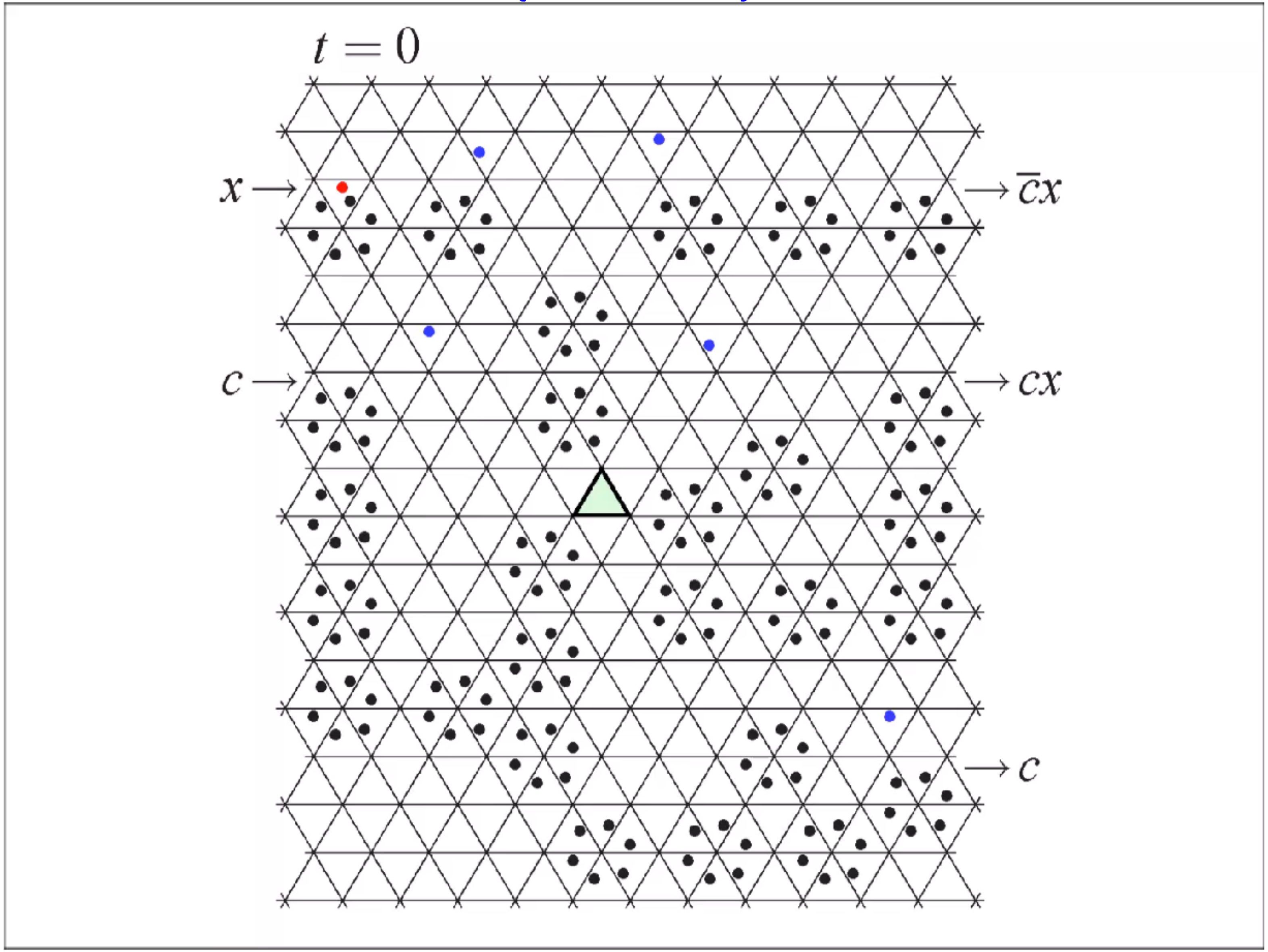
Switch gate module in T_{RL} ($c = 1,\ x = 1$) (Movie)



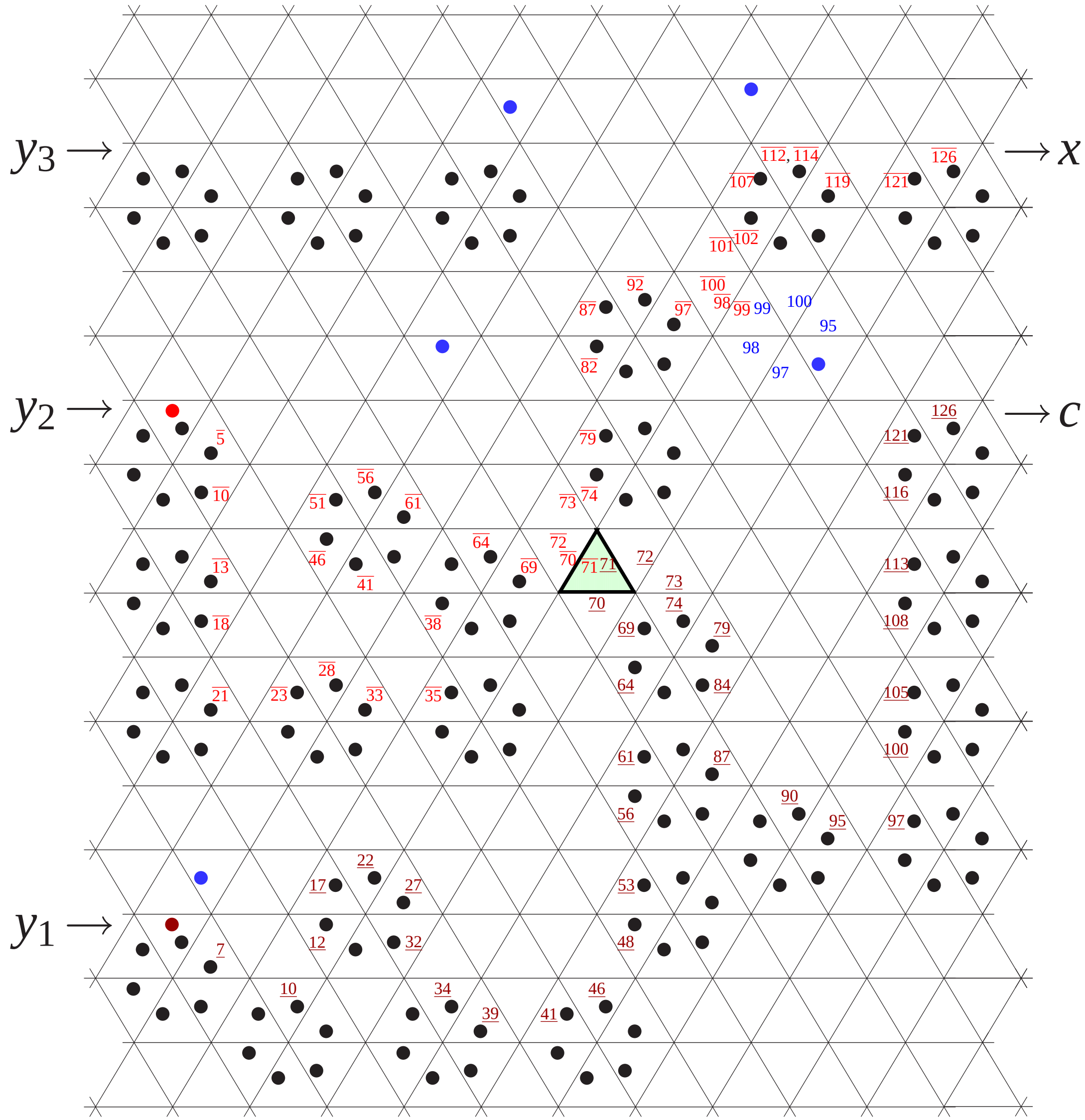
Switch gate module in T_{RL} $(c = 0,\ x = 1)$



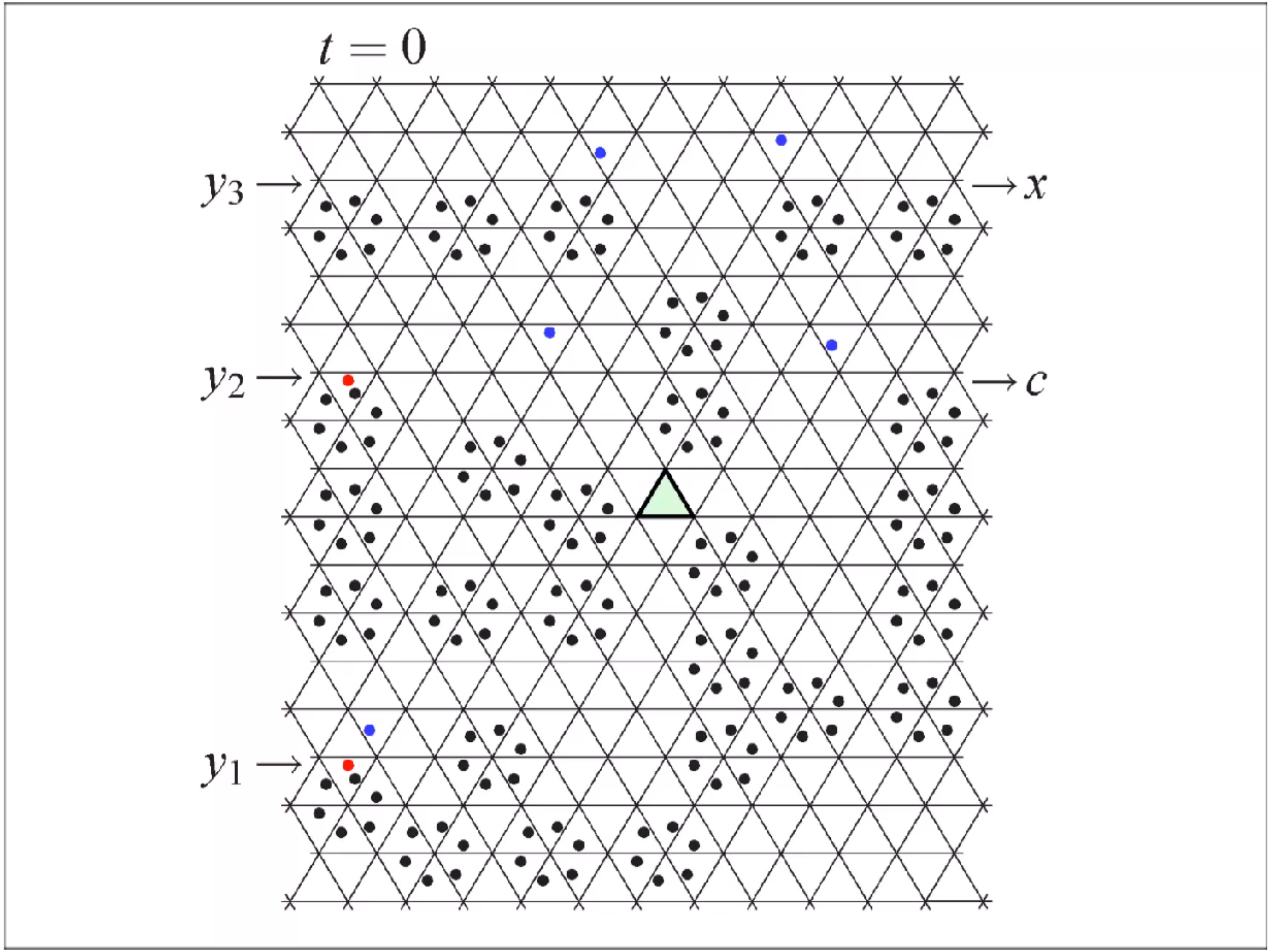
Switch gate module in T_{RL} ($c = 0,\ x = 1$) (Movie)



Inverse switch gate module in T_{RL}
$(y_1 = 1,\ y_2 = 1,\ y_3 = 0)$

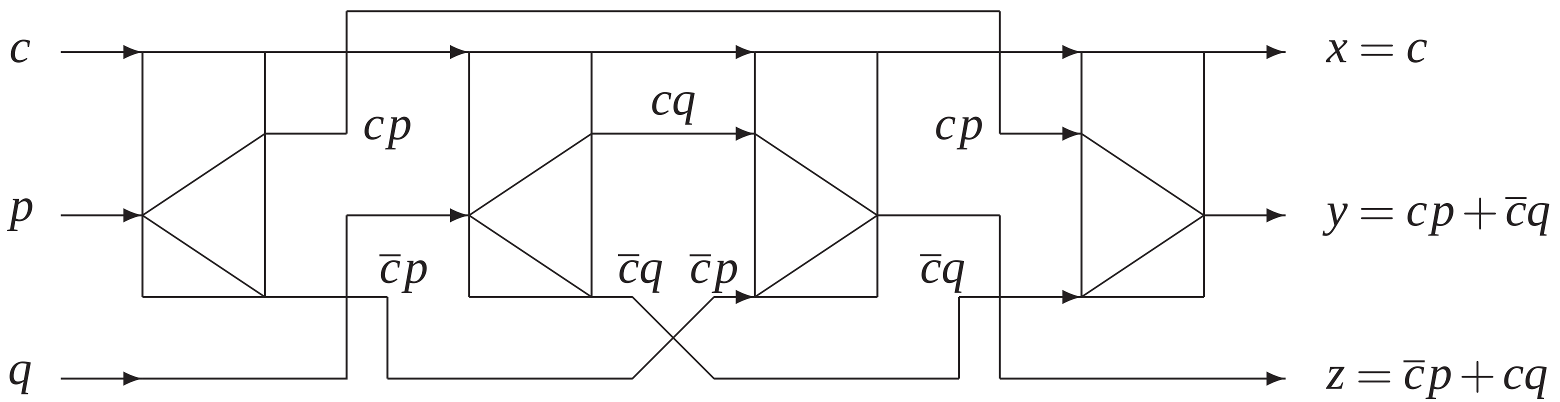


Inverse switch gate module in T_{RL} $(y_1 = 1,\ y_2 = 1,\ y_3 = 0)$ (Movie)



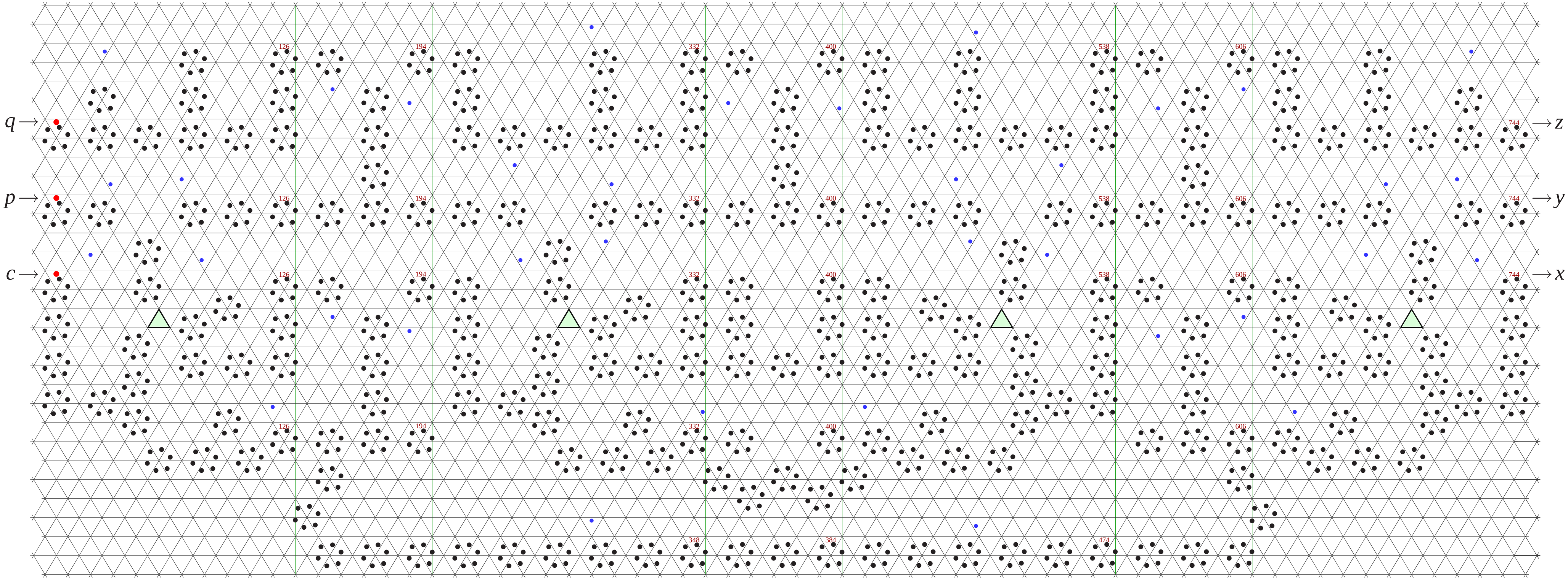
Composing Fredkin gate out of switch gate and inverse switch gate

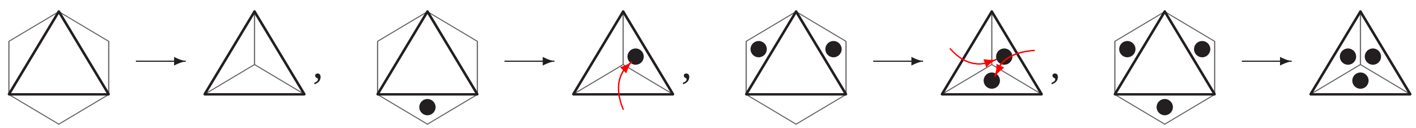
[Fredkin, Toffoli, 1982]



Note: Fredkin gate configuration at the next page realizes the circuit that is *upside down* of the above.

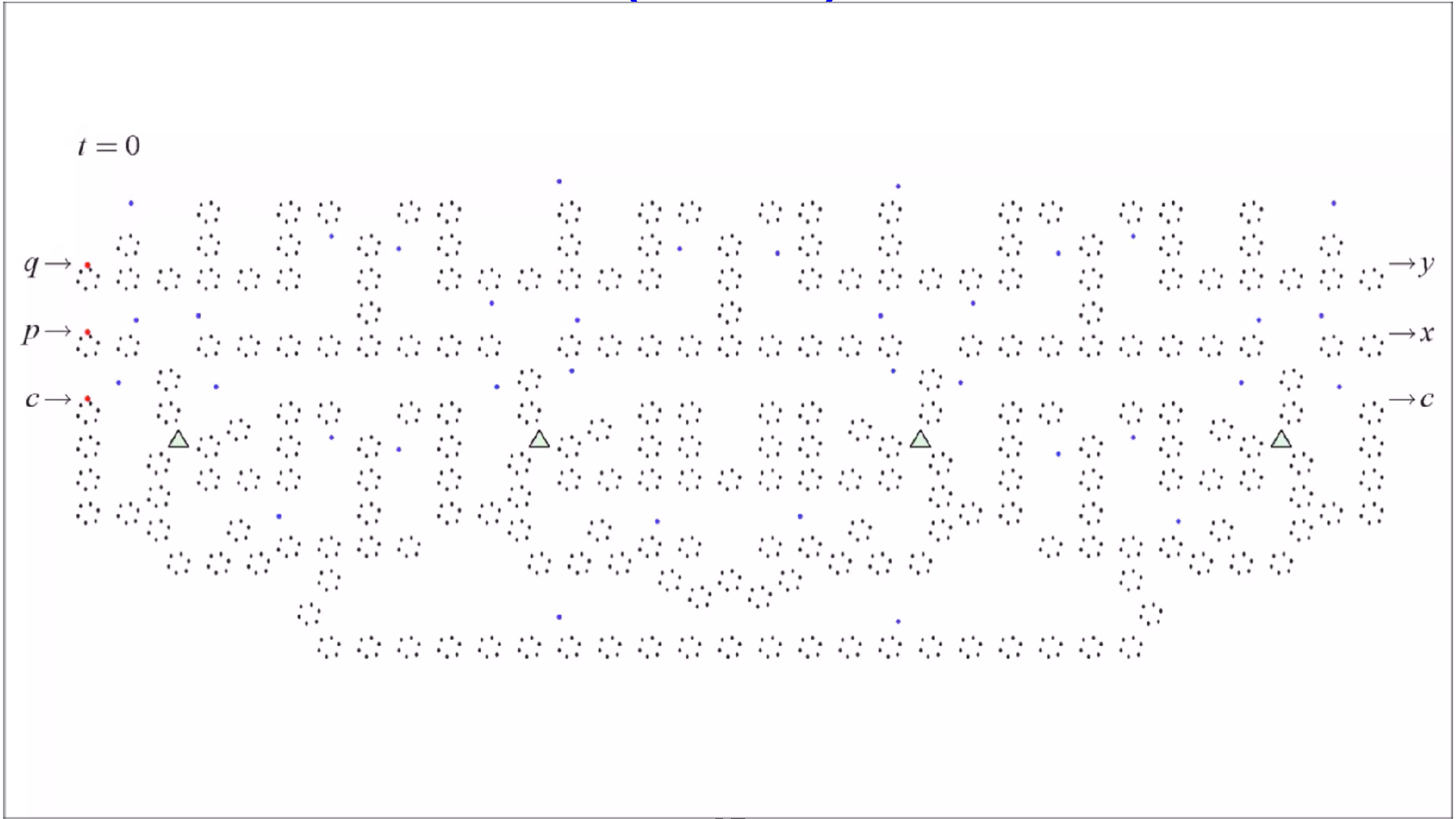
Fredkin gate module in T_{RL}



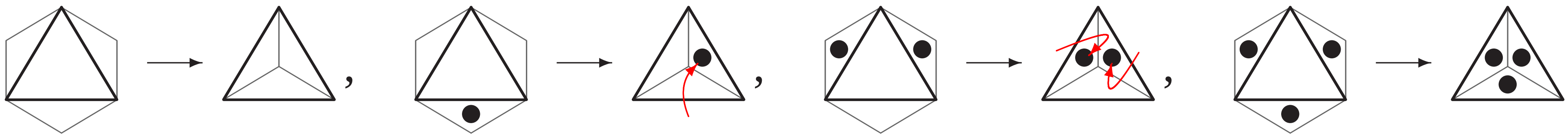


- Configuration size: 30×131
- Delay between input and output: 744 steps

Fredkin gate module in T_{RL} ($c=1,\ p=1,\ q=1$) (Movie)



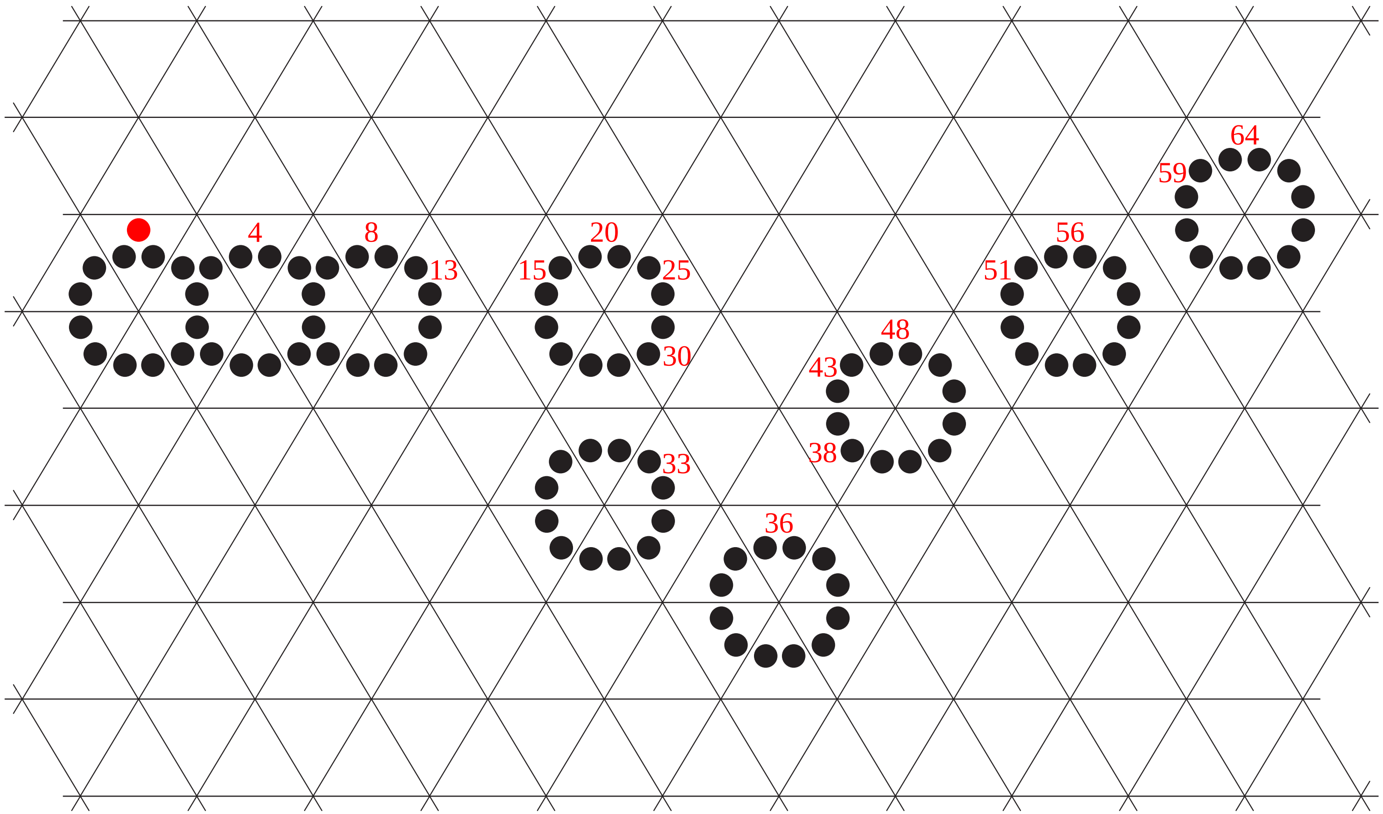
2.2. Universality of RETPCA T_{RU}



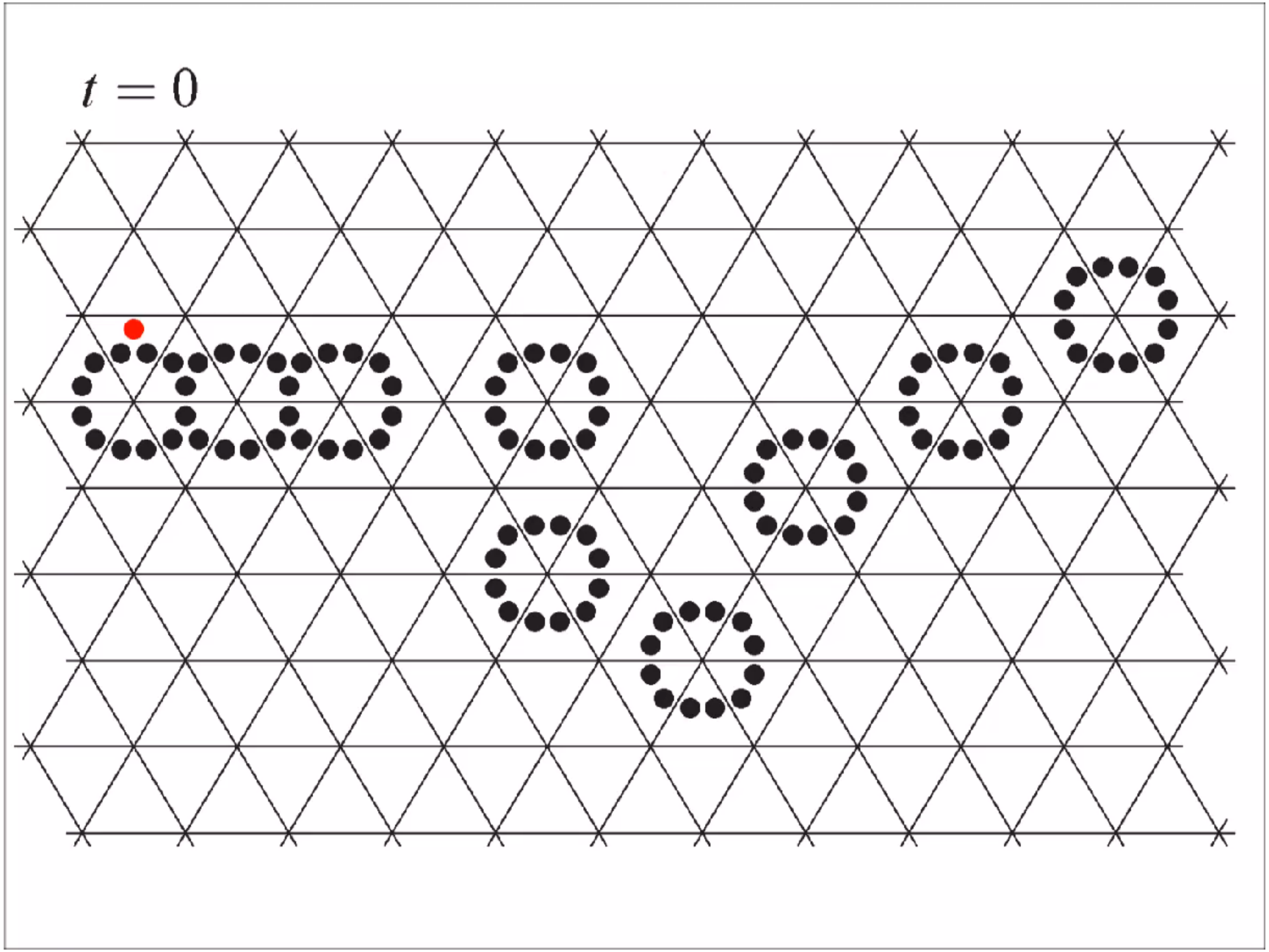
Note: Though its universality has already been shown in [Imai, Morita, 2000], a configuration that simulates Fredkin gate is reconstructed here to reduce its size.

Signal and transmission wire in T_{RU}

- *Signal* is represented by a single particle.
- *Transmission wire* is composed of stable *Blocks*, each of which consists of 12 particles.

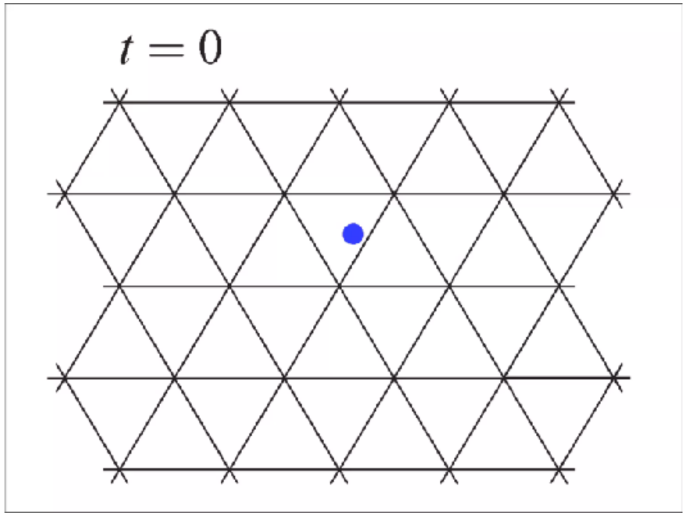


Signal and transmission wire in T_{RU} (Movie)

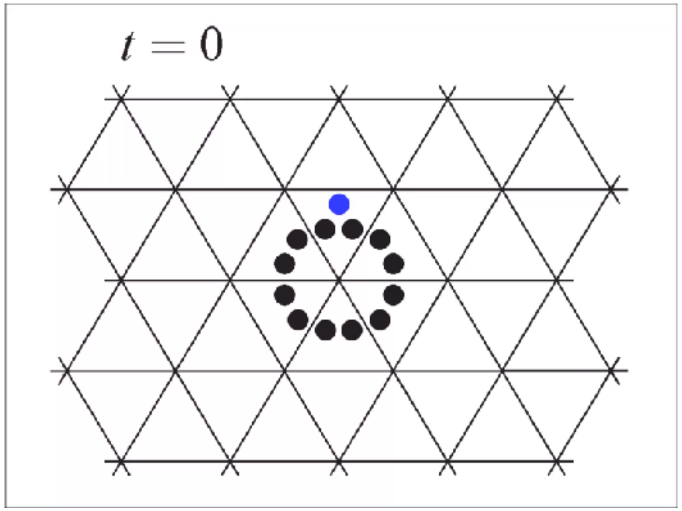


Signal control modules in T_{RU}

- There are two types of signal control modules that can change the trajectory of a signal.
- *Signal control module I* consists only of one particle, whose period is 6.
- *Signal control module II* consists of a particle and a Block, whose period is 30.



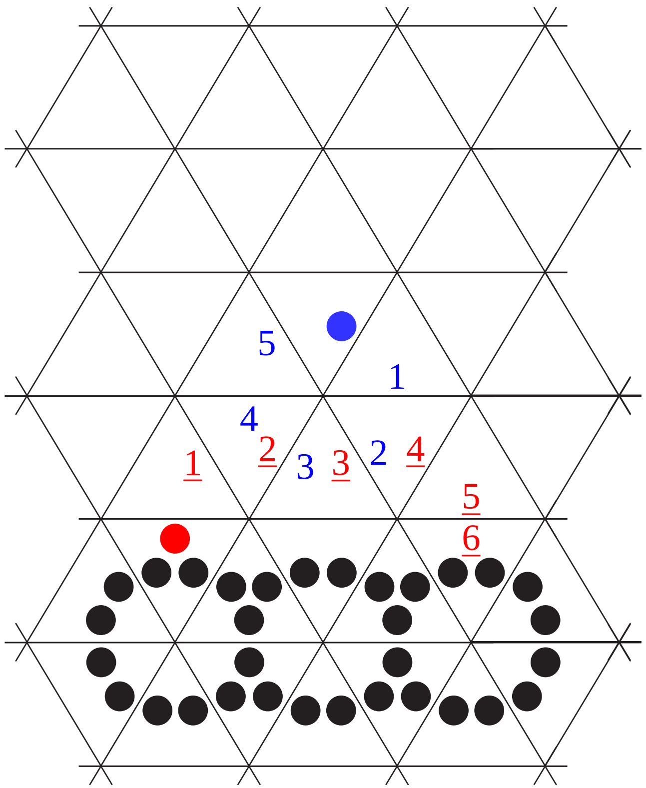
Signal control module I

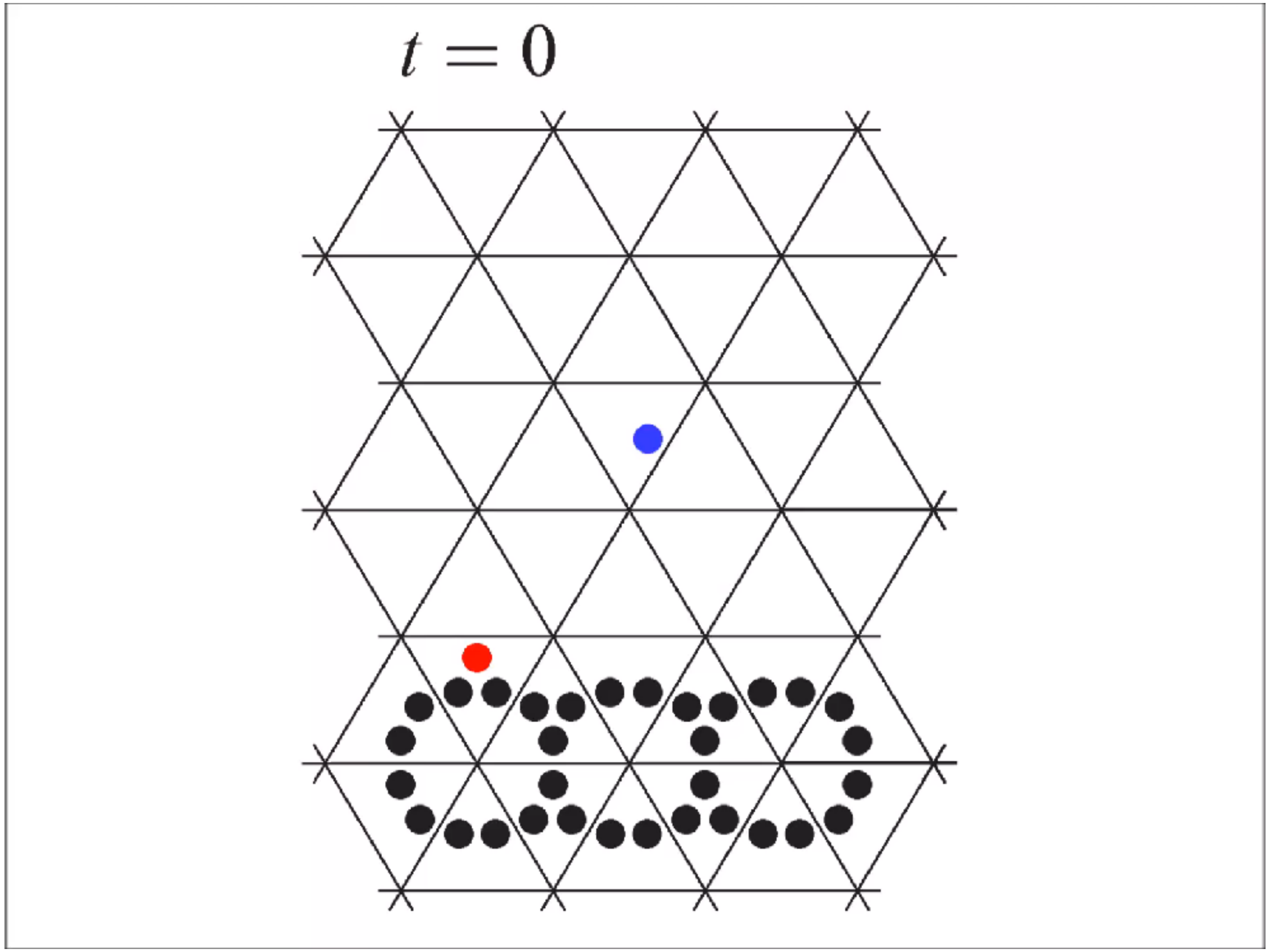


Signal control module II

Delay module of -2 steps in T_{RU}

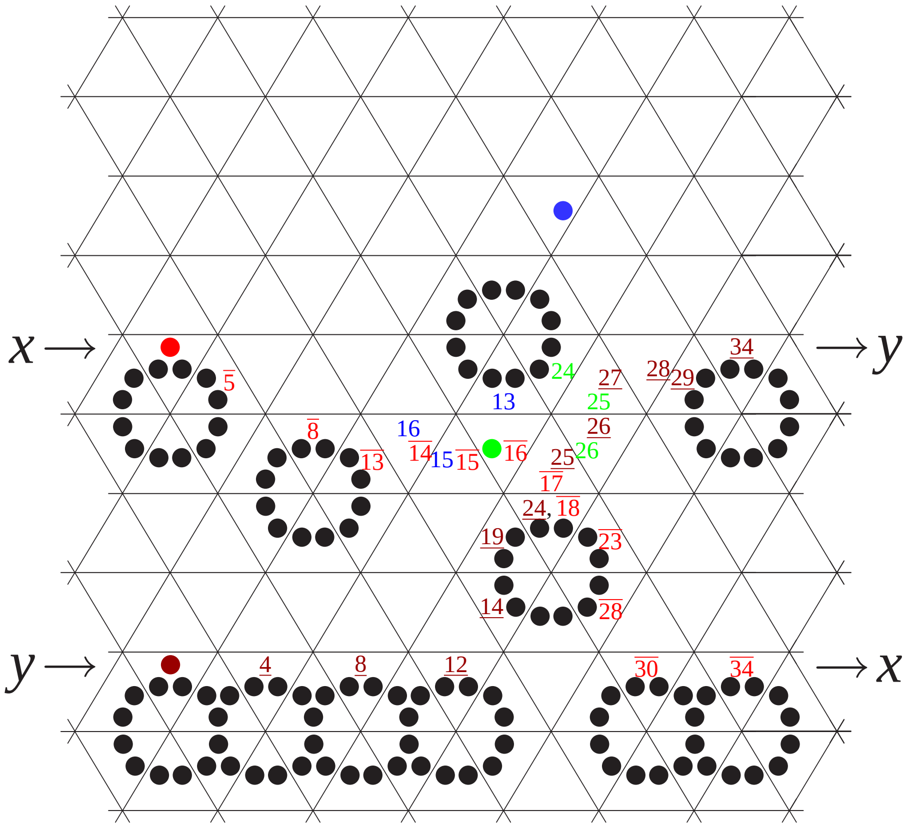
- *Delay module* of -2 steps is implemented by attaching a signal control module I to a transmission wire.



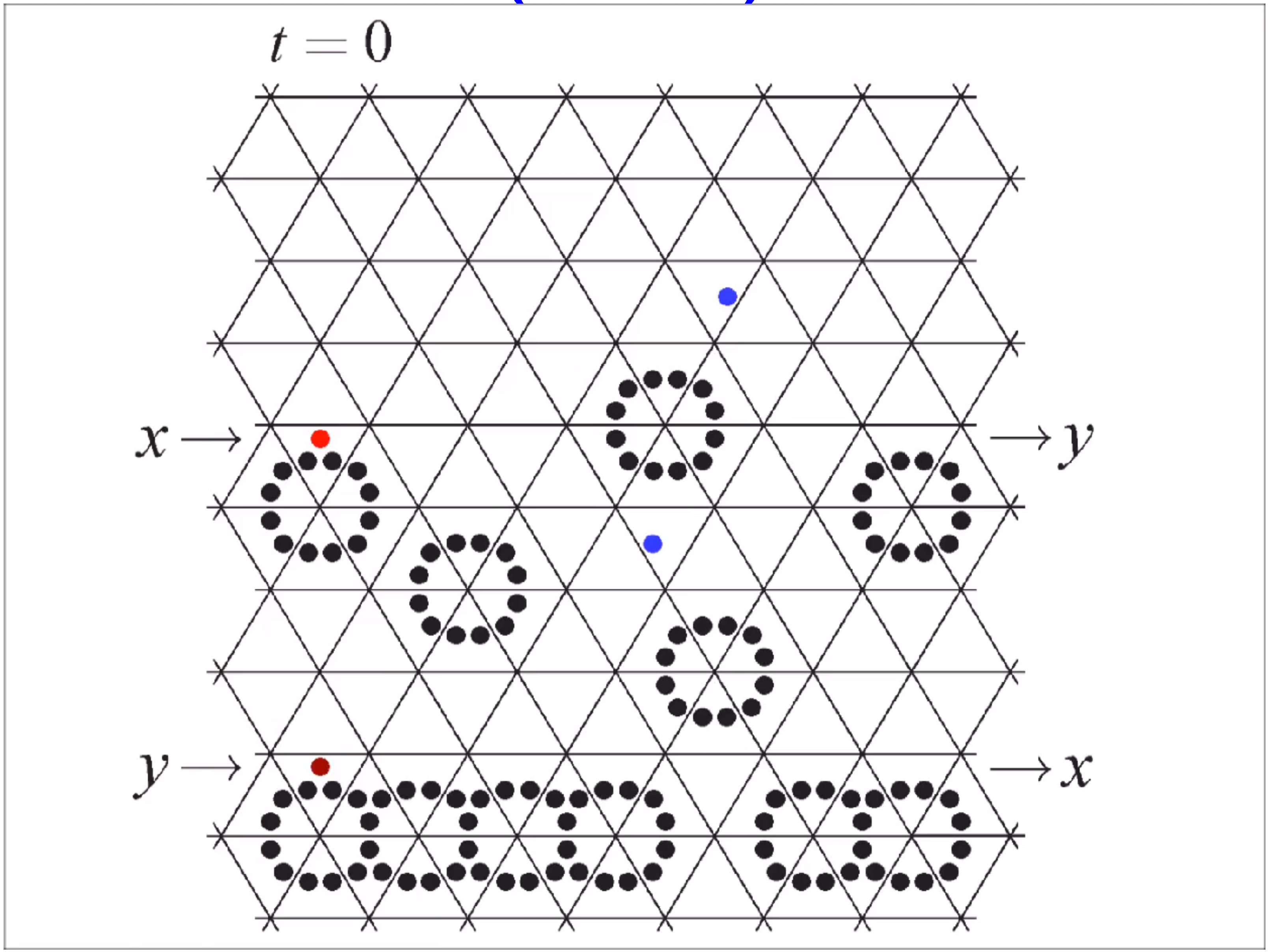


Signal crossing module in T_{RU}

- *Signal crossing module* is composed of a signal control module II with two control signals, and transmission wires.

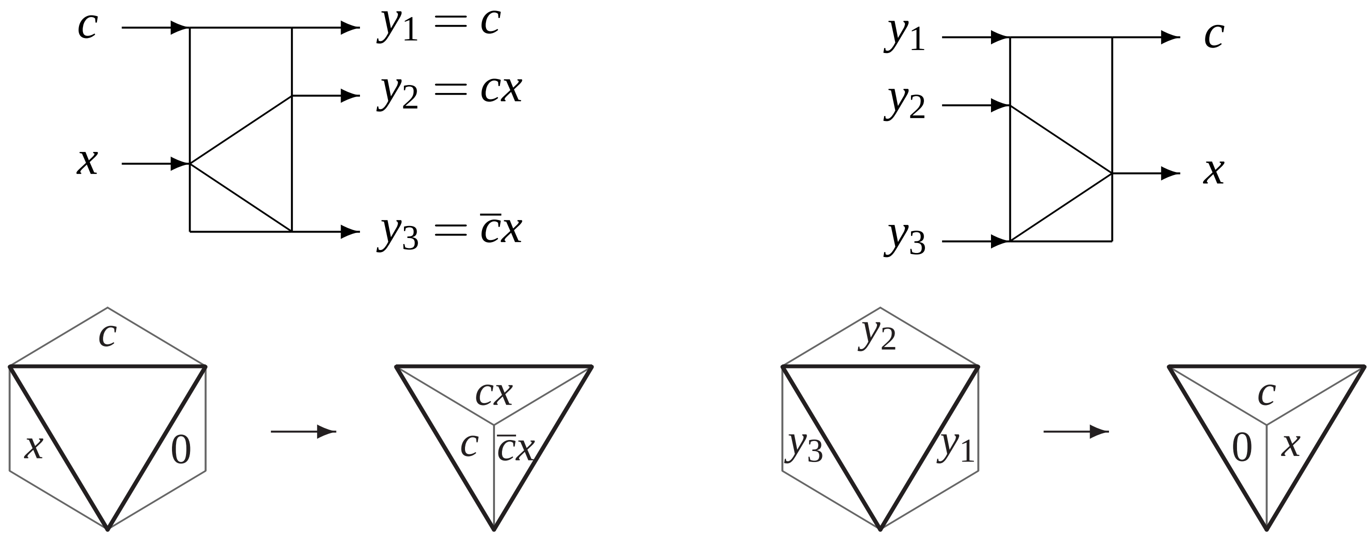


Signal crossing module in T_{RU} (Movie)



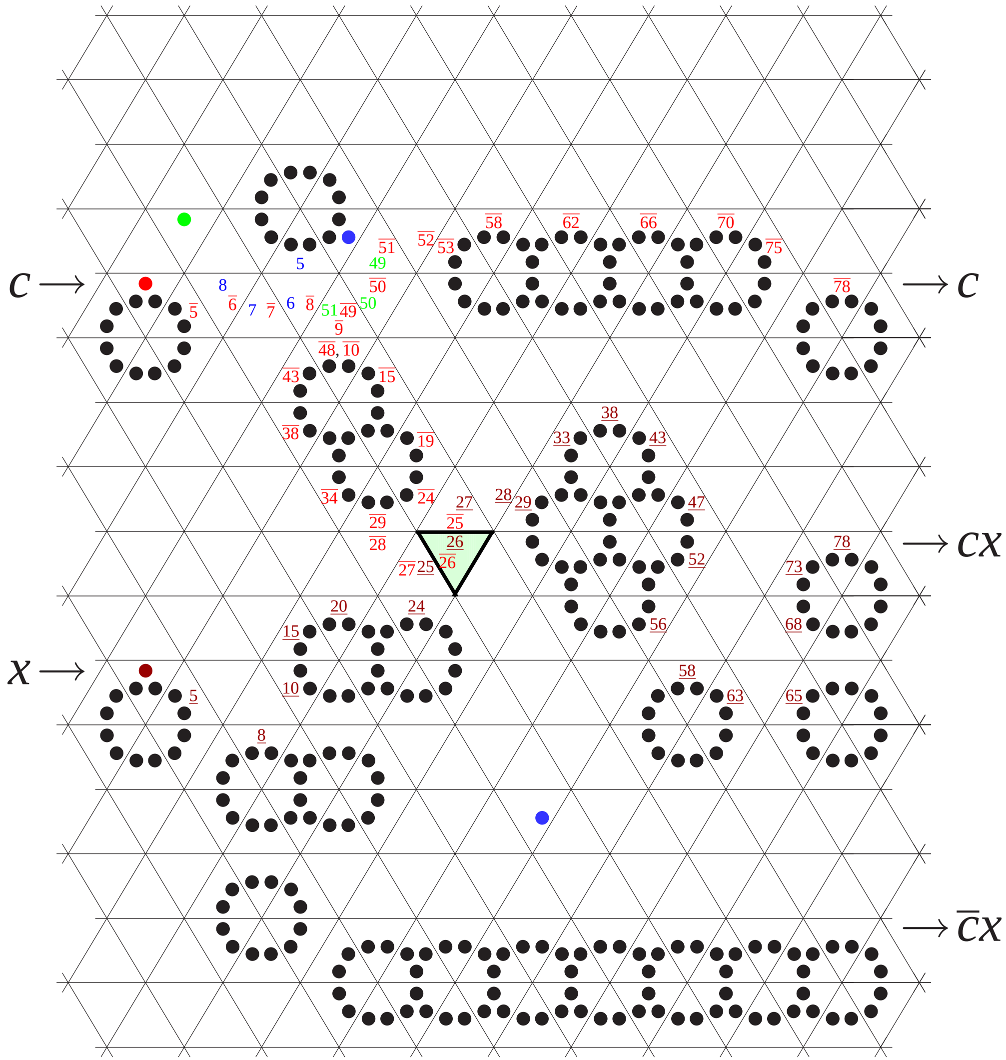
Implementing switch gate and inverse switch gate in T_{RU}

- A *single cell* of T_{RU} works as a switch gate and an inverse switch gate.

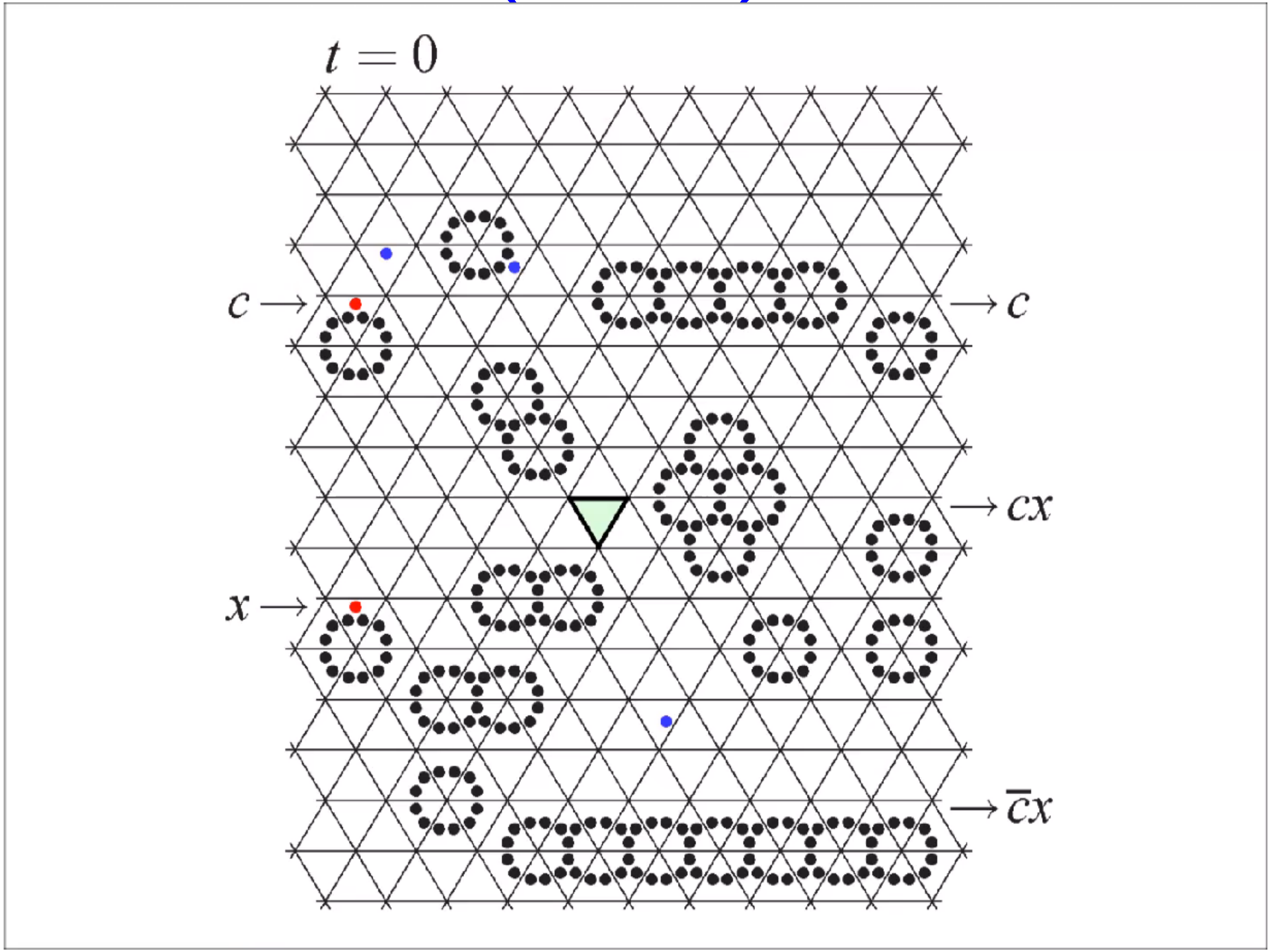


Switch gate module in T_{RU} $(c = 1,\ x = 1)$

- It consists of a signal crossing module, a delay module, and transmission wires.

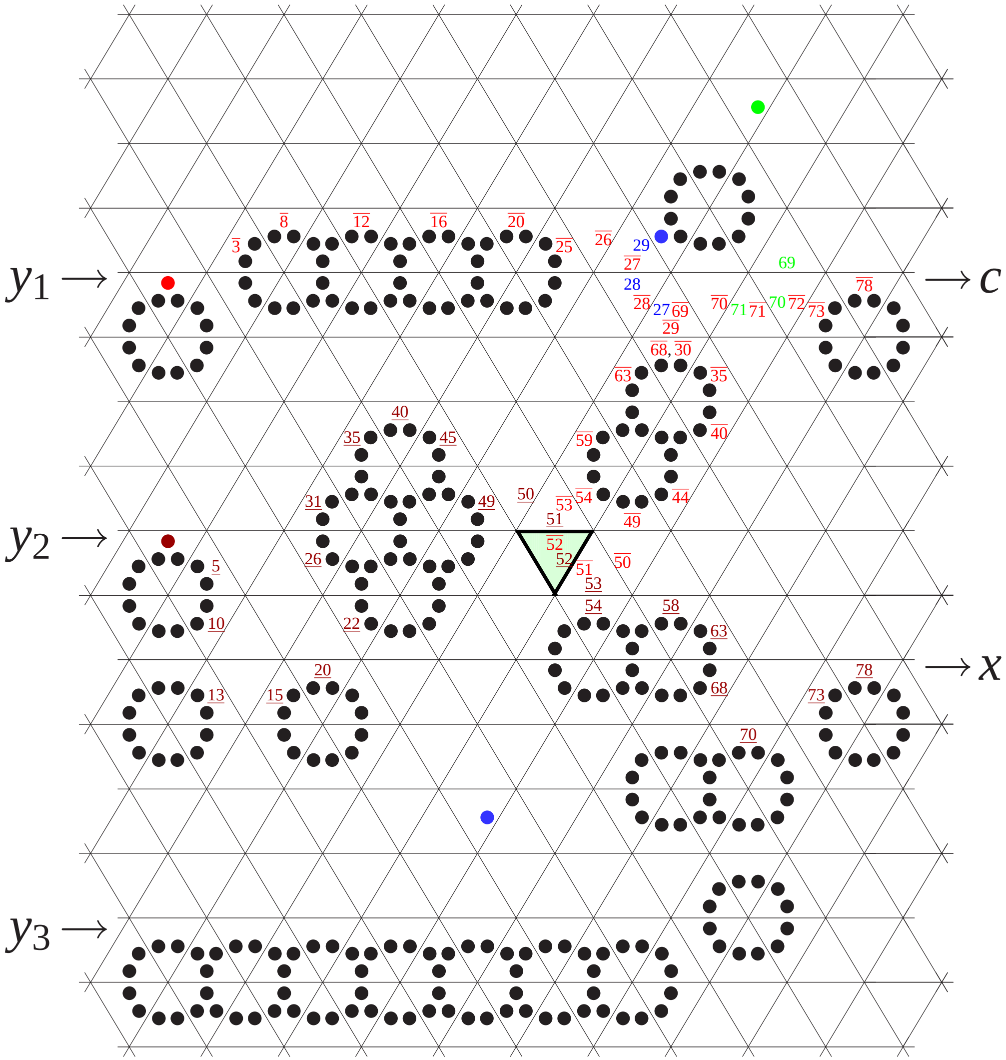


Switch gate module in T_{RU} $(c = 1,\ x = 1)$ (Movie)

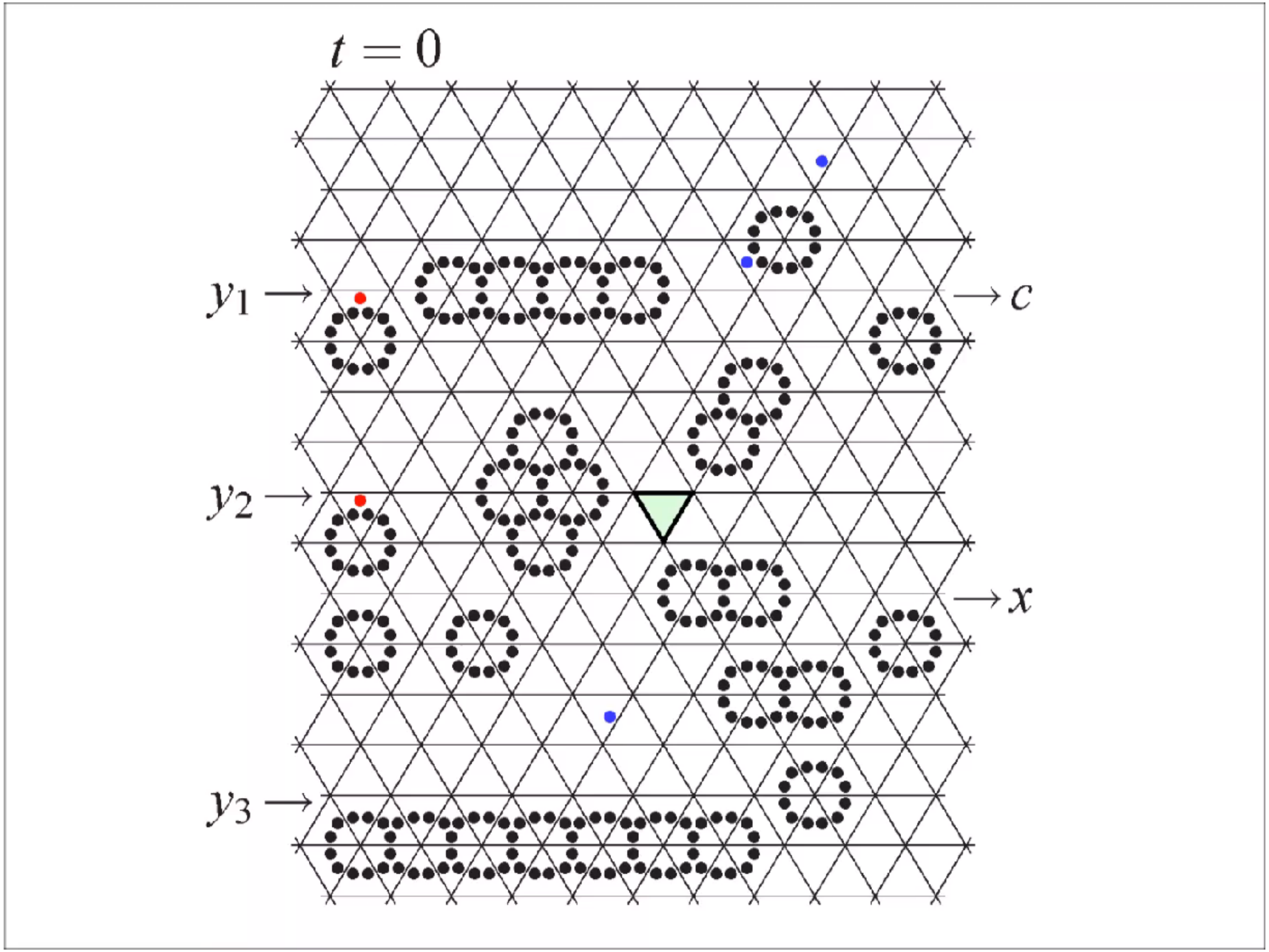


Inverse switch gate module in T_{RU}

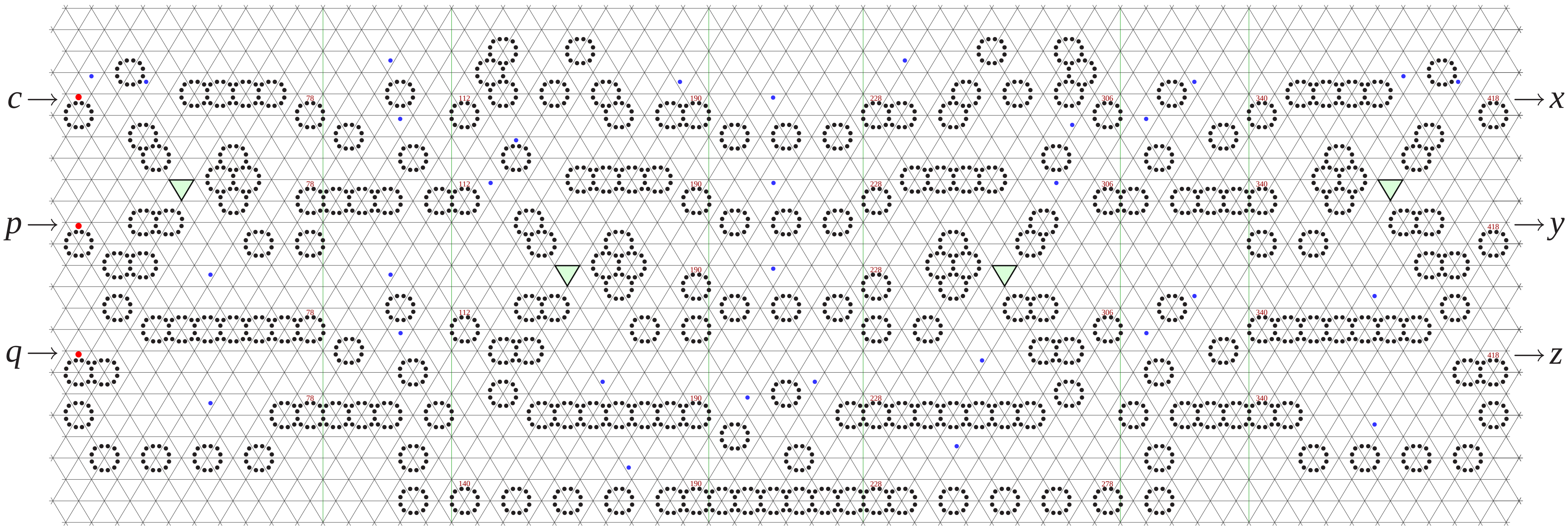
$(y_1 = 1,\ y_2 = 1,\ y_3 = 0)$

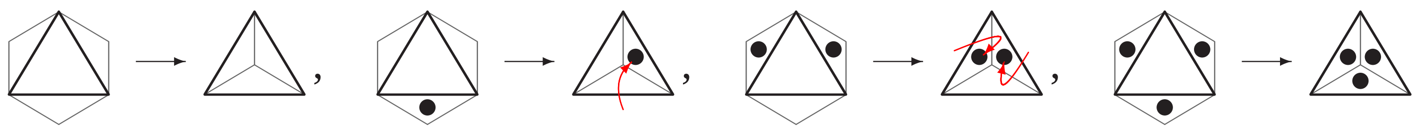


Inverse switch gate module in T_{RU} $(y_1 = 1,\ y_2 = 1,\ y_3 = 0)$ (Movie)



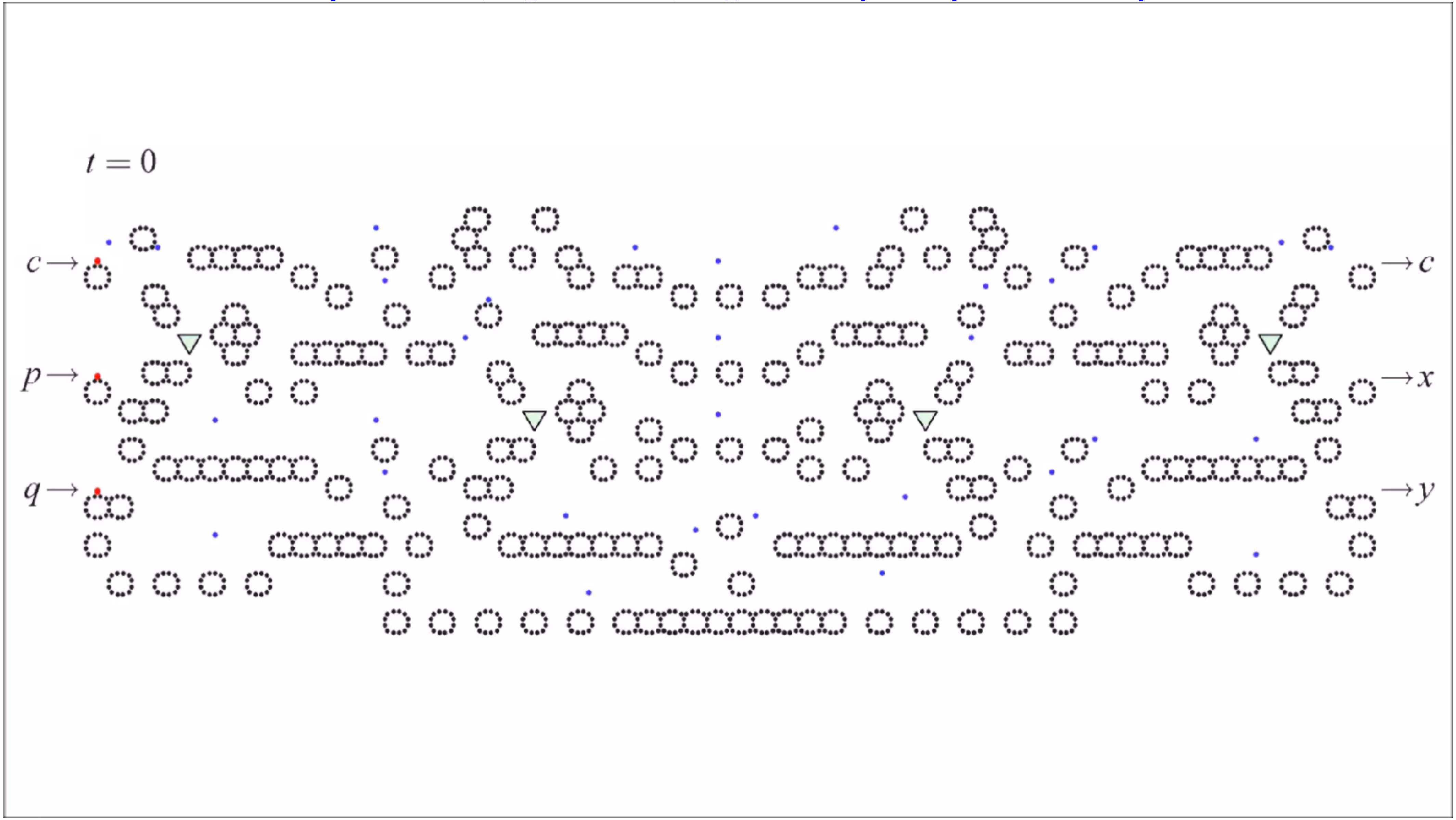
Fredkin gate module in T_{RU}



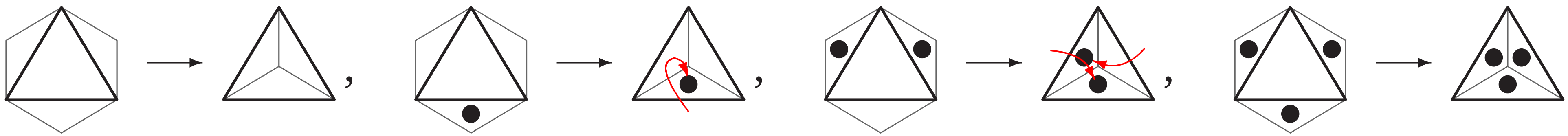


- Configuration size: 24×113
- Delay between input and output: 418 steps

Fredkin gate module in T_{RU} ($c = 1,\ p = 1,\ q = 1$) (Movie)



2.3. Universality of RETPCA T_{UR}



Note: Since $T_{\mathsf{UR}} \underset{\mathsf{conj}}{\longleftrightarrow} T_{\mathsf{RU}}$, universality of T_{UR} follows from that of T_{RU}. But, here we show a *finite configuration* of T_{UR} that simulates Fredkin gate.

How to convert a finite configuration of T_{RU} into a finite configuration of T_{UR}

- Outline of the conversion procedure is as follows.

(a) Assume a finite configuration of T_{RU} consisting of Blocks and signals is given.

(b) Mark the cells such that no signal visits.

(c) Take the complement of the configuration.

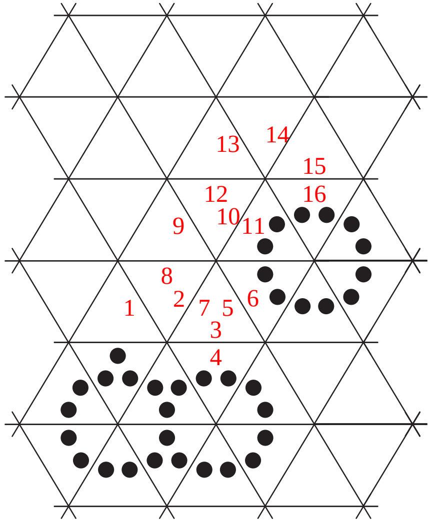
(d) Remove the marked part of the configuration.

(e) Mend the cut edges of cells by putting particles as *patches* to stabilize the configuration.

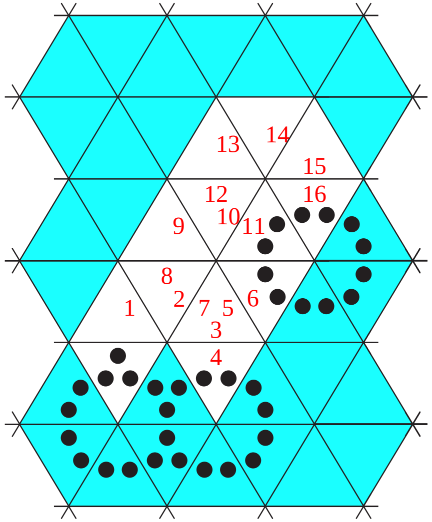
(f) By above, we have a finite configuration of T_{UR}.

Example of conversion from T_{RU} to T_{UR}

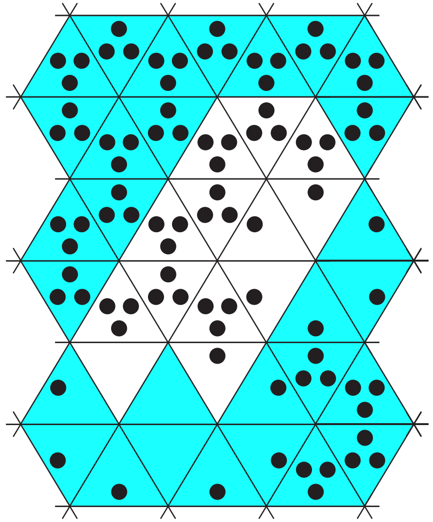
(a) Configuration of T_{RU}



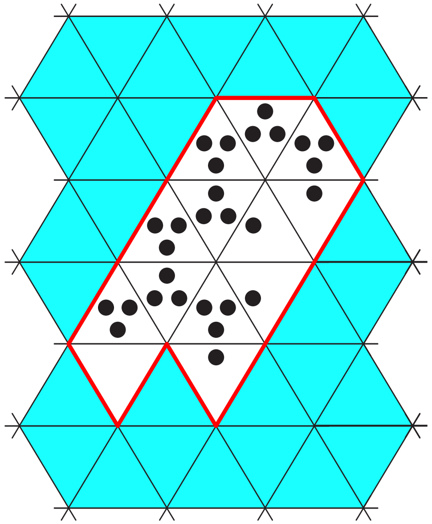
(b) Mark unvisited cells



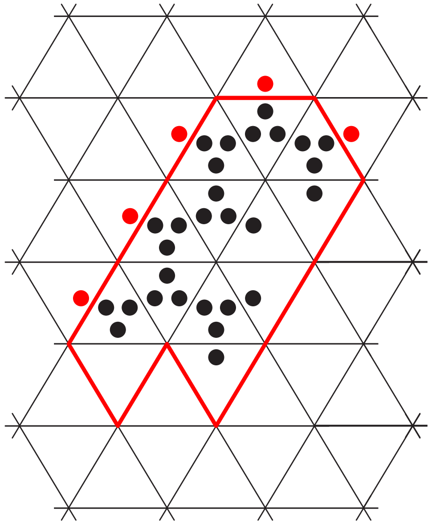
(c) Take the complement



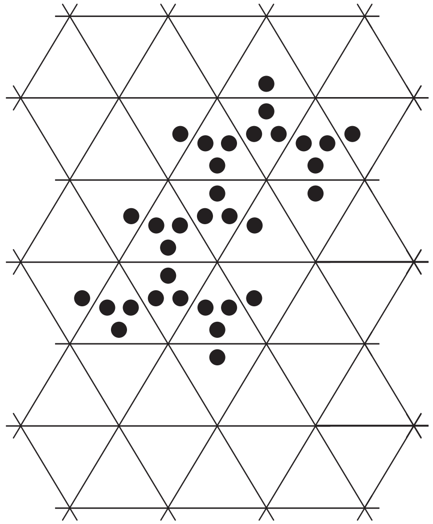
(d) Remove marked part



(e) Put patches (●)

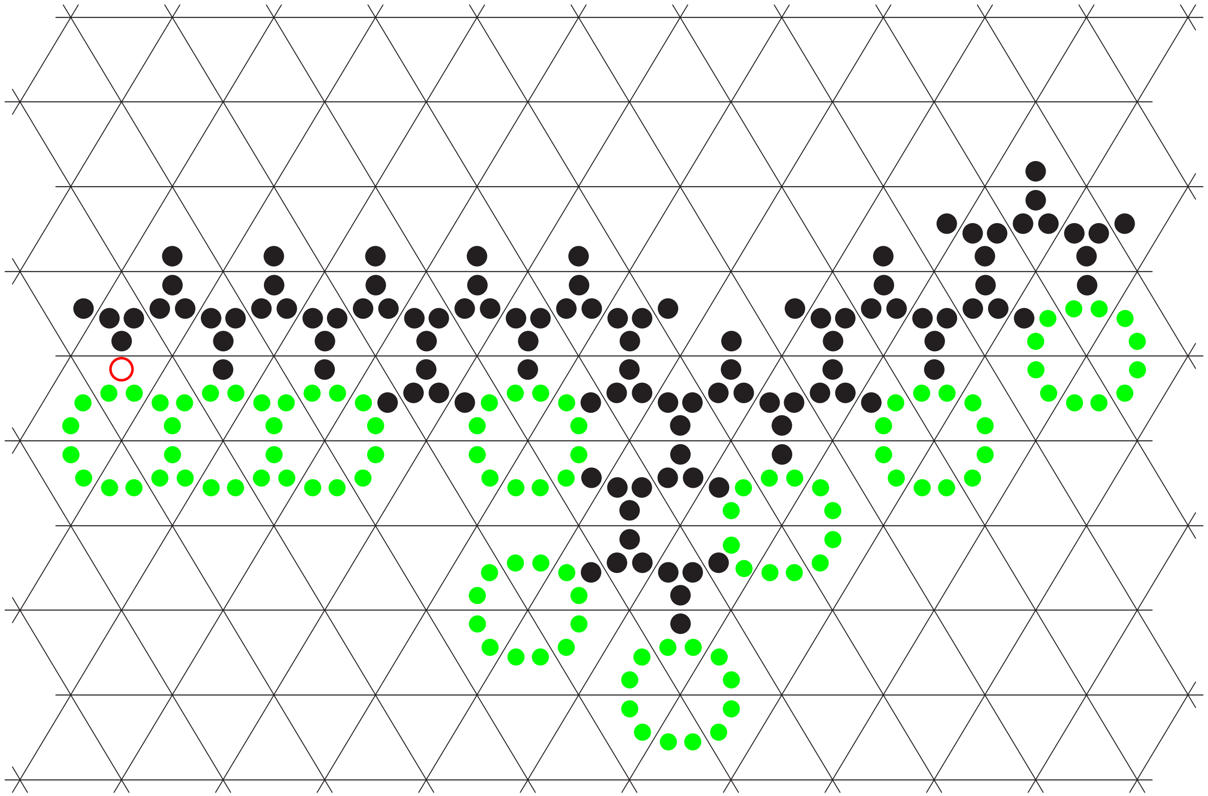


(f) Configuration of T_{UR}



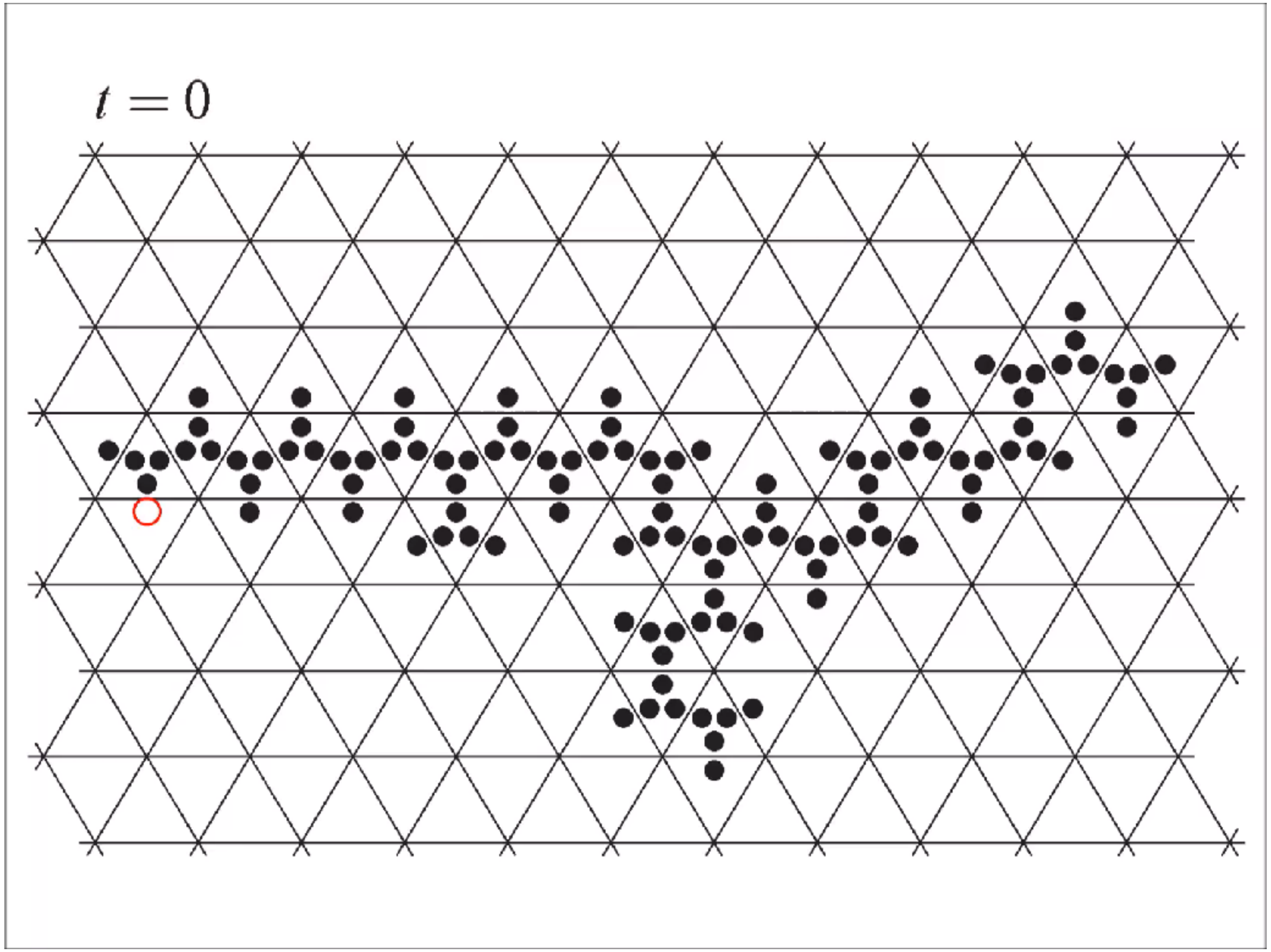
Signal and transmission wire in T_{UR}

- Example of a *transmission wire* is as below.
- *Signal* is represented by a "hole", a blank state.
  Here, it is indicated by ○.



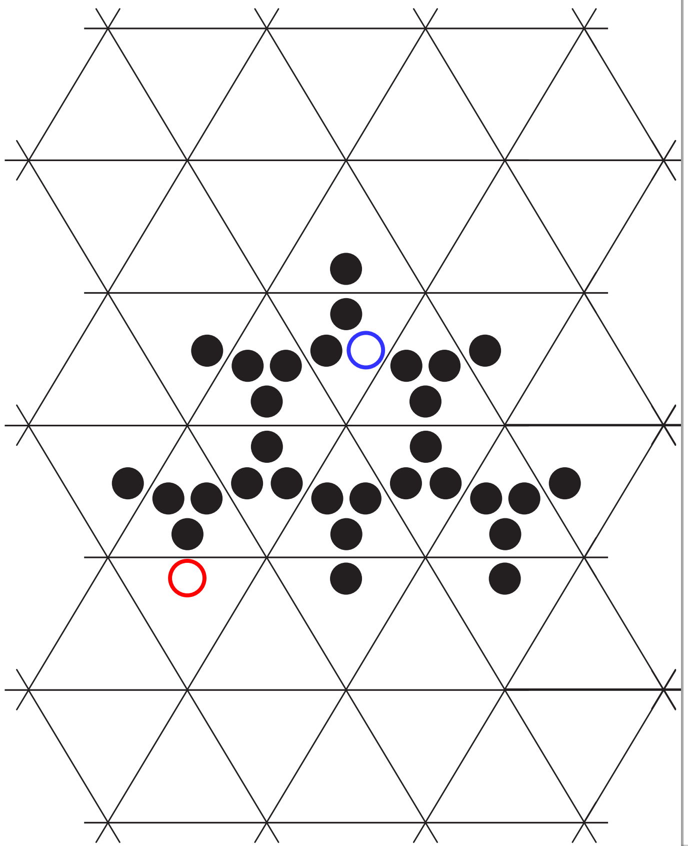
Note: The corresponding wire in T_{RU} is indicated by ●.

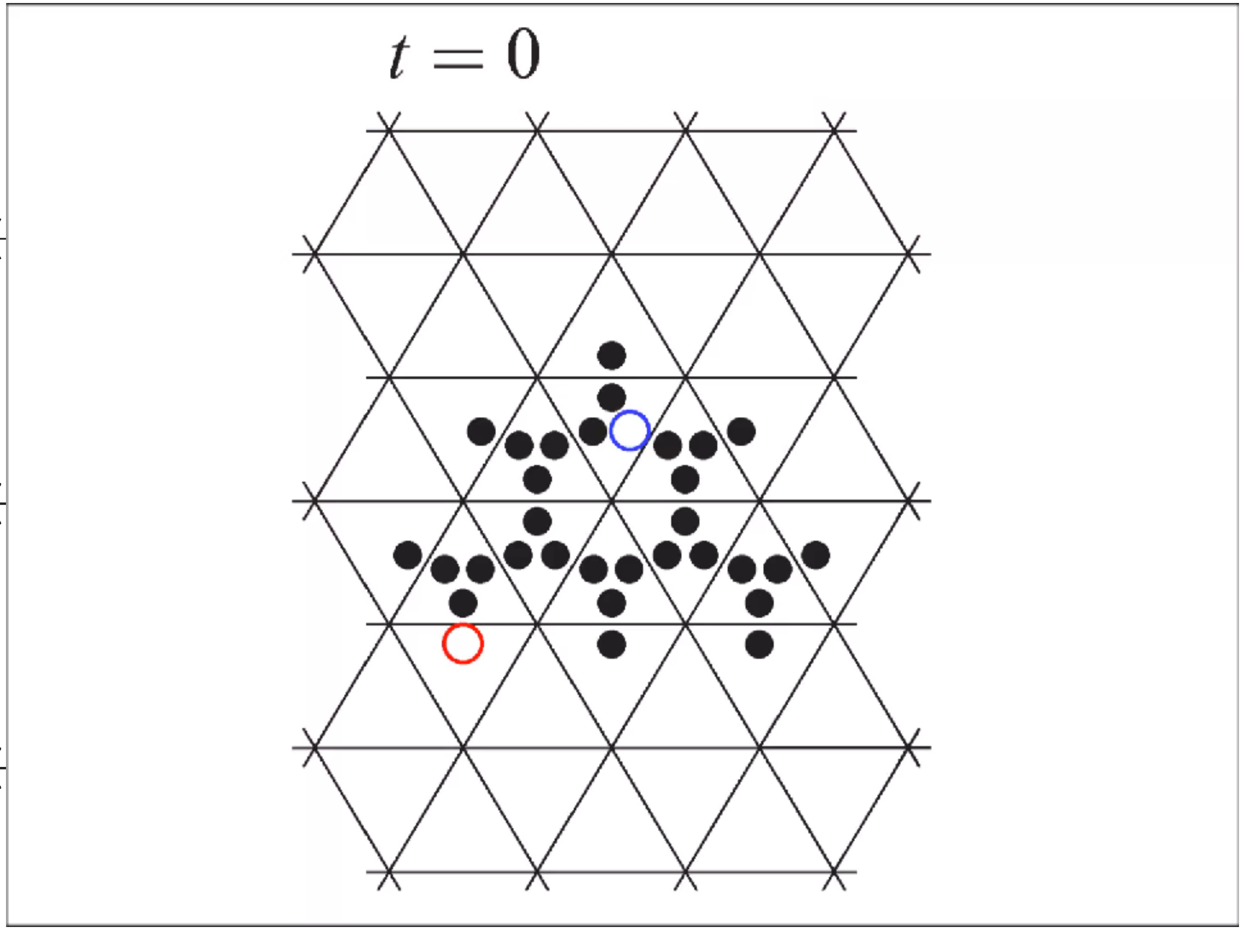
Signal and transmission wire in T_{UR} (Movie)



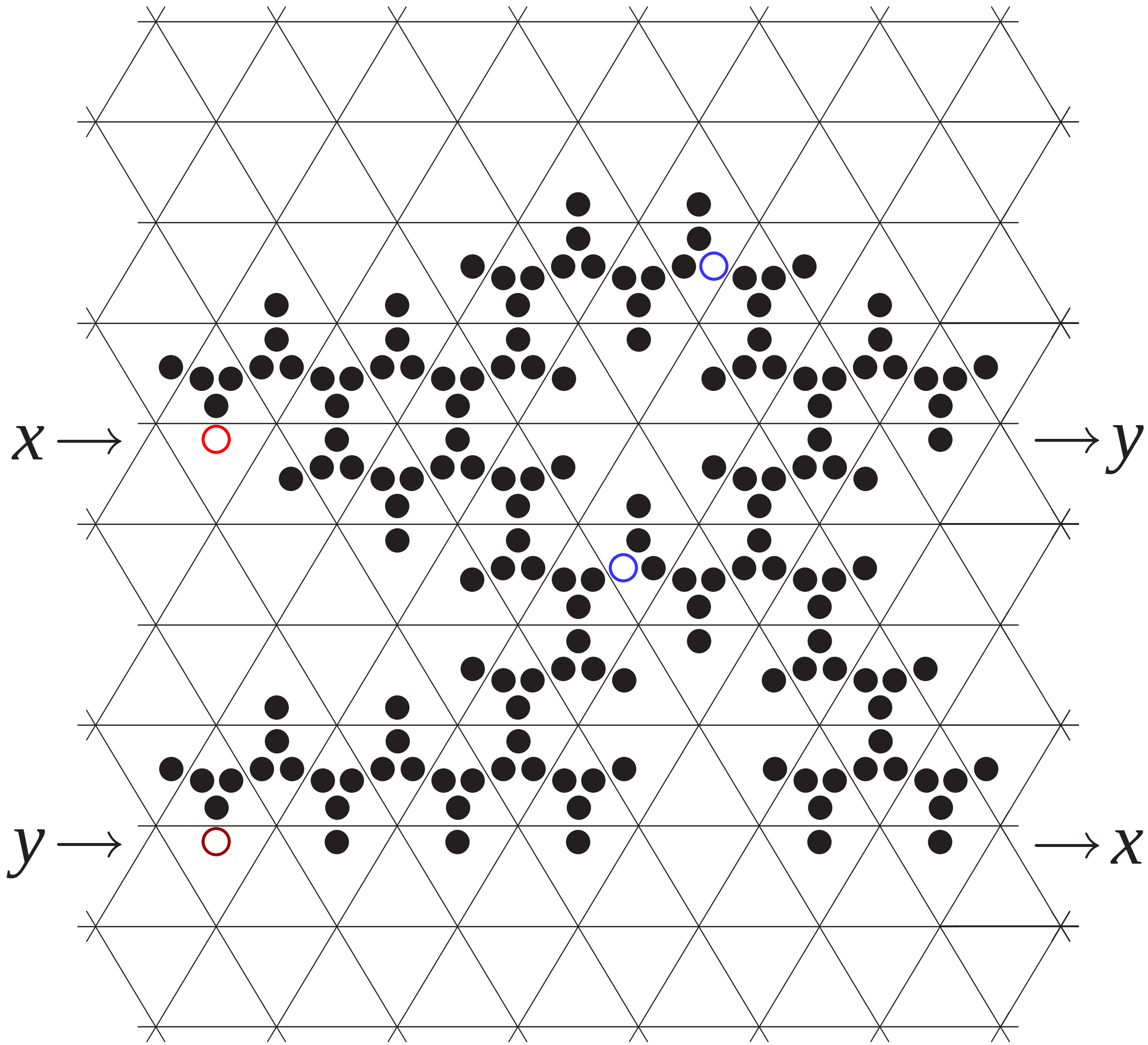
A signal is represented by a colored circle, but it is the same state as a blank.

Delay module of -2 steps in T_{UR}

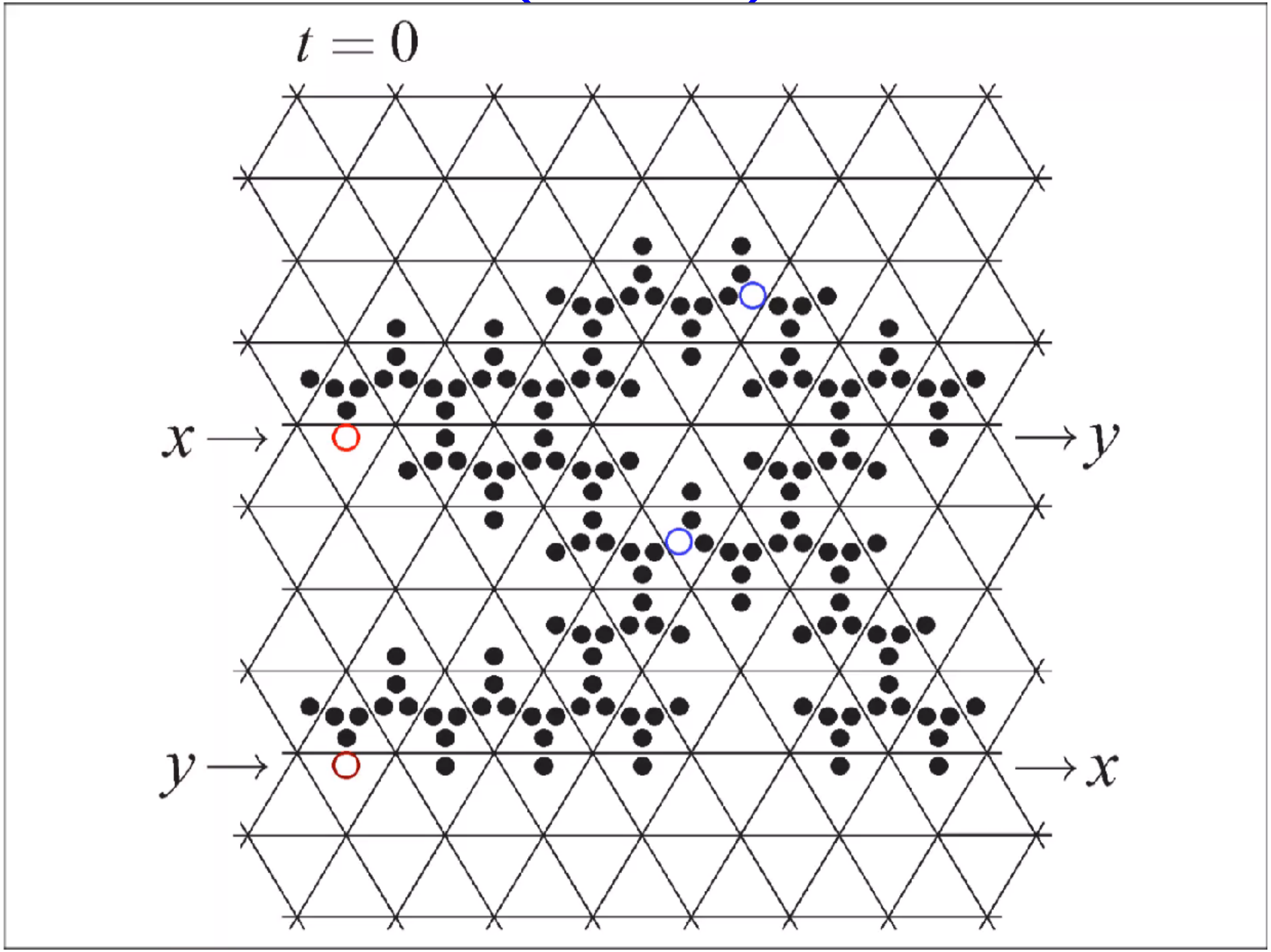




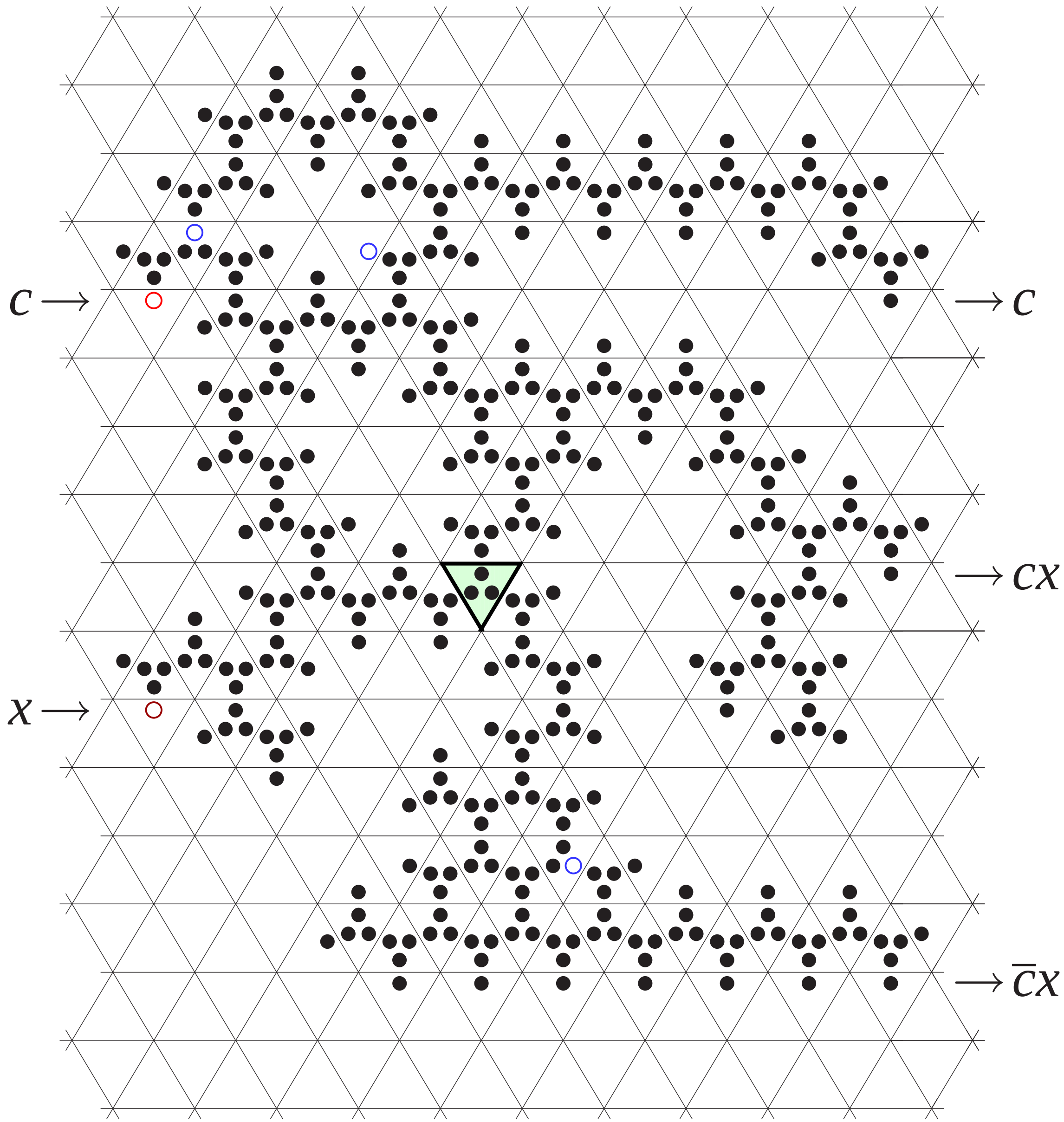
Signal crossing module in T_{UR}



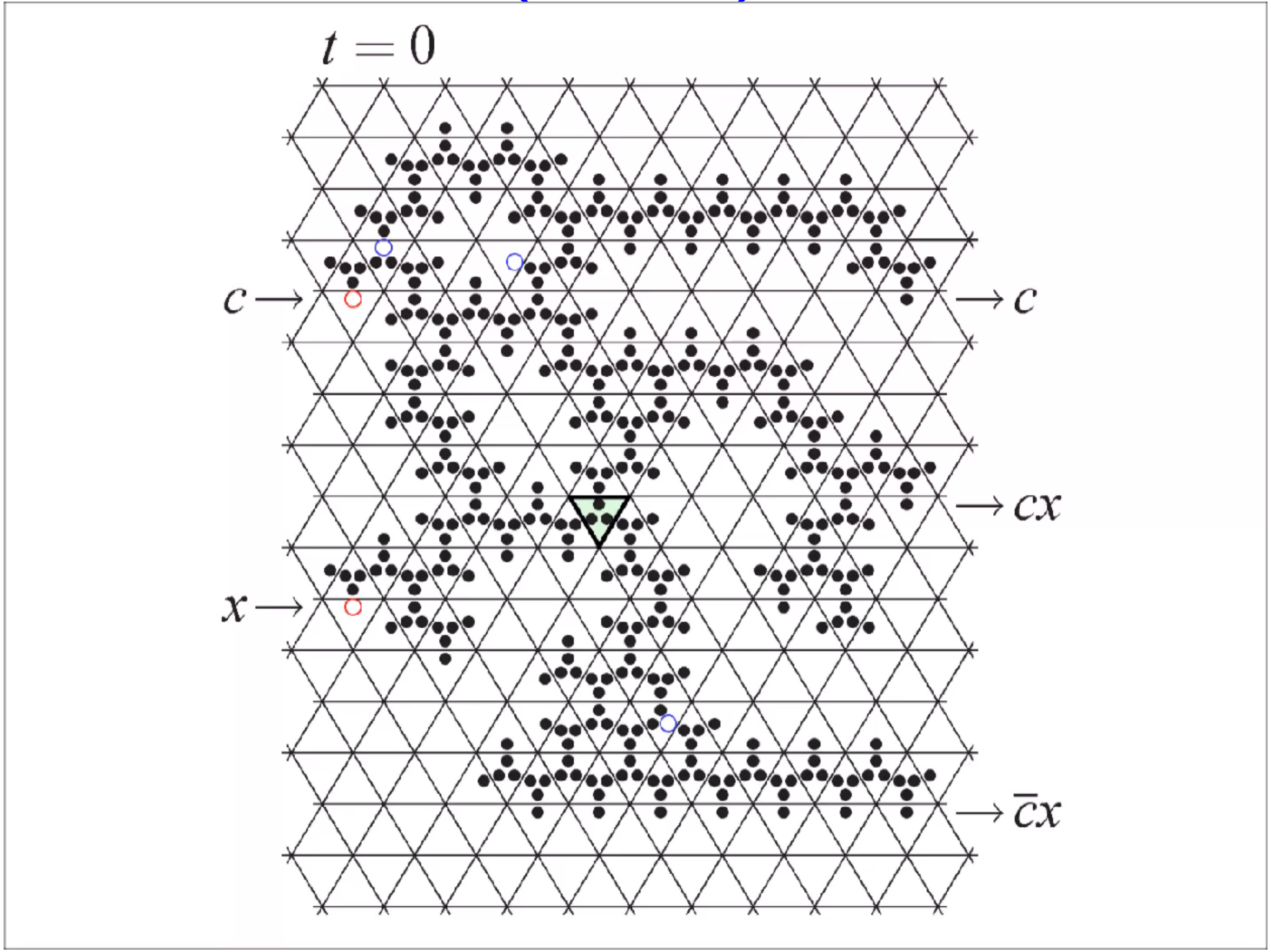
Signal crossing module in T_{UR} (Movie)



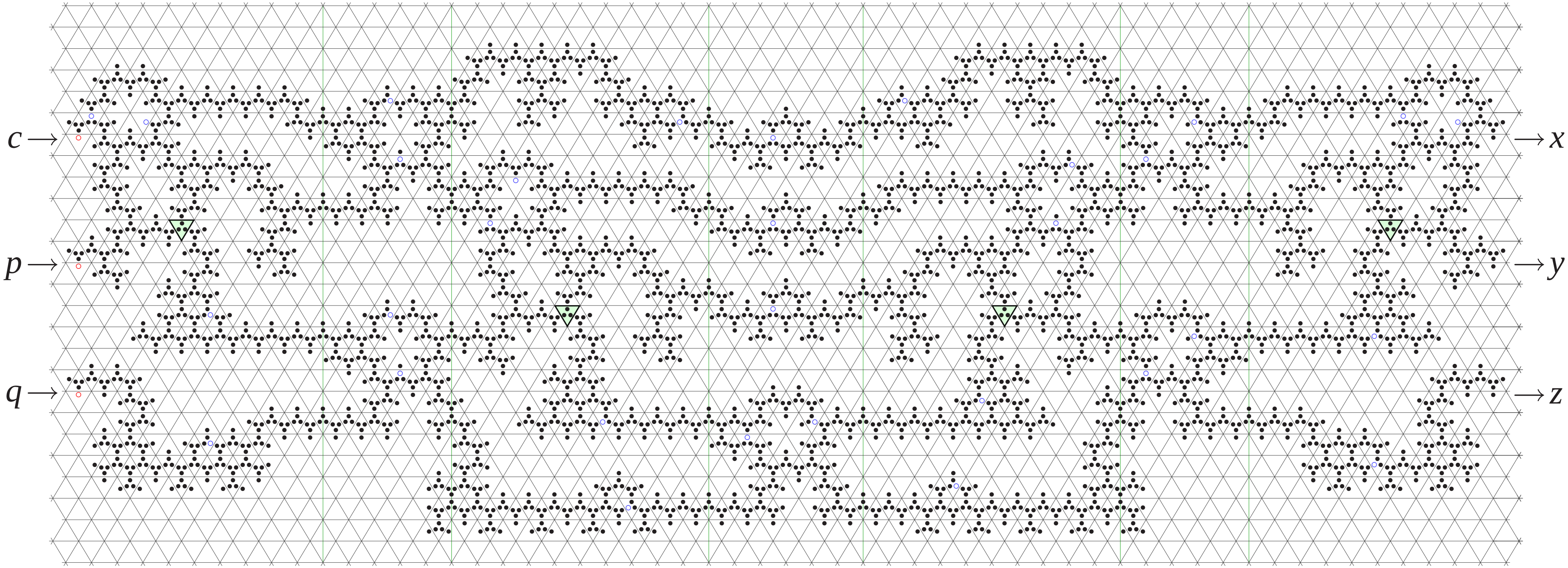
Switch gate module in T_{UR} $(c = 1,\ x = 1)$

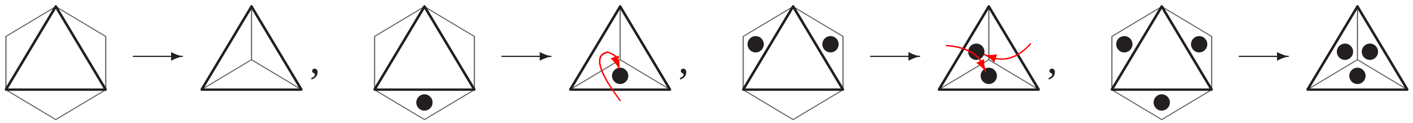


Switch gate module in T_{UR} ($c = 1,\ x = 1$) (Movie)



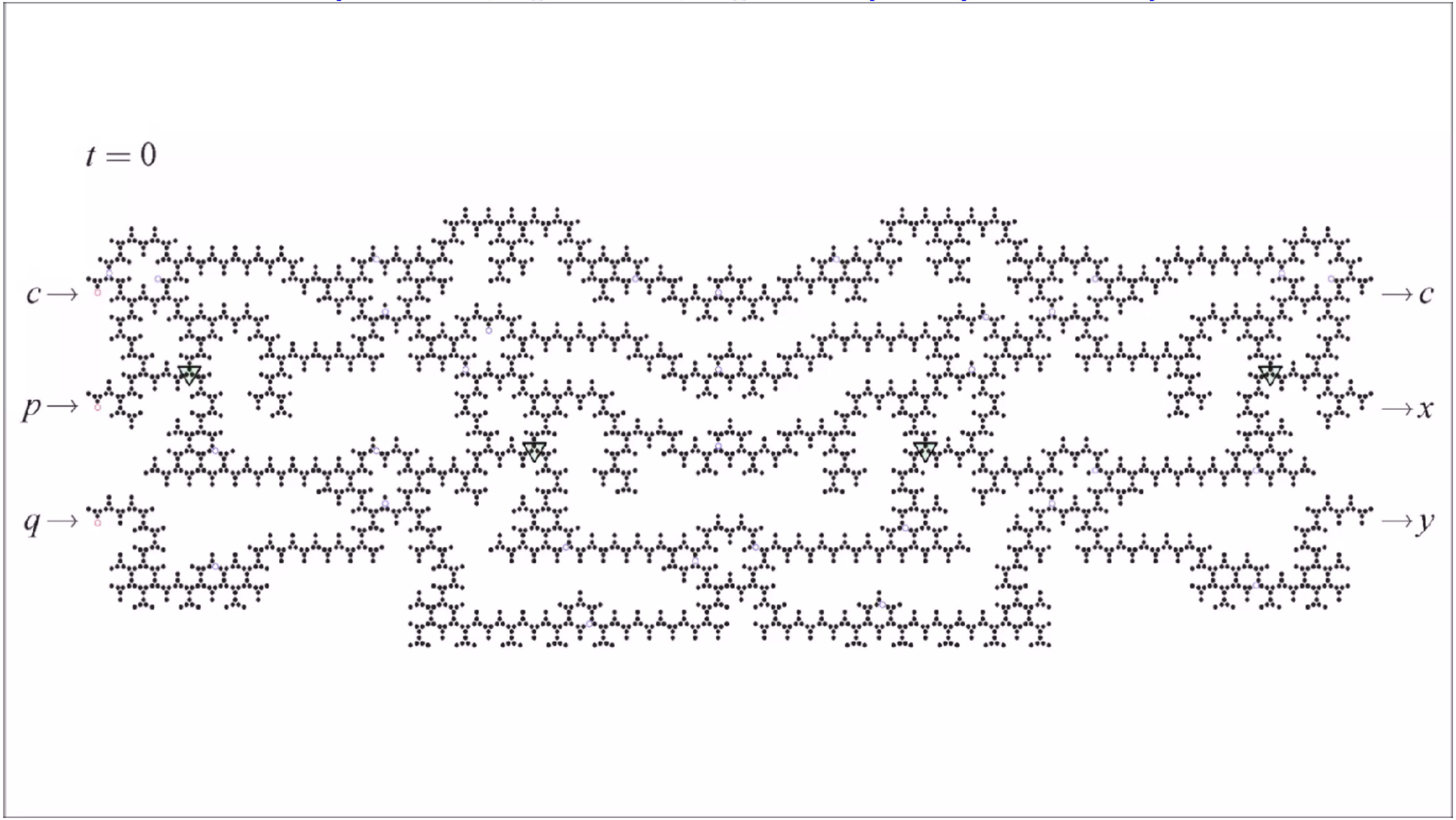
Fredkin gate module in T_{UR}



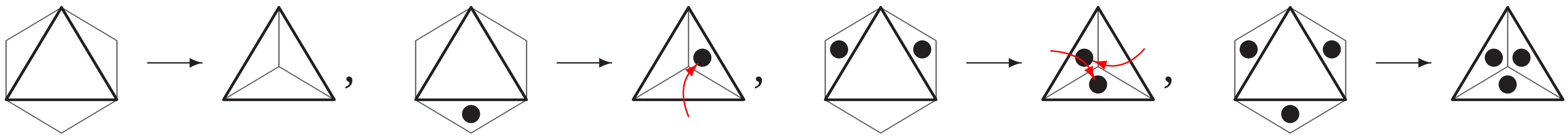


- Configuration size: 24×113
- Delay between input and output: 418 steps

Fredkin gate module in T_{UR}
$(c = 1,\ p = 1,\ q = 1)$ (Movie)

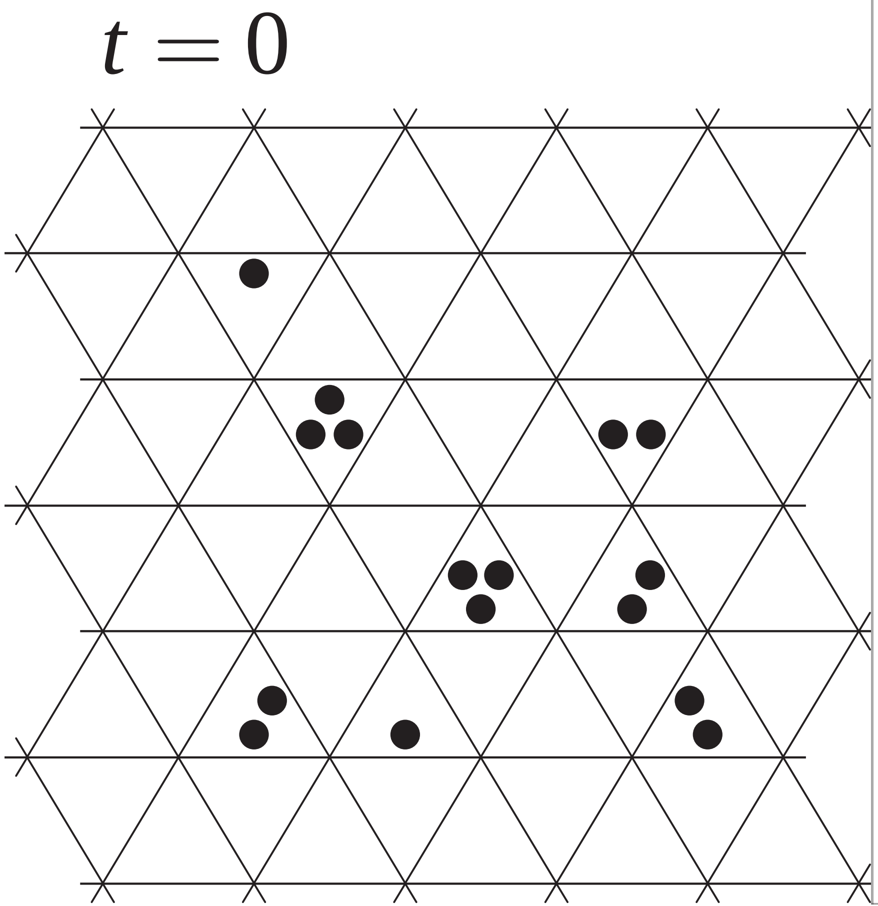


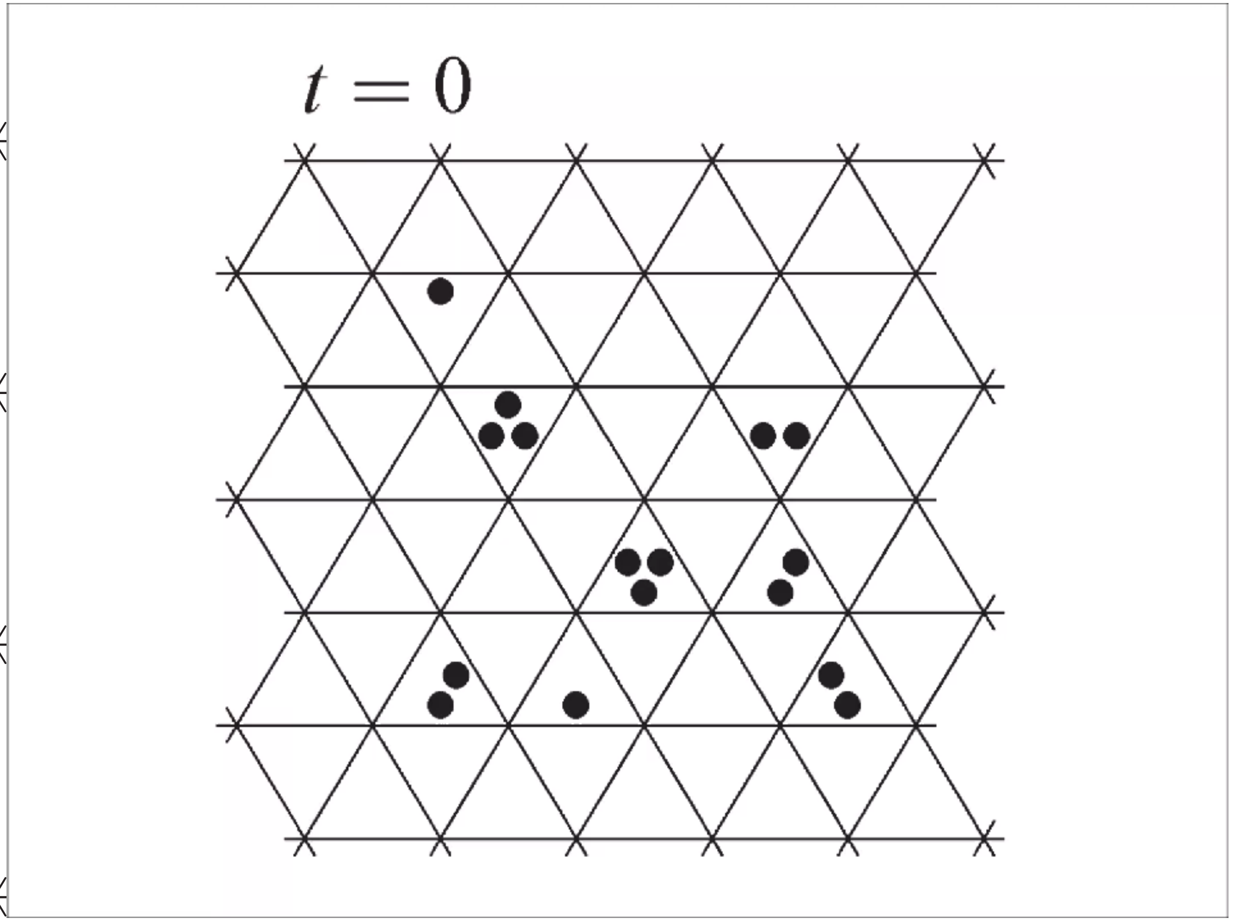
2.4. Non-universality of RETPCA T_{RR}



- We can interpret the above rules to be the ones such that all particles make right-turn.
- Therefore, *every configuration has period 6*, and thus T_{RR} is trivially non-universal.
- Likewise, T_{LL} is also non-universal.

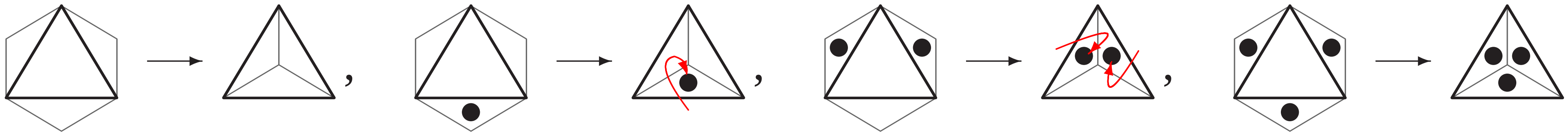
Evolution in RETPCA T_{RR}





Every configuration has period 6

2.5. Non-universality of RETPCA T_{UU}



- We can interpret the above rules to be the ones such that all particles make U-turn.
- Therefore, *every configuration has period 2*, and thus T_{UU} is trivially non-universal.

Results

By the dualities of RETPCAs, we obtain the following theorem.

Theorem 1 *T_{RL}, T_{LR}, $T_{\mathrm{UR}}{}^{\dagger}$, $T_{\mathrm{UL}}{}^{\dagger}$, T_{RU}, and T_{LU} are Turing universal.*

† Universality of T_{UR} and T_{UL} was shown in [Imai, Morita, 2000]

Theorem 2 *T_{RR}, T_{LL}, and T_{UU} are non-universal.*

- By above, universality/non-universality of all 9 kinds of conservative RETPCAs is clarified.

References

1. Bennett, C.H.: Logical reversibility of computation. IBM J. Res. Dev. **17**, 525–532 (1973). DOI 10.1147/rd.176.0525
2. Fredkin, E., Toffoli, T.: Conservative logic. Int. J. Theoret. Phys. **21**, 219–253 (1982). DOI 10.1007/BF01857727
3. Imai, K., Morita, K.: A computation-universal two-dimensional 8-state triangular reversible cellular automaton. Theoret. Comput. Sci. **231**, 181–191 (2000). DOI 10.1016/S0304-3975(99)00099-7
4. Margolus, N.: Physics-like model of computation. Physica D **10**, 81–95 (1984). DOI 10.1016/0167-2789(84)90252-5
5. Morita, K.: A simple construction method of a reversible finite automaton out of Fredkin gates, and its related problem. Trans. IEICE Japan **E-73**, 978–984 (1990)
6. Morita, K.: A simple reversible logic element and cellular automata for reversible computing. In: Proc. MCU 2001 (eds. M. Margenstern, Y. Rogozhin), LNCS 2055, pp. 102–113 (2001). DOI 10.1007/3-540-45132-3_6

7. Morita, K.: Reversible Computing (in Japanese). Kindai Kagaku-sha Co., Ltd., Tokyo, ISBN978-4-7649-0422-4 (2012)
8. Morita, K.: An 8-state simple reversible triangular cellular automaton that exhibits complex behavior. In: Cellular Automata and Discrete Complex Systems (AUTOMATA 2016) (eds. M. Cook, T. Neary), LNCS 9664 (2016). DOI 10.1007/978-3-319-39300-1_14
9. Morita, K.: A reversible elementary triangular partitioned cellular automaton that exhibits complex behavior (slides with simulation movies). Hiroshima University Institutional Repository, http://ir.lib.hiroshima-u.ac.jp/00039321 (2016)
10. Morita, K., Harao, M.: Computation universality of one-dimensional reversible (injective) cellular automata. Trans. IEICE Japan **E72**, 758–762 (1989)
11. Morita, K., Ueno, S.: Computation-universal models of two-dimensional 16-state reversible cellular automata. IEICE Trans. Inf. & Syst. **E75-D**, 141–147 (1992)
12. Wolfram, S.: A New Kind of Science. Wolfram Media Inc. (2002)

Appendices:
List of Attachment Files
— Pdf files of computer simulation results —

- Computer simulation results are given in the pdf files attached in this slide file.
- Open each pdf file, and press the arrow key continuously, then evolving process can be viewed like an animation.

A1. Files of RETPCA T_{RL}

101_T_RL_switch_gate_01.pdf

102_T_RL_switch_gate_11.pdf

103_T_RL_switch_gate_inv_110.pdf

104_T_RL_fredkin_gate_001.pdf

105_T_RL_fredkin_gate_010.pdf

106_T_RL_fredkin_gate_011.pdf

107_T_RL_fredkin_gate_100.pdf

108_T_RL_fredkin_gate_101.pdf

109_T_RL_fredkin_gate_110.pdf

110_T_RL_fredkin_gate_111.pdf

A2. Files of RETPCA T_{RU}

201_T_RU_fredkin_gate_001.pdf

202_T_RU_fredkin_gate_011.pdf

203_T_RU_fredkin_gate_111.pdf

A3. Files of RETPCA T_{UR}

301_T_UR_fredkin_gate_001.pdf

302_T_UR_fredkin_gate_011.pdf

303_T_UR_fredkin_gate_111.pdf